OPTIMAL RELATIONS BETWEEN L^p-NORMS FOR THE HARDY OPERATOR AND ITS DUAL

V.I. KOLYADA

Abstract. We obtain sharp two-sided inequalities between L^p-norms $(1 < p < \infty)$ of functions Hf and H^*f, where H is the Hardy operator, H^* is its dual, and f is a nonnegative measurable function on $(0,\infty)$. In an equivalent form, it gives sharp constants in the two-sided relations between L^p-norms of functions $H\varphi - \varphi$ and φ, where φ is a nonnegative nonincreasing function on $(0,+\infty)$ with $\varphi(+\infty) = 0$. In particular, it provides an alternative proof of a result obtained by N. Kruglyak and E. Setterqvist (2008) for $p = 2k$ $(k \in \mathbb{N})$ and by S. Boza and J. Soria (2011) for all $p \geq 2$, and gives a sharp version of this result for $1 < p < 2$.

1. Introduction and main results

Denote by $\mathcal{M}^+(\mathbb{R}_+)$ the class of all nonnegative measurable functions on $\mathbb{R}_+ \equiv (0,+\infty)$. Let $f \in \mathcal{M}^+(\mathbb{R}_+)$. Set

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

and

$$H^*f(x) = \int_x^\infty \frac{f(t)}{t}\,dt.$$

These equalities define the classical Hardy operator H and its dual operator H^*. By Hardy's inequalities [5, Ch. 9], these operators are bounded in $L^p(\mathbb{R}_+)$ for any $1 < p < \infty$. Furthermore, it is easy to show that for any $f \in \mathcal{M}^+(\mathbb{R}_+)$ and any $1 < p < \infty$ the L^p-norms of Hf and H^*f are equivalent. Indeed, let $f \in \mathcal{M}^+(\mathbb{R}_+)$. By Fubini's theorem,

$$Hf(x) = \frac{1}{x}\int_0^x dt \int_t^x \frac{f(u)}{u}\,du \leq \frac{1}{x}\int_0^x H^*f(t)\,dt.$$

2010 *Mathematics Subject Classification.* Primary 26D10, 26D15; Secondary 46E30.

Key words and phrases. Hardy operator; Dual operator; Best constants.

On the other hand, Fubini's theorem gives that

$$H^*f(x) = \int_x^\infty \frac{du}{u^2}\int_x^u f(t)\,dt \le \int_x^\infty \frac{Hf(u)}{u}\,du.$$

Using these estimates and applying Hardy's inequalities [5, p. 240, 244], we obtain that

$$\frac{1}{p'}||Hf||_p \le ||H^*f||_p \le p||Hf||_p \quad \text{for} \quad 1<p<\infty \tag{1.1}$$

(as usual, $p' = p/(p-1)$).

However, the constants in (1.1) are not optimal. The objective of this paper is to find optimal constants. Our main result is the following theorem.

Theorem 1.1. *Let $f \in \mathcal{M}^+(\mathbb{R}_+)$ and let $1<p<\infty$. Then*

$$(p-1)||Hf||_p \le ||H^*f||_p \le (p-1)^{1/p}||Hf||_p \tag{1.2}$$

if $1<p\le 2$, and

$$(p-1)^{1/p}||Hf||_p \le ||H^*f||_p \le (p-1)||Hf||_p \tag{1.3}$$

if $2\le p<\infty$. All constants in (1.2) and (1.3) are the best possible.

Clearly, the problem on relations between various norms of Hardy operator and its dual is of independent interest (cf. [4]). At the same time, this problem has an equivalent formulation in terms of the difference operator $H\varphi - \varphi$.

Let φ be a nonincreasing and nonnegative function on $\mathbb{R}_+$ such that $\varphi(+\infty)=0$. The quantity $H\varphi-\varphi$ plays an important role in Analysis (see [2], [3], [4], [6], [7] and references therein). It is well known that the norms $||H\varphi-\varphi||_p$ and $||\varphi||_p$ $(1<p<\infty)$ are equivalent (see [1, p. 384]). However, the *sharp* constant is known only in the following inequality.

Let φ be a nonincreasing and nonnegative function on $\mathbb{R}_+$. Then for any $p\ge 2$

$$||H\varphi-\varphi||_p \le (p-1)^{-1/p}||\varphi||_p, \tag{1.4}$$

and the constant is optimal.

This result was obtained in [7] for $p=2k$ $(k\in\mathbb{N})$ and in [2] for all $p\ge 2$ (we observe that (1.4) is a special case of the inequality proved in [2] for weighted L^p-norms).

We shall show that inequality (1.4) is equivalent to the first inequality in (1.3):

$$||Hf||_p \le (p-1)^{-1/p}||H^*f||_p, \quad 2\le p<\infty. \tag{1.5}$$

Thus, (1.5) can be derived from (1.4). However, below we give a simple direct proof of (1.5). Moreover, Theorem 1.1 has the following equivalent form.

Theorem 1.2. *Let φ be a nonincreasing and nonnegative function on $\mathbb{R}_+$ such that $\varphi(+\infty) = 0$ and let $1 < p < \infty$. Then*

$$(p-1)||H\varphi - \varphi||_p \leq ||\varphi||_p \leq (p-1)^{1/p}||H\varphi - \varphi||_p \tag{1.6}$$

if $1 < p \leq 2$, and

$$(p-1)^{1/p}||H\varphi - \varphi||_p \leq ||\varphi||_p \leq (p-1)||H\varphi - \varphi||_p \tag{1.7}$$

if $2 \leq p < \infty$. All constants in (1.6) and (1.7) are the best possible.

2. Proofs of main results

Proof of Theorem 1.1. Taking into account (1.1), we may assume that Hf and H^*f belong to $L^p(\mathbb{R}_+)$. We may also assume that $f(x) > 0$ for all $x \in \mathbb{R}_+$. Denote

$$I_p = \int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx.$$

Since $Hf \in L^p(\mathbb{R}_+)$, we have

$$Hf(x) = o(x^{-1/p}) \quad \text{as} \quad x \to 0+ \quad \text{or} \quad x \to +\infty.$$

Thus, integrating by parts, we obtain

$$I_p = p' \int_0^\infty x^{1-p} f(x) \left(\int_0^x f(t)\, dt \right)^{p-1} dx. \tag{2.1}$$

Further, set

$$I_p^* = \int_0^\infty \left(\int_t^\infty \frac{f(x)}{x}\, dx \right)^p dt. \tag{2.2}$$

First we shall prove that

$$(p-1)I_p \leq I_p^* \quad \text{if} \quad 2 \leq p < \infty \tag{2.3}$$

and

$$I_p^* \leq (p-1)I_p \quad \text{if} \quad 1 < p \leq 2. \tag{2.4}$$

Set

$$\Phi(t,x) = \int_t^x \frac{f(u)}{u}\, du,\ 0 < t \leq x,$$

and $G(t,x) = \Phi(t,x)^p$. Since $G(t,t) = 0$, we have

$$\left(\int_t^\infty \frac{f(x)}{x}\, dx \right)^p = \int_t^\infty G'_x(t,x)\, dx = p \int_t^\infty \frac{f(x)}{x} \Phi(t,x)^{p-1}\, dx.$$

Thus, by Fubini's theorem,

$$I_p^* = p\int_0^\infty \int_t^\infty \frac{f(x)}{x}\Phi(t,x)^{p-1}\,dx\,dt$$
$$= p\int_0^\infty \frac{f(x)}{x}\int_0^x \Phi(t,x)^{p-1}\,dtdx. \tag{2.5}$$

On the other hand, Fubini's theorem gives that

$$\int_0^x f(t)\,dt = \int_0^x \Phi(t,x)\,dt.$$

Hence, by (2.1),

$$I_p = p'\int_0^\infty x^{1-p}f(x)\left(\int_0^x \Phi(t,x)\,dt\right)^{p-1}\,dx. \tag{2.6}$$

Comparing (2.1) with (2.2), we see that $I_2 = I_2^*$. In what follows we assume that $p \neq 2$.

Let $p > 2$. Then by Hölder's inequality

$$\left(\int_0^x \Phi(t,x)\,dt\right)^{p-1} \le x^{p-2}\int_0^x \Phi(t,x)^{p-1}\,dt.$$

Thus, by (2.5) and (2.6),

$$I_p \le p'\int_0^\infty \frac{f(x)}{x}\int_0^x \Phi(t,x)^{p-1}\,dtdx = \frac{I_p^*}{p-1},$$

and we obtain (2.3).

Let now $1 < p < 2$. Applying Hölder's inequality, we get

$$\int_0^x \Phi(t,x)^{p-1}\,dt \le x^{2-p}\left(\int_0^x \Phi(t,x)\,dt\right)^{p-1}.$$

Thus, by (2.5) and (2.6),

$$I_p^* \le p\int_0^\infty x^{1-p}f(x)\left(\int_0^x \Phi(t,x)\,dt\right)^{p-1}\,dx = (p-1)I_p,$$

and we obtain (2.4).

Inequalities (2.3) and (2.4) imply the first inequality in (1.3) and the second inequality in (1.2), respectively.

Now we shall show that

$$I_p^* \le (p-1)^p I_p \quad \text{if} \quad 2 < p < \infty \tag{2.7}$$

and

$$(p-1)^p I_p \le I_p^* \quad \text{if} \quad 1 < p < 2. \tag{2.8}$$

Observe that by our assumption ($f > 0$ and $H^* f \in L^p(\mathbb{R}^*)$),

$$0 < \int_t^\infty \frac{f(x)}{x}\,dx < \infty \quad \text{for all} \quad t > 0.$$

Thus, for any $q > 0$ we have

$$\left(\int_t^\infty \frac{f(x)}{x}\,dx\right)^q = q\int_t^\infty \frac{f(x)}{x}\left(\int_x^\infty \frac{f(u)}{u}\,du\right)^{q-1} dx. \tag{2.9}$$

Applying this equality with $q = p$ in (2.2) and using Fubini's theorem, we obtain

$$I_p^* = p\int_0^\infty f(x)\left(\int_x^\infty \frac{f(u)}{u}\,du\right)^{p-1} dx. \tag{2.10}$$

Further, apply (2.9) for $q = p-1$ and use again Fubini's theorem. This gives

$$\begin{aligned} I_p^* &= p(p-1)\int_0^\infty f(x)\int_x^\infty \frac{f(u)}{u}\left(\int_u^\infty \frac{f(v)}{v}\,dv\right)^{p-2} du\,dx \\ &= p(p-1)\int_0^\infty \frac{f(u)}{u}\left(\int_u^\infty \frac{f(v)}{v}\,dv\right)^{p-2}\int_0^u f(x)\,dx\,du. \end{aligned}$$

Set

$$\varphi(u) = \frac{f(u)^{1/(p-1)}}{u}\int_0^u f(x)\,dx$$

and

$$\psi(u) = f(u)^{(p-2)/(p-1)}\left(\int_u^\infty \frac{f(x)}{x}\,dx\right)^{p-2}$$

(recall that $f > 0$). Then we have

$$I_p^* = p(p-1)\int_0^\infty \varphi(u)\psi(u)\,du. \tag{2.11}$$

Furthermore, by (2.1),

$$\int_0^\infty \varphi(u)^{p-1}\,du = \int_0^\infty \frac{f(u)}{u^{p-1}}\left(\int_0^u f(x)\,dx\right)^{p-1} du = \frac{I_p}{p'}, \tag{2.12}$$

and by (2.10),

$$\int_0^\infty \psi(u)^{(p-1)/(p-2)}\,du = \int_0^\infty f(u)\left(\int_u^\infty \frac{f(x)}{x}\,dx\right)^{p-1} du = \frac{I_p^*}{p} \tag{2.13}$$

for any $p > 1$, $p \neq 2$.

Let $p>2$. Applying in (2.11) Hölder's inequality with the exponent $p-1$ and taking into account equalities (2.12) and (2.13), we obtain

$$I_p^* \le p(p-1)\left(\frac{I_p}{p'}\right)^{1/(p-1)}\left(\frac{I_p^*}{p}\right)^{(p-2)/(p-1)}.$$

This implies (2.7), which is the second inequality in (1.3).

Let now $1<p<2$. Applying in (2.11) Hölder's inequality with the exponent $p-1\in(0,1)$ (see [5, p. 140]), and using equalities (2.12) and (2.13), we get

$$I_p^* \ge p(p-1)\left(\frac{I_p}{p'}\right)^{1/(p-1)}\left(\frac{I_p^*}{p}\right)^{(p-2)/(p-1)}.$$

Thus,

$$(I_p^*)^{1/(p-1)} \ge (p-1)^{p/(p-1)} I_p^{1/(p-1)}.$$

This implies (2.8), which is the first inequality in (1.2).

It remains to show that the constants in (1.2) and (1.3) are optimal. First, set $f_\varepsilon(x)=\chi_{[1,1+\varepsilon]}(x)$ $(\varepsilon>0)$. Then

$$||Hf_\varepsilon||_p^p = \int_1^{1+\varepsilon} x^{-p}(x-1)^p\,dx + \varepsilon^p\int_{1+\varepsilon}^\infty x^{-p}\,dx.$$

Thus,

$$\frac{\varepsilon^p(1+\varepsilon)^{1-p}}{p-1} \le ||Hf_\varepsilon||_p^p \le \frac{\varepsilon^p(1+\varepsilon)^{1-p}}{p-1}+\varepsilon^{p+1}.$$

Further,

$$\begin{aligned}||H^*f_\varepsilon||_p^p &= \int_0^1\left(\int_1^{1+\varepsilon}\frac{dt}{t}\right)^p dx + \int_1^{1+\varepsilon}\left(\int_x^{1+\varepsilon}\frac{dt}{t}\right)^p dx\\ &= (\ln(1+\varepsilon))^p + \int_1^{1+\varepsilon}\left(\ln\frac{1+\varepsilon}{x}\right)^p dx.\end{aligned}$$

Thus,

$$(\ln(1+\varepsilon))^p \le ||H^*f_\varepsilon||_p^p \le (\ln(1+\varepsilon))^p(1+\varepsilon).$$

Using these estimates, we obtain that

$$\lim_{\varepsilon\to 0+}\frac{||Hf_\varepsilon||_p}{||H^*f_\varepsilon||_p} = (p-1)^{-1/p}.$$

It follows that the constants in the right-hand side of (1.2) and the left-hand side of (1.3) cannot be improved.

Let $1<p<2$. Set $f_\varepsilon(x)=x^{\varepsilon-1/p}\chi_{[0,1]}(x)$ $(0<\varepsilon<1/p)$. Then

$$||Hf_\varepsilon||_p^p \ge \int_0^1\left(\frac{1}{x}\int_0^x t^{\varepsilon-1/p}\,dt\right)^p dx = \frac{p^p}{\varepsilon p(p-1+\varepsilon p)^p}.$$

On the other hand,

$$||H^*f_\varepsilon||_p^p \le \left(\frac{1}{p}-\varepsilon\right)^{-p}\int_0^1 x^{(\varepsilon-1/p)p}\,dx = \frac{p^p}{\varepsilon p(1-\varepsilon p)^p}.$$

Hence,

$$\varliminf_{\varepsilon\to 0+}\frac{||Hf_\varepsilon||_p}{||H^*f_\varepsilon||_p} \ge \frac{1}{p-1}.$$

This implies that the constant in the left-hand side of (1.2) is optimal.

Let now $p>2$. Set $f_\varepsilon(x)=x^{-\varepsilon-1/p}\chi_{[1,+\infty)}(x)$ $(0<\varepsilon<1/p')$. Then

$$||H^*f_\varepsilon||_p^p \ge \int_1^\infty\left(\int_x^\infty \frac{dt}{t^{1+1/p+\varepsilon}}\right)^p dx = \frac{p^p}{\varepsilon p(1+\varepsilon p)^p}$$

and

$$||Hf_\varepsilon||_p^p \le \int_1^\infty\left(\frac{1}{x}\int_0^x \frac{dt}{t^{1/p+\varepsilon}}\right)^p dx = \frac{p^p}{\varepsilon p(p-1-\varepsilon p)^p}.$$

Thus,

$$\varliminf_{\varepsilon\to 0+}\frac{||H^*f_\varepsilon||_p}{||Hf_\varepsilon||_p} \ge p-1.$$

This shows that the constant in the right-hand side of (1.3) is the best possible. The proof is completed.

Remark 2.1. We emphasize that in Theorem 1.1 we do not assume that f belongs to $L^p(\mathbb{R}_+)$. It is clear that the condition $Hf\in L^p(\mathbb{R}_+)$ does not imply that $f\in L^p(\mathbb{R}_+)$. For example, let $f(x)=|x-1|^{-1/p}\chi_{[1,2]}(x)$, $p>1$. Then

$$Hf(x)=0 \quad \text{for} \quad x\in[0,1] \quad \text{and} \quad Hf(x)\le\frac{p'}{x} \quad \text{for} \quad x\ge 1.$$

Thus, $Hf\in L^p(\mathbb{R}_+)$, but $f\notin L^p(\mathbb{R}_+)$.

Now we shall show that Theorems 1.1 and 1.2 are equivalent. First we observe that without loss of generality we may assume that a function φ in Theorem 1.2 is locally absolutely continuous on $\mathbb{R}_+$. Indeed, let φ be a nonincreasing and nonnegative function on $\mathbb{R}_+$ such that $\varphi(+\infty)=0$. Set

$$\varphi_n(x)=n\int_x^{x+1/n}\varphi(t)\,dt \quad (n\in\mathbb{N}).$$

Then functions φ_n are nonincreasing, nonnegative, and locally absolutely continuous on $\mathbb{R}_+$. Besides, the sequence $\{\varphi_n(x)\}$ increases for any $x\in\mathbb{R}_+$ and converges to $\varphi(x)$ at every point of continuity of φ. By the monotone convergence theorem, $H\varphi_n(x)\to H\varphi(x)$ as $n\to\infty$ for any $x\in\mathbb{R}_+$, and $||\varphi_n||_p\to||\varphi||_p$. Furthermore, in Theorem 1.2 we may assume that $\varphi \in L^p(\mathbb{R}_+)$ (in conditions of this theorem the norms $||H\varphi - \varphi||_p$ and $||\varphi||_p$ are equivalent [1, p. 384]). Using this assumption, Hardy's inequality, and the dominated convergence theorem, we obtain that $||H\varphi_n - \varphi_n||_p \to ||H\varphi - \varphi||_p$.

Let φ be a nonincreasing, nonnegative, and locally absolutely continuous function on $\mathbb{R}_+$ such that $\varphi(+\infty) = 0$. Then

$$H\varphi(x) - \varphi(x) = \frac{1}{x}\int_0^x [\varphi(t) - \varphi(x)]\, dt$$

$$= \frac{1}{x}\int_0^x \int_t^x |\varphi'(u)|\, du dt = \frac{1}{x}\int_0^x u|\varphi'(u)|\, du.$$

Set $u|\varphi'(u)| = f(u)$. Since $\varphi(+\infty) = 0$, we have

$$\varphi(x) = \int_x^\infty |\varphi'(u)|\, du = \int_x^\infty \frac{f(u)}{u}\, du.$$

Thus,

$$H\varphi(x) - \varphi(x) = \frac{1}{x}\int_0^x f(u)\, du = Hf(x) \tag{2.14}$$

and

$$\varphi(x) = \int_x^\infty \frac{f(u)}{u}\, du = H^* f(x). \tag{2.15}$$

Conversely, if $f \in \mathcal{M}^+(\mathbb{R}_+)$ and

$$\int_0^x f(u)\, du < \infty \quad \text{for any} \quad x > 0,$$

we define φ by (2.15) and then we have equality (2.14). These arguments show the equivalence of Theorems 1.1 and 1.2.

References

[1] C. Bennett and R. Sharpley, *Interpolation of operators,* Academic Press, Boston 1988.

[2] S. Boza and J. Soria, *Solution to a conjecture on the norm of the Hardy operator minus the identity,* J. Funct. Anal. **260** (2011), 1020 – 1028.

[3] M. Carro, A. Gogatishvili, J. Martín and L. Pick, *Functional properties on rearrangement invariant spaces defined in terms of oscillations,* J. Funct. Anal. **229** (2005), 375 – 404.

[4] M. Carro, A. Gogatishvili, J. Martín and L. Pick, *Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces,* J. Operator Theory **59** (2008), 309 – 332.

[5] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities,* 2nd ed., Cambridge University Press, Cambridge, 1967.

[6] V.I. Kolyada, *On embedding theorems*, in: Nonlinear Analysis, Function Spaces and Applications, vol. 8 (Proceedings of the Spring School held in Prague, 2006), Prague, 2007, 35 - 94.

[7] N. Kruglyak and E. Setterqvist, *Sharp estimates for the identity minus Hardy operator on the cone of decreasing functions,* Proc. Amer. Math. Soc. **136** (2008), 2005 – 2013.

Department of Mathematics, Karlstad University, Universitetsgatan 1, 651 88 Karlstad, SWEDEN

E-mail address: viktor.kolyada@kau.se