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REPRESENTATIONS OF THE KAUFFMAN SKEIN ALGEBRA II: PUNCTURED SURFACES

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Abstract. In earlier work [BoW$_3$], we constructed invariants of irreducible representations of the Kauffman skein algebra of a surface. We introduce here an inverse construction, which to a set of possible invariants associates an irreducible representation that realizes these invariants. The current article is restricted to surfaces with at least one puncture, a condition that will be lifted in subsequent work [BoW$_4$] that relies on this one. A step in the proof is of independent interest, and describes the algebraic structure of the Thurston intersection form on the space of integer weight systems for a train track.

This article is a continuation of [BoW$_3$] and is part of the program described in [BoW$_2$], devoted to the analysis and construction of finite-dimensional representations of the skein algebra of a surface.

Let S be an oriented surface of finite topological type without boundary. The *Kauffman skein algebra* $\mathcal{S}^A(S)$ depends on a parameter $A = \mathrm{e}^{\pi \mathrm{i}\hbar} \in \mathbb{C} - \{0\}$, and is defined as follows: one first considers the vector space freely generated by all isotopy classes of framed links in the thickened surface $S \times [0,1]$, and then one takes the quotient of this space by the skein relation that

$$[K_1] = A^{-1}[K_0] + A[K_\infty]$$

whenever the three links K_1, K_0 and $K_\infty \subset S \times [0,1]$ differ only in a little ball where they are as represented in Figure 1. The algebra multiplication is provided by the operation of superposition; see §1.

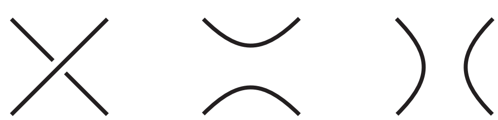

Figure 1. A Kauffman triple

Turaev [Tu], Bullock-Frohman-Kania-Bartoszyńska [BFK$_1$, BFK$_2$] and Przytycki-Sikora [PrS] showed that the skein algebra $\mathcal{S}^A(S)$ provides a quantization of the *character variety*

$$\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) = \{\text{group homomorphisms } r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})\} /\!\!/ \mathrm{SL}_2(\mathbb{C})$$

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where $\mathrm{SL}_2(\mathbb{C})$ acts on homomorphisms by conjugation, and where the double bar indicates that the quotient is to be taken in the sense of geometric invariant theory [MFK]. In fact, if one follows the physical tradition that a quantization of a space X replaces the commutative algebra of functions on X by a non-commutative algebra of operators on a Hilbert space, the actual quantization should be a *representation* of the skein algebra.

In [BoW₃], we identified invariants for irreducible representations $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$, in the case where A^2 is a primitive N–root of unity with N odd. Since $A^{2N} = 1$, then $A^N = \pm 1$ and we distinguish cases accordingly.

When $A^N = -1$, our main invariant is a point of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. This is one more example of a situation where a quantum object determines one of the classical objects that are being quantized. When $A^N = +1$, the invariant belongs to a certain *twisted character variety* $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, precisely defined in §4, whose elements are represented by group homomorphisms $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ endowed with some additional spin information.

Given a point $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and a closed curve K on the surface S, the trace $\mathrm{Tr}\, r(K) \in \mathbb{C}$ is independent of the homomorphism $\pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ that we use to represent r, or of the representative that we choose in the conjugacy class of $\pi_1(S)$ representing K. The definition of $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ is such that, if K is a framed curve in $S \times [0,1]$, there is also a well-defined trace $\mathrm{Tr}\, r(K)$ for every $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$. The spin information and the framing are used to resolve the sign ambiguity in defining the trace of $r(K) \in \mathrm{PSL}_2(\mathbb{C})$; see §4.

The definition of our main invariant involves the N–th normalized Chebyshev polynomial T_N of the first kind, determined by the trigonometric identity that $2\cos N\theta = T_N(2\cos\theta)$. Equivalently, $\mathrm{Tr}\, B^N = T_N(\mathrm{Tr}\, B)$ for every matrix $B \in \mathrm{SL}_2(\mathbb{C})$.

Theorem 1 ([BoW₃])**.** *Suppose that A^2 is a primitive N–root of unity with N odd (so that $A^N = \pm 1$), and let $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ be an irreducible representation of the Kauffman skein algebra.*

(1) *There exists a unique point r_ρ of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if $A^N = -1$, and a unique point r_ρ of the twisted character variety $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ if $A^N = +1$, such that*

$$T_N\big(\rho([K])\big) = -\big(\mathrm{Tr}\, r_\rho(K)\big)\mathrm{Id}_V$$

for every framed knot $K \subset S \times [0,1]$ whose projection to S has no crossing and whose framing is vertical.

(2) *Let P_k be a small loop going around the k–th puncture of S, and consider it as a knot in $S \times [0,1]$ with vertical framing. Then there exists a number $p_k \in \mathbb{C}$ such that $\rho\big([P_k]\big) = p_k \mathrm{Id}_V$.*

(3) *The number p_k of (2) is related to the element $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ of (1) by the property that $T_N(p_k) = -\mathrm{Tr}\, r_\rho(P_k)$.*

The point $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ associated to the irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ by Part (1) of Theorem 1 is the *classical shadow* of ρ. The numbers p_k defined by Part (2) are its *puncture invariants*. Part (3) shows that, once the classical shadow r_ρ is known, there are at most N possible values for each of the puncture invariants p_k.

To an irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ we have thus associated a classical point r_ρ in $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, plus finitely many puncture invariants $p_k \in \mathbb{C}$.

The main result of this article is the following converse statement.

Theorem 2. *Assume that the surface S has at least one puncture, that its Euler characteristic is negative, that A^2 is a primitive N–root of unity with N odd, and that we are given:*

(1) *a point $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ if $A^N = +1$, or a point $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if $A^N = -1$, such that r realizes some ideal triangulation of S in the sense discussed in §5;*
(2) *a number $p_k \in \mathbb{C}$ such that $T_N(p_k) = -\mathrm{Tr}\, r(P_k)$ for each of the punctures of S.*

Then, there exists an irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ whose classical shadow is equal to r and whose puncture invariants are the p_k.

The requirement that r realizes some ideal representation is fairly mild. It can be shown to be satisfied by all points outside of an algebraic subset of complex codimension $2|\chi(S)| - 1$ in the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$.

The sequel [BoW$_4$] to this paper will greatly improve Theorem 2. In particular, it will remove the requirements that r realizes an ideal triangulation, and that S has at least one puncture. More importantly, we will arrange that the representation ρ provided by our proof is independent of the many choices made during the argument, so that the construction is natural, in particular with respect to the action of the mapping class group on the character varieties $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$.

The proof of Theorem 2 uses as a key tool the embedding $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$ of the skein algebra in the quantum Teichmüller space constructed in [BoW$_1$]. The *quantum Teichmüller space* is here incarnated as the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$ of an ideal triangulation λ, and is a quantization of an object that is closely related to the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. It was introduced by L. Chekhov and V. Fock [Fo, ChF$_1$, ChF$_2$] (see also [Ka] for a related construction, and [BoL, Liu] for more discussion), and quantizes a space of homomorphisms $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ endowed with additional fixed point information at each puncture. It is not as natural as the Kauffman skein algebra, but its algebraic structure is very simple. In particular, its representation theory is relatively easy to analyze [BoL]. Composing representations of the quantum Teichmüller space $\mathcal{Z}^\omega(\lambda)$ with the homomorphism $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$ provides an extensive family of representations of the skein algebra $\mathcal{S}^A(S)$.

The main technical challenge in this strategy is to compute the classical shadow of the representations of $\mathcal{S}^A(S)$ so obtained, in terms of the parameters controlling the original representations of $\mathcal{Z}^\omega(\lambda)$. This is provided by the miraculous cancellations discovered in [BoW$_3$]. These properties show that the quantum trace homomorphism $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$ is well-behaved with respect to the Chebyshev homomorphism $\mathcal{S}^\varepsilon(S) \to \mathcal{S}^A(S)$ used to define the classical shadow of a representation of $\mathcal{S}^A(S)$, and with respect to the Frobenius homomorphism $\mathcal{Z}^\iota(\lambda) \to \mathcal{Z}^\omega(\lambda)$ which computes the invariants of representations of $\mathcal{Z}^\omega(\lambda)$.

One of the steps in the proof may be of interest by itself. This statement describes the algebraic structure of the Thurston intersection form on the set $\mathcal{W}(\tau; \mathbb{Z})$ of integer-valued edge weight systems for a train track τ. The result is well-known for real-valued weights. However, the integer valued case has subtler number theoretic properties, resulting in the unexpected simultaneous occurrence of blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$. Because of the ubiquity of the Thurston intersection form in many geometric problems, this statement is probably of interest beyond the quantum topology scope of the current article. Its proof is given in the Appendix.

1. The Kauffman skein algebra

Let S be an oriented surface (without boundary) with finite topological type. Namely, S is obtained by removing finitely many points from a compact oriented surface $\bar{S}$. We consider *framed links* in the thickened surface $S \times [0, 1]$, namely unoriented 1–dimensional submanifolds $K \subset S \times [0, 1]$ endowed with a continuous choice of a vector transverse to K at each point of K. A *framed knot* is a connected framed link.

The *framed link algebra* $\mathcal{K}(S)$ is the vector space (over $\mathbb{C}$, say) freely generated by the isotopy classes of all framed links $K \subset S \times [0, 1]$.

The vector space $\mathcal{K}(S)$ can be endowed with a multiplication, where the product of K_1 and K_2 is defined by the framed link $K_1 \cdot K_2 \subset S \times [0, 1]$ that is the union of K_1 rescaled in $S \times [0, \frac{1}{2}]$ and K_2 rescaled in $S \times [\frac{1}{2}, 1]$. In other words, the product $K_1 \cdot K_2$ is defined by superposition of the framed links K_1 and K_2. This *superposition operation* is compatible with isotopies, and therefore provides a well-defined algebra structure on $\mathcal{K}(S)$.

Three links K_1, K_0 and K_∞ in $S \times [0, 1]$ form a *Kauffman triple* if the only place where they differ is above a small disk in S, where they are as represented in Figure 1 (as seen from above) and where the framing is vertical and pointing upwards (namely the framing is parallel to the $[0, 1]$ factor and points towards 1).

For $A \in \mathbb{C} - \{0\}$, the *Kauffman skein algebra* $\mathcal{S}^A(S)$ is the quotient of the framed link algebra $\mathcal{K}(K)$ by the two-sided ideal generated by all elements $K_1 - A^{-1}K_0 - AK_\infty$ as (K_1, K_0, K_∞) ranges over all Kauffman triples. The superposition operation descends to a multiplication in $\mathcal{S}^A(S)$, endowing $\mathcal{S}^A(S)$ with the structure of an algebra. The class $[\varnothing]$ of the empty link is an identity element in $\mathcal{S}^A(S)$.

An element $[K] \in \mathcal{S}^A(S)$, represented by a framed link $K \subset S \times [0, 1]$, is a *skein* in S. The construction is defined to ensure that the *skein relation*

$$[K_1] = A^{-1}[K_0] + A[K_\infty]$$

holds in $\mathcal{S}^A(S)$ for every Kauffman triple (K_1, K_0, K_∞).

2. The Chekhov-Fock algebra and the quantum trace homomorphism

2.1. **The Chekhov-Fock algebra.**

The Chekhov-Fock algebra (defined in [BoL] and inspired by [ChF$_1$, ChF$_2$, Fo]) is the avatar of the quantum Teichmüller space associated to an ideal triangulation of the surface S. If S is obtained from a compact surface $\bar{S}$ by removing finitely many points v_1, v_2, …, v_s, an *ideal triangulation* of S is a triangulation λ of $\bar{S}$ whose vertex set is exactly $\{v_1, v_2, \dots, v_s\}$. The surface admits an ideal triangulation if and only if it admits at least one puncture, and its Euler characteristic is negative; we will consequently assume these properties satisfied throughout the article. If the surface has genus g and s punctures, an ideal triangulation then has $n = 6g + 3s - 6$ edges and $4g + 2s - 4$ faces.

Let λ_1, λ_2, …, λ_n denote the edges of λ. Let $a_i \in \{0,1,2\}$ be the number of times an end of the edge λ_j immediately succeeds an end of λ_i when going counterclockwise around a puncture of S, and set $\sigma_{ij} = a_{ij} - a_{ji} \in \{-2,-1,0,1,2\}$. The *Chekhov-Fock algebra* $\mathcal{T}^\omega(\lambda)$ of λ is the algebra defined by generators $Z_1^{\pm 1}$, $Z_2^{\pm 1}$, …, $Z_n^{\pm 1}$ associated to the edges λ_1, λ_2, …, λ_n of λ, and by the relations

$$Z_i Z_j = \omega^{2\sigma_{ij}} Z_j Z_i.$$

(The actual Chekhov-Fock algebra $\mathcal{T}^q(\lambda)$ that is at the basis of the quantum Teichmüller space uses the constant $q = \omega^4$ instead of ω. The generators Z_i of $\mathcal{T}^\omega(\lambda)$ appearing here are designed to model square roots of the original generators of $\mathcal{T}^q(\lambda)$.)

An element of the Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$ is a linear combination of monomials $Z_{i_1}^{n_1} Z_{i_2}^{n_2} \dots Z_{i_l}^{n_l}$ in the generators Z_i, with n_1, n_2, …, $n_l \in \mathbb{Z}$. Because of the skew-commutativity relation $Z_i Z_j = \omega^{2\sigma_{ij}} Z_j Z_i$, the order of the variables in such a monomial does matter. It is convenient to use the following symmetrization trick.

The *Weyl quantum ordering* for $Z_{i_1}^{n_1} Z_{i_2}^{n_2} \dots Z_{i_l}^{n_l}$ is the monomial

$$[Z_{i_1}^{n_1} Z_{i_2}^{n_2} \dots Z_{i_l}^{n_l}] = \omega^{-\sum_{u<v} n_u n_v \sigma_{i_u i_v}} Z_{i_1}^{n_1} Z_{i_2}^{n_2} \dots Z_{i_l}^{n_l}.$$

The formula is specially designed that $[Z_{i_1}^{n_1} Z_{i_2}^{n_2} \dots Z_{i_l}^{n_l}]$ is invariant under any permutation of the $Z_{i_u}^{n_u}$.

2.2. **The quantum trace homomorphism.**

Theorem 3 ([BoW$_1$])**.** *For $A = \omega^{-2}$, there exists an injective algebra homomorphism*

$$\mathrm{Tr}_\lambda^\omega \colon \mathcal{S}_{\mathrm{s}}^A(S) \to \mathcal{T}^\omega(\lambda).$$

The specific homomorphism $\mathrm{Tr}_\lambda^\omega$ constructed in [BoW$_1$] is the *quantum trace homomorphism*. It is uniquely determined by certain properties stated in that article, but we will only need to know that it exists and that it satisfies the properties given in §2.3 below.

2.3. **The Chebyshev and Frobenius homomorphisms.** We now assume that A^2 is a primitive N–root of unity with N odd. Set $\varepsilon = A^N = \pm 1$. Recall that T_N denotes the N–th normalized Chebyshev polynomial, defined by the property that $\cos N\theta = \frac{1}{2} T_N(2\cos\theta)$ for every θ.

Theorem 4 ([BoW$_3$])**.** *When A^2 is a primitive N–root of unity with N odd and $\varepsilon = A^N = \pm 1$, there is a unique algebra homomorphism $\mathbf{T}^A \colon \mathcal{S}^\varepsilon(S) \to \mathcal{S}^A(S)$ such that*

$$\mathbf{T}^A\big([K]\big) = T_N\big([K]\big)$$

for every framed knot $K \subset S \times [0,1]$ whose projection to S has no crossing and whose framing is vertical. In addition, the image of $\mathbf{T}^A$ is central in $\mathcal{S}^A(S)$. □

The homomorphism $\mathbf{T}^A$ provided by Proposition 4 is the *Chebyshev homomorphism*. It is a key ingredient in the definition of the invariants of Theorem 1.

There is an analogous and much simpler homomorphism at the level of the Chekhov-Fock algebra, namely the following *Frobenius homomorphism*.

Proposition 5. *If $\iota = \omega^{N^2}$, there is an algebra homomorphism*

$$\mathbf{F}^\omega \colon \mathcal{T}^\iota(\lambda) \to \mathcal{T}^\omega(\lambda)$$

which maps each generator $Z_i \in \mathcal{T}^\iota(\lambda)$ to $Z_i^N \in \mathcal{T}^\omega(\lambda)$, where in the first instance $Z_i \in \mathcal{T}^\iota(\lambda)$ denotes the generator associated to the i–th edge λ_i of λ, whereas the second time $Z_i \in \mathcal{T}^\omega(\lambda)$ denotes the generator of $\mathcal{T}^\omega(\lambda)$ associated to the same edge λ_i. □

The key to our construction is the following compatibility result, which connects the Chebyshev homomorphism to the Frobenius homomorphism through appropriate the quantum trace homomorphism.

Theorem 6 ([BoW$_3$])**.** *The diagram*

$$\begin{array}{ccc} \mathcal{S}^A(S) & \xrightarrow{\mathrm{Tr}_\lambda^\omega} & \mathcal{T}^\omega(\lambda) \\ \mathbf{T}^A \Big\uparrow & & \Big\uparrow \mathbf{F}^\omega \\ \mathcal{S}^\varepsilon(S) & \xrightarrow{\mathrm{Tr}_\lambda^\iota} & \mathcal{T}^\iota(\lambda) \end{array}$$

is commutative. Namely, for every skein $[K] \in \mathcal{S}^\varepsilon(S)$, the quantum trace $\mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big)$ of $[K^{T_N}] = \mathbf{T}^A\big([K]\big)$ is obtained from the classical trace polynomial $\mathrm{Tr}_\lambda^\iota\big([K]\big)$ by replacing each generator $Z_i \in \mathcal{T}^\iota(\lambda)$ by $Z_i^N \in \mathcal{T}^\omega(\lambda)$.

3. The balanced Chekhov-Fock algebra

3.1. **The balanced Chekhov-Fock algebra.** The quantum trace map $\mathrm{Tr}_\lambda^\omega$ of Theorem 3 (and [BoW$_1$]) is far from being surjective. Indeed, for a skein $[K] \in \mathcal{S}^A(S)$, the exponents of the monomials $Z_1^{k_1} Z_2^{k_2} \dots Z_n^{k_n}$ appearing in the expression of $\mathrm{Tr}_\lambda^\omega\big([K]\big)$ are *balanced*, in the sense that they satisfy the following parity condition: for every triangle T_j of the ideal triangulation λ, the sum $k_{i_1} + k_{i_2} + k_{i_3}$ of the exponents of the generators Z_{i_1}, Z_{i_2}, Z_{i_3} associated to the sides of T_j is even.

Let $\mathcal{Z}^\omega(\lambda)$ denote the sub-algebra of $\mathcal{T}^\omega(\lambda)$ generated by all monomials satisfying this parity condition. By definition, $\mathcal{Z}^\omega(\lambda)$ is the *balanced Chekhov-Fock algebra* of the ideal triangulation λ.

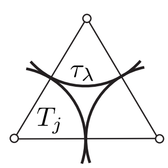

Figure 2.

To keep track of the parity condition, it is convenient to consider a train track τ_λ which, on each triangle T_j of the ideal triangulation λ, looks as in Figure 2. In particular, τ_λ has one switch for each edge of λ, and three edges for each triangle of λ. Let $\mathcal{W}(\tau_\lambda;\mathbb{Z})$ be the set of integer edge weight systems α for τ_λ, assigning a number $\alpha(e) \in \mathbb{Z}$ to each edge e of τ_λ in such a way that, at each switch, the weights of the edges incoming on one side add up to the sum of the weights of the edges outgoing on the other side. There is a natural map $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{Z}^n$ which, given an edge weight system, associates to each of the n switches of τ_λ the sum of the weights of the edges incoming on one side of the switch. Then, an element $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ satisfies the above parity condition if and only if it is in the image of this map. Also, the map $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{Z}^n$ is easily seen to be injective. Since the image of this map has finite index, it follows that $\mathcal{W}(\tau_\lambda;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^n$ as an abelian group.

This enables us to give a different description of $\mathcal{Z}^\omega(\lambda)$. For a weight system $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$, assigning a weight $\alpha_i \in \mathbb{Z}$ to the i–th edge λ_i of λ (= the i–th switch of τ_λ), define

$$Z_\alpha = [Z_1^{\alpha_1} Z_2^{\alpha_2} \dots Z_n^{\alpha_n}] \in \mathcal{Z}^\omega(\lambda)$$

where the bracket [ ] denotes the Weyl quantum ordering defined in §2.1.

The above discussion proves the following fact.

Lemma 7. *As $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ ranges over all weight systems for the train track τ_λ, the associated Z_α form a basis for the vector space $\mathcal{Z}^\omega(\lambda)$.* □

3.2. **The algebraic structure of the balanced Chekhov-Fock algebra.** We first describe the multiplicative structure of the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$ in the context of Lemma 7.

The space $\mathcal{W}(\tau_\lambda;\mathbb{Z})$ carries a very natural antisymmetric bilinear form

$$\Theta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \times \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{Z},$$

the *Thurston intersection form* defined by the property that, for α, $\beta \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$,

$$\Theta(\alpha,\beta) = \sum_{e \text{ right of } e'} \tfrac{1}{2}\big(\alpha(e)\beta(e') - \alpha(e')\beta(e)\big)$$

where the sum is over all pairs (e, e') of edges of τ_λ such that e comes out to the right of e' at some switch of τ_λ. See Lemma 22 in the Appendix for a more conceptual interpretation of Θ, and for a proof that $\Theta(\alpha,\beta)$ is really an integer.

Lemma 8. *For every α, $\beta \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$,*

$$Z_\alpha Z_\beta = \omega^{2\Theta(\alpha,\beta)} Z_{\alpha+\beta}.$$

In particular, $Z_\alpha Z_\beta = \omega^{4\Theta(\alpha,\beta)} Z_\beta Z_\alpha$.

Proof. The second statement, that $Z_\alpha Z_\beta = \omega^{4\Theta(\alpha,\beta)} Z_\beta Z_\alpha$, is a simple computation. The first statement, that $Z_\alpha Z_\beta = \omega^{2\Theta(\alpha,\beta)} Z_{\alpha+\beta}$, then follows by definition of the Weyl quantum ordering. □

This is particularly simple if we replace ω by $\iota = \omega^{N^2}$, with the assumption that $A^{2N} = 1$ so that $\iota^4 = \omega^{4N^2} = A^{-2N^2} = 1$.

Corollary 9. *If $\iota^4 = 1$, the algebra $\mathcal{Z}^\iota(\lambda)$ is commutative.* □

In general, the key to understanding the algebraic structure of $\mathcal{Z}^\omega(\lambda)$ is Lemma 10 below.

For $k = 1, \dots, s$, the k–th puncture of S specifies an element $\eta_k \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$, defined by the property that $\eta_k(e) \in \{0, 1, 2\}$ is the number of sides of the edge e that are adjacent to the same component of $S - \tau_\lambda$ as this puncture.

Recall that the surface S has genus g and s punctures.

Lemma 10. *The lattice $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \cong \mathbb{Z}^n$ admits a basis in which the matrix of the Thurston intersection form Θ is block diagonal with g blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $2g + s - 3$ blocks $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ and s blocks $\begin{pmatrix} 0 \end{pmatrix}$. In addition, the kernel of Θ is freely generated by the elements $\eta_1, \eta_2, \dots, \eta_s \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ associated to the punctures of S.*

Proof. This is a consequence of a more general result given by Theorem 20 in the Appendix, which computes the algebraic structure of the Thurston intersection form for a general train track τ. When applying this result to the train track τ_λ, the numbers h, n_{even} and n_{odd} of Theorem 20 are respectively equal to the genus g of S, to the number s of punctures of S, and to the number $4g+2s-4$ of triangles of the ideal triangulation λ. □

The combination of Lemmas 7, 8 and 10 now provides the complete algebraic structure of the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$. Let $\mathcal{W}^q$ denote the algebra, known as the quantum 2–torus, defined by generators $X^{\pm1}$, $Y^{\pm1}$ and by the relation $XY = qYX$.

Corollary 11. *For $q=\omega^4$, the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$ is isomorphic to*

$$\mathcal{W}_1^q \otimes \mathcal{W}_2^q \otimes \cdots \otimes \mathcal{W}_g^q \otimes \mathcal{W}_{g+1}^{q^2} \otimes \mathcal{W}_{g+2}^{q^2} \otimes \cdots \otimes \mathcal{W}_{3g+s-3}^{q^2} \otimes \mathbb{C}[H_1] \otimes \mathbb{C}[H_2] \otimes \cdots \otimes \mathbb{C}[H_s]$$

where each $\mathcal{W}_i^q$ is a copy of the quantum 2–torus $\mathcal{W}^q$, each $\mathcal{W}_j^{q^2}$ is a copy of $\mathcal{W}^{q^2}$, and each $\mathbb{C}[H_k]$ is a polynomial algebra in the variable H_k. □

In addition, the s generators $H_k = Z_{\eta_k}$ are associated to the punctures of S as in Lemma 10.

3.3. **Representations of the balanced Chekhov-Fock algebra.** The algebraic structure of the balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$ determined in Corollary 11 is relatively simple. This makes it easy to classify its irreducible representations.

As usual, we assume that $A^2=\omega^{-4}$ is a primitive N–root of unity. For every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$, Lemma 8 shows that the element Z_α^N is central in $\mathcal{Z}^\omega(\lambda)$. In particular, if $\rho\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(V)$ is an irreducible representation of $\mathcal{Z}^\omega(\lambda)$, there is a number $\zeta_\rho(\alpha) \in \mathbb{C}^*$ such that $\rho(Z_\alpha^N) = \zeta_\rho(\alpha)\,\mathrm{Id}_V$. This proves:

Lemma 12. *If ω^4 is a primitive N–root of unity, an irreducible representation $\rho\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(V)$ determines a group homomorphism $\zeta_\rho\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$, from $\mathcal{W}(\tau_\lambda;\mathbb{Z})$ to the multiplicative group $\mathbb{C}^* = \mathbb{C} - \{0\}$, by the property that*

$$\rho(Z_\alpha^N) = \zeta_\rho(\alpha)\,\mathrm{Id}_V$$

for every $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$. □

Note that the data of a group homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ is just a fancy way of expressing the fact that we are given a non-zero complex number for each of the n generators of $\mathcal{W}(\tau_\lambda;\mathbb{Z}) \cong \mathbb{Z}^n$.

The balanced Chekhov-Fock algebra $\mathcal{Z}^\omega(\lambda)$ admits s other central elements $H_k = Z_{\eta_k}$, coming from the elements $\eta_k \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ associated to the punctures of S as in Lemma 10 and Corollary 11. For each puncture of the surface S, this determines a number $h_k \in \mathbb{C}^*$ such that

$$\rho(H_k) = h_k\,\mathrm{Id}_V.$$

These numbers h_k are constrained by the group homomorphism $\zeta_\rho\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$, because $h_k^N = \zeta_\rho(\eta_k)$.

Proposition 13. *Assuming that ω^4 is a primitive N–root of unity with N odd, suppose that we are given a group homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ and, for each of the punctures of S, a number $h_k \in \mathbb{C}^*$ such that $h_k^N = \zeta_\rho(\eta_k)$. Then, up to isomorphism, there exists a unique irreducible representation $\rho\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(V)$ such that*

(1) $\zeta_\rho = \zeta$, *namely* $\rho(Z_\alpha^N) = \zeta(\alpha)\,\mathrm{Id}_V$ *for every* $\alpha \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$*;*
(2) $\rho(H_k) = h_k\,\mathrm{Id}_V$ *for* $k = 1, \dots, s$.

In addition, for such a representation, the vector space V *has dimension* N^{3g+s-3}.

Proof. Using elementary linear algebra, this is an immediate consequence of Corollary 11. More precisely, consider the isomorphism

$$\mathcal{Z}^\omega(\lambda) \cong \mathcal{W}_1^q \otimes \cdots \otimes \mathcal{W}_g^q \otimes \mathcal{W}_{g+1}^{q^2} \otimes \cdots \otimes \mathcal{W}_{3g+s-3}^{q^2} \otimes \mathbb{C}[H_1] \otimes \cdots \otimes \mathbb{C}[H_s]$$

provided by Corollary 11.

For $1 \leqslant i \leqslant 3g+s-3$, let $X_i^{\pm1}$ and $Y_i^{\pm1}$ denote the generators of $\mathcal{W}_i^q$ or $\mathcal{W}_i^{q^2}$ (satisfying the relation $X_iY_i = qY_iX_i$ if $1 \leqslant i \leqslant g$ and $X_iY_i = q^2Y_iX_i$ if $g < i \leqslant 3g+s-3$). In particular, these generators are of the form $X_i = Z_{\alpha_i}$ and $Y_i = Z_{\beta_i}$ for some α_i, $\beta_i \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ such that $\Theta(\alpha_i,\beta_j) = 0$ if $i \neq j$, $\Theta(\alpha_i,\beta_i) = 1$ if $1 \leqslant i \leqslant g$, and $\Theta(\alpha_i,\beta_i) = 2$ if $g < i \leqslant 3g+s-3$.

Because N is odd, $q = \omega^4$ and q^2 are both primitive N–root of unity. Arbitrarily pick N–roots $\zeta(\alpha_i)^{\frac{1}{N}}$ and $\zeta(\beta_i)^{\frac{1}{N}}$, and define $\rho_i\colon \mathcal{W}_i^q \to \mathrm{End}(V_i)$ by the property that, if $v_1, v_2, \dots, v_N$ form a basis for $V_i \cong \mathbb{C}^N$,

$$\rho_i(X_i)(v_j) = \zeta(\alpha_i)^{\frac{1}{N}} q^j v_j \text{ and } \rho_i(Y_i)(v_j) = \zeta(\beta_i)^{\frac{1}{N}} v_{j+1} \text{ if } 1 \leqslant i \leqslant g, \text{ and}$$
$$\rho_i(X_i)(v_j) = \zeta(\alpha_i)^{\frac{1}{N}} q^{2j} v_j \text{ and } \rho_i(Y_i)(v_j) = \zeta(\beta_i)^{\frac{1}{N}} v_{j+1} \text{ if } g < i \leqslant 3g+s-3.$$

Then, for $V = V_1 \otimes V_2 \otimes \cdots \otimes V_{3g+s-3}$, define $\rho\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(V)$ by the property that ρ coincides with $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_{3g+s-3}$ on $\mathcal{W}_1^q \otimes \mathcal{W}_2^q \otimes \cdots \otimes \mathcal{W}_{3g+s-3}^q$, and $\rho(H_k) = h_k\,\mathrm{Id}_V$ for every $k = 1, \dots, s$.

It is immediate that ρ satisfies the required properties. The fact that ρ is irreducible, and that every irreducible representation is isomorphic to ρ, is easily proved by elementary linear algebra; see for instance [BoL, §4] for details. □

Remark 14. When $q = \omega^4$ is a primitive N–root of unity with N even, the irreducible representations of $\mathcal{Z}^\omega(\lambda)$ can be classified by similar arguments, but the result is somewhat more complicated to state. Compare [BoL, §4].

4. Character varieties and trace maps

We begin with the character variety

$$\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) = \{\text{group homomorphisms } r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})\} /\!\!/ \mathrm{SL}_2(\mathbb{C})$$

where $\mathrm{SL}_2(\mathbb{C})$ acts on homomorphisms by conjugation, and where the quotient is to be taken in the sense of geometric invariant theory [MFK].

For a group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ and a closed curve $K \subset S \times [0,1]$, the trace $\mathrm{Tr}\, r(K) \in \mathbb{C}$ depends only on the class of r in the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. An observation of Doug Bullock, Charlie Frohman, Jozef Przytycki and Adam Sikora [Bu_1, Bu_2, BFK_1, BFK_2, PrS] then shows that this defines an algebra homomorphism

$$\mathrm{Tr}_r\colon \mathcal{S}^{-1}(S) \to \mathbb{C}$$

by the property that

$$\mathrm{Tr}_r\big([K]\big) = -\mathrm{Tr}\, r(K)$$

for every (connected) framed knot $K \subset S \times [0,1]$. This $\mathrm{Tr}_r\big([K]\big)$ is independent of the framing of K.

The twisted version of the character varieties involves the space $\mathrm{Spin}(S)$ of isotopy classes of spin structures on S. Any two spin structures differ by an obstruction in $H^1(S;\mathbb{Z}_2)$, which defines an action of $H^1(S;\mathbb{Z}_2)$ on $\mathrm{Spin}(S)$. The cohomology group $H^1(S;\mathbb{Z}_2)$ also acts on the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ by the property that, if $\widehat{r} \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and $\alpha \in H^1(S;\mathbb{Z}_2)$, then $\alpha\widehat{r} \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is defined by

$$\alpha\widehat{r}(\gamma) = (-1)^{\alpha(\gamma)}\widehat{r}(\gamma) \in \mathrm{SL}_2(\mathbb{C})$$

for every $\gamma \in \pi_1(S)$.

The *twisted character variety* $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ is then defined as the quotient

$$\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S) = \big(\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \times \mathrm{Spin}(S)\big)/H^1(S;\mathbb{Z}_2).$$

An element $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ represented by $(\widehat{r},\sigma) \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \times \mathrm{Spin}(S)$ then defines an algebra homomorphism

$$\mathrm{Tr}_r \colon \mathcal{S}^{+1}(S) \to \mathbb{C},$$

which to a connected skein $[K] \in \mathcal{S}^{+1}(S)$ associates

$$\mathrm{Tr}_r\big([K]\big) = (-1)^{\sigma(K)}\mathrm{Tr}\,\widehat{r}(K)$$

where $\sigma(K) \in \mathbb{Z}_2$ denotes the monodromy of the framing of K with respect to the spin structure σ. Note that the right-hand side of the above formula is invariant under the action of $H^1(S;\mathbb{Z}_2)$ on the product $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \times \mathrm{Spin}(S)$, so that it depends only on the image $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ of $(\widehat{r},\sigma)$. See [BoW$_1$, §2] or [BoW$_2$, §5.1] for details.

By analogy with the case of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, we will also write

$$\mathrm{Tr}\,r(K) = -\mathrm{Tr}_r\big([K]\big)$$

for every $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ and every framed knot $K \subset S \times [0,1]$.

5. Shear-bend coordinates and their square roots

Let us see what happens if in Proposition 13 we replace ω by $\iota = \omega^{N^2}$. Since $\iota^4 = 1$, this amounts to considering the special case where $N = 1$; in particular, the puncture invariants become irrelevant. Then Proposition 13 associates to any group homomorphism $\zeta \colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ a representation $\rho_\zeta \colon \mathcal{Z}^\iota(\lambda) \to \mathrm{End}(\mathbb{C})$. By composition with the trace homomorphism $\mathrm{Tr}^\iota_\lambda \colon \mathcal{S}^\varepsilon(S) \to \mathcal{Z}^\iota(\lambda)$ of Theorem 3, we now have a homomorphism

$$\rho_\zeta \circ \mathrm{Tr}^\iota_\lambda \colon \mathcal{S}^\varepsilon(S) \to \mathrm{End}(\mathbb{C}) = \mathbb{C}$$

for $\varepsilon = \iota^2 = \pm 1$. This representation of $\mathcal{S}^\varepsilon(S)$ is clearly irreducible since its dimension is 1. We can therefore apply the case $N = 1$ of Theorem 1, which provides a point r_ζ of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, according to whether $\varepsilon = -1$ or $+1$, such that

$$\rho_\zeta \circ \mathrm{Tr}^\iota_\lambda\big([K]\big) = -\mathrm{Tr}\,r_\zeta(K)$$

for every framed knot $K \subset S \times [0,1]$ whose projection to S has no crossing and whose framing is vertical.

It is natural to ask which elements of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ are obtained in this way. The answer involves the following geometric definition.

Let $\widetilde{S}$ be the universal cover of S, and let $\widetilde{\lambda}$ be the ideal triangulation of $\widetilde{S}$ obtained by lifting the edges and faces of λ. Identify $\mathrm{PSL}_2(\mathbb{C})$ to the isometry group of the hyperbolic 3–space $\mathbb{H}^3$. A *pleated surface* with *pleating locus* λ is the data $(\widetilde f, \bar r)$ of a map $\widetilde f\colon \widetilde S \to \mathbb{H}^3$ and a group homomorphism $\bar r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ such that:

(1) $\widetilde f$ homeomorphically sends each edge of $\widetilde\lambda$ to a complete geodesic of the hyperbolic space $\mathbb{H}^3$, and every face of $\widetilde\lambda$ to a totally geodesic ideal triangle of $\mathbb{H}^3$, with vertices on the sphere at infinity;

(2) $\widetilde f$ is $\bar r$–equivariant, in the sense that $\widetilde f(\gamma\widetilde x) = \bar r(\gamma)\big(\widetilde f(\widetilde x)\big)$ for every $\gamma \in \pi_1(S)$ and every $\widetilde x \in \widetilde S$.

A pleated surface $(\widetilde f, \bar r)$ determines, for each edge $\widetilde\lambda_i$ of the ideal triangulation $\widetilde\lambda$ of $\widetilde S$, a complex weight $\widetilde x_i \in \mathbb{C}^*$ defined as follows: If $\widetilde Q_i \subset \widetilde S$ is the quadrilateral formed by the two triangles of $\widetilde\lambda$ meeting along $\widetilde\lambda_i$, then $\widetilde x_i$ is the cross-ratio of the 4 vertices of $\widetilde f(\widetilde Q_i)$ in the sphere at infinity $\mathbb{C}\cup\{\infty\}$ of $\mathbb{H}^3$. There edge weights $\widetilde x_i$ are equivariant under the action of $\pi_1(S)$, and therefore descend to a system of weights x_i for the edges λ_i of λ. The edge weights $x_i \in \mathbb{C}^*$ are the *shear-bend coordinates* of the pleated surface $(\widetilde f, \bar r)$.

Conversely, the pleated surface can be completely reconstructed from the data of these edge weights, up to post-composition by an isometry of $\mathbb{H}^3$ and pre-composition by an isotopy of $\widetilde S$ respecting $\widetilde S$. See for instance [Th, Bo]. In particular, these shear-bend coordinates uniquely determine an element $\bar r \in \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$. We will see in the proof of Proposition 15 that the issue of lifting $\bar r \in \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ to an element $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ is strongly connected to choosing appropriate square roots $\sqrt{x_i}$ for the shear-bend coordinates.

Following the terminology introduced in [Th], we say that the group homomorphism $\bar r : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ *realizes* the ideal triangulation λ if there exists a pleated surface $(\widetilde f, \bar r)$ with pleating locus λ. By extension, a point in the character variety $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ *realizes* λ if it can be represented by a homomorphism $\bar r : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ realizing λ. Finally, a point of one of the character varieties $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ *realizes* λ if, for the canonical projections $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \to \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ and $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S) \to \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, it projects to a point of $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ realizing λ.

We are now ready to state the result promised. At the beginning of this section, we associated to each group homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ a point r_ζ of the twisted character variety $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ if $\varepsilon = +1$, and a point of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if $\varepsilon = -1$.

Proposition 15. *A point of the character varieties $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ or $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, according to whether $\varepsilon = +1$ or -1, is associated to a group homomorphism $\zeta : \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$ as above if and only it realizes the ideal triangulation λ.*

Proof. To avoid having to keep distinguishing cases, let us focus on the case where $\varepsilon = +1$. The case where $\varepsilon = -1$ will be identical.

Consider a point $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ associated to a group homomorphism $\zeta : \mathcal{W}(\tau_\lambda;\mathbb{Z}) \to \mathbb{C}^*$, namely such that the trace homomorphism $\mathrm{Tr}_r : \mathcal{S}^{+1}(S) \to \mathbb{C}$ is equal to $\rho_\zeta \circ \mathrm{Tr}^\iota_\lambda$ for the algebra homomorphism $\rho_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathbb{C}$ determined by ζ.

Recall that an element of $\mathcal{W}(\tau_\lambda;\mathbb{Z})$ can also be interpreted as the assignment of an integer weight to each edge of λ, satisfying a certain parity condition on each face of λ. Let $\alpha_i \in \mathcal{W}(\tau_\lambda;\mathbb{Z})$ correspond to the weight system assigning weight 2 to the i–th edge λ_i and weight 0 to every other edge. Set $x_i = \zeta(\alpha_i) \in \mathbb{C}^*$.

The edge weights $x_i \in \mathbb{C}^*$ determine a pleated surface $(\widetilde{f}, \bar{r})$ whose shear-bend coordinates are these x_i. For the monodromy $\bar{r} \in \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ of this pleated surface and for a closed curve K in $S \times [0,1]$, the trace $\mathrm{Tr}\, \bar{r}(K)$ can be expressed up to sign as an explicit Laurent polynomials in square roots $z_i = \sqrt{x_i}$; see [BoW$_1$, §1.3]. Note that this trace $\mathrm{Tr}\, \bar{r}(K)$ is only defined up to sign because $\bar{r}$ is valued in $\mathrm{PSL}_2(\mathbb{C})$. This is consistent with the ambiguities in the choice of square roots $z_i = \sqrt{x_i}$; indeed, in the explicit formula of [BoW$_1$, §1.3], choosing different square roots multiply all terms of this Laurent polynomial by the same power of -1.

The homomorphism $\mathrm{Tr}^\iota_\lambda \colon \mathcal{S}^{+1}(S) \to \mathcal{Z}^\iota(\lambda)$ was specially designed in [BoW$_1$] to mimic this formula. In particular, $\rho_\zeta \circ \mathrm{Tr}^\iota_\lambda\big([K]\big) = \pm \mathrm{Tr}\, \bar{r}(K)$ for every framed knot K in $S \times [0,1]$. Since $\mathrm{Tr}_r = \rho_\zeta \circ \mathrm{Tr}^\iota_\lambda$, we conclude that the monodromy $\bar{r}$ of the pleated surface $(\widetilde{f}, \bar{r})$ is equal to the projection of $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ to $\mathcal{R}_{PSL)\mathbb{C})}(S)$. By definition, this means that r realizes the ideal triangulation λ, which is what we wanted to prove.

Conversely, let r be a point in $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ that realizes λ. This means that there exists a pleated surface $(\widetilde{f}, \bar{r})$ whose monodromy $\bar{r} \in \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ is the projection of r. Let x_1, x_2, ..., $x_n \in \mathbb{C}^*$ be the shear-bend coordinates of this pleated surface.

Pick square roots $z_i = \sqrt{x_i}$ for each of these shear-bend coordinates. These square roots define a homomorphism $\zeta \colon \mathcal{W}(\tau_\lambda; \mathbb{Z}) \to \mathbb{C}^*$ by the property that, if $\alpha \in \mathcal{W}(\tau_\lambda; \mathbb{Z})$ associates a weight $\alpha(s_i) \in \mathbb{Z}$ to the switch s_i of τ_λ corresponding to the i–th edge λ_i of λ, then $\zeta(\alpha) = z_1^{\alpha(s_1)} z_2^{\alpha(s_2)} \dots z_n^{\alpha(s_n)}$. This group homomorphism defines an algebra homomorphism $\rho_\zeta \colon \mathcal{Z}^\iota(\lambda) \to \mathbb{C}$, which in turn determines an element $r_\zeta \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ such that the two homomorphisms $\rho_\zeta \circ \mathrm{Tr}^\iota_\lambda$, $\mathrm{Tr}_{r_\rho} \colon \mathcal{S}^{+1}(S) \to \mathbb{C}$ are equal.

Again, by design of the homomorphism $\mathrm{Tr}^\iota_\lambda$, $\rho_\zeta \circ \mathrm{Tr}^\iota_\lambda\big([K]\big) = \pm \mathrm{Tr}\, \bar{r}(K)$ for every framed knot K in $S \times [0,1]$. As a consequence, the projection of $r_\zeta \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ to $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ is also equal to $\bar{r}$, and the two elements r, $r_\zeta \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ only differ by an obstruction $o \in H^1(S; \mathbb{Z}_2)$. More precisely, if we use the same spin structure $\sigma \in \mathrm{Spin}(S)$ to represent r and r_ζ, respectively, by $(\widehat{r}, \sigma)$ and $(\widehat{r}_\zeta, \sigma)$ with $\widehat{r}$ and $\widehat{r}_\zeta \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, then $\widehat{r}(\gamma) = (-1)^{o(\gamma)} \widehat{r}_\zeta(\gamma)$ for every $\gamma \in \pi_1(S)$.

We now define a new homomorphism $\zeta' \colon \mathcal{W}(\tau_\lambda; \mathbb{Z}) \to \mathbb{C}^*$ by the property that, for every $\alpha \in \mathcal{W}(\tau_\lambda; \mathbb{Z})$, $\zeta'(\alpha) = (-1)^{o([\alpha])} \zeta(\alpha)$, where $[\alpha] \in H_1(S; \mathbb{Z}_2)$ is the homology class represented by τ_λ endowed with the edge multiplicities defined by α. Now $\widehat{r} = \widehat{r}_{\zeta'}$, which proves that r is associated to the homomorphism $\zeta' \colon \mathcal{W}(\tau_\lambda; \mathbb{Z}) \to \mathbb{C}^*$.

This concludes the proof of Proposition 15 when $\varepsilon = +1$. The case where $\varepsilon = -1$ is essentially identical. □

6. Representations of the skein algebra

We are now ready to prove Theorem 2.

We begin with an elementary lemma about the Chebyshev polynomials T_n. Remember that the polynomial T_n is defined by the property that $\mathrm{Tr}\, M^n = T_n(\mathrm{Tr}\, M)$ for every $M \in \mathrm{SL}_2(\mathbb{C})$.

Lemma 16.

(1) *If $x = a + a^{-1}$, then $T_n(x) = a^n + a^{-n}$;*
(2) *If $y = b + b^{-1}$, the set of solutions to the equation $T_n(x) = y$ consists of the numbers $x = a + a^{-1}$ as a ranges over all n–roots of b.*

Proof. For a matrix $M \in \mathrm{SL}_2(\mathbb{C})$, the data of its trace x is equivalent to the data of its spectrum $\{a, a^{-1}\}$. The first property is then a straightfoward consequence of the fact that $\mathrm{Tr}\, M^n = T_n(\mathrm{Tr}\, M)$. The second property immediately follows. □

Recall from §4 that, for a point r in the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ and for a framed knot $K \subset S \times [0,1]$, there is a well-defined trace $\mathrm{Tr}\, r(K) \in \mathbb{C}$. When $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, this trace is the usual trace, and is independent of the framing of K. However, when $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, the definition $\mathrm{Tr}\, r(K)$ uses both the framing of K and the spin information of r.

In particular, $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ associates a number $\mathrm{Tr}\, r(P_k)$ to the k–th puncture of S, where P_k is a small loop going around the puncture and endowed with the vertical framing.

Theorem 17. *Assume that the surface S has at least one puncture, that A^2 is a primitive N–root of unity with N odd, and that we are given:*

(1) *a point $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ if $A^N = +1$, or a point $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if $A^N = -1$, such that r realizes some ideal triangulation of S;*
(2) *a number $p_k \in \mathbb{C}$ such that $T_N(p_k) = -\mathrm{Tr}\, r(P_k)$ for each of the punctures of S.*

Then, there exists an irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ whose classical shadow is equal to r and whose puncture invariants are the p_k.

Proof. Again, let us first focus on the case where $A^N = +1$.

Given $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ realizing some ideal triangulation λ, Proposition 15 provides a homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda; \mathbb{Z}) \to \mathbb{C}^*$, linearly extending to a representation $\rho_\zeta\colon \mathcal{Z}^\iota(\lambda) \to \mathrm{End}(\mathbb{C}) = \mathbb{C}$, such that the composition $\rho_\zeta \circ \mathrm{Tr}^\iota_\lambda\colon \mathcal{S}^{+1}(S) \to \mathbb{C}$ coincides with the trace homomorphism $\mathrm{Tr}_r\colon \mathcal{S}^{+1}(S) \to \mathbb{C}$ associated to r.

In Theorem 1 and in §3.3, we associated to the k–th puncture of S a skein $[P_k] \in \mathcal{S}^A(S)$ and an element $H_k \in \mathcal{Z}^\omega(\lambda)$ of the balanced Chekhov-Fock algebra. We will use the same letters for the similarly defined elements $[P_k] \in \mathcal{S}^{+1}(S)$ and $H_k \in \mathcal{Z}^\iota(\lambda)$.

The image of $[P_k] \in \mathcal{S}^A(S)$ under the quantum trace homomorphism $\mathrm{Tr}^\omega_\lambda$: $\mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$ is easily computed by using Theorem 3, and $\mathrm{Tr}^\omega_\lambda\big([P_k]\big) = H_k + H_k^{-1}$ in $\mathcal{Z}^\omega(\lambda)$. Similarly, $\mathrm{Tr}^\iota_\lambda\big([P_k]\big) = H_k + H_k^{-1}$ in $\mathcal{Z}^\iota(\lambda)$. (Beware that we are using the same symbols to denote different objects in different spaces.)

Then, for $[P_k] \in \mathcal{S}^{+1}(S)$,

$$\mathrm{Tr}_r\big([P_k]\big) = \rho_\zeta \circ \mathrm{Tr}^\iota_\lambda([P_k]) = \rho_\zeta(H_k + H_k^{-1}) = g_k + g_k^{-1}$$

if $g_k = \rho_\zeta(H_k) \in \mathrm{End}(\mathbb{C}) = \mathbb{C}$. If we are given a number $p_k \in \mathbb{C}$ such that $T_N(p_k) = \mathrm{Tr}_r\big([P_k]\big)$, Lemma 16 then implies that there exists an N–root h_k of $g_k = \rho_\zeta(H_k)$ such that $p_k = h_k + h_k^{-1}$.

Proposition 13 now associates to the homomorphism $\zeta\colon \mathcal{W}(\tau_\lambda; \mathbb{Z}) \to \mathbb{C}^*$ and to the N–root h_k of $\rho_\zeta(H_k)$ an irreducible representation $\mu\colon \mathcal{Z}^\omega(\lambda) \to \mathrm{End}(V)$ such that

(1) $\mu(Z_\alpha^N) = \zeta(\alpha)\,\mathrm{Id}_V$ for every $\alpha \in \mathcal{W}(\tau_\lambda; \mathbb{Z})$;
(2) $\mu(H_k) = h_k\,\mathrm{Id}_V$ for every $k = 1, \dots, s$.

The first property can also expressed in terms of the Frobenius homomorphism $\mathbf{F}^\omega\colon \mathcal{T}^\iota(\lambda) \to \mathcal{T}^\omega(\lambda)$ of §2.3 as follows: $\mu \circ \mathbf{F}^\omega(Z) = \rho_\zeta(Z)\,\mathrm{Id}_V$ for every $Z \in \mathcal{Z}^\iota(\lambda)$.

Composing with the quantum trace map $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$, we now obtain a representation

$$\rho = \mu \circ \mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S) \to \mathrm{End}(V).$$

If we already knew that ρ was irreducible, its non-quantum shadow would be computed by composing ρ with the Chebyshev homomorphism $\mathbf{T}^A\colon \mathcal{S}^\varepsilon(S) \to \mathcal{S}^A(S)$. For a skein $[K] \in \mathcal{S}^A(S)$, Theorem 6 shows that

$$\begin{aligned}\rho \circ \mathbf{T}^A\big([K]\big) &= \mu \circ \mathrm{Tr}^\omega_\lambda \circ \mathbf{T}^A\big([K]\big) = \mu \circ \mathbf{F}^\omega \circ \mathrm{Tr}^\iota_\lambda\big([K]\big)\\ &= \rho_\zeta \circ \mathrm{Tr}^\iota_\lambda\big([K]\big)\,\mathrm{Id}_V = \mathrm{Tr}_r\big([K]\big)\,\mathrm{Id}_V.\end{aligned}$$

In particular, if K is a knot whose projection to S has no crossing and whose framing is vertical,

$$T_N\big(\rho([K])\big) = \rho\big(T_N([K])\big) = \rho \circ \mathbf{T}^A\big([K]\big) = \mathrm{Tr}_r\big([K]\big)\,\mathrm{Id}_V = -\mathrm{Tr}\,r(K)\,\mathrm{Id}_V$$

where the first equality comes from the fact that ρ is an algebra homomorphism. By definition of the classical shadow in Theorem 1, this is exactly what was needed to show that the classical shadow of ρ is equal to r.

Similarly,

$$\rho([P_k]) = \mu \circ \mathrm{Tr}^\omega_\lambda([P_k]) = \mu(H_k + H_k^{-1}) = (h_k + h_k^{-1})\,\mathrm{Id}_V = p_k\,\mathrm{Id}_V.$$

Therefore, if ρ is irreducible, it has the classical shadow and cusp invariants required.

When ρ is not irreducible, it suffices to consider an irreducible component $\rho'\colon \mathcal{S}^A(S) \to \mathrm{End}(W)$ with $W \subset V$. Restricting the above computations to W show that ρ' has classical shadow r and puncture invariants the p_k.

This concludes the proof when $\varepsilon = A^{N^2}$ is equal to $+1$. The case when $\varepsilon = -1$ is identical. □

Remark 18. We conjecture that, when r is sufficiently generic, the representation $\rho = \mu \circ \mathrm{Tr}^\omega_\lambda$ used in the proof of Theorem 17 is already irreducible, and that there is no need to restrict to an irreducible factor.

Remark 19. In the very non-generic case there $r(P_k)$ is the identity for some punctures, the representation $\rho = \mu \circ \mathrm{Tr}^\omega_\lambda$ is definitely reducible. This is at the basis of the “filling in the punctures” process of [BoW$_4$].

Appendix A. The Thurston intersection form of a train track

Let τ be a train track in the surface S, and let $\mathcal{W}(\tau;\mathbb{Z})$ be the space of integer valued edge weights for τ. This abelian group comes with an additional structure provided by the Thurston intersection form

$$\Theta\colon \mathcal{W}(\tau;\mathbb{Z}) \times \mathcal{W}(\tau;\mathbb{Z}) \to \mathbb{Z}$$

defined as in §3.2. Namely,

$$\Theta(\alpha,\beta) = \tfrac{1}{2} \sum_{e \text{ right of } e'} \big(\alpha(e)\beta(e') - \alpha(e')\beta(e)\big)$$

where the sum is over all pairs (e,e') where e and e' are two “germs of edges” emerging on the same side of a switch of τ with e to the right of e' (e and e' are not necessarily adjacent at that switch). At this point, $\Theta(\alpha,\beta)$ is only a half-integer, but Theorem 20 below will prove that it is indeed an integer.

We want to determine the algebraic structure of $\mathcal{W}(\tau;\mathbb{Z})$ endowed with Θ. This is a classical property in the case of real-valued edge weights (see for instance [PeH, §3.2] or [Bo, §3]), but the subtleties of the integer-valued case seem less well known. The result is of independent interest because, beyond the scope of this article, integer-valued edge weight do occur in geometric situations where the Thurston intersection form is also relevant. One such instance arises for general pleated surfaces where the pleating locus is allowed to have uncountably many leaves, as opposed to the simpler pleated surfaces considered in §5. The bending of such a pleated surface is measured by an edge weight system valued in $\mathbb{R}/2\pi\mathbb{Z}$ for a train track carrying the pleating locus, and this edge weight system is related to rotation numbers by the Thurston intersection form [Bo].

The complement $S-\tau$ of the train track τ admits a certain number of "spikes", each locally delimited by two edges of τ that approach the same side of a switch of τ. Thicken τ to a subsurface $U\subset S$ that deformation retracts to τ. Each component of $U-\tau$ is then an annulus that contains one component of ∂U and a certain number of spikes of $S-\tau$. We can then consider the genus h of U, and the number n_{even} (resp. n_{odd}) of components of $U-\tau$ that contain an even (resp. odd) number of spikes.

A component U_1 of $U-\tau$ that contains an even number $n_1>0$ of spikes of $S-\tau$ determines, up to sign, an element of $\mathcal{W}(\tau;\mathbb{Z})$ as follows. The core of U_1 is homotopic to a closed curve γ_1 in τ that is made up of arcs k_1, k_2, …, k_{n_1}, $k_{n_1+1}=k_1$, in this order, such that each arc k_1 is immersed in τ and such that two consecutive arcs k_i and k_{i+1} locally bound a spike of U_1 at their common end point. For each edge e of τ, we can then consider

$$\alpha(e)=\sum_{i=1}^{n_1}(-1)^i\alpha_i(e)\in\mathbb{Z}$$

where $\alpha_i(e)\in\{0,1,2\}$ is the number of times the arc k_i passes over the edge e. Because the signs $(-1)^i$ alternate at the spikes of U_1 (using the fact that n_1 is even for $i=n_1$), one easily sees that these edge weights $\alpha(e)$ satisfy the switch conditions, and therefore define an edge weight system $\alpha\in\mathcal{W}(\tau;\mathbb{Z})$.

A component U_1 of $U-\tau$ that contains no spike similarly determines an edge weight system $\alpha\in\mathcal{W}(\tau;\mathbb{Z})$. The core of U_1 is now homotopic to a closed curve γ_1 immersed in τ, and α associates to each edge e the number $\alpha(e)$ of times γ_1 passes over e.

Theorem 20. *For a connected train track τ in the surface S, let the numbers h, n_{even} and n_{odd} be defined as above. Then, the lattice $\mathcal{W}(\tau;\mathbb{Z})$ of integer valued edge weight systems for τ admits a basis in which the Thurston intersection form Θ is block diagonal with*

- *h blocks $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $\frac{1}{2}n_{\mathrm{odd}}-1$ blocks $\begin{pmatrix}0&2\\-2&0\end{pmatrix}$, and n_{even} blocks $\begin{pmatrix}0\end{pmatrix}$ if $n_{\mathrm{odd}}>0$;*
- *h blocks $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ and n_{even} blocks $\begin{pmatrix}0\end{pmatrix}$ if $n_{\mathrm{odd}}=0$ and τ is non-orientable;*
- *h blocks $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ and $n_{\mathrm{even}}-1$ blocks $\begin{pmatrix}0\end{pmatrix}$ if $n_{\mathrm{odd}}=0$ and τ is orientable.*

In addition, in all cases, we can choose the base elements corresponding to the blocks $\big(0\big)$ *to be edge weight systems associated as above to components of* $U-\tau$ *that contain an even number of spikes.*

Proof of Theorem 20. We will subdivide the proof into several lemmas. The reader may recognize many analogies with the arguments used in the proof of [BoL, Prop. 5].

We first discuss a classical homological interpretation of the elements of $\mathcal{W}(\tau;\mathbb{Z})$ and of the Thurston intersection form Θ.

Because the edges of τ are not oriented, an edge weight system does not directly define a homology class in $H_1(\tau;\mathbb{Z})$. Instead consider the 2–fold orientation covering $\widehat\tau$ of τ, consisting of all pairs (x,o) where $x\in\tau$ and o is a local orientation of the train track τ at x. Note that $\widehat\tau$ is a canonically oriented train track, and that the covering involution $\sigma\colon\widehat\tau\to\widehat\tau$ that exchanges the two sheets of the covering reverses the orientation of $\widehat\tau$.

An edge weight system $\alpha\in\mathcal{W}(\tau;\mathbb{Z})$ lifts to a weight system $\widehat\alpha\in\mathcal{W}(\widehat\tau;\mathbb{Z})$. Endowing each (oriented) edge of $\widehat\tau$ with the weight assigned by $\widehat\alpha$ defines a chain, which is closed because of the switch condition and therefore defines a homology class $[\widehat\alpha]\in H_1(\widehat\tau;\mathbb{Z})$. Note that $\sigma_*\big([\widehat\alpha]\big)=-[\widehat\alpha]$ since the covering involution σ reverses the canonical orientation of $\widehat\tau$.

Conversely, each homology class $[\widehat\alpha]\in H_1(\widehat\tau;\mathbb{Z})$ is represented by a unique linear combination of the edges of $\widehat\tau$, and therefore determines an edge weight system $\widehat\alpha\in\mathcal{W}(\widehat\tau;\mathbb{Z})$. Assuming in addition that $\sigma_*\big([\widehat\alpha]\big)=-[\widehat\alpha]$, this edge weight system is invariant under the action of σ, and therefore comes from an edge weight system $\alpha\in\mathcal{W}(\tau;\mathbb{Z})$. This proves:

Lemma 21. *The above correspondence identifies the space* $\mathcal{W}(\tau;\mathbb{Z})$ *of edge weight systems to the eigenspace*

$$H_1(\widehat\tau;\mathbb{Z})^-=\big\{[\widehat\alpha]\in H_1(\widehat\tau;\mathbb{Z});\sigma_*\big([\widehat\alpha]\big)=-[\widehat\alpha]\big\}\subset H_1(\widehat\tau;\mathbb{Z})$$

of the homomorphism $\sigma_*\colon H_1(\widehat\tau;\mathbb{Z})\to H_1(\widehat\tau;\mathbb{Z})$ *induced by* σ. □

To describe the Thurston intersection form in this homological framework, consider the subsurface U deformation retracting to τ_λ. The covering $\widehat\tau\to\tau$ uniquely extends to a 2–fold covering $\widehat U\to U$, whose covering involution $\sigma\colon\widehat U\to\widehat U$ extends our previous involution σ.

Lemma 22. *If* $[\widehat\alpha]$, $[\widehat\beta]\in H_1(\widehat\tau)^-$ *are associated to the edge weight systems* α, $\beta\in\mathcal{W}(\tau_\lambda;\mathbb{Z})$,

$$\Theta(\alpha,\beta)=\tfrac12\,[\widehat\alpha]\cdot[\widehat\beta]$$

where $\cdot$ *denotes the algebraic intersection number of classes of* $H_1(\widehat U;\mathbb{Z})\cong H_1(\widehat\tau;\mathbb{Z})$. *In addition,* $[\widehat\alpha]\cdot[\widehat\beta]$ *is even, and* $\Theta(\alpha,\beta)$ *is an integer.*

Proof. To prove the first statement push the oriented train track $\widehat\tau$ to its left to obtain a train track $\widehat\tau'\subset\widehat U$ that is transverse to $\widehat\tau$, realize the homology class $[\widehat\alpha]$ by $\widehat\tau$ endowed with the edge multiplicities coming from α, realize $[\widehat\beta]$ by $\widehat\tau'$ endowed with the edge multiplicities coming from β, and use this setup to compute the algebraic intersection number $[\widehat\alpha]\cdot[\widehat\beta]$. Evaluating the contribution to $[\widehat\alpha]\cdot[\widehat\beta]$ of each point of $\widehat\tau\cap\widehat\tau'$ then shows that this algebraic intersection number is equal to $2\Theta(\alpha,\beta)$.

The second statement is obtained by a similar but different computation of $[\widehat{\alpha}]\cdot[\widehat{\beta}]$. Perturb τ to a train track τ'' that is transverse to τ, and let $\widehat{\tau}''$ be the pre-image of τ'' in $\widehat{U}$. Now compute $[\widehat{\alpha}]\cdot[\widehat{\beta}]$ by realizing the homology class $[\widehat{\beta}]$ by $\widehat{\tau}''$ endowed with the edge multiplicities coming from β, while still realizing $[\widehat{\alpha}]$ by $\widehat{\tau}$ endowed with the edge multiplicities coming from α. The intersection $\widehat{\tau}\cap\widehat{\tau}''$ splits into pairs of points interchanged by the covering involution σ, and the two points in each pair have the same contribution to $[\widehat{\alpha}]\cdot[\widehat{\beta}]$. It follows that $[\widehat{\alpha}]\cdot[\widehat{\beta}]$ is even. □

We now need to better understand the action of σ_* on the homology group $H_1(\widehat{U};\mathbb{Z})$.

It will be convenient to systematically use a notation which already appeared in Lemma 21. If V is a space where some restriction of the covering involution σ induces a homomorphism σ_*, then

$$V^- = \{\alpha\in V; \sigma_*(\alpha) = -\alpha\}.$$

For instance, Lemma 21 provides a natural isomorphism $\mathcal{W}(\tau;\mathbb{Z})\cong H_1(\widehat{U};\mathbb{Z})^-$.

Let $\partial_{\mathrm{even}}U$ be the union of the n_{odd} components of ∂U that are adjacent to an even number of spikes to $S-\tau$, and set $\partial_{\mathrm{odd}}U = \partial U - \partial_{\mathrm{even}}U$.

Lemma 23. *Let γ_1 be a component of $\partial_{\mathrm{even}}U$, and let $\widehat{\gamma}_1$ be its pre-image in $\widehat{U}$. Then $H_1(\widehat{\gamma}_1);\mathbb{Z})^-\cong\mathbb{Z}$, and the image in $H_1(\widehat{U};\mathbb{Z})^-\cong\mathcal{W}(\tau;\mathbb{Z})$ of one of its generators coincides up to sign with the edge weight system that we associated right before Theorem* 20 *to the component U_1 of $U-\tau$ that contains γ_1.*

Proof. As right above Theorem 20, the curve γ_1 is homotopic to a closed curve γ_1' in τ that is made up of n_1 arcs k_1, k_2, …, k_{n_1}, $k_{n_1+1}=k_1$, in this order, such that each arc k_1 is immersed in τ and such that two consecutive arcs k_i and k_{i+1} locally bound a spike of U_1 at their common end point. Because n_1 is even, there are two possible ways to orient these arcs in such a way that consecutive arcs have opposite orientations. This shows that γ_1' has two distinct lifts to $\widehat{\tau}$, and therefore that the pre-image $\widehat{\gamma}_1$ of γ_1 in $\widehat{U}$ consists of two components of $\partial\widehat{U}$ that are exchanged by the covering involution. This provides an isomorphism $H_1(\widehat{\gamma}_1;\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}$ where σ_* exchanges the two factors. It immediately follows that $H_1(\widehat{\gamma}_1;\mathbb{Z})^-\cong\mathbb{Z}$.

If $\widehat{\gamma}_1'\subset\widehat{\tau}$ denotes one of the two lifts of γ_1' to $\widehat{\tau}$, the image of $H_1(\widehat{\gamma}_1;\mathbb{Z})^-$ in $H_1(\widehat{U};\mathbb{Z})^-\cong H_1(\widehat{\tau};\mathbb{Z})^-\cong\mathcal{W}(\tau;\mathbb{Z})$ is generated by $[\widehat{\gamma}_1'] - \sigma_*\big([\widehat{\gamma}_1']\big)$. The second statement easily follows. □

We will first restrict attention to the case where $n_{\mathrm{odd}}>0$.

We just saw that the restriction of the covering $\widehat{U}\to U$ above $\partial_{\mathrm{even}}U$ is trivial; similarly, its restriction above each component of $\partial_{\mathrm{odd}}U$ is non-trivial. Therefore, the covering $\widehat{U}\to U$ is classified by a cohomology class in $H^1(U;\mathbb{Z}_2)$ which evaluates to 0 on the elements of $\partial_{\mathrm{even}}U$ and to 1 on the components of $\partial_{\mathrm{odd}}U$.

Since the subset $\partial_{\mathrm{odd}}U$ is non-empty, and can therefore realize the cohomology class classifying the covering $\widehat{U}\to U$ as the Poincaré dual of a family $K\subset U$ of disjoint arcs whose boundary $\partial K = K\cap\partial U$ consists of one point in each component of $\partial_{\mathrm{odd}}U$.

Split U along a simple closed curve γ to isolate K inside of a planar surface $U_1\subset S$ with boundary $\partial U_1 = \gamma\cup\partial_{\mathrm{odd}}U$, while the closure U_2 of $U-U_1$ has genus

h and boundary $\partial U_2 = \gamma \cup \partial_{\mathrm{even}} U$. Let $\widehat U_1$ and $\widehat U_2$ be the respective pre-images of U_1 and U_2 in $\widehat U$.

Since K is disjoint from U_2, the covering $\widehat U_2 \to U_2$ is trivial, and $\widehat U_2$ consists of two disjoint copies of the surface U_2 which are exchanged by σ.

The covering $\widehat U_1 \to U_1$ is non-trivial above each component of $\partial_{\mathrm{odd}} U$ and trivial above γ. Since the surface U_1 is planar, an Euler characteristic computation shows that $\widehat U_1$ has genus $\frac12 n_{\mathrm{odd}} - 1$ and has $n_{\mathrm{odd}} + 2$ boundary components.

Consider the Mayer-Vietoris exact sequence

$$0 \to H_1(\widehat\gamma;\mathbb Z) \to H_1(\widehat U_1;\mathbb Z) \oplus H_1(\widehat U_2;\mathbb Z) \to H_1(\widehat U;\mathbb Z) \to 0$$

where $\widehat\gamma$ denotes the pre-image of γ in $\widehat U$. (To explain the 0 on the right, note that the map $H_0(\widehat\gamma;\mathbb Z) \to H_0(\widehat U_2;\mathbb Z)$ is injective.)

Lemma 24. *Remembering that V^- denotes the (-1)–eigenspace of the action of σ_* over a space V, the above exact sequence induces another exact sequence*

$$0 \to H_1(\widehat\gamma;\mathbb Z)^- \to H_1(\widehat U_1;\mathbb Z)^- \oplus H_1(\widehat U_2;\mathbb Z)^- \to H_1(\widehat U;\mathbb Z)^- \to 0.$$

Proof. The only point that requires some thought is the fact that the third homomorphism is surjective.

Given $u \in H_1(\widehat U;\mathbb Z)^-$, the first exact sequence provides $u_1 \in H_1(\widehat U_1;\mathbb Z)$ and $u_2 \in H_1(\widehat U_2;\mathbb Z)$ such that $u = u_1 + u_2$ in $H_1(\widehat U;\mathbb Z)$. Since $\alpha_*(u) = -u$, we conclude that there exists $v \in H_1(\widehat\gamma;\mathbb Z)$ such that $\sigma_*(u_1) = -u_1 + v$ in $H_1(\widehat U_1;\mathbb Z)$ and $\sigma_*(u_2) = -u_2 - v$ in $H_1(\widehat U_2;\mathbb Z)$. Note that $v \in H_1(\widehat\gamma;\mathbb Z)$ is invariant under α_*. Therefore, for the isomorphism $H_1(\widehat\gamma;\mathbb Z) \cong H_1(\gamma;\mathbb Z) \oplus H_1(\gamma;\mathbb Z)$ coming from the fact that each of the two components of $\widehat\gamma$ is naturally identified to γ, $v = (w,w)$ for some $w \in H_1(\gamma;\mathbb Z)$. If we replace u_1 by $u_1' = u_1 - (w,0)$ and u_2 by $u_2' = u_2 + (w,0)$, we now have that $u = u_1' + u_2'$ with $\sigma_*(u_1') = -u_1'$ and $\sigma_*(u_2') = -u_2'$, as requested. □

We now analyze the terms of the exact sequence of Lemma 24.

The space $H_1(\widehat U_2;\mathbb Z)^-$ is easy to understand, because $\widehat U_2$ is made up of two disjoint copies of U_2, which are exchanged by the covering involution σ. Therefore, $H_1(\widehat U_2;\mathbb Z) \cong H_1(U_2;\mathbb Z) \oplus H_1(U_2;\mathbb Z)$ and, for this isomorphism, $H_1(\widehat U_2;\mathbb Z)^-$ corresponds to $\{(\alpha,-\alpha); \alpha \in H_1(U_2;\mathbb Z)\}$. This defines an isomorphism $H_1(\widehat U_2;\mathbb Z)^- \cong H_1(U_2;\mathbb Z)$, for which the intersection form of $H_1(\widehat U_2;\mathbb Z)^-$ corresponds to twice the intersection form of $H_1(U_2;\mathbb Z)$.

Lemma 25. *There exists a basis for $H_1(\widehat U_2;\mathbb Z)^-$ in which the intersection form is block diagonal with h blocks $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ and n_{even} blocks $\begin{pmatrix} 0 \end{pmatrix}$.*

In addition, we can arrange that the basis elements corresponding to blocks $\begin{pmatrix} 0 \end{pmatrix}$ are the images of generators of $H(\widehat\alpha;\mathbb Z)^- \cong \mathbb Z$ as $\widehat\alpha \subset \widehat U_2$ ranges over all preimages of components α of $\partial_{\mathrm{even}} U$, and that a generator of $H_1(\widehat\gamma;\mathbb Z)^- \cong \mathbb Z$ is sent to the sum of these elements.

Proof. The surface U_2 has genus h and has $n_{\mathrm{even}} + 1$ boundary components, and γ is one of these boundary components. We can therefore find a basis for $H_1(U_2;\mathbb Z)$ in which the intersection form is block diagonal with h blocks $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and n_{even} blocks $\begin{pmatrix} 0 \end{pmatrix}$. In addition, since $\partial U_2 = \gamma \cup \partial_{\mathrm{even}} U$, we can arrange that the basis elements corresponding to the blocks $\begin{pmatrix}0\end{pmatrix}$ are the images of generators of $H_1(\alpha;\mathbb{Z})$ as α ranges over all components of $\partial_{\text{even}}U$, while the image of a generator of $H_1(\gamma;\mathbb{Z})$ is sent to the sum of these elements.

The result then follows by considering the isomorphism $H_1(\widehat{U}_2;\mathbb{Z})^- \cong H_1(U_2;\mathbb{Z})$ mentioned above. □

We now consider $H_1(\widehat{U}_1;\mathbb{Z})^-$.

Lemma 26. *There exists a basis for $H_1(\widehat{U}_1;\mathbb{Z})^-$ in which the intersection form is block diagonal with $\frac{1}{2}n_{\text{odd}}-1$ blocks $\begin{pmatrix}0 & 4\\ -4 & 0\end{pmatrix}$ and with one block $\begin{pmatrix}0\end{pmatrix}$. In addition, the block $\begin{pmatrix}0\end{pmatrix}$ corresponds to the image of the homomorphism $H_1(\widehat{\gamma};\mathbb{Z})^- \to H_1(\widehat{U}_1;\mathbb{Z})^-$ induced by the inclusion map.*

Proof. We will use an explicit description of the covering $\widehat{U}_1 \to U_1$, with a specific basis for $H_1(\widehat{U}_1;\mathbb{Z})$.

Recall that this covering is classified by a cohomology class in $H^1(U_1;\mathbb{Z}_2)$ that is dual to a family $K \subset U_1$ of $\frac{1}{2}n_{\text{odd}}$ disjoint arcs, with one boundary point in each component of $\partial_{\text{odd}}U$. Index the components of $\partial_{\text{odd}}U$ as $\alpha_1, \alpha_2, \dots, \alpha_{n_{\text{odd}}}$ and the components of K as $k_1, k_3, k_5, \dots, k_{n_{\text{odd}}-1}$ in such a way that k_{2i-1} joins α_{2i-1} to α_{2i}. Add to K a family of disjoint arcs $k_2, k_4, \dots, k_{n_{\text{odd}}-2}$, disjoint from the k_{2i-1}, such that each k_{2i} joins α_{2i} to α_{2i+1}. See Figure 3.

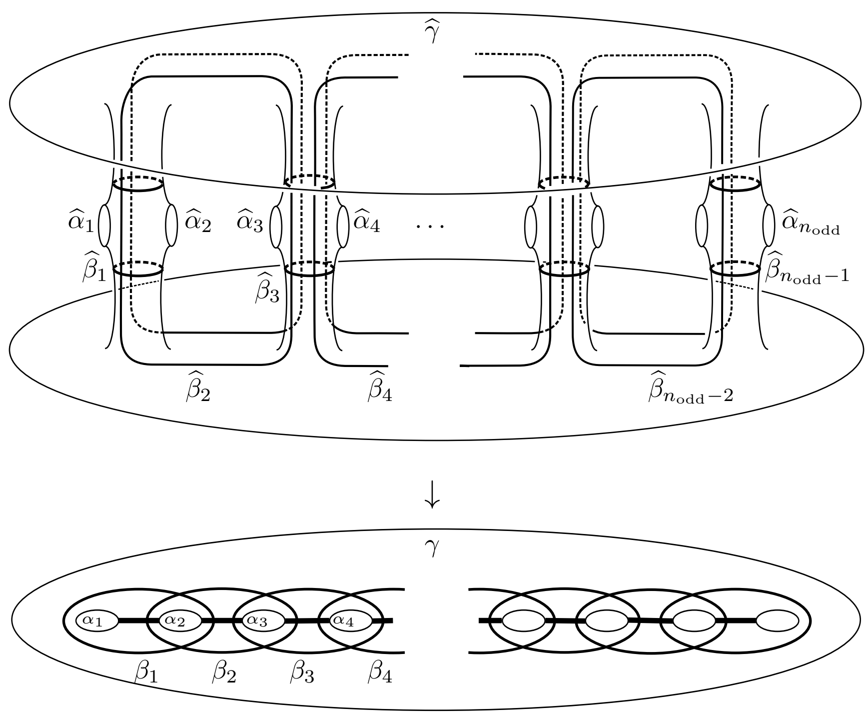

FIGURE 3.

For $i = 1, 2, \dots, n_{\text{odd}}-1$, consider a small regular neighborhood of $k_i \cup \alpha_i \cup \alpha_{i+1}$ in U_1 and let β_i be the boundary component of this neighborhood which is neither

α_i nor α_{i+1}; endow β_i by the corresponding boundary orientation. Orient each curve α_i by the boundary orientation of $\partial_{\text{odd}} U$.

Finally, the pre-image of β_i in $\widehat{U}_1$ consists of two disjoint curves. Arbitrarily choose one of these curves $\widehat{\beta}_i$ and orient it by the orientation of β_i. Similarly orient the curve γ by the boundary orientation of ∂U_1, and lift it to a curve $\widehat{\gamma}$ in $\widehat{U}_1$. The pre-image of β_j is a single curve $\widehat{\beta}_j$. Then, the $[\widehat{\alpha}_i]$ and $[\widehat{\beta}_j]$ form a basis for $H_1(\widehat{U}_1; \mathbb{Z})$. See Figure 3.

Consider an element $u \in H_1(\widehat{U}_1; \mathbb{Z})$, uniquely expressed in this basis as

$$u = \sum_{i=1}^{n_{\text{odd}}} a_i [\widehat{\alpha}_i] + \sum_{j=1}^{n_{\text{odd}}-1} b_j [\widehat{\beta}_j]$$

with all α_i, $\gamma_j \in \mathbb{Z}$. By construction of the curves $\widehat{\alpha}_i$ and $\widehat{\beta}_j$,

$$\sigma_*\big([\widehat{\alpha}_i]\big) = [\widehat{\alpha}_i] \text{ and } \sigma_*\big([\widehat{\beta}_j]\big) = -[\widehat{\beta}_j] - [\widehat{\alpha}_j] - [\widehat{\alpha}_{j+1}].$$

If u belongs to $H_1(\widehat{U}_1; \mathbb{Z})^-$, namely if $\sigma_*(u) = -u$, we necessarily have that

$$\begin{aligned} b_1 &= 2a_1, \\ b_i + b_{i-1} &= 2a_i \text{ for every } i \text{ with } 2 \leqslant i \leqslant n_{\text{odd}} - 1, \\ \text{and } b_{n_{\text{odd}}-1} &= 2a_{n_{\text{odd}}} \end{aligned}$$

by considering the coefficients of each $[\widehat{\alpha}_i]$. In particular, the coefficients b_j are all even, and

$$u = \frac{u - \sigma_*(u)}{2} = \sum_{j=1}^{n_{\text{odd}}-1} \frac{b_j}{2} \big([\widehat{\beta}_j] - \sigma_*([\widehat{\beta}_j])\big).$$

Therefore, the elements $[\widehat{\beta}_j] - \sigma_*\big([\widehat{\beta}_j]\big)$ generate $H_1(\widehat{U}_1; \mathbb{Z})^-$. Since these elements $[\widehat{\beta}_j] - \sigma_*\big([\widehat{\beta}_j]\big) = 2[\widehat{\beta}_j] + [\widehat{\alpha}_j] + [\widehat{\alpha}_{j+1}]$ are linearly independent, they form a basis for $H_1(\widehat{U}_1; \mathbb{Z})^-$.

Note that $[\widehat{\beta}_j] \cdot [\widehat{\beta}_{j'}] = 0$ if $|j - j'| > 1$, and $[\widehat{\beta}_j] \cdot [\widehat{\beta}_{j+1}] = \varepsilon_j = \pm 1$, where the sign depend on which lift of β_j we chose for $\widehat{\beta}_j$. Also,

$$\sigma_*\big([\widehat{\beta}_j]\big) \cdot [\widehat{\beta}_{j'}] = [\widehat{\beta}_j] \cdot \sigma_*\big([\widehat{\beta}_{j'}]\big) = -\sigma_*\big([\widehat{\beta}_j]\big) \cdot \sigma_*\big([\widehat{\beta}_{j'}]\big) = -[\widehat{\beta}_j] \cdot [\widehat{\beta}_{j'}].$$

It follows that, in the basis of $H_1(\widehat{U}_1; \mathbb{Z})^-$ formed by the $[\widehat{\beta}_j] - \sigma_*\big([\widehat{\beta}_j]\big)$, the intersection form has matrix

$$\begin{pmatrix} 0 & 4\varepsilon_1 & 0 & 0 & \dots & 0 & 0 \\ -4\varepsilon_1 & 0 & 4\varepsilon_2 & 0 & \dots & 0 & 0 \\ 0 & -4\varepsilon_2 & 0 & 4\varepsilon_3 & \dots & 0 & 0 \\ 0 & 0 & -4\varepsilon_3 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 4\varepsilon_{n_{\text{odd}}-2} \\ 0 & 0 & 0 & 0 & \dots & -4\varepsilon_{n_{\text{odd}}-2} & 0 \end{pmatrix}$$

By block diagonalizing this matrix, a final modification of the basis provides a new basis for $H_1(\widehat{U}_1; \mathbb{Z})^-$ in which the intersection form is block diagonal with $\frac{1}{2} n_{\text{odd}} - 1$ blocks $\begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}$ and with one block $\big(0\big)$.

There remains to show that the block $\big(0\big)$ corresponds to the image of $H_1(\widehat{\gamma}; \mathbb{Z})^-$. This could be seen by explicitly analyzing the block diagonalization process of the above matrix. However, it is easier to note that $H_1(\widehat\gamma;\mathbb{Z})^- \cong \mathbb{Z}$ is generated by $[\widehat\gamma_1]-[\widehat\gamma_2]$, where $\widehat\gamma_1$ and $\widehat\gamma_2$ are the two components of the pre-image $\widehat\gamma$ of γ and are oriented by the boundary orientation of $\partial\widehat U_1$. Then, $[\widehat\gamma_1]-[\widehat\gamma_2]$ is in the kernel of the intersection form of $H_1(\widehat U_1;\mathbb{Z})^-$, since $\widehat\gamma_1$ and $\widehat\gamma_2$ are in the boundary of $\widehat U_1$, and generate this kernel since it is isomorphic to $\mathbb{Z}$ and since $[\widehat\gamma_1]-[\widehat\gamma_2]$ is indivisible in $H_1(\widehat U_1;\mathbb{Z})$. □

We now only need to combine the computations of Lemmas 24, 25 and 26 to obtain a basis of $H_1(\widehat U;\mathbb{Z})^-$ in which the intersection form is block diagonal with h blocks $\begin{pmatrix}0&-2\\2&0\end{pmatrix}$, $\frac12 n_{\mathrm{odd}}-1$ blocks $\begin{pmatrix}0&-4\\4&0\end{pmatrix}$, and n_{even} blocks $\begin{pmatrix}0\end{pmatrix}$.

Applying Lemmas 21 and 22 to connect this to the Thurston intersection form on the edge weight space $\mathcal{W}(\tau;\mathbb{Z})$, we conclude that $\mathcal{W}(\tau;\mathbb{Z})$ admits a basis in which the intersection form is block diagonal with h blocks $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, $\frac12 n_{\mathrm{odd}}-1$ blocks $\begin{pmatrix}0&-2\\2&0\end{pmatrix}$, and n_{even} blocks $\begin{pmatrix}0\end{pmatrix}$. In addition, by the second half of Lemma 25 and using Lemma 23, the generators corresponding to the blocks $\begin{pmatrix}0\end{pmatrix}$ can be assumed to correspond to the elements of $\mathcal{W}(\tau;\mathbb{Z})$ associated to the components of $\partial_{\mathrm{even}}U$.

This proves Theorem 20, under our assumption that $n_{\mathrm{odd}}>0$.

We now consider the case where $n_{\mathrm{odd}}=0$, namely where $\partial_{\mathrm{odd}}U=\varnothing$, and where the train-track τ is non-orientable. This second property is equivalent to the property that the covering $\widehat U\to U$ is non-trivial. We can then realize the cohomology class of $H^1(U;\mathbb{Z}_2)$ classifying the covering $\widehat U\to U$ as the Poincaré dual of a non-separating simple closed curve K. Let $U_1\subset U$ be a surface of genus 1 containing K and bounded by a simple closed curve γ, and let U_2 be the closure of $U-U_1$. As before, let $\widehat U_1$, $\widehat U_2$, $\widehat\gamma$ denote the respective pre-images of U_1, U_2, γ in $\widehat U$.

The computation of Lemma 25 applies to this case as well, and provides a basis for $H_1(\widehat U_2;\mathbb{Z})^-$ in which the intersection form is block diagonal with h blocks $\begin{pmatrix}0&-2\\2&0\end{pmatrix}$ and n_{even} blocks $\begin{pmatrix}0\end{pmatrix}$.

The surface $\widehat U_1$ is a twice-punctured torus. A simple analysis of the covering $\widehat U_1\to U_1$ shows that $H_1(\widehat U_1;\mathbb{Z})^-\cong\mathbb{Z}$ is equal to the image of $H_1(\widehat\gamma;\mathbb{Z})^-$. The intersection form of $H_1(\widehat U_1;\mathbb{Z})^-$ is then 0.

Again, combining these computations with the exact sequence

$$0\to H_1(\widehat\gamma;\mathbb{Z})^-\to H_1(\widehat U_1;\mathbb{Z})^-\oplus H_1(\widehat U_2;\mathbb{Z})^-\to H_1(\widehat U;\mathbb{Z})^-\to 0.$$

provides in this case a basis for $H_1(\widehat U;\mathbb{Z})^-$ in which the intersection form is block diagonal with h blocks $\begin{pmatrix}0&-2\\2&0\end{pmatrix}$ and n_{even} blocks $\begin{pmatrix}0\end{pmatrix}$. Using Lemmas 21 and 22, this provides the result promised in Theorem 20 in this case as well. The fact that the generators corresponding to the blocks $\begin{pmatrix}0\end{pmatrix}$ can be chosen to be the elements associated to the components of $\partial_{\mathrm{even}}U$ is a byproduct of the proof as in the previous case.

Finally, we need to consider the case where $n_{\mathrm{odd}}=0$ and the train-track τ is orientable. Then the covering $\widehat U\to U$ is trivial, so that $H_1(\widehat U;\mathbb{Z})^-\cong H_1(U;\mathbb{Z})$ in such a way that the intersection form of $H_1(\widehat{U};\mathbb{Z})^-$ corresponds to twice the intersection form of $H_1(U;\mathbb{Z})$. By Lemma 22, the last case of Theorem 20 immediately follows. □

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