REPRESENTATIONS OF THE KAUFFMAN SKEIN ALGEBRA I: INVARIANTS AND MIRACULOUS CANCELLATIONS

FRANCIS BONAHON AND HELEN WONG

ABSTRACT. We study finite-dimensional representations of the Kauffman skein algebra of a surface S. In particular, we construct invariants of such irreducible representations when the underlying parameter $q = \mathrm{e}^{2\pi \mathrm{i}\hbar}$ is a root of unity. The main one of these invariants is a point in the character variety consisting of group homomorphisms from the fundamental group $\pi_1(S)$ to $\mathrm{SL}_2(\mathbb{C})$, or in a twisted version of this character variety. The proof relies on certain miraculous cancellations that occur for the quantum trace homomorphism constructed by the authors. These miraculous cancellations also play a fundamental role in subsequent work of the authors, where novel examples of representations of the skein algebra are constructed.

For an oriented surface S of finite topological type and for a Lie group G, many areas of mathematics involve the *character variety*

$$\mathcal{R}_G(S) = \{\text{group homomorphisms } \pi_1(S) \to G\} /\!\!/ G,$$

where G acts on homomorphisms by conjugation. For $G = \mathrm{SL}_2(\mathbb{C})$, Turaev [Tu$_1$] showed that the corresponding character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ can be quantized by the Kauffman skein algebra of the surface; see also [BuFK$_1$, BuFK$_2$, PrS]. In fact, if one follows the physical tradition that a quantization of a space X replaces the commutative algebra of functions on X by a non-commutative algebra of operators on a Hilbert space, the actual quantization of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ should be a *representation* of the skein algebra.

This article studies finite-dimensional representations of the skein algebra of a surface. The *Kauffman skein algebra* $\mathcal{S}^A(S)$ depends on a parameter $A = \mathrm{e}^{\pi \mathrm{i}\hbar} \in \mathbb{C} - \{0\}$, and is defined as follows: one first considers the vector space freely generated by all isotopy classes of framed links in the thickened surface $S \times [0,1]$, and then one takes the quotient of this space by the skein relation that

$$[K_1] = A^{-1}[K_0] + A[K_\infty]$$

whenever the three links K_1, K_0 and $K_\infty \subset S \times [0,1]$ differ only in a little ball where they are as represented on Figure 1. The algebra multiplication is defined by superposition of skeins. See §2 for details.

Our goal is to study representations of the skein algebra, namely algebra homomorphisms $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ where V is a finite-dimensional vector space over $\mathbb{C}$. See [BoW$_2$] for an interpretation of such representations as generalizations of the Kauffman bracket invariant of framed links in $\mathbb{R}^3$. When A is a root of unity, a classical example of a finite-dimensional representation of the skein algebra

Date: June 11, 2012.

This research was partially supported by grants DMS-0604866, DMS-1105402 and DMS-1105692 from the National Science Foundation, and by a mentoring grant from the Association for Women in Mathematics.

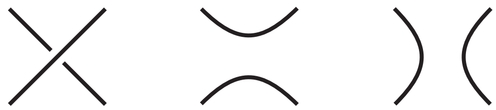

Figure 1. A Kauffman triple

$\mathcal{S}^A(S)$ also arises from the Witten-Reshetikhin-Turaev topological quantum field theory associated to the fundamental representation of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$ [ReT, BHMV, Tu$_2$].

To analyze such representations, we need to find a way to distinguish them. We thus introduce invariants for these representations.

We focus attention on the case where A^2 is a root of unity, and more precisely where A^2 is a primitive N–root of unity with N odd. While the restriction to roots of unity is natural, the parity condition for N is forced on us by the mathematics.

Surprisingly, the main invariant comes from Chebyshev polynomials of the first kind. The *n–th normalized Chebyshev polynomial of the first kind* is the polynomial $T_n(x)$ determined by the trigonometric identity that $2\cos n\theta = T_n(2\cos\theta)$.

More precisely, if A^2 is a primitive N-root of unity with N odd, we consider the Chebyshev polynomial T_N. We separate the discussion into two cases, depending on whether $A^N = +1$ or $A^N = -1$.

Theorem 1. *Suppose that A^2 is a primitive N–root of unity with N odd, and that $A^N = -1$. Then, for every irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ of the skein algebra, there exists a unique point r_ρ of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ such that*

$$T_N\big(\rho([K])\big) = -\big(\mathrm{Tr}\, r_\rho(K)\big)\mathrm{Id}_V$$

for every framed knot $K \subset S\times[0,1]$ whose projection to S has no crossing and whose framing is vertical. (Here, $\mathrm{Tr}\, r_\rho(K) \in \mathbb{C}$ denotes the trace of $r_\rho(K) \in \mathrm{SL}_2(\mathbb{C})$.)

In particular, interpreting the elements of $\mathrm{End}(V)$ as matrices, many "miraculous cancellations" occur in the entries of $T_N\big(\rho([K])\big)$ when one evaluates the Chebyshev polynomial T_N over the endomorphism $\rho\big([K]\big) \in \mathrm{End}(V)$. A more general version of Theorem 1 and of these miraculous cancellations, valid for all framed links in $S \times [0,1]$, is provided by Theorem 16 in §5.1 below.

In general, researchers working on the Kauffman skein algebra (or on the representation theory of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$) are more familiar with the normalized Chebyshev polynomials of the second kind, defined as the polynomials $S_n(x)$ such that $\sin(n+1)\theta = \sin\theta\, S_n(2\cos\theta)$. The occurrence of the other Chebyshev polynomials $T_n(x)$ is here somewhat surprising[1].

There is a companion statement to Theorem 1 when $A^N = +1$, except that it now involves a twisted product $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S) = \mathcal{R}^0_{\mathrm{PSL}_2(\mathbb{C})}(S)\widetilde{\times}\mathrm{Spin}(S)$ of a component of the character variety $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ with the set $\mathrm{Spin}(S)$ of isotopy classes of spin structures on S. This twisted character variety $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ is more natural than would appear at first glance; for instance, the monodromy of a hyperbolic metric on $S \times (0,1)$ determines an element of $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$. See §5.1 for details.

[1] However, see [FrG] for an earlier occurrence, as well as [HaP] for a related but different context.

The definition is designed so that, for every point $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ and for every framed knot $K \subset S \times [0,1]$, there is a well-defined trace $\mathrm{Tr}\, r(K) \in \mathbb{C}$.

Theorem 2. *Suppose that A^2 is a primitive N–root of unity with N odd, and that $A^N = +1$. Then, for every irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ of the skein algebra, there exists a unique point r_ρ of the twisted character variety $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ such that*

$$T_N\big(\rho([K])\big) = -\big(\mathrm{Tr}\, r_\rho(K)\big)\mathrm{Id}_V$$

for every framed knot $K \subset S \times [0,1]$ whose projection to S has no crossing and with vertical framing.

Again, a more general version of Theorem 2 for all framed links in $S \times [0,1]$ is provided by Theorem 16 in §5.1.

For an irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$, Theorems 1 and 2 both associate to this quantum object a point in the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, which in particular is a classical (= non-quantum) geometric object. We call $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ the *classical shadow* of the representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$.

For instance, when $A^N = -1$, we can consider the finite-dimensional representation $\rho_{\mathrm{WRT}}\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ provided by the $\mathrm{SO}(3)$ version of the Witten-Reshetikhin-Turaev topological quantum field theory [ReT, BHMV, Tu$_2$]. An easy variation [BoW$_5$] of the arguments of [Rob] shows that this representation ρ_{WRT} is irreducible. In [BoW$_5$], we show that its classical shadow is the point of $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ corresponding to the trivial homomorphism $\pi_1(S) \to \{\mathrm{Id}\} \subset \mathrm{SL}_2(\mathbb{C})$.

The key to the proof of Theorems 1 and 2 is that, for every knot $K \subset S \times [0,1]$ with no crossing and with vertical framing, the evaluation $T_N\big([K]\big)$ of the Chebyshev polynomial T_N at the element $[K] \in \mathcal{S}^A(S)$ is central in $\mathcal{S}^A(S)$.

When the surface S is non-compact, there are central elements that are easier to identify, and provide more invariants. Let P_k be a small simple loop that goes around the k–th puncture in S. Consider P_k as a knot in $S \times [0,1]$, and endow it with the vertical framing. It is immediate that $[P_k] \in \mathcal{S}^A(S)$ is central in $\mathcal{S}^A(S)$, and the following result easily follows.

Proposition 3. *For every irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$, there exists a number $p_k \in \mathbb{C}$ such that $\rho\big([P_k]\big) = p_k\, \mathrm{Id}_V$.*

Suppose in addition that A^2 is a primitive N–root of unity with N odd, and that $r_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ is the classical shadow associated to ρ by Theorems 1 or 2. Then, $T_N(p_k) = -\mathrm{Tr}\, r_\rho(P_k)$.

The numbers $p_1, p_2, \dots, p_s$ thus associated to the representation ρ are the *puncture invariants* of ρ. The second part of Proposition 3 shows that, up to finitely many choices, these puncture invariants are essentially determined by the classical shadow of ρ.

Thus, we have extracted two types of invariants from an irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ of the skein algebra: the classical shadow of ρ in $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, according to whether $A^N = -1$ or $+1$; the puncture invariants $p_k \in \mathbb{C}$.

The articles [BoW$_3$, BoW$_4$], which are the natural continuation of this one (see also the expository article [BoW$_2$]), provide a converse statement. More precisely, suppose that A^2 is a primitive N–root of unity with N odd and that we are given the following data: a point $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if $A^N = -1$, or a point $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ if $A^N = +1$; a number $p_k \in \mathbb{C}$ for each puncture of S such that $T_N(p_k) = -\mathrm{Tr}\, r(P_k)$, where P_k is a small loop going around the puncture endowed with the vertical framing. Then [BoW$_3$, BoW$_4$] provide an irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ whose classical shadow is equal to r and whose puncture invariants are equal to the p_k.

The construction of irreducible representations in [BoW$_3$, BoW$_4$] uses the *quantum trace homomorphism* $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$ constructed in [BoW$_1$] which, for $\omega = A^{-\frac{1}{2}}$, embeds the skein algebra $\mathcal{S}^A(S)$ in an incarnation $\mathcal{Z}^\omega(\lambda)$ of the quantum Teichmüller space of Chekhov, Fock [Fo, ChF$_1$, ChF$_2$] and Kashaev [Kash]. The algebraic structure of the quantum Teichmüller space $\mathcal{Z}^\omega(\lambda)$ is relatively simple, and its irreducible representations are easily classified [BoL]. Composing these representations $\mathcal{Z}^\omega(\lambda) \to \mathrm{End}(V)$ with the quantum trace homomorphism $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S) \to \mathcal{Z}^\omega(\lambda)$, one obtains many representations $\mathcal{S}^A(S) \to \mathrm{End}(V)$.

This motivates the second part of the present article (in addition to the fact that this second part is used to prove the results of the first part). When constructing representations of the skein algebra from representations of the quantum Teichmüller space, the challenge is to control the classical shadow of the representations so constructed. Recall that this classical shadow is defined using the Chebyshev polynomial T_N. It turns out that the quantum trace homomorphism is extremely well behaved with respect to T_N. This is expressed in the following result, which again provides more miraculous cancellations.

Theorem 4. *Suppose that A^2 is a primitive N–root of unity with N odd, and set $\varepsilon = A^N = \pm 1$ and $\iota = \sqrt{\varepsilon}$. Then, for every knot $K \subset S \times [0,1]$ whose projection to S has no crossing and with vertical framing, the element $\mathrm{Tr}^\omega_\lambda\big(T_N([K])\big) \in \mathcal{Z}^\omega(\lambda)$ is obtained from the classical trace polynomial $\mathrm{Tr}^\iota_\lambda\big([K]\big) \in \mathcal{Z}^\iota(\lambda)$ (expressing the trace $\mathrm{Tr}\, r(K)$ as a function of the shear-bend coordinates of $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$) by replacing each generator Z_i by Z_i^N.*

Since $\mathrm{Tr}^\omega_\lambda$ is an algebra homomorphism, $\mathrm{Tr}^\omega_\lambda\big(T_N([K])\big)$ is also equal to the evaluation $T_N\big(\mathrm{Tr}^\omega_\lambda([K])\big)$ of the Chebyshev polynomial T_N over the quantum trace $\mathrm{Tr}^\omega_\lambda([K])$.

When A is not a root of unity, $\mathrm{Tr}^\omega_\lambda\big(T_N([K])\big)$ is a Laurent polynomial in noncommuting variables Z_i and, as N tends to ∞, the number of its terms grows as cN^k for appropriate constants c, $k > 0$. A consequence of Theorem 4 is that, if we specialize A to an appropriate root of unity, most of these terms disappear and we are just left with a constant number of monomials, all of which involve only N–th powers of the generators Z_i.

To some extent, the miraculous cancellations of Theorem 4 are even more surprising than those of Theorems 1 and 2, which at least are conceptually explained by the fact that certain elements of $\mathcal{S}^A(S)$ are central. We use special cases of Theorem 4 to prove Theorems 1 and 2. Our proof of Theorem 4 essentially relies on brute force, and is quite unsatisfactory in this regard. It seems that this result is quite likely to be grounded on deeper facts about the representation theory of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$.

1. Constants

The article involves various fractional powers of a non-zero complex number $q = \mathrm{e}^{2\pi \mathrm{i}\hbar} \in \mathbb{C}$. Systematically writing these constants as powers of q would lead to somewhat cumbersome notation, such as $\mathcal{S}^{q^{-\frac{1}{2}}}(S)$ or $\mathcal{Z}^{q^{\frac{1}{4}N^2}}(\lambda)$. It is more convenient (and consistent with the weight of history) to introduce a few more constants related to q. While we will regularly remind the reader of the definition of these constants, we collect these definitions here so that this section can be used as an easy reference if needed.

Again, $q \in \mathbb{C} - \{0\}$ is a non-zero complex number. We choose successive square roots $A = \sqrt{q^{-1}}$ and $\omega = \sqrt{A^{-1}} = q^{\frac{1}{4}}$.

For most of the article, q is assumed to be a root of unity, with various restrictions. The strongest restriction used, under which all properties hold, is that q is a primitive N–root of unity with N odd. This is sometimes eased to the weaker assumption that $q^2 = A^{-4}$ is a primitive N–root of unity, with no parity condition on N.

Under any of the above hypotheses, we will set $\varepsilon = A^{N^2}$ and $\iota = \omega^{N^2}$. Note that $\varepsilon = \pm 1$ if $A^{2N} = 1$, and that $\iota^2 = \varepsilon^{-1}$.

2. The Kauffman skein algebra

Let S be an oriented surface (without boundary) with finite topological type. Namely, S is obtained by removing finitely many points from a compact oriented surface $\bar{S}$. We consider *framed links* in the thickened surface $S \times [0,1]$, namely unoriented 1–dimensional submanifolds $K \subset S \times [0,1]$ endowed with a continuous choice of a vector transverse to K at each point of K. A *framed knot* is a connected framed link.

The *framed link algebra* $\mathcal{K}(S)$ is the vector space (over $\mathbb{C}$, say) freely generated by the isotopy classes of all framed links $K \subset S \times [0,1]$.

The vector space $\mathcal{K}(S)$ can be endowed with a multiplication, where the product of K_1 and K_2 is defined by the framed link $K_1 \cdot K_2 \subset S \times [0,1]$ that is the union of K_1 rescaled in $S \times [0, \frac{1}{2}]$ and K_2 rescaled in $S \times [\frac{1}{2}, 1]$. In other words, the product $K_1 \cdot K_2$ is defined by superposition of the framed links K_1 and K_2. This *superposition operation* is compatible with isotopies, and therefore provides a well-defined algebra structure on $\mathcal{K}(S)$.

Three framed links K_1, K_0 and K_∞ in $S \times [0,1]$ form a *Kauffman triple* if the only place where they differ is above a small disk in S, where they are as represented in Figure 1 (as seen from above) and where the framing is vertical and pointing upwards (namely the framing is parallel to the $[0,1]$ factor and points towards 1).

For $A \in \mathbb{C} - \{0\}$, the *Kauffman skein algebra* $\mathcal{S}^A(S)$ is the quotient of the framed link algebra $\mathcal{K}(K)$ by linear subspace generated by all elements $K_1 - A^{-1}K_0 - AK_\infty$ as (K_1, K_0, K_∞) ranges over all Kauffman triples. The superposition operation descends to a multiplication in $\mathcal{S}^A(S)$, endowing $\mathcal{S}^A(S)$ with the structure of an algebra. The class $[\varnothing]$ of the empty link is an identity element in $\mathcal{S}^A(S)$.

An element $[K] \in \mathcal{S}^A(S)$, represented by a framed link $K \subset S \times [0,1]$, is a *skein* in S. The construction is defined to ensure that the *skein relation*

$$[K_1] = A^{-1}[K_0] + A[K_\infty]$$

holds in $\mathcal{S}^A(S)$ for every Kauffman triple (K_1, K_0, K_∞).

3. Central elements of the skein algebra

3.1. **Central elements coming from the punctures.** The simplest central elements of the skein algebra $\mathcal{S}^A(S)$ come from the punctures of S, if any.

Proposition 5. *If P_k is a small embedded loop in S going once around the k–th puncture. Considering P_k as a knot in $S \times [0,1]$ endowed with the vertical framing, the skein $[P_k]$ is central in $\mathcal{S}^A(S)$, for every value of $A \in \mathbb{C} - \{0\}$.* □

These are somewhat obvious elements of the center of $\mathcal{S}^A(S)$. When A is a root of unity, this center contains other elements which are much less evident, defined using Chebyshev polynomials.

3.2. **Chebyshev polynomials.** Chebyshev polynomials of the first and second kind play a prominent rôle in this article. Both types are polynomials $P_n(x)$ that satisfy the recurrence relation

$$P_n(x) = xP_{n-1}(x) - P_{n-2}(x).$$

The (normalized) *Chebyshev polynomials of the first kind* are the polynomials $T_n(x)$ defined by this recurrence relation and by the initial conditions that $T_0(x) = 2$ and $T_1(x) = x$. The (normalized) *Chebyshev polynomials of the second kind* $S_n(x)$ are similarly defined by the above recurrence relation and by the same initial condition $S_1(x) = x$, but differ in the other initial condition $S_0(x) = 1$.

For instance, $T_2(x) = x^2 - 2$, $T_3(x) = x^3 - 3x$ and $T_4(x) = x^4 - 4x^2 + 2$, while $S_2(x) = x^2 - 1$, $S_3(x) = x^3 - 2x$ and $S_4(x) = x^4 - 3x^2 + 1$.

One reason why the occurrence of Chebyshev polynomials in this work is not completely unexpected is that both types are closely related to the trace function in $\mathrm{SL}_2(\mathbb{C})$, through the following classical properties.

Lemma 6. *For any $M \in \mathrm{SL}_2(\mathbb{C})$,*

1. $\mathrm{Tr}\, M^n = T_n(\mathrm{Tr}\, M)$*;*
2. *If $\rho_n \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$ is the unique $(n+1)$–dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$, then $\mathrm{Tr}\, \rho_n(M) = S_n(\mathrm{Tr}\, M)$.* □

Applying Lemma 6 to the matrix $M = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}$ yields the more classical relations that $\cos n\theta = \frac{1}{2} T_n(2\cos\theta)$ and $\sin n\theta = \sin\theta\, S_{n-1}(2\cos\theta)$.

For future reference, we note the following two elementary properties.

Lemma 7. *For $n \geqslant 2$,*

$$T_n(x) = S_n(x) - S_{n-2}(x).$$

Proof. The difference $T_n(x) - S_n(x) + S_{n-2}(x)$ satisfies the linear recurrence relation $P_n(x) = xP_{n-1}(x) - P_{n-2}(x)$, and is equal to 0 for $n = 2$ and $n = 3$. □

Lemma 8.

1. *If $x = a + a^{-1}$, then $T_n(x) = a^n + a^{-n}$;*
2. *If $y = b + b^{-1}$, the set of solutions to the equation $T_n(x) = y$ consists of the numbers $x = a + a^{-1}$ as a ranges over all n–roots of b.*

Proof. For a matrix $M \in \mathrm{SL}_2(\mathbb{C})$, the data of its trace x is equivalent to the data of its spectrum $\{a, a^{-1}\}$. The first property is then an immediate consequence of the fact that $\mathrm{Tr}\, M^n = T_n(\mathrm{Tr}\, M)$. The second property then follows. □

3.3. **Threading a Chebyshev polynomial along a framed link.** Chebyshev polynomials of the first kind surprisingly provide central elements of the skein algebra $\mathcal{S}^A(S)$.

We first introduce some notation. Consider a polynomial

$$P(x) = \sum_{i=0}^{n} a_i x^i.$$

We can then associate to each framed knot K in $S \times [0,1]$ the linear combination

$$[K^P] = \sum_{i=1}^{n} a_i \, [K^{(i)}] \in \mathcal{S}^A(S)$$

where, for each i, $K^{(i)}$ is the framed link obtained by taking i parallel copies of K in the direction indicated by the framing.

More generally, if K is a framed link with components $K_1, K_2, \ldots, K_l$,

$$[K^P] = \sum_{1 \leqslant i_1, \, i_2, \ldots, \, i_l \leqslant n} a_{i_1} a_{i_2} \ldots a_{i_l} \, [K_1^{(i_1)} \cup K_2^{(i_2)} \cup \cdots \cup K_l^{(i_l)}].$$

We will say that the element $[K^P] \in \mathcal{S}^A(S)$ is obtained by *threading the polynomial P along the framed link K*.

When $K \subset S \times [0,1]$ projects to a simple closed curve in S and is endowed with the vertical framing, $[K^P]$ is equal to the evaluation $P\big([K]\big) \in \mathcal{S}^A(S)$ of the polynomial P at the skein $[K]$, for the algebra structure of $\mathcal{S}^A(S)$. More generally, if K, with components $K_1, K_2, \ldots, K_l$, projects to an embedded submanifold of S and is endowed with the vertical framing, then $[K^P] = P\big([K_1]\big) P\big([K_2]\big) \ldots P\big([K_l]\big)$. Beware that this is in general false for a knot or link whose projection to S admits crossings, or whose framing is different from the vertical framing.

Theorem 9. *If A^2 is a primitive N–root of unity, threading the Chebyshev polynomial T_N along a framed link produces a central element of the skein algebra, namely $[K^{T_N}]$ is central in $\mathcal{S}^A(S)$ for every framed link $K \subset S \times [0,1]$.*

Theorem 9 holds with no parity condition on N.

The key to the proof of Theorem 9 is the following special case in the once-punctured torus T, represented in Figure 2 as a square with its corners removed and with opposite sides identified.

Lemma 10. *In the once-punctured torus T, let L_0 and L_∞ be the two curves represented in Figure 2, and consider these curves as framed knots with vertical framing in $T \times [0,1]$. If A^2 is a primitive N–root of unity, then*

$$[L_0^{T_N}][L_\infty] = [L_\infty][L_0^{T_N}]$$

in the skein algebra $\mathcal{S}^A(T)$.

There is an elementary proof of Lemma 10, borrowed from the proof of the Product-to-Sum Formula of [FrG] for the non-punctured torus. See also the (much less elementary) combination of [BuP, §2] and [HaP, Lemma 2]. We provide another proof in §7.9, where it is proved at the same time as Lemma 12 below.

Proof of Theorem 9, assuming Lemma 10. We need to show that $[K^{T_N}][L] = [L][K^{T_N}]$ for any two framed links K and L.

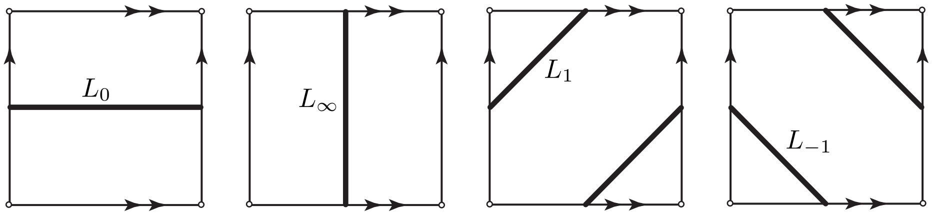

FIGURE 2. The curves L_0, L_∞, L_1 and L_{-1} in the once-punctured torus

Let $K_1 \subset S \times [0,1]$ be isotopic to K, and contained in a small neighborhood of $S \times \{0\}$ so that $[K_1^{T_N} \cup L] = [K^{T_N}][L]$ in $\mathcal{S}^A(S)$ (where $[K_1^{T_N} \cup L]$ denotes the element of $\mathcal{S}^A(S)$ obtained by threading the Chebyshev polynomial T_N along the components of K_1, and leaving L untouched).

By progressively changing the undercrossings of K with L to overcrossings, we construct a sequence of framed links $K_1, K_2, \dots, K_n$ such that each K_{i+1} is obtained from K_i by an isotopy crossing L exactly once, and such that K_n can be isotoped without crossing L into a small neighborhood of $S \times \{1\}$. In particular, $[K_n^{T_N} \cup L] = [L][K^{T_N}]$ in $\mathcal{S}^A(S)$. It therefore suffices to show that $[K_i^{T_N} \cup L] = [K_{i+1}^{T_N} \cup L]$ for every i.

We now consider an embedded thickened once-punctured torus which "swallows" the crossing between K_i and K_{i+1} in question and "follows" the two curves otherwise. We then apply Lemma 10 to exchange the swallowed undercrossing to an overcrossing. More precisely, by construction of K_{i+1} from K_i there is an embedding $\varphi\colon T \times [0,1] \to S \times [0,1]$ for which, with the notation of Lemma 10, the intersections $K_i \cap \varphi\big(T \times [0,1]\big)$, $L \cap \varphi\big(T \times [0,1]\big)$ and $K_{i+1} \cap \varphi\big(T \times [0,1]\big)$ respectively correspond to the framed knots $L_\infty \times \{\frac14\}$, $L_0 \times \{\frac12\}$ and $L_\infty \times \{\frac34\}$ in $T \times [0,1]$, after isotopy in $T \times [0,1]$. Applying Lemma 10, and composing with the homomorphism $\mathcal{S}^A(T) \to \mathcal{S}^A(S)$ induced by φ, shows that $[K_i^{T_N} \cup L] = [K_{i+1}^{T_N} \cup L]$.

By iteration, this concludes the proof. □

4. The Chebyshev Homomorphism

Theorem 11. *If A^4 is a primitive N–root of unity and if $\varepsilon = A^{N^2}$, there is a unique algebra homomorphism $\mathbf{T}^A\colon \mathcal{S}^\varepsilon(S) \to \mathcal{S}^A(S)$ which, for every framed link K in $S \times [0,1]$, associates $[K^{T_N}] \in \mathcal{S}^A(S)$ to the skein $[K] \in \mathcal{S}^\varepsilon(S)$.*

A key step in the proof of Theorem 11 is the following computation.

Lemma 12. *Suppose that A^4 is a primitive N–root of unity. In the once-punctured torus T, let L_0, L_∞, L_1 and L_{-1} be the curves represented in Figure 2. Considering these curves as knots in $T \times [0,1]$ and endowing them with the vertical framing,*

$$[L_0^{T_N}][L_\infty^{T_N}] = A^{-N^2}[L_1^{T_N}] + A^{N^2}[L_{-1}^{T_N}]$$

in the skein algebra $\mathcal{S}^A(T)$.

We postpone the proof of Lemma 12 to §7.9. This result is very reminiscent of the Product-to-Sum Formula of [FrG], which holds in the unpunctured torus but for all values of A. Unlike that formula, Lemma 12 only holds when A^4 is an N–root of unity. It would probably be more satisfying to find a simpler proof of Lemma 12, perhaps similar to or using the arguments of [FrG].

FIGURE 3. The 1–submanifolds L_1, L_0, L_∞ and L_{-1} in the twice-punctured plane

The proof of Theorem 11 also uses the following similar computation.

Lemma 13. *Suppose that A^4 is a primitive N–root of unity. For the twice-punctured plane U, let L_1, L_0, L_∞ and L_{-1} be the links in $U \times [0,1]$ represented in Figure 3, endowed with the vertical framing. Then,*

$$[L_1^{T_N}] = A^{-N^2}[L_0^{T_N}] + A^{N^2}[L_\infty^{T_N}]$$
$$\text{and } [L_{-1}^{T_N}] = A^{N^2}[L_0^{T_N}] + A^{-N^2}[L_\infty^{T_N}]$$

in the skein algebra $\mathcal{S}^A(U)$.

Again, we postpone the proof of Lemma 13 to §7.9.

Proof of Theorem 11 (assuming Lemmas 12 and 13). We have to check that the Chebyshev threads $[K^{T_n}] \in \mathcal{S}^A(S)$ satisfy the skein relation with A replaced by ε.

Let the framed links K_1, K_0, $K_\infty \subset S \times [0,1]$ form a Kauffman triple, as in Figure 1. Let $\widehat{K}_1$, $\widehat{K}_0$, $\widehat{K}_\infty$ denote the union of the (one or two) components of K_1, K_0, K_∞, respectively, that appear in Figure 1. Considering the way the strands represented connect outside of the picture, we distinguish three cases, according to the number of components of $\widehat{K}_1$, $\widehat{K}_0$ and $\widehat{K}_\infty$. One easily sees that exactly two of these three links are connected, while the remaining one has two components.

If $\widehat{K}_1$ is disconnected, there exists an embedding $\varphi\colon T \times [0,1] \to S \times [0,1]$ of the thickened punctured torus $T \times [0,1]$ such that, using the notation of Lemma 12, $\varphi^{-1}(K_1)$ is isotopic to the union of $L_0 \times \{\frac14\}$ and of $L_\infty \times \{\frac34\}$ in $T \times [0,1]$, while $\varphi^{-1}(K_0)$ and $\varphi^{-1}(K_\infty)$ are respectively isotopic to $L_1 \times \{\frac12\}$ and $L_{-1} \times \{\frac12\}$. Applying Lemma 12 and composing with the homomorphism $\mathcal{S}^A(T) \to \mathcal{S}^A(S)$ induced by φ, one finds that

$$[\widehat{K}_1^{T_N}] = A^{-N^2}[\widehat{K}_0^{T_N}] + A^{N^2}[\widehat{K}_\infty^{T_N}] = \varepsilon^{-1}[\widehat{K}_0^{T_N}] + \varepsilon[\widehat{K}_\infty^{T_N}].$$

Since K_1, K_0, K_∞ coincide outside of $\varphi(T \times [0,1]$, it follows that

$$[K_1^{T_N}] = \varepsilon^{-1}[K_0^{T_N}] + \varepsilon[K_\infty^{T_N}].$$

in $\mathcal{S}^A(S)$.

The other two cases are similar, but using Lemma 13 this time. If $\widehat{K}_0$ is connected, then there exists an embedding $\varphi\colon U \times [0,1] \to S \times [0,1]$ of the thickened twice-punctured plane $U \times [0,1]$ such that, using the notation of Lemma 13, $\varphi^{-1}(K_1) = L_1$, $\varphi^{-1}(K_0) = L_0$ and $\varphi^{-1}(K_\infty) = L_\infty$. Applying the first identity of Lemma 13 shows that

$$[\widehat{K}_1^{T_N}] = \varepsilon^{-1}[\widehat{K}_0^{T_N}] + \varepsilon[\widehat{K}_\infty^{T_N}]$$

in this case as well, so that $[K_1^{T_N}] = \varepsilon^{-1}[K_0^{T_N}] + \varepsilon[K_\infty^{T_N}]$ in $\mathcal{S}^A(S)$.

Finally, if K_∞ is connected, we use an embedding $\varphi\colon U \times [0,1] \to S \times [0,1]$ of the thickened twice-punctured plane $U \times [0,1]$ such that $\varphi^{-1}(K_1) = L_{-1}$, $\varphi^{-1}(K_0) =$

L_∞ and $\varphi^{-1}(K_\infty) = L_0$. The second identity of Lemma 13 then provides the relation sought.

Therefore, $[K_1^{T_N}] = \varepsilon^{-1}[K_0^{T_N}] + \varepsilon[K_\infty^{T_N}]$ in $\mathcal{S}^A(S)$ for every Kauffman triple K_1, K_0, K_∞. This proves that the threading map $K \mapsto [K^{T_N}] \in \mathcal{S}^A(S)$ induces a linear map $\mathbf{T}^A\colon \mathcal{S}^\varepsilon(S) \to \mathcal{S}^A(S)$. It is immediate that $\mathbf{T}^A$ is an algebra homomorphism. □

5. Invariants of irreducible representations

Let $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ be an irreducible representation of the skein algebra $\mathcal{S}^A(S)$. We want to construct invariants of this representation.

To take advantage of Theorems 9 and 11, we are here going to need that A^4 is a primitive N–root of unity and $A^{2N} = 1$. This is equivalent to the property that A^2 is a primitive N–root of unity with N odd. In particular, $\varepsilon = A^{N^2} = A^N = \pm 1$.

5.1. **The classical shadow.** Composing the representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$ with the Chebyshev homomorphism $\mathbf{T}^A\colon \mathcal{S}^\varepsilon(S) \to \mathcal{S}^A(S)$ of Theorem 11, we obtain a homomorphism $\mathcal{S}^\varepsilon(S) \to \mathrm{End}(V)$. By Theorem 9, this homomorphism factors through the center of $\mathcal{S}^A(S)$ and therefore, by irreducibility of ρ and by Schur's lemma, there exists an algebra homomorphism

$$\kappa_\rho\colon \mathcal{S}^\varepsilon(S) \to \mathbb{C}$$

such that

$$\rho \circ \mathbf{T}^A\big([K]\big) = \rho\big([K^{T_N}]\big) = \kappa_\rho\big([K]\big)\mathrm{Id}_V$$

for every skein $[K] \in \mathcal{S}^A(S)$.

Because $\varepsilon = \pm 1$, such a homomorphism $\kappa_\rho\colon \mathcal{S}^\varepsilon(S) \to \mathbb{C}$ has a geometric interpretation.

More precisely, consider the character variety

$$\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) = \{\text{group homomorphisms } r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})\} /\!\!/ \mathrm{SL}_2(\mathbb{C})$$

where $\mathrm{SL}_2(\mathbb{C})$ acts on homomorphisms by conjugation, and where the double bar $/\!\!/$ indicates that the quotient is taken in the sense of geometric invariant theory [MuFK] in algebraic geometry. For a group homomorphism $r\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ and a closed curve $K \subset S \times [0,1]$, the trace $\mathrm{Tr}\, r(K) \in \mathbb{C}$ depends only on the class of r in the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$. An observation of Doug Bullock, Charlie Frohman, Jozef Przytycki and Adam Sikora [Bu_1, Bu_2, BuFK_1, BuFK_2, PrS] then shows that this defines an algebra homomorphism

$$\mathrm{Tr}_r\colon \mathcal{S}^{-1}(S) \to \mathbb{C}$$

by the property that

$$\mathrm{Tr}_r\big([K]\big) = -\mathrm{Tr}\, r(K)$$

for every (connected) framed knot $K \subset S \times [0,1]$. Note that $\mathrm{Tr}_r\big([K]\big)$ is independent of the framing of K.

There exists a twisted version of this for $\mathcal{S}^{+1}(S)$. Let $\mathrm{Spin}(S)$ denote the space of isotopy classes of spin structures on S. Given a spin structure $\sigma \in \mathrm{Spin}(S)$, John Barrett [Ba] constructs an algebra homomorphism $B_\sigma\colon \mathcal{S}^{+1}(S) \to \mathcal{S}^{-1}(S)$, which to a skein $[K] \in \mathcal{S}^{+1}(S)$ associates $(-1)^{k+\sigma(k)}[K] \in \mathcal{S}^{-1}(S)$, where k is the number of components of the link K and where $\sigma(K) \in \mathbb{Z}_2$ is the monodromy of the framing of K with respect to the spin structure σ. A pair $r = (\widehat{r}, \sigma)$, consisting of a group homomorphism $\widehat{r}\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ and a spin structure $\sigma \in \mathrm{Spin}(S)$, then defines a trace map

$$\mathrm{Tr}_r = \mathrm{Tr}_{\widehat{r}} \circ B_\sigma\colon \mathcal{S}^{+1}(S) \to \mathbb{C},$$

which to a connected skein $[K] \in \mathcal{S}^{+1}(S)$ associates

$$\mathrm{Tr}_r\big([K]\big) = (-1)^{\sigma(K)}\mathrm{Tr}\,\widehat{r}(K).$$

This trace map Tr_r is unchanged under certain modifications of the pair $(\widehat{r}, \sigma)$. Indeed, two spin structures in $\mathrm{Spin}(S)$ differ by an obstruction in $H^1(S;\mathbb{Z}_2)$; this defines an action of $H^1(S;\mathbb{Z}_2)$ on $\mathrm{Spin}(S)$. The cohomology group $H^1(S;\mathbb{Z}_2)$ also acts on the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ by the property that, if $\widehat{r} \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and $\alpha \in H^1(S;\mathbb{Z}_2)$, then $\alpha\widehat{r} \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ is defined by

$$\alpha\widehat{r}(\gamma) = (-1)^{\alpha(\gamma)}\widehat{r}(\gamma) \in \mathrm{SL}_2(\mathbb{C})$$

for every $\gamma \in \pi_1(S)$. Then the trace map T_r depends only on the image of $r = (\widehat{r}, \sigma)$ in the quotient

$$\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S) = \left(\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) \times \mathrm{Spin}(S)\right)/H^1(S;\mathbb{Z}_2).$$

To explain the notation, consider the character variety

$$\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S) = \{\text{group homomorphisms } r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})\}/\!\!/\mathrm{PSL}_2(\mathbb{C})$$

and the subset $\mathcal{R}^0_{\mathrm{PSL}_2(\mathbb{C})}(S) = \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)/H^1(S;\mathbb{Z}_2)$ of $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ consisting of those homomorphisms $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ that lift to $\mathrm{SL}_2(\mathbb{C})$. Bill Goldman [Go] showed that $\mathcal{R}^0_{\mathrm{PSL}_2(\mathbb{C})}(S)$ is equal to the whole character variety $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ when S is non-compact, and is equal to one of the two components of $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ when S is compact. Then, $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ can be seen as a twisted product of $\mathcal{R}^0_{\mathrm{PSL}_2(\mathbb{C})}(S)$ with $\mathrm{Spin}(S)$. In particular, both $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ and $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ are coverings of $\mathcal{R}^0_{\mathrm{PSL}_2(\mathbb{C})}(S)$ with fiber $\mathrm{Spin}(S) \cong H^1(S;\mathbb{Z}_2)$.

Conversely, all algebra homomorphisms $\mathcal{S}^\varepsilon(S) \to \mathbb{C}$ but one are obtained in this way. This exceptional homomorphism is the *trivial homomorphism*, defined by the property that $\kappa([\varnothing]) = 1$ and $\kappa([K]) = 0$ for every nonempty framed link K in $S \times [0,1]$.

Proposition 14. *Every non-trivial homomorphism $\kappa : \mathcal{S}^\varepsilon(S) \to \mathbb{C}$ is equal to the trace homomorphism Tr_r associated to a unique point $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if $\varepsilon = -1$, and $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ if $\varepsilon = +1$.*

Proof. The case $\varepsilon = -1$ is a result of H. Helling [He, Prop. 1]. The case $\varepsilon = +1$ easily follows by using Barrett's isomorphism $B_\sigma\colon \mathcal{S}^{+1}(S) \to \mathcal{S}^{-1}(S)$. □

Let us now return to our analysis of an irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$, and to its associated homomorphism $\kappa_\rho\colon \mathcal{S}^\varepsilon(S) \to \mathbb{C}$ defined by the property that $\rho\big([K^{T_N}]\big) = \kappa_\rho\big([K]\big)\mathrm{Id}_V$ for every skein $[K] \in \mathcal{S}^A(S)$.

Lemma 15. *The homomorphism $\kappa_\rho\colon \mathcal{S}^\varepsilon(S) \to \mathbb{C}$ is non-trivial.*

Proof. Let O be the boundary of a small disk in $S \times \{\frac{1}{2}\}$, endowed with the vertical framing.

The skein $[O]$ is clearly in the center of $\mathcal{S}^A(S)$. By irreducibility of ρ and by Schur's lemma, there consequently exists a number $o \in \mathbb{C}$ such that $\rho\big([O]\big) = o\,\mathrm{Id}_V$. A classical consequence of the invariance of skeins under isotopy (and in particular under the second Reidemeister move) is that $\big([O]+A^2+A^{-2}\big)[K]=0$ in $\mathcal{S}^A(S)$ for every skein $[K]\in\mathcal{S}^A(S)$ represented by a non-empty link $K\subset S\times[0,1]$; see for instance [Lic₁, Lemmas 3.2 and 3.3]. As a consequence, $\big(o+A^2+A^{-2}\big)\rho\big([K]\big)=0$ for every non-empty skein $[K]\in\mathcal{S}^A(S)$. Applying this to a non-empty skein $[K]\in\mathcal{S}^A(S)$ such that $\rho\big([K]\big)\neq 0$, which exists by irreducibility of ρ, this proves that $o=-\big(A^2+A^{-2}\big)$, and therefore that

$$\rho\big([O]\big)=-\big(A^2+A^{-2}\big)\,\mathrm{Id}_V.$$

Then, because O has no crossing and is endowed with the vertical framing,

$$\begin{aligned}\rho\big([O^{T_N}]\big)&=\rho\big(T_N([O])\big)=T_N\big(\rho([O])\big)=T_N\big((-A^2-A^{-2})\,\mathrm{Id}_V\big)\\&=T_N(-A^2-A^{-2})\,\mathrm{Id}_V=\big((-A^2)^N+(-A^{-2})^N\big)\,\mathrm{Id}_V\\&=-2\,\mathrm{Id}_V\end{aligned}$$

where the second equality comes from the fact that ρ is an algebra homomorphism, while the fifth identity is a consequence of Lemma 8 and the last equality uses the fact that $A^{2N}=1$ and N is odd. In particular, $\kappa_\rho\big([O]\big)=-2\neq 0$, and κ_ρ is non-trivial. □

Combining the definition of the algebra homomorphism $\kappa_\rho\colon\mathcal{S}^\varepsilon(S)\to\mathbb{C}$ with Proposition 14 and Lemma 15 provides the following statement.

Theorem 16. *Suppose that A^2 is a primitive N–root of unity with N odd, so that $\varepsilon=A^{N^2}$ is equal to ±1. Then, for every irreducible representation $\rho\colon\mathcal{S}^A(S)\to\mathrm{End}(V)$ of the skein algebra $\mathcal{S}^A(S)$, there exists a unique point $r_\rho\in\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ if $A^N=-1$, or a unique $r_\rho\in\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ if $A^N=+1$, such that*

$$\rho\big([K^{T_N}]\big)=\big(\mathrm{Tr}_{r_\rho}([K])\big)\mathrm{Id}_V$$

for every skein $[K]\in\mathcal{S}^A(S)$. □

In particular, Theorem 16 implies Theorems 1 and 2 stated in the introduction, since $T_N\big(\rho([K])\big)=\rho\big([K^{T_N}]\big)$ for a knot $K\subset S\times[0,1]$ with no crossing and with vertical framing.

We call the point $r_\rho\in\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$, associated to κ_ρ by Theorem 16, the *classical shadow* of the irreducible representation $\rho\colon\mathcal{S}^A(S)\to\mathrm{End}(V)$.

5.2. **Puncture invariants.** The central elements associated to the punctures of S provide similar invariants for the irreducible representation $\rho\colon\pi_1(S)\to\mathrm{End}(V)$. More precisely, let $P_k\subset S\times\{\frac12\}\subset S\times[0,1]$ be a small simple loop going around the k–th puncture, endowed with the vertical framing. Proposition 5 shows that the corresponding skein $[P_k]\in\mathcal{S}^A(S)$ is central. Therefore, by irreducibility of ρ, there exists a number $p_k\in\mathbb{C}$ such that

$$\rho\big([P_k]\big)=p_k\,\mathrm{Id}_V.$$

This number $p_k\in\mathbb{C}$ is the *k–th puncture invariant* of the irreducible representation ρ.

These numbers p_k are clearly constrained by the classical shadow $r_\rho\in\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ of ρ, in terms of the Chebyshev polynomial T_N. Indeed:

Lemma 17.

$$T_N(p_k)=\mathrm{Tr}_{r_\rho}\big([P_k]\big).$$

Proof. By definition of r_ρ,

$$\begin{aligned}\mathrm{Tr}_{r_\rho}\big([P_k]\big)\mathrm{Id}_V &= \rho\big(\big[P_k^{T_N}\big]\big) = \rho\big(T_N([P_k])\big)\\ &= T_N\big(\rho([P_k])\big) = T_N(p_k\mathrm{Id}_V) = T_N(p_k)\mathrm{Id}_V\end{aligned}$$

where the second equality uses the fact that P_k has no crossings and has vertical framing, and where the third equality holds because ρ is an algebra homomorphism. □

In particular, Lemma 8 shows that, once the classical shadow r_ρ is given, there are at most N possibilities for each puncture invariant p_k.

6. The quantum Teichmüller space and the quantum trace homomorphism

In [BoW$_3$, BoW$_4$], we prove a converse to Theorem 16 and Lemma 17, by showing that every point $r \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ or $\mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ and any set of puncture weights p_k with $T_N(p_k) = \mathrm{Tr}_{r_\rho}\big([P_k]\big)$ can be realized as the classical shadow and the puncture invariants of an irreducible representation $\rho\colon \mathcal{S}^A(S) \to \mathrm{End}(V)$.

The proof of this result relies on the quantum Teichmüller space of [Fo, ChF$_1$, ChF$_2$] and on the quantum trace homomorphism constructed in [BoW$_1$]. A crucial step in the argument is a compatibility property between the quantum trace homomorphism and the Chebyshev homomorphism, proved here as Theorem 20. This section is devoted to this statement. This result is of interest by itself, because of the surprising "miraculous cancellations" that it involves. It is also used in the proof of Lemmas 10, 12 and 13, which we had temporarily postponed.

6.1. **The Chekhov-Fock algebra.** The Chekhov-Fock algebra is the avatar of the quantum Teichmüller space associated to an ideal triangulation of the surface S. If S is obtained from a compact surface $\bar S$ by removing finitely many points v_1, v_2, …, v_s, an *ideal triangulation* of S is a triangulation λ of $\bar S$ whose vertex set is exactly $\{v_1, v_2, \dots, v_s\}$.

Let λ_1, λ_2, …, λ_n denote the edges of λ. Let $a_{ij} \in \{0,1,2\}$ be the number of times an end of the edge λ_j immediately succeeds an end of λ_i when going counterclockwise around a puncture of S, and set $\sigma_{ij} = a_{ij} - a_{ji} \in \{-2,-1,0,1,2\}$. The *Chekhov-Fock algebra* $\mathcal{T}^\omega(\lambda)$ of λ is the algebra defined by generators $Z_1^{\pm1}$, $Z_2^{\pm1}$, …, $Z_n^{\pm1}$ associated to the edges λ_1, λ_2, …, λ_n of λ, and by the relations

$$Z_iZ_j = \omega^{2\sigma_{ij}}Z_jZ_i.$$

(The actual Chekhov-Fock algebra $\mathcal{T}^q(\lambda)$ that is at the basis of the quantum Teichmüller space uses the constant $q = \omega^4$ instead of ω. The generators Z_i of $\mathcal{T}^\omega(\lambda)$ appearing here are designed to model square roots of the original generators of $\mathcal{T}^q(\lambda)$.)

An element of the Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$ is a linear combination of monomials $Z_{i_1}^{n_1}Z_{i_2}^{n_2}\dots Z_{i_l}^{n_l}$ in the generators Z_i, with n_1, n_2, …, $n_l \in \mathbb{Z}$. Because of the skew-commutativity relation $Z_iZ_j = \omega^{2\sigma_{ij}}Z_jZ_i$, the order of the variables in such a monomial does matter. It is convenient to use the following symmetrization trick.

The *Weyl quantum ordering* for $Z_{i_1}^{n_1}Z_{i_2}^{n_2}\dots Z_{i_l}^{n_l}$ is the monomial

$$[Z_{i_1}^{n_1}Z_{i_2}^{n_2}\dots Z_{i_l}^{n_l}] = \omega^{-\sum_{u<v} n_un_v\sigma_{i_ui_v}}Z_{i_1}^{n_1}Z_{i_2}^{n_2}\dots Z_{i_l}^{n_l}.$$

The formula is specially designed so that $[Z_{i_1}^{n_1} Z_{i_2}^{n_2} \dots Z_{i_l}^{n_l}]$ is invariant under any permutation of the $Z_{i_u}^{n_u}$.

6.2. **The Frobenius homomorphism.** The following homomorphism plays a fundamental role in the classification of irreducible representations of the quantum Teichmüller space [BoL].

Proposition 18. *If $\iota = \omega^{N^2}$, there is an algebra homomorphism*

$$\mathbf{F}^\omega \colon \mathcal{T}^\iota(\lambda) \to \mathcal{T}^\omega(\lambda)$$

which maps each generator $Z_i \in \mathcal{T}^\iota(\lambda)$ to $Z_i^N \in \mathcal{T}^\omega(\lambda)$, where in the first instance $Z_i \in \mathcal{T}^\iota(\lambda)$ denotes the generator associated to the i–th edge λ_i of λ, whereas the second time $Z_i \in \mathcal{T}^\omega(\lambda)$ denotes the generator of $\mathcal{T}^\omega(\lambda)$ associated to the same edge λ_i.

Proof. If $Z_i Z_j = \omega^{2\sigma_{ij}} Z_j Z_i$ for generators Z_i, Z_j of $\mathcal{T}^\omega(\lambda)$, then $Z_i^N Z_j^N = \omega^{2\sigma_{ij}N^2} Z_j^N Z_i^N = \iota^{2\sigma_{ij}} Z_j^N Z_i^N$. □

Borrowing the terminology from [FoG], the algebra homomorphism $\mathbf{F}^\omega \colon \mathcal{T}^\iota(\lambda) \to \mathcal{T}^\omega(\lambda)$ is the *Frobenius homomorphism* of $\mathcal{T}^\omega(\lambda)$. When $\omega^N = (-1)^{N+1}$, this Frobenius is remarkably well-behaved with respect to coordinate changes in the quantum Teichmüller space; see [ChF$_1$, BoW$_1$, FoG]. However, we won't need this property here, although it somewhat foreshadows the compatibility property of Theorem 20.

6.3. **The quantum trace homomorphism.** Another key ingredient is the embedding of $\mathcal{S}^A(S)$ in $\mathcal{T}^\omega(\lambda)$ constructed in [BoW$_1$].

The computations of §7 will (unfortunately) require familiarity with the details of this embedding. We therefore need to briefly summarize the main features of its construction.

In particular, we need to extend the definition of the skein algebra to allow S to be a punctured surface with boundary, obtained by removing finitely many points $v_1, v_2, \dots, v_s$ from a compact connected oriented surface $\bar{S}$ with boundary $\partial\bar{S}$. We require that each component of $\partial\bar{S}$ contains at least one puncture v_k, that there is at least one puncture, and that the Euler characteristic of S is less that half the number of components of ∂S. In particular, the boundary ∂S consists of disjoint open intervals. These topological restrictions are equivalent to the existence of an ideal triangulation λ for S. The Chekhov-Fock algebra $\mathcal{T}^\omega(\lambda)$ is then defined as before, with skew-commuting generators corresponding to the edges of λ (including the components of ∂S).

A *framed link* K in $S\times[0,1]$ is a 1–dimensional framed submanifold $K \subset S\times[0,1]$ such that:

1. $\partial K = K \cap \partial(S\times[0,1])$ consists of finitely many points in $(\partial S)\times[0,1]$;
2. at every point of ∂K, the framing is vertical, namely parallel to the $[0,1]$ factor and pointing in the direction of 1;
3. for every component k of ∂S, the points of ∂K that are in $k\times[0,1]$ sit at different elevations, namely have different $[0,1]$–coordinates.

Isotopies of framed links are required to respect all three conditions above. The third condition is particular important for the gluing construction described below.

The *skein algebra* $\mathcal{S}^A(S)$ is then defined as before, first considering the vector space $\mathcal{K}(S)$ freely generated by all isotopy classes of framed links, and then taking its quotient by the skein relation. The multiplication of the algebra structure is provided by the usual superposition operation.

The extension to surfaces with boundary enables us to glue surfaces and skeins. Given two surfaces S_1 and S_2 and two boundary components $k_1 \subset \partial S_1$ and $k_2 \subset \partial S_2$, we can glue S_1 and S_2 by identifying k_1 and k_2 to obtain a new oriented surface S. There is a unique way to perform this gluing so that the orientations of S_1 and S_2 match to give an orientation of S. We allow the "self-gluing" case, where the surfaces S_1 and S_2 are equal as long as the boundary components k_1 and k_2 are distinct. If we are given an ideal triangulation λ_1 of S_1 and an ideal triangulation λ_2 of S_2, these two triangulations fit together to give an ideal triangulation λ of the glued surface S.

Now, suppose in addition that we are given skeins $[K_1] \in \mathcal{S}^A(S_1)$ and $[K_2] \in \mathcal{S}^A(S_2)$, represented by framed links K_1 and K_2 such that $K_1 \cap (k_1 \times [0,1])$ and $K_2 \cap (k_2 \times [0,1])$ have the same number of points. We can then arrange by an isotopy of framed links that K_1 and K_2 fit together to give a framed link $K \subset S \times [0,1]$; note that it is here important that the framings be vertical pointing upwards on the boundary, so that they fit together to give a framing of K. By our hypothesis that the points of $K_1 \cap (k_1 \times [0,1])$ (and of $K_2 \cap (k_2 \times [0,1])$ sit at different elevations, the framed link K is now uniquely determined up to isotopy. Also, this operation is well behaved with respect to the skein relations, so that K represents a well-defined element $[K] \in \mathcal{S}^A(S)$. We say that $[K] \in \mathcal{S}^A(S)$ is *obtained by gluing* the two skeins $[K_1] \in \mathcal{S}^A(S_1)$ and $[K_2] \in \mathcal{S}^A(S_2)$.

The gluing operation is also well-behaved with respect to Chekhov-Fock algebras. Indeed, if S is obtained by gluing two surfaces S_1 and S_2 along boundary components $k_1 \subset \partial S_1$ and $k_2 \subset \partial S_2$, and if the ideal triangulations λ_1 of S_1 and λ_2 of S_2 are glued to provide an ideal triangulation λ of S, there is a canonical embedding $\Phi\colon \mathcal{T}^\omega(\lambda) \to \mathcal{T}^\omega(\lambda_1) \otimes \mathcal{T}^\omega(\lambda_2)$ defined as follows. If Z_k is a generator of $\mathcal{T}^\omega(\lambda)$ associated to an edge k of λ:

(1) $\Phi(Z_k) = Z_{k_1} \otimes Z_{k_2}$ if k is the edge coming from the boundary components $k_1 \subset \partial S_1$ and $k_2 \subset \partial S_2$ that are glued together, and if $Z_{k_1} \in \mathcal{T}^\omega(\lambda_1)$ and $Z_{k_2} \in \mathcal{T}^\omega(\lambda_2)$ are the generators associated to these edges k_1, k_2 of λ_1, λ_2;
(2) $\Phi(Z_k) = Z_{k_1'} \otimes 1$ if k corresponds to an edge k_1' of λ_1 that is not glued to an edge of λ_2;
(3) $\Phi(Z_k) = 1 \otimes Z_{k_2'}$ if k corresponds to an edge k_2' of λ_2 that is not glued to an edge of λ_1.

There is a similar embedding $\mathcal{T}^\omega(\lambda) \to \mathcal{T}^\omega(\lambda_1)$ in the case of self-gluing, when the surface S and the ideal triangulation λ are obtained by gluing two distinct boundary components of a surface S_1 and an ideal triangulation λ_1 of S_1.

Finally, a *state* for a skein $[K] \in \mathcal{S}^A(S)$ is the assignment $s\colon \partial K \to \{+,-\}$ of a sign $\pm$ to each point of ∂K. Let $\mathcal{S}^A_{\mathrm{s}}(S)$ be the algebra consisting of linear combinations of *stated skeins*, namely of skeins endowed with states. In particular, $\mathcal{S}^A_{\mathrm{s}}(S) = \mathcal{S}^A(S)$ when S has empty boundary.

In the case where $K \in \mathcal{S}^A(S)$ is obtained by gluing the two skeins $K_1 \in \mathcal{S}^A(S_1)$ and $K_2 \in \mathcal{S}^A(S_2)$, the states $s\colon \partial K \to \{+,-\}$, $s_1\colon \partial K_1 \to \{+,-\}$, $s_2\colon \partial K_2 \to \{+,-\}$ are *compatible* if s_1 and s_2 coincide on $\partial K_1 \cap (k_1 \times [0,1]) = \partial K_2 \cap (k_2 \times [0,1])$ for the identification given by the gluing, and if s coincides with the restrictions of s_1 and s_2 on $\partial K \subset \partial K_1 \cup \partial K_2$.

Theorem 19 ([BoW$_1$]). *For $A = \omega^{-2}$, there is a unique family of injective algebra homomorphisms*

$$\mathrm{Tr}_\lambda^\omega \colon \mathcal{S}_s^A(S) \to \mathcal{T}^\omega(\lambda),$$

defined for each punctured surface S with boundary and for each ideal triangulation λ of S, such that:

(1) (State Sum Property) *If the surface S is obtained by gluing S_1 to S_2, if the ideal triangulation λ of S is obtained by combining the ideal triangulations λ_1 of S_1 and λ_2 of S_2, and if the skeins $[K_1] \in \mathcal{S}^A(S_1)$ and $[K_2] \in \mathcal{S}^A(S_2)$ are glued together to give $[K] \in \mathcal{S}^A(S)$, then*

$$\mathrm{Tr}_\lambda^\omega\big([K,s]\big) = \sum_{\textit{compatible } s_1, s_2} \mathrm{Tr}_{\lambda_1}^\omega\big([K_1,s_1]\big) \otimes \mathrm{Tr}_{\lambda_2}^\omega\big([K_2,s_2]\big)$$

where the sum is over all states $s_1 \colon \partial K_1 \to \{+,-\}$ and $s_2 \colon \partial K_2 \to \{+,-\}$ that are compatible with $s \colon \partial K \to \{+,-\}$ and with each other. Similarly if the surface S, the ideal triangulation λ of S, and the skein $[K] \in \mathcal{S}^A(S)$ are obtained by gluing the surface S_1, the ideal triangulation λ_1 of S_1, and the skein $[K_1] \in \mathcal{S}^A(S_1)$, respectively, to themselves, then

$$\mathrm{Tr}_\lambda^\omega\big([K,s]\big) = \sum_{\textit{compatible } s_1} \mathrm{Tr}_{\lambda_1}^\omega\big([K_1,s_1]\big).$$

(2) (Elementary Cases) *When S is a triangle and K projects to a single arc embedded in S, with vertical framing, then*

(a) *in the case of Figure* 4(a), *where ε_1, $\varepsilon_2 = \pm$ are the signs associated by the state s to the end points of K,*

$$\mathrm{Tr}_\lambda^\omega\big([K,s]\big) = \begin{cases} 0 \textit{ if } \varepsilon_1 = - \textit{ and } \varepsilon_2 = + \\ [Z_1^{\varepsilon_1} Z_2^{\varepsilon_2}] \textit{ if } \varepsilon_1 \neq - \textit{ or } \varepsilon_2 \neq + \end{cases}$$

where Z_1 and Z_2 are the generators of $\mathcal{Z}^\omega(\lambda)$ associated to the sides λ_1 and λ_2 of S indicated, and where $[Z_1^{\varepsilon_1} Z_2^{\varepsilon_2}] = \omega^{-\varepsilon_1\varepsilon_2} Z_1^{\varepsilon_1} Z_2^{\varepsilon_2} = \omega^{\varepsilon_1\varepsilon_2} Z_2^{\varepsilon_2} Z_1^{\varepsilon_1}$ (identifying the sign $\varepsilon = \pm$ to the exponent $\varepsilon = \pm 1$) is the Weyl quantum ordering defined in §6.1;

(b) *in the case of Figure* 4(b), *where the end point of K marked by ε_1 is higher in $\partial S \times [0,1]$ than the point marked by ε_2,*

$$\mathrm{Tr}_\lambda^\omega\big([K,s]\big) = \begin{cases} 0 \textit{ if } \varepsilon_1 = \varepsilon_2 \\ -\omega^{-5} \textit{ if } \varepsilon_1 = + \textit{ and } \varepsilon_2 = - \\ \omega^{-1} \textit{ if } \varepsilon_1 = - \textit{ and } \varepsilon_2 = + \end{cases}$$

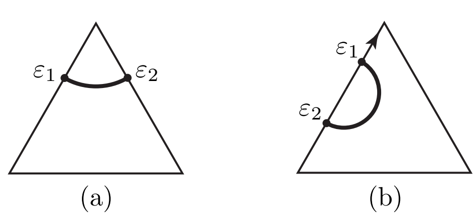

Figure 4. Elementary skeins in the triangle

When $A = \pm 1$ and S has no boundary, $\mathrm{Tr}^\omega_\lambda\big([K]\big)$ is just the Laurent polynomial which, for an element $r \in \mathcal{R}^{\mathrm{Spin}}_{\mathrm{PSL}_2(\mathbb{C})}(S)$ or $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, expresses the trace $\mathrm{Tr}_r\big([K]\big)$ in terms of (the square roots of) the shear coordinates of r with respect to λ. See [BoW$_1$].

When gluing skeins, it is crucial to keep track of the ordering of boundary points by their elevation. Figure 4(b) illustrates a convenient method to describe this ordering in pictures representing a link K in $S \times [0,1]$. For a component k of ∂S, we first choose an orientation of k and indicate it by an arrow. We then modify K by an isotopy (respecting the elevations of boundary points) so that the ordering of the points of $\partial K \cap (k \times [0,1])$ by increasing elevation exactly corresponds to the ordering of their projections to S for the chosen orientation of k. In addition, we will always assume in pictures that the framing is vertical.

6.4. **Compatibility between the Chebyshev, quantum trace and Frobenius homomorphisms.** Theorem 20 below shows that the Chebyshev homomorphism $\mathbf{T}^A\colon \mathcal{S}^\varepsilon(S) \to \mathcal{S}^A(S)$, the Frobenius homomorphism $\mathbf{F}^A\colon \mathcal{T}^\iota(\lambda) \to \mathcal{T}^\omega(\lambda)$, and the quantum trace homomorphisms $\mathrm{Tr}^\omega_\lambda\colon \mathcal{S}^A(S) \to \mathcal{T}^\omega(\lambda)$ and $\mathrm{Tr}^\iota_\lambda\colon \mathcal{S}^\varepsilon(S) \to \mathcal{T}^\iota(\lambda)$ are remarkably compatible with each other. Theorem 4 in the introduction is an immediate consequence of this statement.

Theorem 20. *Let S be a punctured surface with no boundary, and let λ be an ideal triangulation of S. Then, if A^4 is a primitive N–root of unity, $A = \omega^{-2}$, $\varepsilon = A^{N^2}$ and $\iota = \omega^{N^2}$, the diagram*

$$\begin{array}{ccc} \mathcal{S}^A(S) & \xrightarrow{\mathrm{Tr}^\omega_\lambda} & \mathcal{T}^\omega(\lambda) \\ \mathbf{T}^A \big\uparrow & & \big\uparrow \mathbf{F}^\omega \\ \mathcal{S}^\varepsilon(S) & \xrightarrow{\mathrm{Tr}^\iota_\lambda} & \mathcal{T}^\iota(\lambda) \end{array}$$

is commutative. Namely, for every skein $[K] \in \mathcal{S}^\varepsilon(S)$, the quantum trace $\mathrm{Tr}^\omega_\lambda\big([K^{T_N}]\big)$ of $[K^{T_N}] = \mathbf{T}^A\big([K]\big)$ is obtained from the classical trace polynomial $\mathrm{Tr}^\iota_\lambda\big([K]\big)$ by replacing each generator $Z_i \in \mathcal{T}^\iota(\lambda)$ by $Z_i^N \in \mathcal{T}^\omega(\lambda)$.

The proof of Theorem 20 will occupy all of the next section §7. As should already be apparent from the statement, it involves a large number of "miraculous cancellations".

7. Quantum traces of Chebyshev skeins

7.1. **Algebraic preliminaries.** For a parameter $a \in \mathbb{C} - \{0\}$, we will make use of two types of quantum integers,

$$(n)_a = \frac{a^n - 1}{a - 1} = a^{n-1} + a^{n-2} + a^{n-3} + \cdots + 1,$$

and

$$[n]_a = \frac{a^n - a^{-n}}{a - a^{-1}} = a^{n-1} + a^{n-3} + a^{n-5} + \cdots + a^{-(n-1)} = a^{-(n-1)}\,(n)_{a^2}\,.$$

In particular, the first type of quantum integers will occur in the quantum factorials

$$(n)_a! = (n)_a\,(n-1)_a \dots (2)_a\,(1)_a\,,$$

and in the quantum binomial coefficients

$$\binom{n}{p}_a = \frac{(n)_a\,(n-1)_a \dots (n-p+2)_a\,(n-p+1)_a}{(p)_a\,(p-1)_a \dots (2)_a\,(1)_a} = \frac{(n)_a!}{(p)_a!\,(n-p)_a!}.$$

Although this definition of the quantum binomial coefficient $\binom{n}{p}_a$ requires that $a^k \neq 1$ for every $k \leqslant \min\{p, n-p\}$, see Lemmas 21 or 22 below to make sense of it in all cases.

Quantum binomial coefficients naturally arise in the following classical formula (see for instance [Kass, §IV.2]).

Lemma 21 (Quantum Binomial Formula)**.** *If the quantities X and Y are such that $YX = aXY$, then*

$$(X+Y)^n = \sum_{p=0}^{n} \binom{n}{p}_a X^{n-p} Y^p. \qquad \square$$

We will encounter quantum binomial coefficients in a more combinatorial setting, where they occur as generating functions. Let P be a linearly ordered set of n points. A *state* for P is a map $s\colon P \to \{+,-\}$ assigning a sign $s(x) = +$ or $-$ to each point $x \in P$. An *inversion* for s is a pair (x, x') of two points x, $x' \in P$ such that $x < x'$ for the ordering of P, but such that $s(x) > s(x')$ (namely $s(x) = +$ and $s(x') = -$). Let $\iota(s)$ denote the number of inversions of s, and let $|s| = \#\{x \in P; s(x) = +\}$ be the number of $+$ signs in s.

Lemma 22.

$$\binom{n}{p}_a = \sum_{|s|=p} a^{\iota(s)}$$

where the sum is over all states $s\colon \{1, 2, \dots, n\} \to \{+,-\}$ with $|s| = p$.

Proof. This essentially is a rephrasing of the Quantum Binomial Formula. Indeed, the expansion of $(X+Y)^n$ is the sum of all monomials $Z_1 Z_2 \dots Z_n$ where each Z_i is equal to X or Y. Such a monomial $Z_1 Z_2 \dots Z_n$ can be described by a state $s\colon \{1, 2, \dots, n\} \to \{+,-\}$, defined by the property that $s(i) = -$ if $Z_i = X$ and $s(i) = +$ when $Z_i = Y$. Then $Z_1 Z_2 \dots Z_n = a^{\iota(s)} X^{n-|s|} Y^{|s|}$ after reordering the terms. The result now follows from the Quantum Binomial Formula of Lemma 21 by considering the coefficient of $X^{n-p}Y^p$. $\square$

7.2. **Quantum traces in the biangle.** In practice, computing the quantum trace of Theorem 19 is made easier by including *biangles* among allowable surfaces, namely infinite strips diffeomorphic to $[0,1] \times \mathbb{R}$.

Proposition 23 (Proposition 13 and Lemma 14 of [BoW$_1$])**.** *For $A = \omega^{-2}$, there is a unique family of algebra homomorphisms*

$$\mathrm{Tr}_B^\omega\colon \mathcal{S}_{\mathrm{s}}^A(B) \to \mathbb{C},$$

defined for all oriented biangles B, such that:

(1) (State Sum Property) *if the biangle B is obtained by gluing together two distinct biangles B_1 and B_2, and if $[K_1] \in \mathcal{S}^A(B_1)$ and $[K_2] \in \mathcal{S}^A(B_2)$ are glued together to give $[K] \in \mathcal{S}^A(B)$, then*

$$\mathrm{Tr}_B^\omega\big([K,s]\big) = \sum_{\textit{compatible } s_1, s_2} \mathrm{Tr}_{B_1}^\omega\big([K_1, s_1]\big)\mathrm{Tr}_{B_2}^\omega\big([K_2, s_2]\big),$$

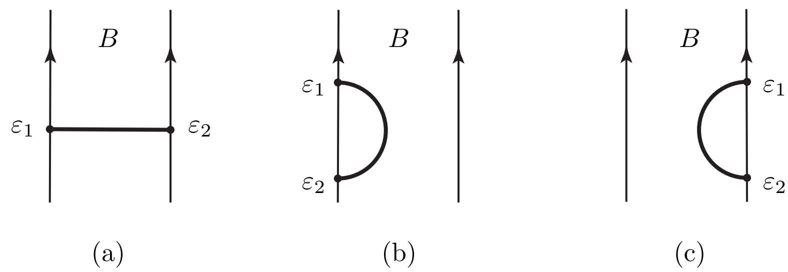

FIGURE 5. Elementary skeins in the biangle

where the sum is over all states $s_1\colon \partial K_1 \to \{+,-\}$ *and* $s_2\colon \partial K_2 \to \{+,-\}$ *that are compatible with* $s\colon \partial K \to \{+,-\}$ *and with each other;*

(2) (Elementary Cases) *if the biangle* B *is represented by a vertical strip in the plane as in Figure* 5 *and if* K *projects to a single arc embedded in* B, *then*

(a) *in the case of Figure* 5(a), *where* ε_1, $\varepsilon_2 = \pm$ *are the signs associated by the state* s *to the end points of* K,

$$\mathrm{Tr}_B^\omega\big([K,s]\big) = \begin{cases} 1 \textit{ if } \varepsilon_1 = \varepsilon_2 \\ 0 \textit{ if } \varepsilon_1 \neq \varepsilon_2; \end{cases}$$

(b) *in the case of Figure* 5(b),

$$\mathrm{Tr}_B^\omega\big([K,s]\big) = \begin{cases} 0 \textit{ if } \varepsilon_1 = \varepsilon_2 \\ -\omega^{-5} \textit{ if } \varepsilon_1 = + \textit{ and } \varepsilon_2 = - \\ \omega^{-1} \textit{ if } \varepsilon_1 = - \textit{ and } \varepsilon_2 = +. \end{cases}$$

(c) *in the case of Figure* 5(c),

$$\mathrm{Tr}_B^\omega\big([K,s]\big) = \begin{cases} 0 \textit{ if } \varepsilon_1 = \varepsilon_2 \\ \omega \textit{ if } \varepsilon_1 = + \textit{ and } \varepsilon_2 = - \\ -\omega^{5} \textit{ if } \varepsilon_1 = - \textit{ and } \varepsilon_2 = +. \end{cases} \qquad \square$$

Note that Figure 5 uses the picture conventions indicated at the end of §6.3. In particular, framings are everywhere vertical and, in Figures 5(b) and (c), the end point labelled by ε_1 sits at a higher elevation than the point labelled by ε_2.

The quantum trace Tr_B^ω of Proposition 23 is just a version of the classical Kauffman bracket for tangles.

The relationship between the quantum traces of Proposition 23 and Theorem 19 is the following. If we glue one side of a biangle B_1 to one side of a triangle T_2, the resulting surface is a triangle T. Each side of T is naturally associated to a side of T_2, so that there is a natural isomorphism $\mathcal{T}^\omega(\lambda) \cong \mathcal{T}^\omega(\lambda_2)$ between their associated Chekhov-Fock algebras.

The following is an immediate consequence of the construction of the quantum trace homomorphism in [BoW$_1$, §6] (which uses biangles as well as triangles).

Proposition 24. *If the triangle* T *is obtained by gluing a biangle* B_1 *to a triangle* T_2, *and if* $[K_1] \in \mathcal{S}^A(B_1)$ *and* $[K_2] \in \mathcal{S}^A(T_2)$ *are glued together to give* $[K] \in \mathcal{S}^A(T)$, *then for* $A = \omega^{-2}$

$$\mathrm{Tr}_T^\omega\big([K,s]\big) = \sum_{\textit{compatible } s_1,s_2} \mathrm{Tr}_{B_1}^\omega\big([K_1,s_1]\big)\mathrm{Tr}_{T_2}^\omega\big([K_2,s_2]\big) \in \mathcal{T}^\omega(T_2) \cong \mathcal{T}^\omega(T),$$

where the sum is over all states $s_1\colon \partial K_1 \to \{+,-\}$ and $s_2\colon \partial K_2 \to \{+,-\}$ that are compatible with $s\colon \partial K \to \{+,-\}$ and with each other. □

Note that, because a triangle T has a unique triangulation λ, we write Tr^ω_T and $\mathcal{T}^\omega(T)$ instead of $\mathrm{Tr}^\omega_\lambda$ and $\mathcal{T}^\omega(\lambda)$, by a slight abuse of notation.

7.3. **Jones-Wenzl idempotents.** The *Jones-Wenzl idempotents* are special elements $JW_n \in \mathcal{S}^A(B)$ of the skein algebra of the biangle $B = [0,1]\times\mathbb{R}$. The n–th Jones Wenzl idempotent JW_n is a linear combination of skeins, each of which meets each of the two components of $\partial B \times [0,1]$ in n points.

It is convenient (and traditional) to represent JW_n by a box with n strands emerging on each side, as on the left-hand side of Figure 6. In this case, it is important to remember that the box does not represent a single link, but a linear combination of links. When several boxes appear on a picture, multiple sums need to be taken.

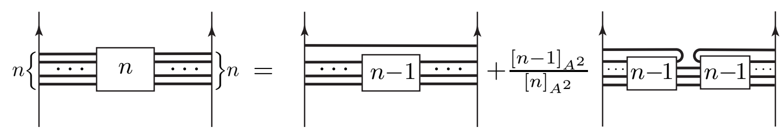

FIGURE 6. The Jones-Wenzl recurrence relation

The Jones-Wenzl idempotents are then inductively defined by the property that JW_1 is the skein consisting of a single unknotted arc with vertical framing, as in Figure 5(a), and by the recurrence relation indicated in Figure 6. In particular, JW_n is defined as long as $[k]_{A^2} \neq 0$, or equivalently $A^{4k} \neq 1$, for every $k \leqslant n$.

The book chapter [Lic$_1$, Chap. 13] or the survey article [Lic$_2$] are convenient references for Jones-Wenzl idempotents. We will use the relations in $\mathcal{S}^A(B)$ summarized in Figures 7, 8 and 9.

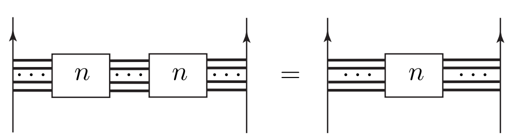

FIGURE 7. The Jones-Wenzl idempotent property

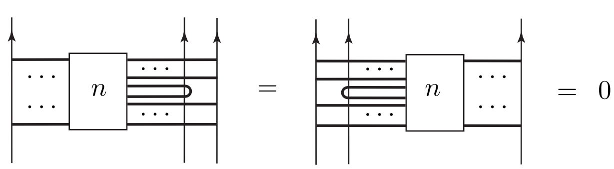

FIGURE 8. Gluing a U-turn to a Jones-Wenzl idempotent

What makes Jones-Wenzl idempotents convenient for us is the following property. If we glue the two boundary components of the biangle B together, we obtain a cylinder C.

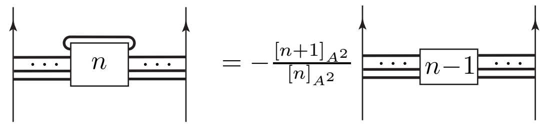

FIGURE 9.

Lemma 25. *Let the cylinder $C \cong S^1 \times \mathbb{R}$ be obtained by gluing together the two sides of the biangle $B = [0,1] \times \mathbb{R}$. Then, the element of $\mathcal{S}^A(C)$ obtained by applying this gluing to the Jones-Wenzl idempotent $JW_n \in \mathcal{S}^A(B)$ is equal to the evaluation $S_n\big([K]\big)$ of the Chebyshev polynomial S_n at the skein $[K] \in \mathcal{S}^A(C)$ represented by an embedded loop $K \subset S \times \{\frac{1}{2}\}$ going around the cylinder, endowed with the vertical framing.* □

See for instance [Lic$_1$, §13] or [Lic$_2$, p. 715] for a proof.

We will use Lemma 25 to compute the quantum traces $\mathrm{Tr}_\lambda^\omega\big([K^{S_N}]\big)$ and $\mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big)$ of the elements $[K^{S_N}]$, $[K^{T_N}] \in \mathcal{S}^A(S)$ obtained by threading the Chebyshev polynomials S_N and T_N along a framed link K. A particularly useful feature is the idempotent property of Figure 7, which will enable us to localize the computations.

Jones-Wenzl idempotents however have the drawback that not all of them are defined when A is a root of unity (and in particular that JW_N is undefined when $A^{4N} = 1$, which is precisely that case we are interested in). Consequently, we will temporary assume that A is *generic*, namely is not a root of unity, to ensure that all Jones-Wenzl idempotents are defined. We will then let A tend to an appropriate root of unity, and analyze what happens in the limit to complete our computation.

7.4. **Quantum traces of Jones-Wenzl idempotents in the biangle.** As indicated above, we assume for a while that A is generic. Let us fix an integer $N \geqslant 1$.

Let s be a state for the Jones-Wenzl idempotent $JW_N \in \mathcal{S}^A(B)$. This means that we are assigning the same state s to each of the links that appear in the expression of JW_N. It is here important that all these links have the same boundary, which we denote by ∂JW_N.

Let λ_1 and λ_2 be the two boundary components of the biangle B, oriented in a parallel way and so that λ_1 is the component on the left. The state s for JW_N consists of a state s_1 on $\lambda_1 \cap \partial JW_N$ and a state s_2 on $\lambda_2 \cap \partial JW_N$ (where $\lambda_i \cap \partial JW_N$ is a shorthand notation for $(\lambda_1 \times [0,1]) \cap \partial JW_N$). By convention, we order the points of $\lambda_i \cap \partial JW_N$ from bottom to top, in increasing order vertically upwards along the $[0,1]$–direction. As in §7.1, let $|s_i|$ and $\iota(s_i)$ respectively denote the number of $+$ signs and the number of inversions of s_i.

Proposition 26. *For a generic $A \in \mathbb{C} - \{0\}$,*

$$\mathrm{Tr}_B(JW_N, s) = 0 \qquad \text{if } |s_1| \neq |s_2|, \text{ and}$$

$$\mathrm{Tr}_B(JW_N, s) = \frac{A^{2\iota(s_1)} A^{2\iota(s_2)}}{\binom{N}{p}_{A^4}} \qquad \text{if } |s_1| = |s_2| = p.$$

Proof. The first property is an easy consequence of the definition of the quantum trace of Proposition 23; see [BoW$_1$, Lemma 21].

For the second property, we first reduce our computation to the case where there is no inversion.

Suppose that s_1 has at least one inversion, corresponding to two points x, $x' \in \lambda_1 \cap \partial JW_N$ such that $x < x'$ for the ordering of $\lambda_1 \cap \partial JW_N$, $s(x) = +$ and $s(x') = -$. Without loss of generality, we can assume that x and x' are next to each other for the order on $\lambda_1 \cap \partial JW_N$. Let the state s' be obtained from s by exchanging $s(x)$ and $s(x')$. In particular, $\iota(s_1') = \iota(s_1) - 1$.

Glue to B another biangle B_1 along λ_1, and glue to $[JW_N] \in \mathcal{S}^\omega(B)$ along the points x and x' a U-turn K_1 as in Figure 8. By the identity described by Figure 8, the resulting element of $\mathcal{S}^A(B \cup B_1)$ is trivial. Combining the State Sum Property of Proposition 23 with the values of the quantum trace for the skein of Figure 5(c),

$$0 = \omega \operatorname{Tr}_B(JW_N, s') - \omega^5 \operatorname{Tr}_B(JW_N, s)$$

and therefore

$$\operatorname{Tr}_B(JW_N, s) = \omega^{-4} \operatorname{Tr}_B(JW_N, s') = A^2 \operatorname{Tr}_B(JW_N, s').$$

The same formula holds when s' is obtained from s by removing an inversion of consecutive points in λ_2. This proves that

$$\operatorname{Tr}_B(JW_N, s) = A^{2\iota(s_1)} A^{2\iota(s_2)} \operatorname{Tr}_B(JW_N, s_0)$$

where s_0 is the state that has no inversion, and has p signs $+$ on $\lambda_1 \cap \partial JW_N$ and on $\lambda_2 \cap \partial JW_N$.

It therefore suffices to show that $C(N,p) = \operatorname{Tr}_B(JW_N, s_0)$ is equal to $\binom{N}{p}_{A^4}^{-1}$. We will prove this by induction on the number $N - p$ of minus signs in the states s_1 and s_2.

Using the recurrence relation of Figure 6 to expand $[JW_N] \in \mathcal{S}^A(B)$ as a linear combination of skeins, the only skein in this expansion that does not contain a U-turn as in Figure 5(b) is the one consisting of N parallel strands as in Figure 5(a), and the coefficient of this term in the expansion is equal to 1. It follows that $C(N,N) = 1$, which proves the initial step of the induction, when $N - p = 0$.

For the general case, suppose the property proved for every N', p' with $N' - p' < N - p$. Consider the identity in $\mathcal{S}^A(B)$ represented in Figure 9 (see [Lic$_1$, p. 137] or [Lic$_2$, p. 714] for a proof). Endow both sides with the state that has no inversion, and had p signs $+$ on each edge of B. If we use the State Sum Property to compute the quantum trace of the right hand side, only two terms have non-trivial contributions, and the relation of Figure 9 gives

$$[N+1]_{A^2}\, C(N-1,p) = -\,[N]_{A^2}\left((-\omega^5) C(N,p+1)\omega^{-1} + \omega A^{2p} A^{2p} C(N,p)(-\omega^{-5})\right).$$

Remembering that $[N]_{A^2} = A^{-2(N-1)}\,(N)_{A^4}$ and $A = \omega^{-2}$, and using the induction hypothesis,

$$\begin{aligned}
C(N,p) &= A^{-4(p+1)}\frac{(N+1)_{A^4}}{(N)_{A^4}}C(N-1,p) - A^{-4(p+1)}C(N,p+1)\\
&= A^{-4(p+1)}\frac{(N+1)_{A^4}}{(N)_{A^4}}\,\frac{(p)_{A^4}\,(p-1)_{A^4}\dots(1)_{A^4}}{(N-1)_{A^4}\,(N-2)_{A^4}\dots(N-p)_{A^4}}\\
&\qquad\qquad - A^{-4(p+1)}\frac{(p+1)_{A^4}\,(p)_{A^4}\dots(1)_{A^4}}{(N)_{A^4}\,(N-1)_{A^4}\dots(N-p)_{A^4}}\\
&= A^{-4(p+1)}\frac{(N+1)_{A^4}-(p+1)_{A^4}}{(N-p)_{A^4}}\,\frac{(p)_{A^4}\,(p-1)_{A^4}\dots(1)_{A^4}}{(N)_{A^4}\,(N-1)_{A^4}\dots(N-p+1)_{A^4}}\\
&= \binom{N}{p}_{A^4}^{-1}.
\end{aligned}$$

The last step uses the property that $(N+1)_{A^4} - (p+1)_{A^4} = A^{4(p+1)}(N-p)_{A^4}$, which is an immediate consequence of the definition of the quantum integers $(k)_{A^4}$.

This concludes the proof by induction that $C(N,p) = \binom{N}{p}_{A^4}^{-1}$, and therefore that

$$\mathrm{Tr}_B(JW_N, s) = A^{2\iota(s_1)}A^{2\iota(s_2)}\mathrm{Tr}_B(JW_N, s_0) = A^{2\iota(s_1)}A^{2\iota(s_2)}\binom{N}{p}_{A^4}^{-1}$$

when $|s_1| = |s_2| = p$. □

7.5. **Quantum traces of Jones-Wenzl skeins in the triangle.** We now consider a Jones-Wenzl idempotent JW_N in a triangle T, as represented in Figure 10(a).

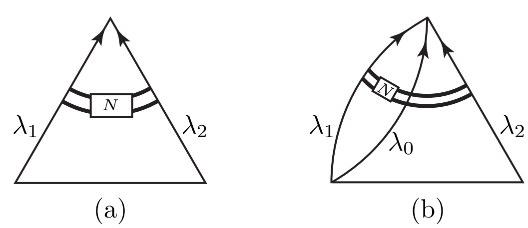

FIGURE 10.

Proposition 27. *In the triangle T, let $JW_N \in \mathcal{S}^A(T)$ be the Jones-Wenzl idempotent represented in Figure 10(a). Let λ_1 and λ_2 be the two sides of T indicated in Figure 10(a), and let Z_1 and Z_2 denote the corresponding generators of $\mathcal{T}^\omega(T)$. Let s be a state for JW_N, consisting of a state s_1 on $\lambda_1 \cap JW_N$ and a state s_2 on $\lambda_2 \cap JW_N$, let $\iota(s_1)$ and $\iota(s_2)$ be the respective numbers of inversions of s_1 and s_2, and set $p_1 = |s_1|$ and $p_2 = |s_2|$.*

Then $\mathrm{Tr}^\omega_T\big([JW_N, s]\big) = 0$ if $p_2 > p_1$, and otherwise

$$\mathrm{Tr}^\omega_T\big([JW_N, s]\big) = A^{2\iota(s_1)}A^{2\iota(s_2)}\frac{(N-p_2)_{A^4}!\,(p_1)_{A^4}!}{(N)_{A^4}!\,(p_1-p_2)_{A^4}!}\,A^{-(p_1-p_2)(N-p_1+p_2)}\big[Z_1^{2p_1-N}Z_2^{2p_2-N}\big]$$

where $\big[Z_1^{2p_1-N}Z_2^{2p_2-N}\big]$ denotes the Weyl quantum ordering for the monomial $Z_1^{2p_1-N}Z_2^{2p_2-N}$, as defined in §6.1.

Proof. The same U-turn trick as in the proof of Proposition 26 reduces the computation to the case where there are no inversions. Therefore, we henceforth assume that s_1 and s_2 have no inversion.

Split the triangle T into a biangle B_1 and a triangle T_2 as in Figure 10(b), and let λ_0 be their common edge $B_1 \cap T_2$. In particular, the Jones-Wenzl idempotent JW_N in T splits into a Jones-Wenzl idempotent JW_N^1 in the biangle B_1 and into a family K_2 of parallel strands in the triangle T_2. Applying the State Sum Property of Proposition 24,

$$\mathrm{Tr}^\omega_T(JW_N, s) = \sum_{s_0} \mathrm{Tr}^\omega_{B_1}(JW_N, s_1 \cup s_0)\mathrm{Tr}^\omega_{T_2}(K_2, s_2 \cup s_0)$$

where the sum is over all states s_0 for $\lambda_0 \cap JW_N$.

By Proposition 26, a state s_0 with a non-trivial contribution to the above sum is such that $|s_0| = p_1$, and in this case

$$\mathrm{Tr}^\omega_{B_1}(JW_N, s_1 \cup s_0) = A^{2\iota(s_0)} \binom{N}{p_1}_{A^4}^{-1}.$$

We now consider the terms coming from the triangle T_2. Let $s_j(i) \in \{-,+\}$ denote the sign assigned by the state s_j to the i–th point of $\lambda_j \cap JW_n$. In particular, since s_2 has no inversion, $s_2(i) = +$ if and only if $i > N - p_2$. Therefore, by Case 2(a) of Theorem 19, if $\mathrm{Tr}^\omega_{T_2}(K_2, s_2 \cup s_0) \neq 0$ then necessarily $s_0(i) = +$ for every $i > N - p_2$. In addition, identifying the sign $\pm$ to the number ± 1 in the exponents, this contribution is then equal to

$$\mathrm{Tr}^\omega_{T_2}(K_2, s_2 \cup s_0) = [Z_1^{s_0(1)} Z_2^{-1}][Z_1^{s_0(2)} Z_2^{-1}] \dots [Z_1^{s_0(N-p_2)} Z_2^{-1}][Z_1 Z_2]^{p_2}.$$

Using the property that

$$[Z_1 Z_2^{-1}][Z_1^{-1} Z_2^{-1}] = \omega^{-4}[Z_1^{-1} Z_2^{-1}][Z_1 Z_2^{-1}] = A^2[Z_1^{-1} Z_2^{-1}][Z_1 Z_2^{-1}],$$

the terms in this contribution can be reordered as

$$\mathrm{Tr}^\omega_{T_2}(K_2, s_2 \cup s_0) = A^{2\iota(s_0)}[Z_1^{-1} Z_2^{-1}]^{N-|s_0|}[Z_1 Z_2^{-1}]^{|s_0|-p_2}[Z_1 Z_2]^{p_2}$$

Combining this with Lemma 22, we obtain

$$\begin{aligned}
\mathrm{Tr}^\omega_T(JW_N, s) &= \sum_{\substack{s_0(i) = + \text{ if } i > N-p_2 \\ |s_0| = p_1}} A^{4\iota(s_0)} \binom{N}{p_1}_{A^4}^{-1} [Z_1^{-1} Z_2^{-1}]^{N-p_1}[Z_1 Z_2^{-1}]^{p_1-p_2}[Z_1 Z_2]^{p_2} \\
&= \binom{N-p_2}{p_1-p_2}_{A^4} \binom{N}{p_1}_{A^4}^{-1} [Z_1^{-1} Z_2^{-1}]^{N-p_1}[Z_1 Z_2^{-1}]^{p_1-p_2}[Z_1 Z_2]^{p_2} \\
&= \frac{(N-p_2)_{A^4}!\,(p_1)_{A^4}!}{(N)_{A^4}!\,(p_1-p_2)_{A^4}!} [Z_1^{-1} Z_2^{-1}]^{N-p_1}[Z_1 Z_2^{-1}]^{p_1-p_2}[Z_1 Z_2]^{p_2} \\
&= \frac{(N-p_2)_{A^4}!\,(p_1)_{A^4}!}{(N)_{A^4}!\,(p_1-p_2)_{A^4}!} A^{-(p_1-p_2)(N-p_1+p_2)}[Z_1^{2p_1-N} Z_2^{2p_2-N}]
\end{aligned}$$

after a final grouping of terms.

This concludes the proof of Proposition 27. ∎

7.6. **Evaluation of Chebyshev threads of the second kind.** Let $[K] \in \mathcal{S}^A(S)$ be a skein in a surface S without boundary, and let λ be an ideal triangulation of S. We will restrict attention to the case where $K \subset S \times [0,1]$ is a λ–*simple knot*, in the sense that it projects to a simple closed curve in S that meets each triangle of λ in at most one arc, and that its framing is everywhere vertical.

This λ–simple condition is not essential, but it will greatly simplify our exposition since we will not have to worry about the order in which we take terms in the contribution of each triangle. It will also be sufficient to restrict attention to this case, because λ–simple skeins generate $\mathcal{S}^A(S)$ as an algebra [Bu$_3$].

Arbitrarily pick an orientation for K. Let $\lambda_1, \lambda_2, \dots, \lambda_k, \lambda_{k+1} = \lambda_1$ denote, in this order, the edges of λ that are crossed by the projection of K to S. Similarly, let $T_1, T_2, \dots T_k$ be the triangles of λ that are crossed by K, in such a way that K crosses T_i between λ_{i-1} and λ_i.

When K crosses λ_i, the orientations of K and S determine a left and a right endpoint for λ_i, and there are four possible configurations according to whether λ_{i-1} and λ_{i+1} are respectively adjacent to the left or right endpoint of λ_i. We will say that K crosses λ_i in a *left-left, left-right, right-left* or *right-right pattern* accordingly. For instance K crosses λ_i in a left-right pattern if λ_{i-1} is adjacent to the left endpoint of λ_i, and λ_{i+1} is adjacent to its right end point.

Let $[K^{S_N}] \in \mathcal{S}^A(S)$ be obtained by threading the Chebyshev polynomial of the second kind S_N along K.

Proposition 28. *Let $[K] \in \mathcal{S}^A(S)$ be a λ–simple skein in the surface S. Then, for a generic A,*

$$\mathrm{Tr}_\lambda^\omega\big([K^{S_N}]\big) = \sum_{p_1,p_2,\dots,p_k} a_0 b_1 b_2 \dots b_k\, [Z_1^{2p_1-N} Z_2^{2p_2-N} \dots Z_k^{2p_k-N}]$$

where the sum is over all integers p_i with $0 \leqslant p_i \leqslant N$, where

$$a_0 = \prod_{i=1}^{k} \frac{A^{(p_i-p_{i+1})^2}}{(|p_i - p_{i+1}|)_{A^4}!}$$

and where

$$b_i = \begin{cases} 1 \ \textit{if } p_{i-1} \geqslant p_i \geqslant p_{i+1} \\ 0 \ \textit{otherwise} \end{cases} \textit{when } K \textit{ crosses } \lambda_i \textit{ in a left-left pattern,}$$

$$b_i = \begin{cases} 1 \ \textit{if } p_{i-1} \leqslant p_i \leqslant p_{i+1} \\ 0 \ \textit{otherwise} \end{cases} \textit{when } K \textit{ crosses } \lambda_i \textit{ in a right-right pattern,}$$

$$b_i = \begin{cases} A^{2Np_i} \frac{(N-p_i)_{A^4}!}{(p_i)_{A^4}!} \ \textit{if } p_{i-1} \geqslant p_i \leqslant p_{i+1} \\ 0 \ \textit{otherwise} \end{cases} \textit{when } K \textit{ crosses } \lambda_i \textit{ in a left-right}$$

pattern, and

$$b_i = \begin{cases} A^{-2Np_i} \frac{(p_i)_{A^4}!}{(N-p_i)_{A^4}!} \ \textit{if } p_{i-1} \leqslant p_i \geqslant p_{i+1} \\ 0 \ \textit{otherwise} \end{cases} \textit{when } K \textit{ crosses } \lambda_i \textit{ in a right-}$$

left pattern.

Proof. We will use Jones-Wenzl idempotents.

By Lemma 25, the element $[K^{S_N}] \in \mathcal{S}^A(S)$ can be obtained by threading the Jones-Wenzl idempotent JW_N along K (using a thickened annulus with core K

embedded in $S\times[0,1]$). Using the idempotent property of Figure 7, we can even put a Jones-Wenzl idempotent above each triangle T_j of the ideal triangulation λ.

Consequently, for each triangle T_j, replace the arc $K\cap\big(T_j\times[0,1]\big)$ by a Jones-Wenzl idempotent $JW_N^{(j)}$ as in Figure 10(a). Our discussion above shows that $[K^{S_N}]$ is equal to the element of $\mathcal{S}^A(S)$ obtained by gluing the $JW_N^{(i)}$ together.

We can then apply the State Sum Property, which gives

$$\mathrm{Tr}_\lambda^\omega\big([K^{S_N}]\big)=\sum_{s_1,s_2,\dots,s_i}\prod_{i=1}^k \mathrm{Tr}_{T_i}^\omega(JW_N^{(i)},s_{i-1}\cup s_i)$$

where the s_i range over all states for $\lambda_i\cap K^{S_N}$. Letting $p_i=|s_i|$ be the number of $+$ signs in s_i, Proposition 27 computes the contribution of each family of states s_1, s_2, …, s_k as

$$\prod_{i=1}^k \mathrm{Tr}_{T_i}^\omega(JW_N^{(i)},s_{i-1}\cup s_i)=c_1\dots c_k\,A^{4\iota(s_1)}\dots A^{4\iota(s_k)}\,[Z_1^{2p_1-N}\dots Z_k^{2p_k-N}]$$

where

$$c_i=\begin{cases}0 & \text{if } p_{i-1}<p_i\\ \frac{(N-p_i)_{A^4}!(p_{i-1})_{A^4}!}{(N)_{A^4}!(p_{i-1}-p_i)_{A^4}!}\,A^{(p_{i-1}-p_i)^2+N(p_i-p_{i-1})} & \text{if } p_{i-1}\geqslant p_i\end{cases}$$

when λ_{i-1} and λ_i are adjacent to the left, and

$$c_i=\begin{cases}\frac{(N-p_{i-1})_{A^4}!(p_i)_{A^4}!}{(N)_{A^4}!(p_i-p_{i-1})_{A^4}!}\,A^{(p_{i-1}-p_i)^2+N(p_{i-1}-p_i)} & \text{if } p_{i-1}\leqslant p_i\\ 0 & \text{if } p_{i-1}>p_i\end{cases}$$

when λ_{i-1} and λ_i are adjacent to the right. Note that we are here using the property that K is λ–simple, namely that K crosses each triangle at most once, so that the contributions of the various $i=1,\dots,k$ commute. In particular, the quantum ordering $[Z_1^{2p_1-N}\dots Z_k^{2p_k-N}]$ is equal to the product of the quantum orderings of these k contributions.

If we fix the numbers p_1, p_2, …, p_k and sum over all states s_i with $|s_i|=p_i$, Lemma 22 shows that

$$\sum_{|s_i|=p_i}A^{4\iota(s_1)}A^{4\iota(s_2)}\dots A^{4\iota(s_k)}=\binom{N}{p_1}_{A^4}\binom{N}{p_2}_{A^4}\cdots\binom{N}{p_k}_{A^4}.$$

Therefore, the contribution of the states s_i with $|s_i|=p_i$ is equal to

$$c_1\dots c_k\binom{N}{p_1}_{A^4}\cdots\binom{N}{p_k}_{A^4}[Z_1^{2p_1-N}\dots Z_k^{2p_k-N}].$$

This product is often 0. When it is not, many of the quantum factorials involved in the coefficients c_i and $\binom{N}{p_i}_{A^4}=\frac{(N)_{A^4}!}{(N-p_i)_{A^4}!(p_i)_{A^4}!}$ cancel out. For instance, all terms $(N)_{A^4}!$ disappear. The remaining terms are then easily grouped as in the statement of Proposition 28. □

7.7. **Evaluation of Chebyshev threads of the first kind.** We now turn to Chebyshev polynomials of the first kind T_N. Remembering from Lemma 7 that $T_N=S_N-S_{N-2}$, we now want to evaluate

$$\mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big)=\mathrm{Tr}_\lambda^\omega\big([K^{S_N}]\big)-\mathrm{Tr}_\lambda^\omega\big([K^{S_{N-2}}]\big)$$

for a λ–simple skein $[K]\in\mathcal{S}^A(S)$.

For this, it is convenient to rephrase the formula of Proposition 28 by putting more emphasis on the powers $n_i = 2p_i - N$ of the generators Z_i. Note that $-N \leqslant n_i \leqslant N$, and that n_i has the same parity as N.

We will say that a sequence $n_1, n_2, \dots, n_k$ is *admissible* if each of the corresponding $p_i = \frac{N+n_i}{2}$ contributes a non-trivial term to the formula of Proposition 28, namely if $-N \leqslant n_i \leqslant N$ for every i, if each n_i has the same parity as N, and if

$n_{i-1} \geqslant n_i$ when λ_{i-1} and λ_i are adjacent to the left, and
$n_{i-1} \leqslant n_i$ when λ_{i-1} and λ_i are adjacent to the right.

Then Proposition 28 can be rephrased as

Proposition 29. *For a generic A,*

$$\mathrm{Tr}^\omega_\lambda\big([K^{S_N}]\big) = \sum_{\textit{admissible } n_1, n_2, \dots, n_k} a_0 b_1 b_2 \dots b_k \, [Z_1^{n_1} Z_2^{n_2} \dots Z_k^{n_k}]$$

where

$$a_0 = \prod_{i=1}^k \frac{A^{(\frac{n_i - n_{i+1}}{2})^2}}{\left(\frac{|n_i - n_{i+1}|}{2}\right)_{A^4}!}$$

and where

$b_i = 1$ *when K crosses λ_i in a left-left or right-right pattern,*
$b_i = A^{N(N+n_i)} \dfrac{\left(\frac{N-n_i}{2}\right)_{A^4}!}{\left(\frac{N+n_i}{2}\right)_{A^4}!}$ *when K crosses λ_i in a left-right pattern, and*
$b_i = A^{-N(N+n_i)} \dfrac{\left(\frac{N+n_i}{2}\right)_{A^4}!}{\left(\frac{N-n_i}{2}\right)_{A^4}!}$ *when K crosses λ_i in a right-left pattern.* $\square$

An almost identical formula holds for $\mathrm{Tr}^\omega_\lambda\big([K^{S_{N-2}}]\big)$, except that the admissible sequences $n_1, n_2, \dots, n_k$ are further constrained by the condition that $-N+2 \leqslant n_i \leqslant N-2$ for every i. Because of the parity condition, this is equivalent to $-N < n_i < N$.

More precisely,

$$\mathrm{Tr}^\omega_\lambda\big([K^{S_{N-2}}]\big) = \sum_{\substack{\text{admissible } n_1, n_2, \dots, n_k \\ \text{with } -N < n_i < N}} a_0 b'_1 b'_2 \dots b'_k \, [Z_1^{n_1} Z_2^{n_2} \dots Z_k^{n_k}]$$

where a_0 is defined as in Proposition 29, and

$b'_i = 1$ when K crosses λ_i in a left-left or right-right pattern,
$b'_i = A^{(N-2)(N-2+n_i)} \dfrac{\left(\frac{N-2-n_i}{2}\right)_{A^4}!}{\left(\frac{N-2+n_i}{2}\right)_{A^4}!}$ when K crosses λ_i in a left-right pattern,
$b'_i = A^{-(N-2)(N-2+n_i)} \dfrac{\left(\frac{N-2+n_i}{2}\right)_{A^4}!}{\left(\frac{N-2-n_i}{2}\right)_{A^4}!}$ when K crosses λ_i in a right-left pattern.

So far, our computations assumed that A was generic. We will see that many cancellations occur in this expression of $\mathrm{Tr}^\omega_\lambda\big([K^{T_N}]\big) = \mathrm{Tr}^\omega_\lambda\big([K^{S_N}]\big) - \mathrm{Tr}^\omega_\lambda\big([K^{S_{N-2}}]\big)$ when A^4 is a primitive N–root of unity. However, because $(N)_{A^4} = 0$ in this case, we have to be careful in the definition of the quantities considered, and make sure that we never attempt to divide by 0. We will make sense of these properties by a limiting process.

Lemma 30. *Let $n_1, n_2, \dots, n_k$ be an admissible sequence such that $-N < n_i < N$ for every i. Then, the respective contributions*

$$a_0 b_1 b_2 \dots b_k \, [Z_1^{n_1} Z_2^{n_2} \dots Z_k^{n_k}]$$

and

$$a_0 b_1' b_2' \dots b_k' \, [Z_1^{n_1} Z_2^{n_2} \dots Z_k^{n_k}]$$

of this admissible sequence to $\mathrm{Tr}_\lambda^\omega\big([K^{S_N}]\big)$ *and* $\mathrm{Tr}_\lambda^\omega\big([K^{S_{N-2}}]\big)$ *have the same limit as* A^4 *tends to a primitive* N*–root of unity.*

Proof. Because of the assumption that $-N < n_i < N$, all quantum integers involved in these contributions are different from $(N)_{A^4}$ and we do not really have to worry about taking limits here. By continuity, choosing A^4 to be a primitive N–root of unity will be sufficient. We therefore need to show that

$$b_1 b_2 \dots b_k = b_1' b_2' \dots b_k'$$

under this hypothesis on A.

Let us compare the coefficients b_i and b_i'. When K crosses λ_i in a left-left or right-right patterns, we of course have that $b_i = b_i' = 1$.

When K crosses λ_i in a left-right pattern,

$$\begin{aligned} b_i &= A^{N(N+n_i)} \frac{\left(\frac{N-n_i}{2}\right)_{A^4}!}{\left(\frac{N+n_i}{2}\right)_{A^4}!} = A^{4N+2n_i-4} A^{(N-2)(N-2+n_i)} \frac{\left(\frac{N-n_i}{2}\right)_{A^4}}{\left(\frac{N+n_i}{2}\right)_{A^4}} \frac{\left(\frac{N-n_i}{2}-1\right)_{A^4}!}{\left(\frac{N+n_i}{2}-1\right)_{A^4}!} \\ &= A^{4N+2n_i-4} \frac{\left(\frac{N-n_i}{2}\right)_{A^4}}{\left(\frac{N+n_i}{2}\right)_{A^4}} b_i' = A^{4N+2n_i-4} \frac{A^{2N-2n_i}-1}{A^{2N+2n_i}-1} b_i' = -A^{2N-4} b_i', \end{aligned}$$

using the fact that $A^{4N} = 1$ for the last equality.

When K crosses λ_i in a tight-left pattern, a similar computation gives

$$b_i = -A^{-(2N-4)} b_i'.$$

Note that K crosses as many λ_i in a left-right pattern as in a right-left pattern. Therefore, as we compute the product of the b_i, the $-A^{\pm(2N-4)}$ terms cancel out, and

$$b_1 b_2 \dots b_k = b_1' b_2' \dots b_k'. \qquad \square$$

A consequence of Lemma 30 is that, as we let A^4 tend to a primitive N–root of unity, all the terms of $\mathrm{Tr}_\lambda^\omega\big([K^{S_{N-2}}]\big)$ cancel out with terms of $\mathrm{Tr}_\lambda^\omega\big([K^{S_N}]\big)$ in the difference $\mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big) = \mathrm{Tr}_\lambda^\omega\big([K^{S_N}]\big) - \mathrm{Tr}_\lambda^\omega\big([K^{S_{N-2}}]\big)$.

We now consider the remaining terms of $\mathrm{Tr}_\lambda^\omega\big([K^{S_N}]\big)$.

Lemma 31. *Let* $n_1, n_2, \dots, n_k$ *be an admissible sequence such that at least one* n_i *is equal to* $\pm N$ *and at least one* n_j *is* not *equal to* $\pm N$*. Then, the contribution*

$$a_0 b_1 b_2 \dots b_k \, [Z_1^{n_1} Z_2^{n_2} \dots Z_k^{n_k}]$$

of this admissible sequence to $\mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big)$ *converges to* 0 *as* A^4 *tends to a primitive* N*–root of unity.*

Proof. In the expression of $a_0 b_1 b_2 \dots b_k$, the only quantum integers that can tend to 0 are the terms $(N)_{A^4}$. We therefore have to show that more quantum integers $(N)_{A^4}$ occur in the numerator of this expression than in the denominator.

The number of terms $(N)_{A^4}$ in the denominator of a_0 is equal to the number of indices i where n_i switches from $n_i = \pm N$ to $n_{i+1} = \mp N$.

A factor $(N)_{A^4}$ occurs in the numerator of a coefficient b_i exactly when

$n_i = -N$ and K crosses λ_i in a left-right pattern, or
$n_i = +N$ and K crosses λ_i in a right-left pattern.

Similarly, a factor $(N)_{A^4}$ occurs in the denominator of a coefficient b_i exactly when

$n_i = +N$ and K crosses λ_i in a left-right pattern, or
$n_i = -N$ and K crosses λ_i in a right-left pattern.

Consider a maximal sequence of consecutive $n_i = +N$ namely, considering indices modulo k, two indices i_1, i_2 such that $n_i = +N$ whenever $i_1 \leqslant i \leqslant i_2$, and $n_{i_1-1} \neq +N$ and $n_{i_2+1} \neq +N$. Note that, since the sequence of n_i is admissible, λ_{i_1-1} and λ_{i_1} are necessarily adjacent to the right of K, whereas λ_{i_2} and λ_{i_2+1} are adjacent to the left. Therefore, if we examine how K crosses λ_i when $i_1 \leqslant i \leqslant i_2$, we see one more right-left pattern than left-right patterns. It follows that this maximal sequence of consecutive $n_i = +N$ contributes one more $(N)_{A^4}$ to the numerator than to the denominator of the product of the corresponding b_i.

Similarly, a maximal sequence of consecutive $n_i = -N$ contributes one more $(N)_{A^4}$ to the numerator than to the denominator of the product of the corresponding b_i.

Because of our assumption that there exists at least one $n_j \neq \pm N$, the number of i where n_i switches from $n_i = \pm N$ to $n_{i+1} = \mp N$ is strictly less that the total number of maximal sequences of consecutive $n_i = +N$ and of maximal sequences of consecutive $n_i = -N$. It follows that there is at least one more $(N)_{A^4}$ in the numerator of $a_0 b_1 b_2 \dots b_k$ than in the denominator. This term therefore converges to 0 as A^4 tends to a primitive N–root of unity. □

Lemma 32. *Let n_1, n_2, …, n_k be an admissible sequence where each n_i is equal to $\pm N$. Then the contribution*

$$a_0 b_1 b_2 \dots b_k \, [Z_1^{n_1} Z_2^{n_2} \dots Z_k^{n_k}]$$

of this admissible sequence to $\mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big)$ is equal to $[Z_1^{n_1} Z_2^{n_2} \dots Z_k^{n_k}]$.

Proof. The proof of Lemma 31 shows that, in this case, we have exactly as many quantum factorials $(N)_{A^4}!$ in the numerator as in the denominator of $a_0 b_1 b_2 \dots b_k$. These quantum factorials therefore cancel out.

All remaining quantum factorials are equal to $(0)_{A^4}! = 1$.

We therefore only have to worry about the powers of A that occur in $a_0 b_1 b_2 \dots b_k$. Going back to the definition of the constants a_0 and b_i in Proposition 29, one obtains that

$$a_0 b_1 b_2 \dots b_k = A^{2N^2(\alpha+\beta-\beta')}$$

where

$$\alpha = \tfrac{1}{2} \sum_{i=1}^{k} \left(\tfrac{n_i - n_{i+1}}{2N} \right)^2 = \tfrac{1}{2} \#\{i; n_i \neq n_{i+1}\}$$

$$\beta = \sum_{\text{left-right pattern at } \lambda_i} \tfrac{N+n_i}{N} = \#\{i; n_i = +N \text{ and left-right pattern at } \lambda_i\}$$

$$\text{and } \beta' = \sum_{\text{right-left pattern at } \lambda_i} \tfrac{N+n_i}{N} = \#\{i; n_i = +N \text{ and right-left pattern at } \lambda_i\}.$$

If all n_i are equal to each other, then $\alpha = 0$ and $\beta = \beta'$, so that $\alpha + \beta - \beta' = 0$.

Otherwise, α is equal to the number of intervals $I = \{u, u+1, \dots, v-1, v\}$ in the index set (counting indices modulo k) where $n_i = +N$ for every $i \in I$ while $n_{u-1} = n_{v+1} = -N$. For such an interval I, the $i \in I$ contribute a total of -1 to

$\beta - \beta'$. Since there are α such intervals I, we conclude that $\alpha + \beta - \beta' = 0$ in this case as well.

This proves that $a_0 b_1 b_2 \dots b_k = 1$ is all cases. $\square$

If we combine Proposition 29 and Lemmas 30, 31 and 32, we now have the following computation.

Proposition 33. *Suppose that A^4 is a primitive N–root of unity. Then, for every λ–simple skein $[K] \in \mathcal{S}^A(S)$,*

$$\mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big) = \sum_{\substack{\textit{admissible } n_1, n_2, \dots, n_k \\ n_i = \pm N}} [Z_1^{n_1} Z_2^{n_2} \dots Z_k^{n_k}]$$

where the sum is over all admissible sequences n_1, n_2, $\dots$, n_k with all $n_i = \pm N$, and where the Z_i and the notion of admissibility are defined as in §7.6 *and at the beginning of this* §7.7. $\square$

Note the dramatic difference between the number of terms in the expression of $\mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big)$ provided by Proposition 33, and that in the formula for generic A given in Proposition 29. Indeed, the number of monomials in the formula of Proposition 33 is independent of N. On the other hand, the number of terms in the expression of Proposition 29 is a polynomial in N of degree k (it is the Ehrhart polynomial of a certain k–dimensional polytope determined by the admissibility conditions).

7.8. **Proof of Theorem 20.** Assuming that A^4 is a primitive N–root of unity, we want to prove that the diagram

$$\begin{tikzcd} \mathcal{S}^A(S) \arrow[r, "\mathrm{Tr}_\lambda^\omega"] & \mathcal{T}^\omega(\lambda) \\ \mathcal{S}^\varepsilon(S) \arrow[u, "\mathbf{T}^A"] \arrow[r, "\mathrm{Tr}_\lambda^\iota"] & \mathcal{T}^\iota(\lambda) \arrow[u, "\mathbf{F}^\omega"'] \end{tikzcd}$$

is commutative.

By [Bu$_3$], the skein algebra $\mathcal{S}^\varepsilon(S)$ is generated as an algebra by λ–simple skeins $[K]$. Since the maps involved are algebra homomorphisms, it therefore suffices to show that

$$\mathrm{Tr}_\lambda^\omega\big(\mathbf{T}^A([K])\big) = \mathbf{F}^\omega\big(\mathrm{Tr}_\lambda^\iota([K])\big)$$

for every λ–simple skein $[K] \in \mathcal{S}^\varepsilon(S)$.

The quantum trace $\mathrm{Tr}_\lambda^\omega\big(\mathbf{T}^A([K])\big) = \mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big)$ is computed by Proposition 33. With the notation of that statement,

$$\mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big) = \sum_{\substack{\text{admissible } n_1, n_2, \dots, n_k \\ n_i = \pm N}} [Z_1^{n_1} Z_2^{n_2} \dots Z_k^{n_k}]$$

Similarly, by Theorem 19 or Proposition 33,

$$\mathrm{Tr}_\lambda^\iota\big([K]\big) = \sum_{\substack{\text{admissible } \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \\ \varepsilon_i = \pm 1}} [Z_1^{\varepsilon_1} Z_2^{\varepsilon_2} \dots Z_k^{\varepsilon_k}].$$

Finally , $\mathbf{F}^\omega\big(\mathrm{Tr}_\lambda^\iota([K])\big)$ is obtained from $\mathrm{Tr}_\lambda^\iota\big([K]\big)$ by replacing each occurrence of Z_i by Z_i^N, and is therefore equal to $\mathrm{Tr}_\lambda^\omega\big([K^{T_N}]\big) = \mathrm{Tr}_\lambda^\omega\big(\mathbf{T}^A([K])\big)$ by comparison of the above formulas. $\square$

7.9. **The cases of the once-punctured torus and the twice-punctured plane.** At this point, there is a gap that needs to be filled in. Indeed, our arguments so far assumed that the Chebyshev homomorphism $\mathbf{T}^A\colon \mathcal{S}^\varepsilon(S)\to\mathcal{S}^A(S)$ is a well-defined algebra homomorphism. This property relied on Lemmas 12 and 13, whose proofs we temporarily postponed. We now prove these two statements, using the computations of §7.7. At the same time, we will prove Lemma 10, which was used in Theorem 9 to show that Chebyshev threads $[K^{T_N}]$ are central in $\mathcal{S}^A(S)$.

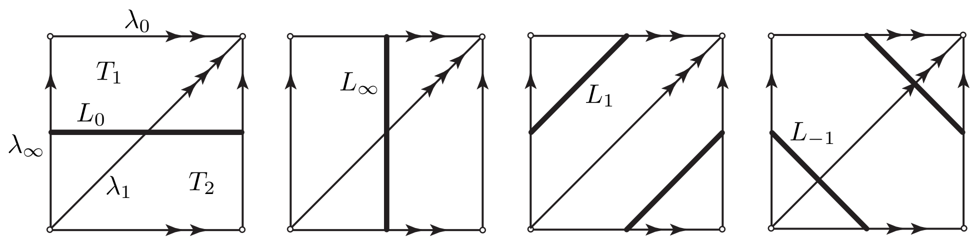

FIGURE 11. The curves L_0, L_∞, L_1 and L_{-1} in the once-punctured torus, with an ideal triangulation λ

We begin with Lemma 10, which we repeat for the convenience of the reader.

Lemma 34. *In the once-punctured torus T, let L_0 and L_∞ be the two curves represented in Figure 11 (or Figure 2), and consider these curves as framed knots with vertical framing in $T\times[0,1]$. If A^2 is a primitive N–root of unity, then*

$$[L_0^{T_N}][L_\infty]=[L_\infty][L_0^{T_N}]$$

in $\mathcal{S}^A(T)$.

Proof. Consider the ideal triangulation λ indicated in Figure 11, with edges λ_0, λ_1 and λ_∞. We are going to use the fact that the trace map $\mathrm{Tr}_\lambda^\omega\colon\mathcal{S}^A(S)\to\mathcal{T}^\omega(\lambda)$ is an injective algebra homomorphism [BoW$_1$, Prop. 29]. It then suffices to check that

$$\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big)\mathrm{Tr}_\lambda^\omega\big([L_\infty]\big)=\mathrm{Tr}_\lambda^\omega\big([L_\infty]\big)\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big).$$

First consider the case where N is odd. Then, A^4 is also a primitive N–root of unity, and we can use the computations of §7.7. Let Z_0, Z_1, Z_∞ be the generators of $\mathcal{T}^\omega(\lambda)$ respectively associated to the edges λ_0, λ_1, λ_∞ of λ. Proposition 33 then shows that the quantum trace $\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big)$ is a Laurent polynomial in the variables Z_1^N and Z_∞^N.

Note that, when $i\neq j$, the two ends of the edges λ_i are adjacent to the two ends of λ_j, and that $Z_iZ_j=\omega^{\pm4}Z_jZ_i$. In particular, Z_i^N commutes with Z_j since $\omega^{4N}=A^{-2N}=1$. As a consequence, $\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big)$ is central in $\mathcal{Z}^\omega(\lambda)$, and in particular commutes with $\mathrm{Tr}_\lambda^\omega\big([L_\infty]\big)$.

By injectivity of the quantum trace homomorphism $\mathrm{Tr}_\lambda^\omega\colon\mathcal{S}^A(S)\to\mathcal{T}^\omega(\lambda)$, this completes the proof when N is odd.

When N is even, A^4 is a primitive $\frac{N}{2}$–root of unity. Proposition 33 now shows that $\mathrm{Tr}_\lambda^\omega\big([L_0^{T_{N/2}}]\big)=T_{\frac{N}{2}}\big(\mathrm{Tr}_\lambda^\omega([L_0])\big)$ is a linear combination of monomials $Z_1^{n_1}Z_\infty^{n_2}$ with n_1, $n_2\in\{-\frac{N}{2},+\frac{N}{2}\}$. By definition of Chebyshev polynomials, $T_N=T_2\circ T_{\frac{N}{2}}$ (use for instance Lemma 6) and $T_2(x)=x^2-2$. Therefore

$$\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big)=T_N\big(\mathrm{Tr}_\lambda^\omega([L_0])\big)=T_2\Big(T_{\frac{N}{2}}\big(\mathrm{Tr}_\lambda^\omega([L_0])\big)\Big)=\Big(\mathrm{Tr}_\lambda^\omega\big([L_0^{T_{N/2}}]\big)\Big)^2-2$$

is a linear combination of monomials $Z_1^{m_1} Z_\infty^{m_2}$ with m_1, $m_2 \in \{-N, 0, +N\}$. As a consequence, $\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big)$ is again central in $\mathcal{Z}^\omega(\lambda)$, which proves the property needed by injectivity of $\mathrm{Tr}_\lambda^\omega \colon \mathcal{S}^A(S) \to \mathcal{T}^\omega(\lambda)$. □

We now address Lemma 12.

Lemma 35. *Suppose that A^4 is a primitive N–root of unity. In the once-punctured torus T, let L_0, L_∞, L_1 and L_{-1} be the curves represented in Figure* 11 *(or Figure* 2*). Considering these curves as knots in $T \times [0,1]$ and endowing them with the vertical framing,*

$$[L_0^{T_N}][L_\infty^{T_N}] = A^{-N^2}[L_1^{T_N}] + A^{N^2}[L_{-1}^{T_N}]$$

in the skein algebra $\mathcal{S}^A(T)$.

Proof. As usual, set $\varepsilon = A^{N^2}$ and $\iota = \omega^{N^2}$. By injectivity of the trace homomorphism $\mathrm{Tr}_\lambda^\omega \colon \mathcal{S}^A(S) \to \mathcal{T}^\omega(\lambda)$, it again suffices to check that

$$\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big)\,\mathrm{Tr}_\lambda^\omega\big([L_\infty^{T_N}]\big) = \varepsilon^{-1}\,\mathrm{Tr}_\lambda^\omega\big([L_1^{T_N}]\big) + \varepsilon\,\mathrm{Tr}_\lambda^\omega\big([L_{-1}^{T_N}]\big)$$

in $\mathcal{T}^\omega(\lambda)$, for the ideal triangulation λ indicated in Figure 11.

For this, we need to compute these $\mathrm{Tr}_\lambda^\omega\big([L_i^{T_N}]\big) \in \mathcal{S}^A(S)$, and we will compare them to $\mathrm{Tr}_\lambda^\iota\big([L_i^{T_N}]\big) \in \mathcal{S}^\varepsilon(S)$ through the Frobenius homomorphism $\mathbf{F}^A : \mathcal{T}^\iota(\lambda) \to \mathcal{T}^\omega(\lambda)$. Note that we cannot use here the full force of Theorem 20, because we are trying to prove one of the steps in the proof of that statement, namely that threading Chebyshev polynomials T_N provides an algebra homomorphism $\mathbf{T}^\omega \colon \mathcal{S}^\varepsilon(S) \to \mathcal{S}^A(S)$! However, we can rely on the computations of §7.7.

The framed knots L_0, L_∞ and L_1 are λ–simple. So we can use Proposition 33, which gives

$$\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big) = \sum_{\substack{n_1,\, n_2 \,=\, \pm N \\ n_1 \leqslant n_2}} [Z_\infty^{n_1} Z_1^{n_2}] = [Z_\infty^{-N} Z_1^{-N}] + [Z_\infty^{-N} Z_1^{N}] + [Z_\infty^{N} Z_1^{N}].$$

The definition of the quantum trace in Theorem 19 also shows that

$$\mathrm{Tr}_\lambda^\iota\big([L_0]\big) = [Z_\infty^{-1} Z_1^{-1}] + [Z_\infty^{-1} Z_1] + [Z_\infty Z_1]$$

in $\mathcal{T}^\iota(\lambda)$. Comparing the two formulas, we see that $\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big) = \mathbf{F}^\omega\big(\mathrm{Tr}_\lambda^\iota([L_0])\big)$. Since L_∞ and L_1 are also λ–simple, the same argument shows that $\mathrm{Tr}_\lambda^\omega\big([L_\infty^{T_N}]\big) = \mathbf{F}^\omega\big(\mathrm{Tr}_\lambda^\iota([L_\infty])\big)$ and $\mathrm{Tr}_\lambda^\omega\big([L_1^{T_N}]\big) = \mathbf{F}^\omega\big(\mathrm{Tr}_\lambda^\iota([L_1])\big)$.

Unfortunately, L_{-1} is not λ–simple, so that the computation of $\mathrm{Tr}_\lambda^\omega\big([L_{-1}^{T_N}]\big)$ requires more care. The arguments will actually end up being the same, and the only new twist in this case is that the notation is clumsier.

Let T_1 and T_2 denote the two triangle faces of λ. The torus T is obtained by gluing these two triangles along their sides. As indicated in §6.3, there is a canonical embedding $\mathcal{T}^\omega(\lambda) \to \mathcal{T}^\omega(T_1) \otimes \mathcal{T}^\omega(T_2)$ that sends the generator Z_i associated to an edge of λ to the tensor product $Z_{i,1} \otimes Z_{i_2} \in \mathcal{T}^\omega(T_1) \otimes \mathcal{T}^\omega(T_2)$ of the generators of $\mathcal{T}^\omega(T_1)$ and $\mathcal{T}^\omega(T_2)$ corresponding to the same edge.

We can arrange L_{-1} in $T \times [0,1]$ so that, above each triangle T_j, the component of $L_{-1} \cap \big(T_j \times [0,1]\big)$ that is closest to the upper right corner of the square in Figure 11 sits at a higher elevation than the second component. This is consistent with the orientation of the edges of λ indicated by the arrows in Figure 11.

Then, the arguments used in the proof of Proposition 33 carry over to this case, except that we have to keep track of the order of the contributions of each components of a $K_{-1} \cap \big(U_j \times [0,1]\big)$. In our case, this gives

$$\mathrm{Tr}^\omega_\lambda\big([L_{-1}^{T_N}]\big) = \sum_{\substack{n_1 \leqslant n_2 \leqslant n_3 \geqslant n_4 \geqslant n_1 \\ n_i = \pm N}} [Z_{\infty,1}^{n_1} Z_{1,1}^{n_2}][Z_{0,1}^{n_3} Z_{1,1}^{n_4}] \otimes [Z_{1,2}^{n_2} Z_{0,2}^{n_3}][Z_{1,2}^{n_4} Z_{\infty,2}^{n_1}]$$

where the order of the terms is dictated by the respective elevations of the components of each $K_{-1} \cap \big(T_j \times [0,1]\big)$. Listing all 6 possibilities for (n_1, n_2, n_3, n_4), reordering the terms in each monomial and remembering that Z_i corresponds to $Z_{i,1} \otimes Z_{i_2}$ for the embedding $\mathcal{T}^\omega(\lambda) \to \mathcal{T}^\omega(T_1) \otimes \mathcal{T}^\omega(T_2)$,

$$\begin{aligned}\mathrm{Tr}^\omega_\lambda\big([L_{-1}^{T_N}]\big) = [Z_\infty^{-N} Z_1^{-2N} Z_0^{-N}] + [Z_\infty^{-N} Z_1^{-2N} Z_0^{N}] + (\omega^{4N^2} + \omega^{-4N^2})[Z_\infty^{-N} Z_0^{N}] \\ + [Z_\infty^{-N} Z_1^{2N} Z_0^{N}] + [Z_\infty^{N} Z_1^{2N} Z_0^{N}]\end{aligned}$$

in $\mathcal{T}^\omega(\lambda)$

Replacing ω by ι and applying the definition of the quantum trace by Theorem 19 gives

$$\begin{aligned}\mathrm{Tr}^\iota_\lambda\big([L_{-1}]\big) &= \sum_{\substack{n_1 \leqslant n_2 \leqslant n_3 \geqslant n_4 \geqslant n_1 \\ n_i = \pm 1}} [Z_{\infty,1}^{n_1} Z_{1,1}^{n_2}][Z_{0,1}^{n_3} Z_{1,1}^{n_4}] \otimes [Z_{1,2}^{n_2} Z_{0,2}^{n_3}][Z_{1,2}^{n_4} Z_{\infty,2}^{n_1}] \\ &= [Z_\infty^{-1} Z_1^{-2} Z_0^{-1}] + [Z_\infty^{-1} Z_1^{-2} Z_0] + (\iota^4 + \iota^{-4})[Z_\infty^{-1} Z_0] \\ &\qquad + [Z_\infty^{-1} Z_1^{2} Z_0] + [Z_\infty Z_1^{2} Z_0]\end{aligned}$$

in $\mathcal{T}^\iota(\lambda)$. Therefore $\mathrm{Tr}^\omega_\iota\big([L_{-1}]\big) = \mathbf{F}^\omega\big(\mathrm{Tr}^\iota_\lambda([L_{-1}])\big)$ in this case as well, since $\iota = \omega^{N^2}$.

Note that $[L_0][L_\infty] = \varepsilon^{-1}[L_1] + \varepsilon[L_{-1}]$ in $\mathcal{S}^\varepsilon(S)$, by the skein relation. Because $\mathrm{Tr}^\iota_\lambda \colon \mathcal{S}^\varepsilon(S) \to \mathcal{T}^\iota(\lambda)$ is an algebra homomorphism, it follows that

$$\mathrm{Tr}^\iota_\lambda\big([L_0]\big)\,\mathrm{Tr}^\iota_\lambda\big([L_\infty]\big) = \varepsilon^{-1}\mathrm{Tr}^\iota_\lambda\big([L_1]\big) + \varepsilon\,\mathrm{Tr}^\iota_\lambda\big([L_{-1}]\big).$$

Applying the algebra homomorphism $\mathbf{F}^\omega$ on both sides, and using the fact that $\mathrm{Tr}^\omega_\iota\big([L_i]\big) = \mathbf{F}^\omega\big(\mathrm{Tr}^\iota_\lambda([L_i])\big)$ for $i = 0$, ∞, 1 and -1, we conclude that

$$\mathrm{Tr}^\omega_\lambda\big([L_0^{T_N}]\big)\,\mathrm{Tr}^\omega_\lambda\big([L_\infty^{T_N}]\big) = \varepsilon^{-1}\mathrm{Tr}^\omega_\lambda\big([L_1^{T_N}]\big) + \varepsilon\,\mathrm{Tr}^\omega_\lambda\big([L_{-1}^{T_N}]\big).$$

(Of course, after our computation of the $\mathrm{Tr}^\omega_\lambda\big([L_i^{T_N}]\big)$, we could also have reached this identity by a brute force computation. However, this approach seems a little more conceptual.)

This concludes the proof, since the algebra homomorphism $\mathrm{Tr}^\omega_\lambda \colon \mathcal{S}^A(S) \to \mathcal{T}^\omega(\lambda)$ is injective. □

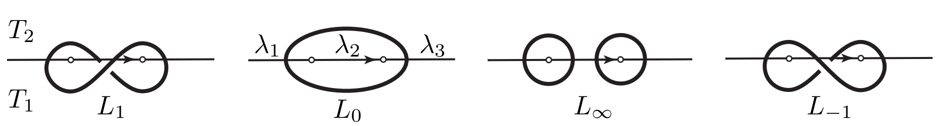

FIGURE 12. The 1–submanifolds L_1, L_0 and L_∞ in the twice-punctured plane, with an ideal triangulation λ

The proof of Lemma 13 is very similar.

Lemma 36. *Suppose that A^2 is a primitive N–root of unity with N odd. In the twice-punctured plane U, let L_1, L_{-1}, L_0 and L_∞ be the 1–submanifolds represented in Figure 12 (or Figure 3). Considering these submanifolds as links in $U\times[0,1]$ and endowing them with the vertical framing,*

$$[L_1^{T_N}] = A^{-N^2}[L_0^{T_N}] + A^{N^2}[L_\infty^{T_N}]$$
$$\text{and } [L_{-1}^{T_N}] = A^{N^2}[L_0^{T_N}] + A^{-N^2}[L_\infty^{T_N}]$$

in the skein algebra $\mathcal{S}^A(U)$.

Proof. Consider the ideal triangulation λ of U, with edges λ_1, λ_2, λ_3 and faces T_1, T_2, represented in Figure 12. As usual, let $Z_i = Z_{i,1}\otimes Z_{i_2}$ denote the generator of $\mathcal{T}^\omega(\lambda)\subset \mathcal{T}^\omega(T_1)\otimes\mathcal{T}^\omega(T_2)$ associated to λ_i.

As for Lemma 35, using the arguments of the proof of Proposition 33 enables us to compute

$$\begin{aligned}
\mathrm{Tr}_\lambda^\omega\big([L_1^{T_N}]\big) &= \sum_{\substack{n_1\geqslant n_2\geqslant n_3\leqslant n_4\leqslant n_1\\ n_i=\pm N}} [Z_{2,1}^{n_1}Z_{3,1}^{n_2}][Z_{2,1}^{n_3}Z_{1,1}^{n_4}]\otimes[Z_{1,2}^{n_4}Z_{2,2}^{n_1}][Z_{3,2}^{n_2}Z_{2,2}^{n_3}]\\
&= \omega^{-2N^2}[Z_1^{-N}Z_2^{-2N}Z_3^{-N}] + \omega^{2N^2}[Z_1^{-N}Z_3^{-N}] + \omega^{-2N^2}[Z_1^{-N}Z_3^{+N}]\\
&\qquad + \omega^{-2N^2}[Z_1^{N}Z_3^{-N}] + \omega^{2N^2}[Z_1^{N}Z_3^{N}] + \omega^{-2N^2}[Z_1^{N}Z_2^{2N}Z_3^{N}],\\
\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big) &= [Z_1^{-N}Z_3^{-N}] + [Z_1^{N}Z_3^{N}]
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Tr}_\lambda^\omega\big([L_\infty^{T_N}]\big) &= \big([Z_1^{-N}Z_2^{-N}] + [Z_1^{N}Z_2^{N}]\big)\big([Z_2^{-N}Z_3^{-N}] + [Z_2^{N}Z_3^{N}]\big)\\
&= [Z_1^{-N}Z_2^{-2N}Z_3^{-N}] + [Z_1^{-N}Z_3^{N}] + [Z_1^{N}Z_3^{-N}] + [Z_1^{N}Z_2^{2N}Z_3^{N}].
\end{aligned}$$

Remembering that $A=\omega^{-2}$, we immediately see that

$$\mathrm{Tr}_\lambda^\omega\big([L_1^{T_N}]\big) = A^{-N^2}\mathrm{Tr}_\lambda^\omega\big([L_0^{T_N}]\big) + A^{N^2}\mathrm{Tr}_\lambda^\omega\big([L_\infty^{T_N}]\big).$$

Since the quantum trace homomorphism $\mathrm{Tr}_\lambda^\omega\colon \mathcal{S}^A(U)\to\mathcal{T}^\omega(\lambda)$ is injective, this proves the first statement of the lemma.

The second statement is proved by the same method. □

This takes care of the proofs that we had postponed up to this point.

References

[Ba] John W. Barrett, *Skein spaces and spin structures*, Math. Proc. Cambridge Philos. Soc. 126 (1999), 267–275.

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. *Topological quantum field theories derived from the Kauffman bracket*, Topology 34 (1995), 883–927.

[BoL] Francis Bonahon, Xiaobo Liu, *Representations of the quantum Teichmüller space and invariants of surface diffeomorphisms*, Geom. Topol. 11 (2007), 889–938.

[BoW$_1$] Francis Bonahon, Helen Wong, *Quantum traces for representations of surface groups in* $\mathrm{SL}_2(\mathbb{C})$, Geometry & Topology 15 (2011), 1569–1615.

[BoW$_2$] Francis Bonahon, Helen Wong, *Kauffman brackets, character varieties and triangulations of surfaces*, in *Topology and Geometry in Dimension Three: Triangulations, Invariants, and Geometric Structures* (W. Li, L.Bartolini, J. Johnson, F. Luo, R. Myers, J. H. Rubinstein eds.), Contemporary Mathematics 560, American Math. Society, 2011.

[BoW$_3$] Francis Bonahon, Helen Wong, *Representations of the Kauffman skein algebra II: punctured surfaces*, in preparation.

[BoW$_4$] Francis Bonahon, Helen Wong, *Representations of the Kauffman skein algebra III: closed surfaces and naturality*, in preparation.

[BoW_5] Francis Bonahon, Helen Wong, *The classical shadow of the Witten-Reshetikhin-Turaev representation of the Kauffman skein algebra*, in preparation.

[Bu_1] Doug Bullock, *Estimating a skein module with* $SL_2(\mathbb{C})$ *characters*, Proc. Amer. Math. Soc. 125 (1997), 1835–1839.

[Bu_2] Doug Bullock, *Rings of* $SL_2(\mathbb{C})$*–characters and the Kauffman bracket skein module*, Comment. Math. Helv. 72 (1997), 521–542.

[Bu_3] Doug Bullock, *A finite set of generators for the Kauffman bracket skein algebra*, Math. Z. 231 (1999), 91–101.

[$BuFK_1$] Doug Bullock, Charles Frohman, Joanna Kania-Bartoszyńska, *Understanding the Kauffman bracket skein module*, J. Knot Theory Ramifications 8 (1999), 265–277.

[$BuFK_2$] Doug Bullock, Charles Frohman, Joanna Kania-Bartoszyńska, *The Kauffman bracket skein as an algebra of observables*, Proc. Amer. Math. Soc. 130 (2002), 2479–2485.

[BuP] Doug Bullock, Józef H. Przytycki, *Multiplicative structure of Kauffman bracket skein module quantizations*, Proc. Amer. Math. Soc. 128 (2000), 923-931.

[ChF_1] Leonid O. Chekhov, Vladimir V. Fock, *Quantum Teichmüller spaces*, Theor.Math.Phys. 120 (1999), 1245–1259.

[ChF_2] Leonid O. Chekhov, Vladimir V. Fock, *Observables in 3D gravity and geodesic algebras*, in: *Quantum groups and integrable systems* (Prague, 2000), Czechoslovak J. Phys. 50 (2000), 1201–1208.

[Fo] Vladimir V. Fock, *Dual Teichmüller spaces*, unpublished preprint, 1997, `arXiv:Math/dg-ga/9702018` .

[FoG] Vladimir V. Fock, Alexander B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Ann. Sci. Éc. Norm. Supér. 42 (2009), 865–930.

[FrG] Charles Frohman, Răzvan Gelca, *Skein modules and the noncommutative torus*, Trans. Amer. Math. Soc. 352 (2000), 4877–4888.

[Go] William M. Goldman, *Topological components of spaces of representations*, Invent. Math. 93 (1988), 557–607.

[HaP] Miloslav Havlíček, Severin Pošta, *On the classification of irreducible finite-dimensional representations of* $U'_q(\mathrm{so}_3)$ *algebra*, J. Math. Phys. 42 (2001), 472–500.

[He] Heinz Helling, *Diskrete Untergruppen von* $SL_2(\mathbb{R})$, Invent. Math. 17 (1972), 217–229.

[Kash] Rinat Kashaev, *Quantization of Teichmüller spaces and the quantum dilogarithm*, Lett. Math. Phys. 43 (1998), 105–115.

[Kass] Christian Kassel, *Quantum groups*, Graduate Texts in Mathematics vol. 155, Springer-Verlag, New York, 1995.

[Lic_1] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Math. 175, Springer-Verlag, 1997.

[Lic_2] W. B. Raymond Lickorish, *Quantum invariants of* 3*–manifolds*, in: *Handbook of geometric topology*, 707734, North-Holland, Amsterdam, 2002.

[MuFK] David Mumford, John Fogarty, Frances Kirwan, *Geometric invariant theory. Third edition*, Ergebnisse der Mathematik und ihrer Grenzgebiete 34, Springer-Verlag, 1994.

[PrS] Jozef H. Przytycki, Adam S. Sikora, *On skein algebras and* $SL_2(\mathbb{C})$*-character varieties*, Topology 39 (2000), 115–148.

[ReT] Nicolai Y. Reshetikhin and Vladimir G. Turaev, *Invariants of* 3*–manifolds via link polynomials and quantum groups*, Invent. Math. 103 (1991), 547–597.

[Rob] Justin Roberts, *Irreducibility of some quantum representations of mapping class groups*, Knots in Hellas '98, Vol. 3 (Delphi), J. Knot Theory Ramifications 10 (2001), 763–767.

[Tu_1] Vladimir Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. École Norm. Sup. 24 (1991), 635–704.

[Tu_2] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics Vol. 18, Walter de Gruyter & Co., Berlin, 1994.

Department of Mathematics, University of Southern California, Los Angeles CA 90089-2532, U.S.A.

E-mail address: `fbonahon@math.usc.edu`

Department of Mathematics, Carleton College, Northfield MN 55057, U.S.A.

E-mail address: `hwong@carleton.edu`