

# POLYGONS IN MINKOWSKI THREE SPACE AND PARABOLIC HIGGS BUNDLES OF RANK TWO ON $\mathbb{C}\mathbb{P}^1$

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**ABSTRACT.** Consider the moduli space of parabolic Higgs bundles  $(E, \Phi)$  of rank two on  $\mathbb{C}\mathbb{P}^1$  such that the underlying holomorphic vector bundle for the parabolic vector bundle  $E$  is trivial. It is equipped with the natural involution defined by  $(E, \Phi) \mapsto (E, -\Phi)$ . We study the fixed point locus of this involution. In [GM], this moduli space with involution was identified with the moduli space of hyperpolygons equipped with a certain natural involution. Here we identify the fixed point locus with the moduli spaces of polygons in Minkowski 3-space. This identification yields information on the connected components of the fixed point locus.

## 1. INTRODUCTION

Parabolic vector bundles over a compact Riemann surface  $\Sigma$  with  $n$  marked points are holomorphic vector bundles over  $\Sigma$  with a weighted flag structure over each of the marked points. They were introduced by Seshadri, [Se], and are of interest for many reasons. For instance, there is a natural bijective correspondence between the isomorphism classes of polystable parabolic bundles of parabolic degree zero and the equivalence classes of unitary representations of the fundamental group of the  $n$ -punctured surface.

Parabolic Higgs bundles are pairs of the form  $(E, \Phi)$ , where  $E$  is a parabolic vector bundle on  $\Sigma$  and  $\Phi$  is a meromorphic  $End(E)$ -valued 1-form holomorphic outside the  $n$  marked points such that  $\Phi$  has at most a simple pole with nilpotent residue (with respect to the flag) at each of the marked points. There is a natural relationship between the polystable parabolic Higgs bundles of parabolic degree zero and the representations of the fundamental group of the  $n$ -punctured surface in the general linear groups [Si]. Parabolic Higgs bundles have been studied in other works such as [BY, Na1, Ko1, GM].

We will be particularly interested in the case of parabolic Higgs bundles of rank two over a  $n$ -pointed Riemann surface of genus zero.

Consider the split real form  $PGL(2, \mathbb{R})$  of  $PGL(2, \mathbb{C})$  defined by the involution  $A \mapsto \overline{A}$ . It produces the anti-holomorphic involution on the moduli space of representations corresponding to the holomorphic involution

$$(1) \quad \sigma : (E, \Phi) \mapsto (E, -\Phi)$$

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2000 *Mathematics Subject Classification.* 14D20, 14H60, 53C26, 53D20.

*Key words and phrases.* Polygons, Minkowski space, parabolic bundle, Higgs field, hyperpolygons.

Partially supported by FCT projects PTDC/MAT/108921/2008, PTDC/MAT/099275/2008, PTDC/MAT/120411/2010 and FCT grant SFRH/BPD/44041/2008.

of the moduli space of parabolic Higgs bundles [Hi1]. Note that  $\sigma$  is the restriction to  $-1$  of the  $U(1) = S^1$ -action on the moduli space of parabolic Higgs bundles defined by

$$\lambda \cdot (E, \Phi) = (E, \lambda\Phi), \quad \lambda \in S^1.$$

The isomorphism classes of stable parabolic Higgs bundles fixed by this involution correspond to  $SU(2)$  or  $SL(2, \mathbb{R})$  representations, the former corresponding to parabolic Higgs bundles with zero Higgs field; see [Hi1].

We study the fixed points in the special case where the underlying vector bundle is holomorphically trivial. Let  $\mathcal{H}(\beta)$  be the moduli space of parabolic Higgs bundles  $(E, \Phi)$ , where  $E$  is a holomorphically trivial vector bundle over  $\mathbb{C}\mathbb{P}^1$  of rank two with a weighted complete flag structure over each of the  $n$  marked points  $x_1, \dots, x_n$

$$\begin{aligned} E_{x_i,1} \supsetneq E_{x_i,2} \supsetneq 0, \\ 0 \leq \beta_1(x_i) < \beta_2(x_i) < 1. \end{aligned}$$

As shown in [GM], there is an isomorphism between  $\mathcal{H}(\beta)$  and the hyperpolygon space  $X(\alpha)$ , with  $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$ , defined as a hyper-Kähler quotient of  $T^*\mathbb{C}^{2n}$  by

$$K := (U(2) \times U(1)^n)/U(1) = (SU(2) \times U(1)^n)/(\mathbb{Z}/2\mathbb{Z}),$$

where  $\mathbb{Z}/2\mathbb{Z}$  acts by multiplication of each factor by  $-1$ . (See also Sections 2 and 3 for details.)

Using this correspondence between the two moduli spaces, we study in Section 4 the fixed point set of the corresponding involution of  $X(\alpha)$  defined by

$$(2) \quad \sigma : [p, q] \mapsto [-p, q],$$

with  $(p, q) \in T^*\mathbb{C}^{2n}$ . We show that this fixed-point set is formed by  $M(\alpha)$ , the space of polygons in  $\mathbb{R}^3$  obtained when  $p = 0$ , and several other connected components  $Z_S$ , where  $S$  runs over all subsets of  $\{1, \dots, n\}$  with  $|S| \geq 2$  and

$$(3) \quad \sum_{i \in S} \alpha_i < \sum_{i \in S^c} \alpha_i$$

(the complement of  $S$  is denoted by  $S^c$ ). These components  $Z_S$  are all non-compact except when  $|S| = n - 1$ , in which case  $Z_S = \mathbb{C}\mathbb{P}^{n-2}$  and  $M(\alpha)$  is empty. Let  $\mathcal{S}'(\alpha)$  be the collection of all subsets of  $\{1, \dots, n\}$  with  $|S| \geq 2$  satisfying (3).

We describe these sets  $Z_S$  and the corresponding components  $\mathcal{Z}_S$  of the fixed point set of the involution of  $\mathcal{H}(\beta)$  defined in (1); the following theorem is proved (see Section 4).

**Theorem 1.1.** *The fixed-point set of the involution in (1) of the space of parabolic Higgs bundles  $\mathcal{H}(\beta)$  is*

$$\mathcal{H}(\beta)^{\mathbb{Z}/2\mathbb{Z}} = \mathcal{M}_{\beta,2,0} \cup \bigcup_{S \in \mathcal{S}'(\alpha)} \mathcal{Z}_S,$$

with  $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$ , where  $\mathcal{M}_{\beta,2,0}$  is the space of rank two degree zero parabolic vector bundles over  $\mathbb{C}\mathbb{P}^1$ , and where  $\mathcal{Z}_S \subset \mathcal{H}(\beta)$  is formed by parabolic Higgs bundles  $\mathbf{E} = (E, \Phi) \in \mathcal{H}(\beta)$  such that

- (i) the parabolic vector bundle  $E$  admits a direct sum decomposition  $E = L_0 \oplus L_1$ , where  $L_0$  and  $L_1$  are parabolic line bundles where the parabolic weight of  $L_0$  (respectively,  $L_1$ ) at  $x_i \in S^c$  is  $\beta_2(x_i)$  (respectively,  $\beta_1(x_i)$ ), and the parabolic weight of  $L_0$  (respectively,  $L_1$ ) at  $x_i \in S$  is  $\beta_1(x_i)$  (respectively,  $\beta_2(x_i)$ );
- (ii) the residues of the Higgs field  $\Phi$  at the parabolic points  $x_i$  are either upper or lower triangular with respect to the above decomposition, according to whether  $i$  is in  $S$  or in  $S^c$ .

Moreover,  $\mathcal{Z}_S$  is a non-compact manifold of dimension  $2(n-3)$  except when  $|S| = n-1$ , in which case  $\mathcal{Z}_S = \mathcal{M}_S$  is compact and diffeomorphic to  $\mathbb{C}\mathbb{P}^{n-3}$ . In all cases,  $\mathcal{H}(\beta)^{\mathbb{Z}/2\mathbb{Z}}$  has  $2^{(n-1)} - (n+1)$  non-compact components and one compact component.

**Remark 1.1.**

- Since the vector bundle underlying  $E$  is holomorphically trivial, it follows that the holomorphic line bundles underlying  $L_0$  and  $L_1$  are both holomorphically trivial.
- Statement (i) in Theorem 1.1 means that if

$$E_{x_i,1} \supset E_{x_i,2} \supset 0$$

$$0 \leq \beta_1(x_i) < \beta_2(x_i) < 1$$

is the parabolic structure, then  $E_{x_i,2} = E_{x_j,2}$  whenever  $i, j \in S$  or  $i, j \in S^c$ . Note that this condition is independent of the choice of the trivialization of  $E$ .

In Section 6, we show that for any  $S \in \mathcal{S}'(\alpha)$ , the corresponding component of the fixed point sets of the involution of  $X(\alpha)$  (or of  $\mathcal{H}(\beta)$ ) is diffeomorphic to a moduli space of polygons in Minkowski 3-space, meaning  $\mathbb{R}^3$  equipped with the Minkowski inner product

$$v \circ w = -x_1x_2 - y_1y_2 + t_1t_2,$$

for  $v = (x_1, y_1, t_1)$  and  $w = (x_2, y_2, t_2)$ . The surface  $S_R$  in  $\mathbb{R}^3$  defined by the equation  $-x^2 - y^2 + t^2 = R^2$  (a *pseudosphere* of radius  $R$ ) has two connected components:  $S_R^+$ , corresponding to  $t > 0$ , which is called a *future pseudosphere*, and  $S_R^-$ , corresponding to  $t < 0$ , which is called a *past pseudosphere*. The group  $SU(1, 1)$  acts transitively on  $S_R^+$  (respectively,  $S_R^-$ ) since one can think of  $\mathbb{R}^3$  as  $\mathfrak{su}(1, 1)^*$  with  $S_R^+$  (respectively,  $S_R^-$ ) being an elliptic coadjoint orbit. Consequently, both  $S_R^+$  and  $S_R^-$  have the  $SU(1, 1)$ -invariant Kostant–Kirillov symplectic structure of a coadjoint orbit. Fixing two positive integers  $k_1, k_2$  with  $k_1 + k_2 = n$ , we consider closed polygons in Minkowski 3-space with the first  $k_1$  sides lying in future pseudospheres of radii  $\alpha_1, \dots, \alpha_{k_1}$  and the last  $k_2$  sides lying in past pseudospheres of radii  $\alpha_{k_1+1}, \dots, \alpha_n$ . The space of all such closed polygons can be identified with the zero level set of the moment map

$$\mu : \mathcal{O}_1 \times \dots \times \mathcal{O}_n \longrightarrow \mathfrak{su}(1, 1)^*$$

for the diagonal  $SU(1, 1)$ -action, where the coadjoint  $SU(1, 1)$ -orbit  $\mathcal{O}_i \cong S_{\alpha_i}^+$ ,  $1 \leq i \leq k_1$ , is a future pseudosphere of radius  $\alpha_i$ , and  $\mathcal{O}_i \cong S_{\alpha_i}^-$ ,  $k_1+1 \leq i \leq n$ , is a past pseudosphere of radius  $\alpha_i$ , equipped with its Kostant–Kirillov symplectic structure [Fo]. Then the corresponding moduli space of polygons is defined as the symplectic quotient

$$M^{k_1, k_2}(\alpha) := \mu^{-1}(0)/SU(1, 1).$$

We have the following result.

**Theorem 1.2.** *For any  $S \in \mathcal{S}'(\alpha)$ , the components  $\mathcal{Z}_S$  and  $Z_S$ , of the fixed-point sets of the involutions in (1) and (2) respectively, are diffeomorphic to the moduli space*

$$M^{|\mathcal{S}|, |\mathcal{S}^c|}(\alpha)$$

*of closed polygons in Minkowski 3-space.*

This interpretation allows us to see the fixed-point set of the above involutions as a moduli space of another related problem, thus helping us to understand many of its geometrical properties as seen in the example of Section 7.

**Acknowledgements.** We thank O. García-Prada for suggesting the study of the hyper-polygon description of the fixed-point set of the natural involution of the moduli space of parabolic Higgs bundles.

## 2. HYPERPOLYGON SPACES

Let  $\mathcal{Q}$  be the star-shaped quiver with vertices parametrized by  $I \cup \{0\} = \{1, \dots, n\} \cup \{0\}$  and the arrows parametrized by  $I$  such that, for any  $i \in I$ , the tail and the head of the corresponding arrow are  $i$  and 0 respectively. Consider all representations of  $\mathcal{Q}$  with  $V_i = \mathbb{C}$ , for  $i \in I$ , and  $V_0 = \mathbb{C}^2$ . They are parametrized by

$$E(\mathcal{Q}, V) := \bigoplus_{i \in I} \text{Hom}(V_i, V_0) = \mathbb{C}^{2n}.$$

Using the actions of  $U(1)$  and  $U(2)$  on  $\mathbb{C}$  and  $\mathbb{C}^2$  respectively, we construct an action of  $U(2) \times U(1)^n$  on  $E(\mathcal{Q}, V)$ . This action produces an action of  $U(2) \times U(1)^n$  on the cotangent bundle  $T^*E(\mathcal{Q}, V) = T^*\mathbb{C}^{2n}$ . One gets a hyper-Kähler quiver variety by performing the hyper-Kähler reduction on  $T^*E(\mathcal{Q}, V)$  for this action of  $U(2) \times U(1)^n$ . Since the diagonal circle

$$\{(c \cdot \text{Id}_{\mathbb{C}^2}, c, \dots, c) \mid |c| = 1\} \subset U(2) \times U(1)^n$$

acts trivially on  $T^*E(\mathcal{Q}, V)$ , the action factors through the quotient group

$$K := \left( U(2) \times U(1)^n \right) / U(1) = \left( \text{SU}(2) \times U(1)^n \right) / (\mathbb{Z}/2\mathbb{Z}),$$

where  $\mathbb{Z}/2\mathbb{Z}$  acts as multiplication by  $-1$  on each factor. As  $T^*\mathbb{C}^2 = (\mathbb{C}^2)^* \times \mathbb{C}^2$  can be identified with the space of quaternions, the cotangent bundle  $T^*E(\mathcal{Q}, V) = T^*\mathbb{C}^{2n}$  has a natural hyper-Kähler structure (see for example [Ko2, Hi2]). The hyper-Kähler quotient of  $T^*\mathbb{C}^{2n}$  by  $K$  can be explicitly described as follows. Let  $(p, q)$  be coordinates on  $T^*\mathbb{C}^{2n}$ , where  $p = (p_1, \dots, p_n)$  is the  $n$ -tuple of row vectors  $p_i = \begin{pmatrix} a_i & b_i \end{pmatrix} \in (\mathbb{C}^2)^*$  and  $q = (q_1, \dots, q_n)$  is the  $n$ -tuple of column vectors  $q_i = \begin{pmatrix} c_i \\ d_i \end{pmatrix} \in \mathbb{C}^2$ . In terms of these coordinates, the action of  $K$  on  $T^*\mathbb{C}^{2n}$  is given by

$$(p, q) \cdot [A; e_1, \dots, e_n] = \left( (e_1^{-1} p_1 A, \dots, e_n^{-1} p_n A), (A^{-1} q_1 e_1, \dots, A^{-1} q_n e_n) \right).$$

This action is hyper-Hamiltonian with hyper-Kähler moment map

$$\mu_{HK} := \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : T^*\mathbb{C}^{2n} \longrightarrow (\mathfrak{su}(2)^* \oplus (\mathbb{R}^n)^*) \oplus (\mathfrak{sl}(2, \mathbb{C})^* \oplus (\mathbb{C}^n)^*),$$

[Ko2], where the real moment map  $\mu_{\mathbb{R}}$  is given by

$$(4) \quad \mu_{\mathbb{R}}(p, q) = \frac{\sqrt{-1}}{2} \sum_{i=1}^n (q_i q_i^* - p_i^* p_i)_0 \oplus \left( \frac{1}{2}(|q_1|^2 - |p_1|^2), \dots, \frac{1}{2}(|q_n|^2 - |p_n|^2) \right),$$

and the complex moment map  $\mu_{\mathbb{C}}$  is given by

$$(5) \quad \mu_{\mathbb{C}}(p, q) = - \sum_{i=1}^n (q_i p_i)_0 \oplus (\sqrt{-1} p_1 q_1, \dots, \sqrt{-1} p_n q_n).$$

The hyperpolygon space  $X(\alpha)$  is then defined to be the hyper-Kähler quotient

$$(6) \quad X(\alpha) := T^*\mathbb{C}^{2n} \mathbin{///}_{\alpha} K := \left( \mu_{\mathbb{R}}^{-1}(0, \alpha) \cap \mu_{\mathbb{C}}^{-1}(0, 0) \right) / K$$

for  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$ .

An element  $(p, q) \in T^*\mathbb{C}^{2n}$  is in  $\mu_{\mathbb{C}}^{-1}(0, 0)$  if and only if

$$p_i q_i = 0 \quad \text{and} \quad \sum_{i=1}^n (q_i p_i)_0 = 0.$$

In other words, an element  $(p, q)$  of  $T^*\mathbb{C}^{2n}$  is in  $\mu_{\mathbb{C}}^{-1}(0, 0)$  if and only if

$$(7) \quad a_i c_i + b_i d_i = 0$$

and

$$(8) \quad \sum_{i=1}^n a_i c_i - b_i d_i = 0, \quad \sum_{i=1}^n a_i d_i = 0, \quad \sum_{i=1}^n b_i c_i = 0.$$

Similarly,  $(p, q)$  is in  $\mu_{\mathbb{R}}^{-1}(0, \alpha)$  if and only if

$$\frac{1}{2}(|q_i|^2 - |p_i|^2) = \alpha_i \quad \text{and} \quad \sum_{i=1}^n (q_i q_i^* - p_i^* p_i)_0 = 0,$$

i.e., if and only if

$$(9) \quad |c_i|^2 + |d_i|^2 - |a_i|^2 - |b_i|^2 = 2\alpha_i$$

and

$$(10) \quad \sum_{i=1}^n |c_i|^2 - |a_i|^2 + |b_i|^2 - |d_i|^2 = 0, \quad \sum_{i=1}^n a_i \bar{b}_i - \bar{c}_i d_i = 0.$$

An element  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$  is said to be *generic* if and only if

$$(11) \quad \varepsilon_S(\alpha) := \sum_{i \in S} \alpha_i - \sum_{i \in S^c} \alpha_i \neq 0$$

for every subset  $S \subset \{1, \dots, n\}$ . For a generic  $\alpha$ , the hyperpolygon space  $X(\alpha)$  is a non-empty complex manifold of complex dimension  $2(n-3)$  (see [Ko2] for details).

Hyperpolygon spaces can be described from an algebro-geometric point of view as geometric invariant theoretic quotients. To elaborate this, we need the stability criterion, developed by Nakajima [Na2, Na3] for quiver varieties and adapted by Konno [Ko2] to hyperpolygon spaces. We will recall this below.

Let  $\alpha$  be generic. A subset  $S \subset \{1, \dots, n\}$  is called *short* if

$$(12) \quad \varepsilon_S(\alpha) < 0$$

and *long* otherwise (see (11) for the definition of  $\varepsilon_S(\alpha)$ ). Given  $(p, q) \in T^*\mathbb{C}^{2n}$  and a subset  $S \subset \{1, \dots, n\}$ , we say that  $S$  is *straight* at  $(p, q)$  if  $q_i$  is proportional to  $q_j$  for all  $i, j \in S$ .

**Theorem 2.1** ([Ko2]). *Let  $\alpha \in \mathbb{R}_+^n$  be generic. A point  $(p, q) \in T^*\mathbb{C}^{2n}$  is  $\alpha$ -stable if and only if the following two conditions hold:*

- (i)  $q_i \neq 0$  for all  $i$ , and
- (ii) if  $S \subset \{1, \dots, n\}$  is straight at  $(p, q)$  and  $p_j = 0$  for all  $j \in S^c$ , then  $S$  is short.

**Remark 2.1.** Note that it is enough to verify (ii) in Theorem 2.1 for all maximal straight sets, that is, for those that are not contained in any other straight set at  $(p, q)$ .

Let  $\mu_{\mathbb{C}}^{-1}(0, 0)^{\alpha\text{-st}}$  denote the set of points in  $\mu_{\mathbb{C}}^{-1}(0, 0)$  that are  $\alpha$ -stable, and let

$$K^{\mathbb{C}} := (\mathrm{SL}(2, \mathbb{C}) \times (\mathbb{C}^*)^n) / (\mathbb{Z}/2\mathbb{Z})$$

be the complexification of  $K$ .

**Proposition 2.2** ([Ko2]). *Let  $\alpha \in \mathbb{R}_+^n$  be generic. Then*

$$\mu_{HK}^{-1}((0, \alpha), (0, 0)) \subset \mu_{\mathbb{C}}^{-1}(0, 0)^{\alpha\text{-st}},$$

and there exists a natural bijection

$$\iota : \mu_{HK}^{-1}((0, \alpha), (0, 0)) / K \longrightarrow \mu_{\mathbb{C}}^{-1}(0, 0)^{\alpha\text{-st}} / K^{\mathbb{C}}.$$

From Proposition 2.2 and the definition in (6) it follows that

$$X(\alpha) = \mu_{\mathbb{C}}^{-1}(0, 0)^{\alpha\text{-st}} / K^{\mathbb{C}}.$$

Following [HP], we denote the elements in  $\mu_{\mathbb{C}}^{-1}(0, 0)^{\alpha\text{-st}} / K^{\mathbb{C}}$  by  $[p, q]_{\alpha\text{-st}}$ , and denote by  $[p, q]_{\mathbb{R}}$  the elements in  $\mu_{HK}^{-1}((0, \alpha), (0, 0)) / K$ , when we need to make an explicit use of one of the two constructions. In all other cases, we will simply write  $[p, q]$  for a hyperpolygon in  $X(\alpha)$ .

**2.1. A circle action.** Consider the  $S^1$ -action on  $X(\alpha)$  defined by

$$(13) \quad \lambda \cdot [p, q] = [\lambda p, q].$$

This action is Hamiltonian with respect to symplectic structure on  $X(\alpha)$ ; the associated moment map  $\phi : X(\alpha) \longrightarrow \mathbb{R}$  is given by

$$(14) \quad \phi([p, q]) = \frac{1}{2} \sum_{i=1}^n |p_i|^2.$$

This  $\phi$  is a Morse-Bott function that is proper and bounded from below. Following Konno [Ko2], let us consider  $\mathcal{S}(\alpha)$ , namely the collection of short sets for  $\alpha$ , and its subset

$$\mathcal{S}'(\alpha) := \{S \subset \{1, \dots, n\} \mid S \text{ is } \alpha\text{-short, } |S| \geq 2\}.$$

For any  $S \in \mathcal{S}'(\alpha)$ , define

$$X_S := \{[p, q] \in X(\alpha) \mid S \text{ and } S^c \text{ are straight at } (p, q) \text{ and } p_j = 0 \forall j \in S^c\}.$$

Then the fixed-point set of the circle action on  $X(\alpha)$  is the following.

**Theorem 2.3** ([Ko2]). *The fixed point set for the  $S^1$ -action in (13) is*

$$X(\alpha)^{S^1} = M(\alpha) \cup \bigcup_{S \in \mathcal{S}'(\alpha)} X_S.$$

*The fixed-point set component  $X_S$  is diffeomorphic to  $\mathbb{C}\mathbb{P}^{|S|-2}$ , and it has index  $2(n-1-|S|)$ .*

Let us now determine the isotropy weights of the circle action in (13) at different fixed points.

For  $S \in \mathcal{S}'(\alpha)$ , let us fix  $[p', q']_{\alpha\text{-st}} \in X_S$ . We may assume that for each  $i \in S$ ,

$$q'_i = \begin{pmatrix} c_i \\ 0 \end{pmatrix} \text{ and } p'_i = \begin{pmatrix} 0 & b_i \end{pmatrix}, \text{ for } i \in S$$

and for each  $i \in S^c$ ,

$$q'_i = \begin{pmatrix} 0 \\ d_i \end{pmatrix} \text{ and } p'_i = \begin{pmatrix} 0 & 0 \end{pmatrix}, \text{ for } i \in S^c.$$

Moreover, we can assume that  $S = \{1, \dots, l\}$  and that  $b_1, b_2 \neq 0$ . Since  $c_i, d_i \neq 0$  for all  $i$ , there exists a unique element  $h \in K^{\mathbb{C}}$  such that  $(p', q')h = (p^0, q^0) \in \mu_{\mathbb{C}}^{-1}(0, 0)^{\alpha\text{-st}}$ , where for each  $i \in S$ ,

$$q_i^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } p_i^0 = \begin{pmatrix} 0 & b_i^0 \end{pmatrix},$$

and for  $i \in S^c$ ,

$$q_i^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } p_i^0 = \begin{pmatrix} 0 & 0 \end{pmatrix},$$

with  $b_1^0 = 1$  and  $b_2^0 \neq 0$ . There exists an open neighborhood  $U$  of  $(p^0, q^0)$  in  $T^*\mathbb{C}^{2n}$  such that for all  $(p, q) \in U \cap \mu_{\mathbb{C}}^{-1}(0, 0)$ , there is a unique element  $[A; e_1, \dots, e_n] \in K^{\mathbb{C}}$  satisfying the conditions that

$$A^{-1}q_i e_i = \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & r_1 \end{pmatrix}^t, & \text{if } i = 1 \\ \begin{pmatrix} 1 & r_1 \\ 1 & w_i \end{pmatrix}^t, & \text{if } i = 2 \\ \begin{pmatrix} 1 & w_i \\ w_i & 1 \end{pmatrix}^t, & \text{if } i = 3, \dots, l \\ \begin{pmatrix} w_i & 1 \\ 0 & 1 \end{pmatrix}^t, & \text{if } i = l+1, \dots, n-1 \\ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}^t, & \text{if } i = n \end{cases}$$

and

$$e_i^{-1}p_i A = \begin{cases} \begin{pmatrix} 0 & 1 \\ -r_1 r_2 & r_2 \end{pmatrix}, & \text{if } i = 1 \\ \begin{pmatrix} -z_i w_i & z_i \\ z_i & -z_i w_i \end{pmatrix}, & \text{if } i = 3, \dots, l \\ \begin{pmatrix} z_i & -z_i w_i \\ r_3 & 0 \end{pmatrix}, & \text{if } i = l+1, \dots, n-1 \\ \begin{pmatrix} 0 & 1 \\ r_3 & 0 \end{pmatrix}, & \text{if } i = n, \end{cases}$$

where  $r_1, r_2, r_3$  are uniquely determined by

$$(15) \quad \{z_i, w_i \mid i = 3, \dots, n-1\};$$

so the functions in (15) define a local coordinate system in  $X(\alpha)$  around  $[p', q']_{\alpha\text{-st}}$ . Indeed,  $\alpha$ -stability is an open condition so that any  $(p, q) \in \mu_{\mathbb{C}}^{-1}(0, 0)^{\alpha\text{-st}}$  sufficiently close to  $(p', q')$  will be  $\alpha$ -stable. Moreover, there exist unique, up to multiplication by  $\pm I$ ,

$$e_1, e_n \in \mathbb{C} \setminus \{0\} \text{ and } A \in \text{SL}(2, \mathbb{C})$$

such that

$$A^{-1}q_1e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A^{-1}q_n e_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$e_1^{-1}p_1A = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Then one can uniquely determine  $e_2, \dots, e_{n-1}$  such that

$$A^{-1}q_i e_i = \begin{pmatrix} 1 \\ * \end{pmatrix}, \text{ for } i = 2, \dots, l, \quad A^{-1}q_i e_i = \begin{pmatrix} * \\ 1 \end{pmatrix}, \text{ for } i = l+1, \dots, n-1.$$

Now it can be easily shown that the  $S^1$ -action (constructed in (13)) in these local coordinates is given by

$$(16) \quad \lambda \cdot (z_i, w_i) = \begin{cases} (z_i, \lambda w_i), & \text{if } i = 3, \dots, l \\ (\lambda^2 z_i, \lambda^{-1} w_i), & \text{if } i = l+1, \dots, n-1. \end{cases}$$

Let us now consider a fixed point  $[0, q']_{\alpha\text{-st}} \in M(\alpha)$ . Then we may assume that

$$\begin{aligned} q'_1 &= \begin{pmatrix} c_1 \\ 0 \end{pmatrix}, \text{ with } c_1 \neq 0, & q'_2 &= \begin{pmatrix} c_2 \\ d_2 \end{pmatrix}, \text{ with } c_2, d_2 \neq 0, \\ q'_3 &= \begin{pmatrix} c_3 \\ d_3 \end{pmatrix}, \text{ with } d_3 \neq 0, \\ q'_i &= \begin{pmatrix} c_i \\ d_i \end{pmatrix}, \text{ with } c_i \neq 0, \text{ for } i = 4, \dots, n-1, \\ q'_n &= \begin{pmatrix} 0 \\ d_n \end{pmatrix}, \text{ with } d_n \neq 0, \end{aligned}$$

since  $[0, q']_{\alpha\text{-st}}$  is not in any of the sets  $X_S$ . As  $c_i \neq 0$  for all  $i \neq 3, n$ , and  $d_3 \neq 0 \neq d_n$ , there exists a unique element  $h \in K^{\mathbb{C}}$  such that  $(0, q')h = (0, q^0) \in \mu_{\mathbb{C}}^{-1}(0, 0)^{\alpha\text{-st}}$ , where

$$\begin{aligned} q_1^0 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & q_2^0 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & q_3^0 &= \begin{pmatrix} w_3 \\ 1 \end{pmatrix}, \\ q_i^0 &= \begin{pmatrix} 1 \\ w_i \end{pmatrix}, \text{ for } i = 4, \dots, n-1, \\ q_n^0 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$



Then there exists an open neighborhood  $U$  of  $(0, q^0)$  in  $T^*\mathbb{C}^{2n}$  such that for all  $(p, q) \in U \cap \mu_{\mathbb{C}}^{-1}(0, 0)$ , there is a unique element  $[A; e_1, \dots, e_n] \in K^{\mathbb{C}}$  such that

$$A^{-1}q_i e_i = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^t, & \text{if } i = 1 \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^t, & \text{if } i = 2 \\ \begin{pmatrix} w_3 & 1 \\ 0 & 1 \end{pmatrix}^t, & \text{if } i = 3 \\ \begin{pmatrix} 1 & w_i \\ 0 & 1 \end{pmatrix}^t, & \text{if } i = 4, \dots, n-1 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^t, & \text{if } i = n \end{cases}$$

and

$$e_i^{-1}p_i A = \begin{cases} \begin{pmatrix} 0 & r_1 \\ -r_2 & r_2 \end{pmatrix}, & \text{if } i = 1 \\ \begin{pmatrix} -r_2 & r_2 \\ z_3 & -z_3 w_3 \end{pmatrix}, & \text{if } i = 2 \\ \begin{pmatrix} z_3 & -z_3 w_3 \\ -z_i w_i & z_i \end{pmatrix}, & \text{if } i = 3, \dots, l \\ \begin{pmatrix} -z_i w_i & z_i \\ r_3 & 0 \end{pmatrix}, & \text{if } i = 4, \dots, n-1 \\ \begin{pmatrix} r_3 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } i = n, \end{cases}$$

where  $r_1, r_2, r_3$  are uniquely determined by

$$(17) \quad \{z_i, w_i \mid i = 3, \dots, n-1\};$$

so (17) is a local coordinate system around  $[0, q']_{\alpha\text{-st}}$  in  $X(\alpha)$ . Indeed,  $\alpha$ -stability is an open condition and, moreover, there exist unique, up to multiplication by  $\pm I$ ,

$$A \in \text{SL}(2, \mathbb{C}) \quad \text{and} \quad e_1, e_2, e_n \in \mathbb{C} \setminus \{0\}$$

such that

$$A^{-1}q_1 e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A^{-1}q_2 e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad A^{-1}q_n e_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then one can uniquely determine  $e_3, \dots, e_{n-1}$  such that

$$A^{-1}q_3 e_3 = \begin{pmatrix} * \\ 1 \end{pmatrix} \quad \text{and} \quad A^{-1}q_i e_i = \begin{pmatrix} 1 \\ * \end{pmatrix}, \quad \text{for } i = 4, \dots, n-1.$$

It is straight-forward to check that the circle action (see (13)) in these local coordinates is given by

$$(18) \quad \lambda \cdot (z_i, w_i) = (\lambda z_i, w_i) \quad \text{for } i = 3, \dots, n-1.$$

Using (16) and (18) we obtain the following result.

**Theorem 2.4.** *Let  $[p, q]_{\alpha\text{-st}}$  be a point in  $X_S$ . Then the non-zero isotropy weights of the  $S^1$ -representation on  $T_{[p, q]_{\alpha\text{-st}}} X(\alpha)$  are*

- +1 with multiplicity  $|S| - 2$ ;
- -1 with multiplicity  $(n-1) - |S|$ ;
- +2 with multiplicity  $(n-1) - |S|$ .

*Let  $[0, q]_{\alpha\text{-st}}$  be a point of the space  $M(\alpha)$ . Then the non-zero isotropy weights of the  $S^1$ -representation on  $T_{[0, q]_{\alpha\text{-st}}} X(\alpha)$  are*

- +1 with multiplicity  $(n-1) - |S|$ .

## 3. SPACES OF PARABOLIC HIGGS BUNDLES

Let  $\Sigma$  be  $\mathbb{CP}^1$  with  $n$  ordered marked points  $D = \{x_1, \dots, x_n\}$  and let  $E$  be a parabolic vector bundle of rank two over  $\Sigma$  with parabolic structure

$$\begin{aligned} E_x &:= E_{x,1} \supset E_{x,2} \supset 0, \\ 0 &\leq \beta_1(x) < \beta_2(x) < 1 \end{aligned}$$

over each point of  $D$ . Its *parabolic degree* is then

$$\text{par-deg}(E) := \text{degree}(E) + \sum_{x \in D} (\beta_1(x) + \beta_2(x)).$$

We recall that  $E$  is said to be *stable* if  $\text{par-}\mu(E) > \text{par-}\mu(L)$  for every line subbundle  $L$  of  $E$  equipped with the induced parabolic structure, where, for any parabolic vector bundle  $F$ , the *slope*  $\text{par-}\mu(F)$  is defined as  $\text{par-deg}(F)/\text{rank}(F)$ .

Now if  $L$  is a parabolic line subbundle of  $E$ , its induced parabolic structure is given by the trivial flag over each point  $x$  of  $D$ ,

$$L_x \supset 0,$$

with weights

$$\beta^L(x) = \begin{cases} \beta_1(x), & \text{if } L_x \cap E_{x,2} = \{0\}, \\ \beta_2(x), & \text{if } L_x \cap E_{x,2} = \mathbb{C}, \end{cases}$$

and so it has parabolic degree

$$\text{par-deg}(L) = \text{degree}(L) + \sum_{i \in S_L} \beta_2(x_i) + \sum_{i \in S_L^c} \beta_1(x_i),$$

where

$$(19) \quad S_L := \{i \in \{1, \dots, n\} \mid \beta^L(x_i) = \beta_2(x_i)\}.$$

Hence,  $E$  is stable if and only if every parabolic line subbundle  $L$  satisfies the inequality

$$(20) \quad \text{degree}(E) - 2 \cdot \text{degree}(L) > \sum_{i \in S_L} (\beta_2(x_i) - \beta_1(x_i)) - \sum_{i \in S_L^c} (\beta_2(x_i) - \beta_1(x_i)).$$

The holomorphic cotangent bundle of the Riemann surface  $\Sigma$  will be denoted by  $K_\Sigma$ . The line bundle on  $\Sigma$  defined by the divisor  $D$  will be denoted by  $\mathcal{O}_\Sigma(D)$ . A *parabolic Higgs bundle* of rank two is a pair  $\mathbf{E} := (E, \Phi)$ , where  $E$  is a parabolic vector bundle over  $\Sigma$  of rank two, and

$$\Phi \in H^0(\Sigma, \text{SParEnd}(E) \otimes K_\Sigma(D))$$

is a *Higgs field* on  $E$ . Here  $\text{SParEnd}(E)$  denotes the subsheaf of  $\text{End}(E)$  formed by strongly parabolic endomorphisms  $\varphi : E \rightarrow E$ , which, in this situation, simply means that

$$\varphi(E_{x,1}) \subset E_{x,2} \quad \text{and} \quad \varphi(E_{x,2}) = 0, \quad \text{for all } x \in D.$$

Note that  $\Phi$  is then a meromorphic endomorphism-valued one-form with simple poles along  $D$  whose residue at each  $x \in D$  is nilpotent with respect to the flag, i.e.,

$$(\text{Res}_x \Phi)(E_{x,i}) \subset E_{x,i+1}$$

for all  $i = 1, 2$  and  $x \in D$ , with  $E_{x,3} = 0$ . The definition of stability extends to Higgs bundles: a parabolic Higgs bundle  $\mathbf{E} = (E, \Phi)$  is *stable* if  $\text{par-}\mu(E) > \text{par-}\mu(L)$  for all parabolic line subbundles  $L \subset E$  which are preserved by  $\Phi$ .

Let  $\mathcal{H}(\beta)$  be the moduli space of parabolic Higgs bundles of rank two such that the underlying holomorphic vector bundle is holomorphically trivial. In [GM] it is shown that  $\mathcal{H}(\beta)$  is diffeomorphic to the space of hyperpolygons  $X(\alpha)$  with  $\alpha_i = \beta_2(x_i) - \beta_1(x_i)$ . The correspondence between these two spaces is given by the map

$$(21) \quad \begin{aligned} \mathcal{I} : X(\alpha) &\longrightarrow \mathcal{H}(\beta) \\ [p, q]_{\alpha\text{-st}} &\longmapsto (E_{(p,q)}, \Phi_{(p,q)}) =: \mathbf{E}_{(p,q)} \end{aligned}$$

where  $E_{(p,q)}$  is the trivial vector bundle  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}^2 \longrightarrow \mathbb{C}\mathbb{P}^1$  with the parabolic structure consisting of weighted flags

$$\begin{aligned} \mathbb{C}^2 &\supset \langle q_i \rangle \supset 0 \\ 0 &\leq \beta_1(x_i) < \beta_2(x_i) < 1 \end{aligned}$$

over the  $n$  marked points  $\{x_1, \dots, x_n\} = D \subset \mathbb{C}\mathbb{P}^1$  with  $\beta_i(x_j)$  satisfying

$$(22) \quad \beta_2(x_i) - \beta_1(x_i) = \alpha_i,$$

and  $\Phi_{[p,q]} \in H^0(S\text{ParEnd}(E_{(p,q)}) \otimes K_{\mathbb{C}\mathbb{P}^1}(D))$  is the Higgs field uniquely determined by the following condition on the residue:

$$(23) \quad \text{Res}_{x_i} \Phi := (q_i p_i)_0$$

at each  $x_i \in D$ . In particular, the polygon space  $M(\alpha)$  (obtained when  $p = 0$ ) is mapped to the moduli space  $\mathcal{M}_{\beta,2,0}$  of parabolic vector bundles of rank two over  $\Sigma$  such that the underlying holomorphic vector bundle is trivial (this map is obtained by setting  $\Phi = 0$ ).

This isomorphism is equivariant with respect to the circle action on  $X(\alpha)$  (see (13)) and the circle action on  $\mathcal{H}(\beta)$  defined by

$$(24) \quad \lambda \cdot (E, \Phi) = (E, \lambda\Phi), \quad \text{for } \lambda \in S^1.$$

Each connected component  $X_S$  of the fixed point set of the circle action on  $X(\alpha)$  is mapped to a manifold  $\mathcal{M}_S$  formed by the trivial holomorphic bundle  $E$  over  $\Sigma$  equipped with weighted flag structures

$$\begin{aligned} \mathbb{C}^2 &\supset E_{x_i,2} \supset 0 \\ 0 &\leq \beta_1(x_i) < \beta_2(x_i) < 1 \end{aligned}$$

such that  $E_{x_i,2} = E_{x_j,2}$  whenever  $i, j \in S$  or  $i, j \in S^c$ , and an Higgs field with zero residue at all points  $x_i$  with  $i \in S^c$ . Note that this description of the critical sets agrees with the one given by Simpson in [Si]. Indeed, the bundles in  $\mathcal{M}_S$  have a direct sum decomposition  $E = L_0 \oplus L_1$  as parabolic bundles, where the parabolic weight of  $L_0$  (respectively,  $L_1$ ) at  $x_i \in S^c$  is  $\beta_2(x_i)$  (respectively,  $\beta_1(x_i)$ ), and the parabolic weight of  $L_0$  (respectively,

$L_1$ ) at  $x_i \in S$  is  $\beta_1(x_i)$  (respectively,  $\beta_2(x_i)$ ). (Note that the holomorphic line bundles underlying  $L_0$  and  $L_1$  are trivial.) Moreover,  $\Phi$ , being lower triangular with respect to this decomposition, preserves  $L_1$  and induces a nonzero strongly parabolic homomorphism  $\Phi|_{L_0} : L_0 \rightarrow L_1 \otimes K_\Sigma(D)$ ; that this strongly parabolic homomorphism is nonzero follows from the fact that  $X_S \cap M(\alpha) = \emptyset$ .

#### 4. AN INVOLUTION

As before, let  $\mathcal{H}(\beta)$  be the moduli space of  $\beta$ -stable parabolic Higgs bundles of rank two such that the underlying holomorphic vector bundle is trivial. In this section we will restrict ourselves to the action of  $\mathbb{Z}/2\mathbb{Z} \subset S^1$  on  $\mathcal{H}(\beta)$  giving the involution

$$(25) \quad (E, \Phi) \longmapsto (E, -\Phi),$$

and we will study the fixed-point set of this involution.

The parabolic Higgs bundles with zero Higgs field are clearly fixed by the involution in (25), and so the moduli space  $\mathcal{M}_{\beta,2,0}$  of  $\beta$ -stable rank two holomorphically trivial parabolic vector bundles over  $\mathbb{C}\mathbb{P}^1$  is contained in the fixed point set of the involution.

For the remaining fixed points, we will use the isomorphism in (21) and study the fixed point set of the corresponding involution on the hyperpolygon space  $X(\alpha)$  with  $\alpha = (\alpha_1, \dots, \alpha_n)$  satisfying (22). The fixed-point set of the involution

$$(26) \quad (p, q) \longmapsto (-p, q),$$

on the hyperpolygon space  $X(\alpha)$  is the set of points  $X(\alpha)^{\mathbb{Z}/2\mathbb{Z}}$  that are fixed by the action of  $\mathbb{Z}/2\mathbb{Z} \subset S^1$  in (13).

As before,  $M(\alpha)$  denotes the moduli space of polygons in  $\mathbb{R}^3$ . Theorem 4.1 describes the fixed-point set of the  $\mathbb{Z}/2\mathbb{Z}$ -action in (26).

For each element  $S$  of  $\mathcal{S}'(\alpha)$ ,

$$Z_S := \{[p, q] \in X(\alpha) \mid S \text{ and } S^c \text{ are straight at } (p, q)\}.$$

**Theorem 4.1.** *The fixed-point set of the involution in (26) is*

$$X(\alpha)^{\mathbb{Z}/2\mathbb{Z}} = M(\alpha) \cup \bigcup_{S \in \mathcal{S}'(\alpha)} Z_S,$$

where  $Z_S$  is defined above.

Moreover,  $Z_S$  is a non-compact manifold of dimension  $2(n-3)$  except when  $|S| = n-1$ , in which case  $Z_S = X_S$  is compact and diffeomorphic to  $\mathbb{C}\mathbb{P}^{n-3}$ . In all cases,  $X(\alpha)^{\mathbb{Z}/2\mathbb{Z}}$  has  $2^{n-1} - (n+1)$  non-compact components and one compact component.

*Proof.* From the isotropy weights of the  $S^1$ -action given in Theorem 2.4 it follows immediately that, if  $M(\alpha)$  is nonempty, then it is a connected component of the fixed point set of the  $\mathbb{Z}_2/2\mathbb{Z}$ -action. Furthermore, the connected components of the complement  $X(\alpha)^{\mathbb{Z}/2\mathbb{Z}} \setminus M(\alpha)$  are parametrized by the elements  $S$  of  $\mathcal{S}'(\alpha)$  and have dimension

$$\dim Z_S = 2((n-1) - |S|) + \dim X_S = 2(n-3).$$

Therefore, it remains to show that each connected component  $Z_S$  of  $X(\alpha)^{\mathbb{Z}/2\mathbb{Z}} \setminus M(\alpha)$  can be described as

$$Z_S = \{[p, q] \in X(\alpha) \mid S \text{ and } S^c \text{ are straight at } (p, q)\}.$$

Suppose that  $[p, q] \in X(\alpha)^{\mathbb{Z}/2\mathbb{Z}} \setminus M(\alpha)$ . Then there exists an element

$$[A; e_1, \dots, e_n] \in K \setminus \{I\}$$

such that

$$e_i^{-1} p_i A = -p_i \quad \text{and} \quad A^{-1} q_i e_i = q_i, \quad \text{for } i = 1, \dots, n,$$

and so

$$A p_i^* = -e_i p_i^* \quad \text{and} \quad A q_i = e_i q_i \quad \text{for } i = 1, \dots, n.$$

Since  $|q_i|^2 - |p_i|^2 = 2\alpha_i$ , we have  $q_i \neq 0$  for all  $i = 1, \dots, n$  and so  $q_i$  is an eigenvector of  $A$  with eigenvalue  $e_i$ . Moreover, since  $[p, q] \in X(\alpha)^{\mathbb{Z}/2\mathbb{Z}} \setminus M(\alpha)$ , there exists an integer  $i_0 \in \{1, \dots, n\}$  such that  $p_{i_0} \neq 0$ , and so  $p_{i_0}^*$  is an eigenvector of  $A$  with eigenvalue  $-e_{i_0}$ . Hence, assuming that  $A$  is diagonal, we have

$$A = \begin{pmatrix} e_{i_0} & 0 \\ 0 & -e_{i_0} \end{pmatrix}$$

with  $e_{i_0} = \pm\sqrt{-1}$ . We conclude that there exists an index set  $S \subset \{1, \dots, n\}$  with  $i_0 \in S$  such that

$$(27) \quad \begin{aligned} p_i &= \begin{pmatrix} 0 & b_i \end{pmatrix}, & q_i &= \begin{pmatrix} c_i \\ 0 \end{pmatrix}, \quad \forall i \in S \\ p_i &= \begin{pmatrix} a_i & 0 \end{pmatrix}, & q_i &= \begin{pmatrix} 0 \\ d_i \end{pmatrix}, \quad \forall i \in S^c. \end{aligned}$$

Since  $|q_i|^2 - |p_i|^2 = 2\alpha_i$ , we conclude that

$$(28) \quad |c_i|^2 - |b_i|^2 = 2\alpha_i \quad \text{for all } i \in S \quad \text{and} \quad |d_i|^2 - |a_i|^2 = 2\alpha_i \quad \text{for all } i \in S^c.$$

Moreover, since  $\sum_{i=1}^n (q_i q_i^* - p_i^* p_i)_0 = 0$ , we obtain that

$$(29) \quad \sum_{i \in S} (|c_i|^2 + |b_i|^2) - \sum_{i \in S^c} (|d_i|^2 + |a_i|^2) = 0$$

and so, using (28), we get that

$$(30) \quad \sum_{i \in S} \alpha_i - \sum_{i \in S^c} \alpha_i = \sum_{i \in S^c} |a_i|^2 - \sum_{i \in S} |b_i|^2.$$

On the other hand, since  $\sum_{i=1}^n (q_i p_i)_0 = 0$ , we have that

$$(31) \quad \sum_{i \in S} b_i c_i = \sum_{i \in S^c} a_i d_i = 0.$$

If  $S$  is short, then we work with  $S$  and, in particular, since  $c_i \neq 0$  and there exists an  $i_0 \in S$  such that  $b_{i_0} \neq 0$ , from (31) it follows that there exists another  $i_1 \in S$  such that  $b_{i_1} \neq 0$  and we obtain that  $S$  has cardinality at least two. If, on the other hand,  $S$  is long, we consider  $S^c$  instead (which is now short). Since  $S$  is long, from (30) it follows that there is at least one  $i_1 \in S^c$  such that  $p_{i_1} \neq 0$  and then, since  $d_{i_1} \neq 0$ , from (31) it

follows that there is another element  $i_2$  in  $S^c$  with  $p_{i_2} \neq 0$ , implying that the short set  $S^c$  has cardinality at least two.

Finally, since for every subset  $S \subset \{1, \dots, n\}$ , we have that either  $S$  or  $S^c$  is short, the number of short sets for  $\alpha$  is

$$\frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} = 2^{(n-1)} - 1.$$

If there is no short set of cardinality  $n-1$ , then there are exactly  $n$  short sets of cardinality 1 and so

$$|\mathcal{S}'(\alpha)| = 2^{(n-1)} - (n+1).$$

Moreover, in this case, all the components  $Z_S$  are non-compact. If, on the other hand, there is a short set  $\tilde{S}$  of cardinality  $n-1$ , then there are only  $n-1$  short sets of cardinality 1 and then the number of elements in  $\mathcal{S}'(\alpha)$  is

$$|\mathcal{S}'(\alpha)| = 2^{(n-1)} - n.$$

However, in this case,  $M(\alpha)$  is empty and  $Z_{\tilde{S}}$  is compact. We conclude that, in both cases, the number of non-compact components of  $X(\alpha)^{S^1}$  is  $2^{(n-1)} - (n+1)$  and that there is exactly one compact component (which is either  $M(\alpha)$  or  $Z_{\tilde{S}}$ ).  $\square$

Each manifold  $Z_S$ , being a component of  $X(\alpha)^{\mathbb{Z}/2\mathbb{Z}}$ , is symplectic and invariant under the circle action in (13). Hence, whenever  $|S| \neq n-1$ , we obtain an effective Hamiltonian circle action on  $Z_S$  (the action factors through the quotient of  $S^1$  by  $\mathbb{Z}/2\mathbb{Z}$ ). The corresponding moment map then coincides with the restriction of  $\frac{1}{2}\phi$  to  $Z_S$ . The only critical submanifold of this map is  $X_S$  where it attains its minimum value. Consequently, we have the following results.

**Theorem 4.2.** *Each manifold  $X_S \cong \mathbb{C}\mathbb{P}^{|S|-2}$  is a deformation retraction of  $Z_S$ . In particular,  $Z_S$  is simply connected.*

**Theorem 4.3.** *Let  $P_t(M)$  be the Poincaré polynomial of  $M$ . Then*

$$P_t(Z_S) = P_t(X_S) = P_t(\mathbb{C}\mathbb{P}^{|S|-2}) = 1 + t + \dots + t^{2(|S|-2)}.$$

Going back to the space of parabolic Higgs bundles  $\mathcal{H}(\beta)$  and using the isomorphism in (21), we obtain from (23) and (27) that the fixed-point set  $\mathcal{H}(\beta)^{\mathbb{Z}/2\mathbb{Z}}$  of the involution in (25) is described as in Theorem 1.1.

## 5. POLYGONS IN MINKOWSKI 3-SPACE

Let us consider the *Minkowski inner product* on  $\mathbb{R}^3$

$$v \circ w = -v_1w_1 - v_2w_2 + v_3w_3,$$

for  $v = (v_1, v_2, v_3)$  and  $w = (w_1, w_2, w_3)$ . The inner product space consisting of  $\mathbb{R}^3$  together with this signature  $(-, -, +)$ -inner product will be denoted by  $\mathbb{R}^{2,1}$ ; it is called the *Minkowski 3-space*. The *Minkowski norm* of a vector  $v \in \mathbb{R}^{2,1}$  is then defined to be

$$\|v\|_{2,1} = \sqrt{|v \circ v|}.$$

All elements  $v$  of  $\mathbb{R}^{2,1}$  are classified according to the sign of  $v \circ v$ . The set of all  $v$  such that  $v \circ v = 0$  is called the *light cone* of  $\mathbb{R}^3$ ; any vector  $v$  with  $v \circ v = 0$  is said to be *light-like*. If  $v \circ v > 0$ , then  $v$  is called *time-like*, and if  $v \circ v < 0$ , then it is called *space-like*. A time-like vector is said to be lying in *future* (respectively, *past*) if  $v_3 > 0$  (respectively,  $v_3 < 0$ ). Note that the exterior of the light cone consists of all space-like vectors, while its interior consists of all time-like vectors. From now on we will write any  $v \in \mathbb{R}^{2,1}$  as  $v = (x, y, t)$ .

Moduli spaces of polygons in  $\mathbb{R}^{2,1}$  were described by Foth, [Fo], as follows. Consider the surface  $S_R$  in  $\mathbb{R}^3$  defined by the equation  $-x^2 - y^2 + t^2 = R^2$ , which is called a *pseudosphere*. The Minkowski metric on  $\mathbb{R}^{2,1}$  restricts to a constant curvature Riemannian metric on  $S_R$ . It is an hyperboloid of two sheets. The connected component  $S_R^+ \subset S_R$ , corresponding to  $t > 0$ , is called a *future pseudosphere*, and the connected component  $S_R^- \subset S_R$ , corresponding to  $t < 0$ , is called a *past pseudosphere*. The group  $\mathrm{SU}(1, 1)$  acts transitively on each connected component since one can think of  $\mathbb{R}^{2,1} \cong \mathbb{R}^3$  as  $\mathfrak{su}(1, 1)^*$  with  $S_R^+$  and  $S_R^-$  being elliptic coadjoint orbits. Consequently, both  $S_R^+$  and  $S_R^-$  have a natural invariant symplectic structure (the Kostant–Kirillov form on a coadjoint orbit). The Minkowski metric is also invariant (since  $\mathrm{SU}(1, 1)$  acts by isometries) and both connected components are Kähler manifolds; they are in fact isomorphic to the hyperbolic plane  $\mathrm{SU}(1, 1)/\mathrm{U}(1)$ . We will study the geometry of the symplectic quotients of the products of several future and past pseudospheres with respect to the diagonal  $\mathrm{SU}(1, 1)$ -action.

Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  be an  $n$ -tuple of positive real numbers. Let us fix two positive integers  $k_1, k_2$  with  $k_1 + k_2 = n$ . We will consider polygons in Minkowski 3-space that have the first  $k_1$  edges in the future time-like cone and the last  $k_2$  edges in the past time-like cone, such that the Minkowski length of the  $i$ -th edge is  $\alpha_i$ . A closed polygon will then be one whose sum of the first  $k_1$  sides in the future time-like cone coincides with the negative of the sum of the last  $k_2$  sides in the past time-like cone. The space of all such closed polygons can be identified with the zero level set of the moment map

$$(32) \quad \begin{array}{ccc} \mu : \mathcal{O}_1 \times \dots \times \mathcal{O}_n & \longrightarrow & \mathfrak{su}(1, 1)^* \\ (u_1, \dots, u_n) & \longmapsto & u_1 + \dots + u_n \end{array}$$

for the diagonal  $\mathrm{SU}(1, 1)$ -action, where  $\mathcal{O}_i \cong S_{\alpha_i}^+$  is a future pseudosphere of radius  $\alpha_i$  if  $1 \leq i \leq k_1$ , and  $\mathcal{O}_i \cong S_{\alpha_i}^-$  is a past pseudosphere if  $k_1 + 1 \leq i \leq n$ , equipped with the Kostant–Kirillov symplectic form on coadjoint orbits. Hence,

$$M^{k_1, k_2}(\alpha) := \mu^{-1}(0)/\mathrm{SU}(1, 1),$$

which is a quotient of a non-compact space by a non-compact Lie group. For a generic choice of  $\alpha$ , meaning  $M^{k_1, k_2}(\alpha)$  is non-empty with  $\sum_{i=1}^{k_1} \alpha_i \neq \sum_{i=k_1+1}^n \alpha_i$ , every point in  $M^{k_1, k_2}(\alpha)$  represents a polygon with a trivial stabilizer. In that situation, the group  $\mathrm{SU}(1, 1)$  acts freely and properly on  $\mu^{-1}(0)$ . Moreover, 0 is a regular value of the moment map  $\mu$  and so the quotient space  $M^{k_1, k_2}(\alpha)$  is, for a generic  $\alpha$ , a smooth symplectic manifold of dimension  $2(n - 3)$ . Note that the spaces

$$M^{k_1, k_2}(\alpha_1, \dots, \alpha_{k_1}, \alpha_{k_1+1}, \dots, \alpha_n) \quad \text{and} \quad M^{k_2, k_1}(\alpha_{k_1+1}, \dots, \alpha_n, \alpha_1, \dots, \alpha_{k_1})$$

are symplectomorphic by the isomorphism given by the involution of  $\mathbb{R}^{2,1}$  defined by  $(x, y, t) \mapsto (-x, -y, -t)$ .

**Theorem 5.1** ([Fo]). *The space  $M^{k_1, k_2}(\alpha)$  is non-compact, unless  $k_1 = 1$  or  $k_2 = 1$  in which case it is compact.*

We give a brief outline of an argument for Theorem 5.1.

Let us first assume that  $k_2 = 1$ . The last side of the polygon can be represented (after being acted on by an element of  $SU(1, 1)$ ) by a vector in  $\mathbb{R}^{2,1}$  with coordinates  $(0, 0, -\alpha_n)$ . Hence, the sum of the first  $n-1$  future time-like sides of the polygon is  $(0, 0, \alpha_n)$ . The only symmetry left is the circle rotation around the  $t$ -axis. Therefore, this space of polygons is clearly bounded and closed and therefore compact.

The space  $M^{k_1, k_2}(\alpha)$  is non-compact if  $k_2 > 1$ . For example, let us consider the simple case where  $k_1 = k_2 = 2$  and  $\alpha_1 = \alpha_2 = \alpha_4 = 1$ . Again we can assume that the last side is  $(0, 0, -1)$  and the only symmetry left is the circle rotation around the  $t$ -axis. Let  $x_n$  be the closed polygon with sides

$$\begin{aligned} u_1 &= (-1, 0, \sqrt{2}), & u_2 &= (1 - P(n), Q(n), n - \sqrt{2}), \\ u_3 &= (P(n), -Q(n), 1 - n) & \text{and } u_4 &= (0, 0, -1), \end{aligned}$$

where

$$P(n) = \frac{1}{2}(3 + 2(\sqrt{2} - 1)n) \quad \text{and} \quad Q(n) = \sqrt{8(\sqrt{2} - 1)n^2 - 4(3\sqrt{2} - 1)n - 9}.$$

The sequence  $\{x_n\}$  has no limit point in  $M^{2,2}(1, 1, 2, 1)$  and so this space is not compact.

Let us now describe the symplectic structure on  $M^{k_1, k_2}(\alpha)$ . For that, define the *Minkowski cross product*  $\dot{\times}$  as

$$v \dot{\times} w := \det \begin{pmatrix} -e_1 & -e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix},$$

where  $v = (v_1, v_2, v_3)$  and  $w = (w_1, w_2, w_3)$  with  $e_1, e_2, e_3$  being the standard unit vectors in  $\mathbb{R}^3$ . This cross product satisfies the usual properties:

$$\begin{aligned} v \dot{\times} w &= -w \dot{\times} v \\ (u \dot{\times} v) \dot{\times} w + (v \dot{\times} w) \dot{\times} u + (w \dot{\times} u) \dot{\times} v &= 0 \end{aligned}$$

and so  $(\mathbb{R}^3, \dot{\times})$  is a Lie algebra. Moreover, it is isomorphic to  $\mathfrak{su}(1, 1)$  via the map

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} -\sqrt{-1}t & x + \sqrt{-1}y \\ x - \sqrt{-1}y & \sqrt{-1}t \end{pmatrix}.$$

Under this identification, the Minkowski inner product  $\circ$  corresponds to  $(A, B) \mapsto -2 \cdot \text{trace}(AB)$ .

The symplectic form on the pseudosphere  $S_R$  is then given by

$$\omega_u(v, w) = \frac{1}{R^2} u \circ (v \dot{\times} w),$$



where  $u \in S_R$  and  $v, w \in T_u S_R$  (we think of  $T_u S_R$  as the linear subspace of  $\mathbb{R}^{2,1}$  orthogonal to  $u$  with respect to the Minkowski inner product), and the map in (32) is the moment map for the diagonal  $SU(1, 1)$ -action and the product symplectic structure.

## 6. BACK TO THE INVOLUTION

Let us go back to Section 4 and consider a point  $[p, q]$  in some  $Z_S$ . Let  $u_i \in \mathbb{R}^3$  be the vector

$$u_i := \frac{\sqrt{-1}}{2}(p_i^* p_i - q_i q_i^*)_0 + \frac{1}{2}(p_i^* q_i^* + q_i p_i)_0,$$

where we use the identifications,  $\mathfrak{su}(2)^* \cong (\mathbb{R}^3)^* \cong \mathfrak{su}(1, 1)$ . If  $i \in S$  then  $p_i = \begin{pmatrix} 0 & b_i \end{pmatrix}$  and  $q_i = \begin{pmatrix} c_i \\ 0 \end{pmatrix}$ , implying that

$$(33) \quad u_i = \left( \operatorname{Re}(b_i c_i), \operatorname{Im}(b_i c_i), \frac{|b_i|^2 + |c_i|^2}{2} \right)$$

with

$$u_i \circ u_i = (|c_i|^2 - |b_i|^2)^2/4 = \alpha_i^2$$

( $u_i$  has Minkowski norm  $\alpha_i$ ). Similarly, if  $i \in S^c$ , we have  $p_i = \begin{pmatrix} a_i & 0 \end{pmatrix}$  and  $q_i = \begin{pmatrix} 0 \\ d_i \end{pmatrix}$ , yielding

$$(34) \quad u_i = \left( \operatorname{Re}(a_i d_i), -\operatorname{Im}(a_i d_i), -\frac{|a_i|^2 + |d_i|^2}{2} \right)$$

and

$$u_i \circ u_i = (|a_i|^2 - |d_i|^2)^2/4 = \alpha_i^2.$$

Moreover, by (29) and (31) we have that

$$\sum_{i=1}^n u_i = 0.$$

So the vectors  $u_i$  form a closed polygon in Minkowski 3-space with the first  $|S|$  sides in the positive time-like cone and the last  $n - |S|$  sides in the past, with the  $i$ -th side being of Minkowski length  $\alpha_i$ .

**Theorem 6.1.** *For any  $S \in \mathcal{S}'(\alpha)$ , the components  $Z_S$  and  $Z_{S^c}$ , of the fixed-point sets of the involutions in (25) and (26) respectively, are diffeomorphic to the moduli space*

$$M^{|S|, |S^c|}(\alpha)$$

*of closed polygons in Minkowski 3-space.*

*Proof.* Let  $S$  be a short set of cardinality at least two. Consider the map  $\varphi : Z_S \rightarrow M^{|S|, |S^c|}(\alpha)$  defined above, that is,  $\varphi([p, q]_{\mathbb{R}})$  is the element of  $M^{|S|, |S^c|}(\alpha)$  represented by the polygon whose sides are the vectors  $u_i$  given by (33) and (34) for  $i$  in  $S$  and  $S^c$  respectively.

Note that the pseudo-unitary group  $SU(1, 1)$  is generated by the following orientation preserving isometries of the pseudosphere:  $A_\theta$  and  $T_\phi$ , where

$$A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is an Euclidean *rotation* by an angle  $\theta$  in the  $(x, y)$ -plane, and

$$T_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \phi & \sinh \phi \\ 0 & \sinh \phi & \cosh \phi \end{pmatrix}$$

is a *boost* of rapidity  $\phi$  along the  $y$ -direction<sup>1</sup> (cf. [BV] for the details).

Let us first see that  $\varphi$  is well defined. For that, consider two representatives  $(p, q)$  and  $(p', q')$  of the same element  $[p, q]_{\mathbb{R}}$  in  $Z_S$ . Then there exists  $[A; e_1, \dots, e_n] \in K$  such that

$$e_i^{-1} p_i A = p'_i \quad \text{and} \quad A^{-1} q_i e_i = q'_i, \quad i = 1, \dots, n.$$

Since  $p_i = \begin{pmatrix} a_i & b_i \end{pmatrix}$ ,  $p'_i = \begin{pmatrix} a'_i & b'_i \end{pmatrix}$  with  $a_i = a'_i = 0$  for  $i \in S$  and  $b_i = b'_i = 0$  for  $i \in S^c$ , while  $q_i = \begin{pmatrix} c_i & d_i \end{pmatrix}^t$ ,  $q'_i = \begin{pmatrix} c'_i & d'_i \end{pmatrix}^t$  with  $d_i = d'_i = 0$  for  $i \in S$  and  $c_i = c'_i = 0$  for  $i \in S^c$ , we conclude that

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$$

with  $\alpha = e^{\sqrt{-1}\theta_0} \in S^1$ . Then we have

$$\begin{pmatrix} \operatorname{Re}(b'_i c'_i) \\ \operatorname{Im}(b'_i c'_i) \\ \frac{|b'_i|^2 + |c'_i|^2}{2} \end{pmatrix} = A_{-2\theta_0} \begin{pmatrix} \operatorname{Re}(b_i c_i) \\ \operatorname{Im}(b_i c_i) \\ \frac{|b_i|^2 + |c_i|^2}{2} \end{pmatrix} \quad \text{for } i \in S,$$

and

$$\begin{pmatrix} \operatorname{Re}(a'_i d'_i) \\ -\operatorname{Im}(a'_i d'_i) \\ -\frac{|a'_i|^2 + |d'_i|^2}{2} \end{pmatrix} = A_{-2\theta_0} \begin{pmatrix} \operatorname{Re}(a_i d_i) \\ -\operatorname{Im}(a_i d_i) \\ -\frac{|a_i|^2 + |d_i|^2}{2} \end{pmatrix} \quad \text{for } i \in S^c,$$

where  $A_{-2\theta_0}$  is a rotation in  $SU(1, 1)$ . Therefore, it follows that  $\varphi$  is well-defined.

To show that  $\varphi$  is injective, let us consider two points  $[p, q]_{\mathbb{R}}, [p', q']_{\mathbb{R}} \in Z_S$  with  $\varphi([p, q]_{\mathbb{R}}) = \varphi([p', q']_{\mathbb{R}})$ . Then, writing

$$p_i = \begin{pmatrix} 0 & b_i \end{pmatrix}, \quad p'_i = \begin{pmatrix} 0 & b'_i \end{pmatrix} \quad \text{and} \quad q_i = \begin{pmatrix} c_i \\ 0 \end{pmatrix}, \quad q'_i = \begin{pmatrix} c'_i \\ 0 \end{pmatrix}, \quad \text{for } i \in S$$

with  $\sum_{i \in S} b_i c_i = \sum_{i \in S} b'_i c'_i = 0$  (cf. (31)), and

$$p_i = \begin{pmatrix} a_i & 0 \end{pmatrix}, \quad p'_i = \begin{pmatrix} a'_i & 0 \end{pmatrix} \quad \text{and} \quad q_i = \begin{pmatrix} 0 \\ d_i \end{pmatrix}, \quad q'_i = \begin{pmatrix} 0 \\ d'_i \end{pmatrix}, \quad \text{for } i \in S^c,$$

---

<sup>1</sup>In special relativity, the rapidity parameter  $\phi$  is defined by  $\tanh \phi = v/c$ , where  $v$  is the velocity.

with  $\sum_{i \in S^c} a_i d_i = \sum_{i \in S^c} a'_i d'_i = 0$ , there exists an Euclidean rotation  $A_{\theta_0}$  by an angle  $\theta_0$  on the  $(x, y)$ -plane such that

$$(35) \quad \begin{pmatrix} \operatorname{Re}(b'_i c'_i) \\ \operatorname{Im}(b'_i c'_i) \\ \frac{|b'_i|^2 + |c'_i|^2}{2} \end{pmatrix} = A_{\theta_0} \begin{pmatrix} \operatorname{Re}(b_i c_i) \\ \operatorname{Im}(b_i c_i) \\ \frac{|b_i|^2 + |c_i|^2}{2} \end{pmatrix} \quad \text{for } i \in S,$$

and

$$(36) \quad \begin{pmatrix} \operatorname{Re}(a'_i d'_i) \\ -\operatorname{Im}(a'_i d'_i) \\ -\frac{|a'_i|^2 + |d'_i|^2}{2} \end{pmatrix} = A_{\theta_0} \begin{pmatrix} \operatorname{Re}(a_i d_i) \\ -\operatorname{Im}(a_i d_i) \\ -\frac{|a_i|^2 + |d_i|^2}{2} \end{pmatrix} \quad \text{for } i \in S^c.$$

Indeed, if the two vectors on the left-hand side of (35) and (36) were not obtained from the corresponding vectors on the right-hand side by an Euclidean rotation, but by an element of  $\operatorname{SU}(1, 1)$  involving a boost, they would fail to satisfy the condition

$$\sum_{i \in S} b'_i c'_i = \sum_{i \in S^c} a'_i d'_i = 0.$$

We conclude that

$$b'_i c'_i = e^{\sqrt{-1}\theta_0} b_i c_i, \quad \text{and} \quad |b'_i|^2 + |c'_i|^2 = |b_i|^2 + |c_i|^2, \quad \text{for } i \in S,$$

while

$$a'_i d'_i = e^{-\sqrt{-1}\theta_0} a_i d_i, \quad \text{and} \quad |a'_i|^2 + |d'_i|^2 = |a_i|^2 + |d_i|^2, \quad \text{for } i \in S^c,$$

and so

$$p'_i = p_i A \quad \text{and} \quad q'_i = A^{-1} q_i, \quad i = 1, \dots, n$$

with  $A = \begin{pmatrix} e^{-\sqrt{-1}\theta_0/2} & 0 \\ 0 & e^{\sqrt{-1}\theta_0/2} \end{pmatrix}$ , implying that  $[p, q]_{\mathbb{R}} = [p', q']_{\mathbb{R}}$ .

Let us now see that  $\varphi$  is surjective. For that, take any element  $[v] \in M^{|S|, |S^c|}(\alpha)$ . Using the  $\operatorname{SU}(1, 1)$ -action, the  $(k_1 + 1)$ -th vertex can be placed on the  $t$ -axis (so that  $\sum_{i=1}^{k_1} v_i$  is a vector along the  $t$ -axis). Therefore, we may assume that  $[v]$  is represented by a polygon with the first  $|S|$  sides being  $(x_i, y_i, t_i)$  with  $t_i > 0$ , on the positive time-like cone and the last  $n - |S|$  sides being  $(x_i, y_i, -t_i)$  with  $t_i > 0$ , in the past, satisfying the additional conditions

$$\sum_{i=1}^{k_1} x_i = \sum_{i=1}^{k_1} y_i = \sum_{i=k_1+1}^n x_i = \sum_{i=k_1+1}^n y_i = 0.$$

Then  $[v]$  is the image of the hyperpolygon  $[p, q]_{\mathbb{R}}$ , where

$$p_i = \left( 0 \quad \frac{1}{l_i}(x_i + \sqrt{-1}y_i) \right), \quad q_i = \begin{pmatrix} l_i \\ 0 \end{pmatrix} \quad \text{for } i \in S,$$

and

$$p_i = \left( \frac{1}{l_i}(x_i - \sqrt{-1}y_i) \quad 0 \right), \quad q_i = \begin{pmatrix} 0 \\ l_i \end{pmatrix} \quad \text{for } i \in S^c,$$

with

$$l_i = \sqrt{\alpha_i + \sqrt{\alpha^2 + |x_i + \sqrt{-1}y_i|^2}} = \sqrt{\alpha_i + t}, \quad i = 1, \dots, n.$$

Here  $[p, q]_{\mathbb{R}} \in Z_S$  since

$$\sum_{i \in S} b_i c_i = \sum_{i=1}^{k_1} (x_i + \sqrt{-1}y_i) = 0, \quad \sum_{i \in S^c} a_i d_i = \sum_{i=k_1+1}^n (x_i - \sqrt{-1}y_i),$$

and

$$|c_i|^2 - |b_i|^2 = 2\alpha_i \text{ for all } i \in S \quad \text{while} \quad |d_i|^2 - |a_i|^2 = 2\alpha_i \text{ for all } i \in S^c,$$

where as usual we write  $p_i = \begin{pmatrix} a_i & b_i \end{pmatrix}$  and  $q_i = \begin{pmatrix} c_i & d_i \end{pmatrix}^t$ , for  $i = 1, \dots, n$ .

Note that clearly  $\varphi$  and its inverse are differentiable and the theorem follows.  $\square$

**Remark 6.1.** Note that when  $|S| = n - 1$  we obtain that the space  $M^{n-1,1}(\alpha)$ , which we already knew is compact, is, in fact, diffeomorphic to  $\mathbb{C}\mathbb{P}^{n-3}$ .

Theorem 6.1 allows us draw several conclusions on the polygon spaces in Minkowski 3-space which are immediate consequences of Theorem 4.2 and Theorem 4.3.

**Theorem 6.2.** *Let  $M^{k_1, k_2}(\alpha)$  be the moduli space of closed polygons in Minkowski 3-space that have the first  $k_1$  sides in the future time-like cone and the last  $k_2$  in the past, such that the Minkowski length of the  $i$ -th side is  $\alpha_i$ . Assume without loss of generality that  $\sum_{i=1}^{k_1} \alpha_i < \sum_{i=k_1+1}^n \alpha_i$ . Then,*

- (i)  $M^{k_1, k_2}(\alpha)$  admits a deformation retraction to  $\mathbb{C}\mathbb{P}^{k_1-2}$ , and
- (ii) the Poincaré polynomial of  $M^{k_1, k_2}(\alpha)$  is

$$P_t(M^{k_1, k_2}(\alpha)) = P_t(\mathbb{C}\mathbb{P}^{k_1-2}) = 1 + t + \dots + t^{2(|S|-2)}.$$

## 7. AN EXAMPLE

As an example, we consider the case where  $n = 4$ . Let  $\mathcal{H}(\beta)$  be the moduli space of parabolic Higgs bundles  $(E, \Phi)$  of rank two over  $\mathbb{C}\mathbb{P}^1$  with four parabolic points, where the underlying holomorphic vector bundle is trivial and  $\beta$  is generic. Let

$$\alpha_i := \beta_2(x_i) - \beta_1(x_i), \quad i = 1, \dots, 4.$$

Since a subset of  $\{1, 2, 3, 4\}$  is either short or long, we know that there are exactly three short sets of cardinality two for any value of  $\alpha = (\alpha_1, \dots, \alpha_4)$ . Let us denote these sets by  $S_1, S_2$  and  $S_3$ . Then the fixed point set of the involution in (25) has exactly 4 connected components

$$\mathcal{M}_{\beta, 2, 0}, \mathcal{Z}_{S_1}, \mathcal{Z}_{S_2}, \mathcal{Z}_{S_3} \quad \text{or} \quad \mathcal{Z}_{S_1}, \mathcal{Z}_{S_2}, \mathcal{Z}_{S_3}, \mathcal{Z}_{\tilde{S}},$$

according to whether  $\mathcal{M}_{\beta, 2, 0}$  is empty or nonempty, where  $\tilde{S}$  is a short set of cardinality 3 which we know exists exactly when  $\mathcal{M}_{\beta, 2, 0} = \emptyset$  [GM, BY].

If  $\mathcal{M}_{\beta, 2, 0} \neq \emptyset$ , then this space  $\mathcal{M}_{\beta, 2, 0}$  is a compact toric manifold of dimension two, therefore diffeomorphic to  $\mathbb{C}\mathbb{P}^1$ . Indeed, let us assume without loss of generality that  $\alpha_1 \neq \alpha_2$  (note that  $\alpha$  is generic) and consider the diagonal  $d_2 := u_1 + u_2$  connecting the origin to the third vertex of the polygon. For each intermediate value of the length

of  $d_2$ , we have a circle of possible classes of polygons obtained by rotating the first two sides of the polygon around the diagonal, while fixing the other two. The minimum and maximum values of this length are

$$\max\{|\alpha_1 - \alpha_2|, |\alpha_3 - \alpha_4|\} \quad \text{and} \quad \min\{\alpha_1 + \alpha_2, \alpha_3 + \alpha_4\}$$

respectively, in which cases we only have one possible polygon. Note that this length is the moment map for the bending flow obtained by rotating the first two sides of the polygon around the diagonal.

If  $\mathcal{M}_{\beta,2,0} = \emptyset$  then, since  $|\tilde{S}| = 3$ , we have that  $\mathcal{Z}_{\tilde{S}} = \mathcal{M}_{\tilde{S}}$  is a connected component of  $\mathcal{H}(\beta)^{\mathbb{Z}/2\mathbb{Z}}$  diffeomorphic to  $\mathbb{CP}^1$ .

Let us now consider  $\mathcal{Z}_{S_i}$ . By Theorem 6.1 we know that this space is diffeomorphic to  $M^{2,2}(\alpha)$  formed by classes of closed polygons in Minkowski 3-space with the first two sides  $u_1, u_2$  in the future time-like cone and the last two, namely  $u_3$  and  $u_4$ , in the past, where each side  $u_i$  has Minkowski length  $\alpha_i$ . Let us again consider the diagonal  $d_2 = u_1 + u_2$  connecting the origin to the third vertex of the polygon. This vector is also a future time-like vector and we can consider its Minkowski length  $\ell$ . Note that if we place the first vertex at the origin and use the  $SU(1,1)$ -action to place the third vertex on the  $t$ -axis, then the bending flow can be described as a rotation of the vectors  $u_1, u_2$  around the  $t$ -axis with a constant angular speed while fixing the other two vectors. Hence,  $\mathcal{Z}_{S_i}$  is a non-compact toric manifold with moment map  $\ell$ . By the reversed triangle inequality we have that  $\ell$  has the minimum value

$$\max\{\alpha_1 + \alpha_2, \alpha_3 + \alpha_4\}$$

which is attained at just one point (the polygon with two sides aligned along the  $t$ -axis) and has no other critical value. We conclude that  $\mathcal{Z}_{S_i}$  is diffeomorphic to  $\mathbb{C}$ .

In all cases we conclude that  $\mathcal{H}(\beta)^{\mathbb{Z}/2\mathbb{Z}}$  has one compact connected component diffeomorphic to  $\mathbb{CP}^1$  and three non-compact components diffeomorphic to  $\mathbb{C}$ .

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