ON COMPLETE CONSTANT MEAN CURVATURE VERTICAL MULTIGRAPHS IN $\mathbb{E}(\kappa, \tau)$

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ABSTRACT. We prove that any complete surface with constant mean curvature in a homogeneous space $\mathbb{E}(\kappa, \tau)$ which is transversal to the vertical Killing vector field is, in fact, a vertical graph. As a consequence we get that any orientable, parabolic, complete, immersed surface with constant mean curvature H in $\mathbb{E}(\kappa, \tau)$ (different from a horizontal slice in $\mathbb{S}^2 \times \mathbb{R}$) is either a vertical cylinder or a vertical graph (in both cases, it must be $4H^2 + \kappa \leq 0$).

1. INTRODUCTION

The classical theory of constant mean curvature surfaces in 3-manifolds is still an active field of research nowadays. Among all ambient 3-manifolds, the most studied subclass consists of those simply-connected with constant sectional curvature, the so-called *space forms* ($\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$). Their isometry group has dimension 6 and acts transitively on their tangent bundle. Apart from the space forms, the most symmetric ones are those simply-connected whose isometry group has dimension 4. They are usually denoted by $\mathbb{E}(\kappa, \tau)$, where $\kappa, \tau \in \mathbb{R}$ satisfy $\kappa - 4\tau^2 \neq 0$, and include the Riemannian product manifolds $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $\mathbb{H}^2(\kappa) \times \mathbb{R}$ (when $\tau = 0$), the Heisenberg space Nil_3 (for $\kappa = 0$), the universal cover $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ of the special linear group $\mathrm{SL}_2(\mathbb{R})$ endowed with some special left-invariant metrics (for $\kappa < 0$ and $\tau \neq 0$), and the Berger spheres (for $\kappa > 0$ and $\tau \neq 0$). See [D07] for more details.

The spaces $\mathbb{E}(\kappa, \tau)$ are characterized by admitting a Riemannian submersion $\pi : \mathbb{E}(\kappa, \tau) \to \mathbb{M}^2(\kappa)$ with constant bundle curvature τ, where $\mathbb{M}^2(\kappa)$ stands for the simply-connected Riemannian surface with constant curvature κ, such that the fibers of π are the integral curves of a unit Killing vector field in $\mathbb{E}(\kappa, \tau)$ (see [Ma12]). In what follows, we will refer to this field as the *vertical* Killing vector field and it will be denoted by ξ.

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In the theory of constant mean curvature surfaces (H-surfaces in the sequel) in $\mathbb{E}(\kappa,\tau)$, vertical multigraphs play an important role. A surface Σ immersed in $\mathbb{E}(\kappa,\tau)$ is said to be a *vertical multigraph* if the following equivalent conditions hold:

a) Σ is transversal to the vertical Killing vector field ξ.
b) The angle function $\nu = \langle N, \xi\rangle$ has no zeros in Σ, where N is a global unit normal vector field of Σ.
c) The projection $\pi_{|\Sigma} : \Sigma \to \mathbb{M}^2(\kappa)$ is a local diffeomorphism.

The first two conditions are trivially equivalent, whereas the third one follows from the fact that the absolute value of the Jacobian of $\pi_{|\Sigma}$ equals $|\nu|$.

The need of knowing whether a vertical H-multigraph is or not embedded often arises in the study of such surfaces. In this paper we solve this problem by proving that they are always *vertical graphs* (i.e., they intersect at most once each integral curve of ξ), and in particular embedded, when we additionally assume that the vertical H-multigraph is complete. (We observe that the completeness asumption is needed: A counterexample is given, for example, by the helicoid of $\mathbb{H}^2 \times \mathbb{R}$ constructed by Nelli and Rosenberg in [NR02] minus a neighborhood of its axis.) In this line, we recall the following known results:

- Hauswirth, Rosenberg and Spruck proved a half-space theorem for properly embedded $\frac{1}{2}$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and concluded that a complete vertical $\frac{1}{2}$-multigraph in $\mathbb{H}^2 \times \mathbb{R}$ is always an entire vertical graph (see [HRS10]), i.e., it intersects exactly once each integral curve of ξ. Later on, by using another halfspace theorem for properly immersed surfaces in Nil_3, Daniel and Hauswirth [DH09] proved the same result for minimal vertical multigraphs in Nil_3. Finally, using this theorem by Daniel and Hauswirth, the classification theorem by Fernández and Mira in [FM09] and the Daniel correspondence [D07], it is possible to extend the aforementioned results to prove that a complete vertical H-multigraph in $\mathbb{E}(\kappa,\tau)$ satisfying $4H^2 + \kappa = 0$ is an entire vertical graph (see also Corollary 4.6.3 in [DHM09] for a complete reference).
- Espinar and Rosenberg proved in [ER09] that there are no complete vertical H-multigraphs in $\mathbb{H}^2 \times \mathbb{R}$ for $H > \frac{1}{2}$. In a joint work with Joaquín Pérez (see the proof of Theorem 2 in [MaPR11]), the authors proved using a different approach that the only complete vertical H-multigraphs in $\mathbb{E}(\kappa,\tau)$ with $4H^2 + \kappa > 0$ are the horizontal slices $\mathbb{S}^2(\kappa) \times \{t_0\}$ in $\mathbb{S}^2(\kappa) \times \mathbb{R}$, for any $\kappa > 0$ (and $\tau = H = 0$). The latter result has also been proved in [P12].

To obtain the general result stated below, it only remains to study the case $4H^2 + \kappa < 0$. We highlight that the geometry of an H-surface in the homogeneous spaces $\mathbb{E}(\kappa,\tau)$ varies essentially depending on the sign of $4H^2 + \kappa$. For instance, it is known that constant mean curvature spheres exist if, and only if, $4H^2 + \kappa > 0$; see Theorem 2.5.3 in [DHM09].

Theorem 1. *Let Σ be a complete vertical H-multigraph in $\mathbb{E}(\kappa,\tau)$. Then, one of the following statements hold:*

a) $\mathbb{E}(\kappa,\tau)=\mathbb{S}^2(\kappa)\times\mathbb{R}$, $H=0$ *and* $\Sigma=\mathbb{S}^2(\kappa)\times\{t_0\}$, *for some* $t_0\in\mathbb{R}$.
b) $4H^2+\kappa\leq 0$ *and Σ is a vertical graph. Moreover, if the equality holds then the graph is entire.*

We remark that the condition $4H^2+\kappa=0$ in Theorem 1 is not necessary in order to obtain entire vertical H-graphs: Besides horizontal slices, other entire minimal vertical graphs in $\mathbb{H}^2\times\mathbb{R}$ have been constructed by Nelli and Rosenberg in [NR02], by Collin and Rosenberg in [CR10] and by Mazet, Rosenberg and the second author in [MRR11]. The reader can also find some examples of rotationally invariant entire vertical H-graphs in $\mathbb{H}^2\times\mathbb{R}$ for any $0<H\leq\frac{1}{2}$ in [NR06]. We also emphasize that there exist many complete vertical H-graphs in $\mathbb{H}^2\times\mathbb{R}$ which are not entire:

- On the one hand, Sa Earp [SE08] and Abresch gave a explicit complete minimal vertical graph defined on half a hyperbolic plane. In [CR10, MRR11, Ro10], some complete minimal examples in $\mathbb{H}^2\times\mathbb{R}$ (which are vertical graphs over simply connected domains bounded by –possibly infinitely many– complete geodesics where the graph has non-bounded boundary data, and/or finitely many arcs at the infinite boundary of $\mathbb{H}^2$) are given. Melo constructed in [Me] minimal examples in $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ similar to those in [CR10].
- On the other hand, Folha and Melo obtained in [FoM10] vertical H-graphs in $\mathbb{H}^2\times\mathbb{R}$, with $0<H<\frac{1}{2}$, over simply-connected domains bounded by an even number of curves of geodesic curvature $\pm 2H$ (disposed alternately) over which the graphs go to $\pm\infty$.

Let us now explain a consequence of Theorem 1. We consider the *stability operator* of a two-sided H-surface immersed in $\mathbb{E}(\kappa,\tau)$, given by

$$L=\Delta+|A|^2+\mathrm{Ric}(N), \tag{1.1}$$

where Δ stands for the Laplacian with respect to the induced metric on Σ, and A and N denote respectively the shape operator and a unit normal vector field of Σ. The surface Σ is said to be *stable* when $-L$ is a non-negative operator (see [MPR09] for more details on stable H-surfaces). Rosenberg proved in [R06] the non-existence of stable H-surfaces in $\mathbb{E}(\kappa,\tau)$ provided that $3H^2+\kappa>\tau^2$, other than $\mathbb{S}^2(\kappa)\times\{t_0\}$ in $\mathbb{S}^2(\kappa)\times\mathbb{R}$. It is conjectured that the optimal condition for such non-existence result is $4H^2+\kappa>0$. In a joint work with Pérez [MaPR11], the authors slightly improve Rosenberg's bound in the general case and obtain the expected bound under the additional assumption of parabolicity. We recall that a Riemannnian manifold Σ is said to be *parabolic* when the only positive superharmonic functions defined on M are the constant functions. As a direct consequence of Theorem 1 above and Theorem 2 in [MaPR11], we get the following nice classification result:

Corollary 2. *Let Σ be an orientable, parabolic, complete, stable H-surface in $\mathbb{E}(\kappa,\tau)$. Then, one of the following statements hold:*

a) $\mathbb{E}(\kappa,\tau)=\mathbb{S}^2(\kappa)\times\mathbb{R}$, $H=0$ *and* $\Sigma=\mathbb{S}^2(\kappa)\times\{t_0\}$, *for some* $t_0\in\mathbb{R}$.
b) $4H^2+\kappa\leq 0$ *and Σ is either a vertical graph or a vertical cylinder over a complete curve of geodesic curvature $2H$ in $\mathbb{M}^2(\kappa)$.*

2. Preliminaries

From now on, Σ will denote a complete vertical H-multigraph in $\mathbb{E}(\kappa,\tau)$, with $4H^2+\kappa<0$. Let us remark that the latter condition implies $\kappa<0$, so $\mathbb{E}(\kappa,\tau)$ admits a fibration over $\mathbb{H}^2(\kappa)$. By applying a convenient homothety in the metric, there is no loss of generality in supposing that $\kappa=-1$. Hence, we can consider the disk $\mathbb{D}(2)=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<4\}$ and the model $\mathbb{E}(-1,\tau)=\mathbb{D}(2)\times\mathbb{R}$, endowed with the Riemannian metric

$$\mathrm{d}s^2=\lambda^2(\mathrm{d}x^2+\mathrm{d}y^2)+(\mathrm{d}z+\tau\lambda(y\,\mathrm{d}x-x\,\mathrm{d}y))^2,$$

where $\lambda:\mathbb{D}(2)\to\mathbb{R}$ is given by $\lambda(x,y)=\left(1-\frac{1}{4}(x^2+y^2)\right)^{-1}$. In this model, the Riemannian fibration is nothing but $\pi:\mathbb{E}(-1,\tau)\to\mathbb{H}^2$, $\pi(x,y,z)=(x,y)$, when we identify $\mathbb{H}^2\equiv(\mathbb{D}(2),\lambda^2(\mathrm{d}x^2+\mathrm{d}y)^2)$.

Given an open set $\Omega\subset\mathbb{H}^2$ and a function $u\in C^\infty(\Omega)$, the graph associated to u is defined as the surface parametrized by

$$F_u:\Omega\to\mathbb{E}(-1,\tau),\quad F_u(x,y)=(x,y,u(x,y)).$$

Let us also fix the following notation, for any $R>0$:

- Given $x_0\in\mathbb{H}^2$, we denote by $B(x_0,R)$ the ball in $\mathbb{H}^2$ of center x_0 and radius R.
- Given $p_0\in\Sigma$, we denote by $B_\Sigma(p_0,R)$ the intrinsic ball in Σ of center p_0 and radius R.

The proof of Theorem 1 relies on some technical results given originally by Hauswirth, Rosenberg and Spruck in [HRS10]. Although they treat the case $H=\frac{1}{2}$ in $\mathbb{H}^2\times\mathbb{R}$, their arguments can be directly generalized to H-surfaces in $\mathbb{E}(\kappa,\tau)$ with $4H^2+\kappa\leq 0$, giving rise to Lemma 3 below.

Take $x_0\in\mathbb{H}^2$ and $R>0$ such that there exists $u\in C^\infty(B(x_0,R))$ with $F_u(B(x_0,R))\subset\Sigma$. (As $\pi_{|\Sigma}$ is a local diffeomorphism, this can be done for any $x_0\in\pi(\Sigma)$.) We also fix a unit normal vector field N of Σ so that the angle function $\nu=\langle N,\xi\rangle$ is positive. We observe that ν lies in the kernel of the stability operator of Σ, defined in (1.1), see [BCE88]. Since ν has no zeros, we deduce by a theorem given by Fischer-Colbrie [FC85] (see Lemma 2.1 in [MPR09]) that Σ stable. This is an important fact in the proof of Lemma 3.

Suppose that $\hat{x}\in\partial B(x_0,R)$ satisfies that u cannot be extended to any neighborhood of $\hat{x}$ in $\mathbb{H}^2$ as a vertical H-graph. Given a sequence $\{x_n\}\subset B(x_0,R)$ converging to $\hat{x}$ and calling $p_n=F_u(x_n)$, the arguments in [HRS10] can be extended to show that the sequence of surfaces $\{\Sigma_n\}$ (where Σ_n results from translating Σ vertically so that p_n is at height zero), converges to a vertical cylinder $\pi^{-1}(\Gamma)$, for a curve $\Gamma \subset \mathbb{H}^2$ of constant geodesic curvature $2H$ or $-2H$ which is tangent to $\partial B(x_0, R)$ at $\hat{x}$. Note that the condition $4H^2 - 1 < 0$ implies that Γ is non-compact.

Given $\delta > 0$, we denote by $N_\delta(\Gamma)$ the open neighborhood of Γ in $\mathbb{H}^2$ of radius δ. Moreover, given $x \in \mathbb{H}^2 \setminus \Gamma$, we call $N_\delta(\Gamma, x) = N_\delta(\Gamma) \cap U$, where U is the connected component of $\mathbb{H}^2 \setminus \Gamma$ containing x. By coherence, we denote $U = N_\infty(\Gamma, x)$.

Lemma 3 ([HRS10]). *In the setting above, suppose that $\hat{x} \in \partial B(x_0, R)$ satisfies that u cannot be extended to any neighborhood of $\hat{x}$ in $\mathbb{H}^2$ as a vertical H-graph. Then, there exist $\delta > 0$ and a complete curve $\Gamma \subset \mathbb{H}^2$ of constant geodesic curvature $2H$ or $-2H$, tangent to $\partial B(x_0, R)$ at $\hat{x}$, such that u extends as a vertical H-graph to $B(x_0, R) \cup N_\delta(\Gamma, x_0)$ with infinite constant boundary values along Γ.*

Remark 4. In our study, we will always find a dichotomy between geodesic curvature $2H$ or $-2H$, as well as boundary values $+\infty$ or $-\infty$. Although, the arguments below do not depend on these signs, it is worth saying something about the relation between them in order to describe the complete vertical graphs. As mentioned above, it can be shown that the H-multigraphs Σ_n converge uniformly on compact subsets to a vertical cylinder and we are considering the unit normal vector field of Σ which points upwards. By analyzing the normals of Γ in $\mathbb{H}^2$ and of $\pi^{-1}(\Gamma)$ in $\mathbb{E}(-1, \tau)$, it is not difficult to realize that if u tends to $+\infty$ (resp. $-\infty$) along Γ, the geodesic curvature of Γ will be $2H$ (resp. $-2H$) with respect to the normal vector pointing to the domain of definition of the graph. This fact was proved for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ by Nelli and Rosenberg in [NR02]; for H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$, for $0 < H \leq 1/2$, by Hauswirth, Rosenberg and Spruck in [HRS09]; and for minimal surfaces in $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ by Younes in [Y10].

3. The proof of Theorem 1

As explained above, supposed that Σ is a complete vertical H-multigraph in $\mathbb{E}(-1, \tau)$ with $0 \leq H < \frac{1}{2}$, we will show that Σ is a vertical graph. The proof relies on the following result.

Proposition 5. *Let Σ be a complete vertical H-multigraph in $\mathbb{E}(-1, \tau)$. Given $p \in \Sigma$ and $R > 0$, there exist an open set $\Omega \subseteq \mathbb{H}^2$ and a function $u \in C^\infty(\Omega)$ such that $B_\Sigma(p, R) \subseteq F_u(\Omega)$.*

Note that the fact that Σ is indeed a graph follows from Proposition 5: Reasoning by contradiction, if there existed $p, q \in \Sigma$, $p \neq q$, which project by $\pi : \mathbb{E}(-1, \tau) \to \mathbb{H}^2$ on the same point, then we could take R bigger than the intrinsic distance from p to q to reach a contradiction. Thus, the rest of this section will be devoted to prove Proposition 5.

From now on, we fix a point $p \in \Sigma$ and denote $x_0 = \pi(p)$. The following definition will be useful in the sequel.

Definition 6. We say that $\Omega \subset \mathbb{H}^2$ is *admissible* if it is a connected open set containing x_0 for which there exists $u \in C^\infty(\Omega)$ such that $F_u(\Omega) \subset \Sigma$ and $F_u(x_0) = p$.

If $\Omega \subset \mathbb{H}^2$ is admissible, then the function u in the definition above is unique, and we will call it the function *associated* to Ω. The technique we will use to prove Proposition 5 consists of enlarging gradually an initial admissible domain until it eventually contains the projection of an arbitrarily large intrinsic ball.

As $\nu > 0$, there exists a neighborhood of p in Σ which projects one-to-one to a ball $B(x_0, \rho)$, for some $\rho > 0$. In other words, there exists $\rho > 0$ such that $B(x_0, \rho)$ is admissible. Let us consider

$$R_0 = \sup\{\rho > 0 : B(x_0, \rho) \text{ is admissible}\}. \tag{3.1}$$

If $R_0 = +\infty$, Proposition 5 follows trivially for such a point p, so we assume $R_0 < +\infty$. This implies that $B(x_0, R_0)$ is a *maximal admissible ball*, and it will play the role of our initial domain.

Lemma 7. *Let Ω an admissible domain such that $\partial\Omega$ is a piecewise C^2-embedded curve. Suppose that there exists $\hat{x}$ in a regular arc of $\partial\Omega$ such that Ω cannot be extended as an admissible domain to any neighborhood of $\hat{x}$. Then:*

i) *The geodesic curvature of $\partial\Omega$ at $\hat{x}$ with respect to its inner normal vector is at least $-2H$.*
ii) *There exists a curve $\Gamma \subset \mathbb{H}^2$ with constant geodesic curvature $2H$ or $-2H$, tangent to $\partial\Omega$ at $\hat{x}$, and there exist $\delta > 0$, $y \in \Omega$ and a function $v \in C^\infty(N_\delta(\Gamma, y))$ such that:*
   a) *The function v has constant boundary values $+\infty$ or $-\infty$ along Γ.*
   b) *Given $r > 0$, we have $U_r = N_\delta(\Gamma, y) \cap B(\hat{x}, r) \cap \Omega \neq \emptyset$ and there exists $r_0 > 0$ such that $u = v$ in U_r for $0 < r < r_0$.*

Proof of Lemma 7. Let us consider a geodesic ball $B(y, \rho) \subset \mathbb{H}^2$ contained in Ω and tangent to $\partial\Omega$ at $\hat{x}$. Lemma 3 guarantees the existence of a curve $\Gamma \subset \mathbb{H}^2$ with constant geodesic curvature $2H$ or $-2H$, tangent to $B(y, \rho)$ at $\hat{x}$, and $\delta > 0$ such that the H-graph over $B(y, \rho)$ can be extended to an H-graph over $B(y, \rho) \cup N_\delta(\Gamma, y)$. Let us define v as the restriction of such an extension to $N_\delta(\Gamma, y)$. It can be easily shown that Γ, δ, y and v satisfy the conditions in item ii). The uniqueness of prolongation ensures that Ω must be contained in $N_\infty(\Gamma, y)$, from where the estimation for the geodesic curvature of $\partial\Omega$ given in item i) follows. □

We will prove (see Lemma 10 for $R = R_0$ and $\mathcal{C} = \emptyset$) that the set of points in $\partial B(x_0, R_0)$ such that u cannot be extended to any neighborhood of them as a vertical H-graph is finite. So we can denote them as $x_1, \dots, x_r$. Lemma 7 guarantees that, given $j \in \{1, \dots, r\}$, there exist $\delta_j > 0$ and

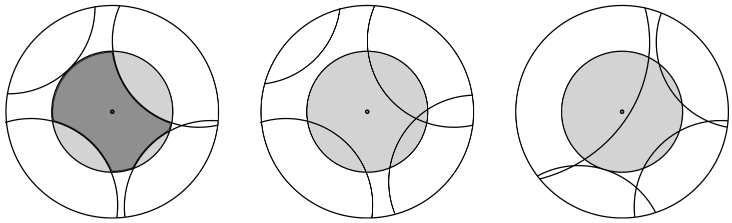

FIGURE 1. Three families of four curves in $\mathbb{H}^2$. The first one is in general position with respect to the shaded ball and the domain $\Omega_{\mathcal{C}}$ associated to this family has been also shaded in a darker grey. On the contrary, the other two families are not in general position: conditions (a) and (c) in the definition fail for the second one, whereas condition (b) does not hold for the third one.

a curve $\Gamma_j \subset \mathbb{H}^2$ of constant geodesic curvature $2H$ or $-2H$ which is tangent to $\partial B(x_0, R_0)$ at x_j, such that u can be extended to $B(x_0, R_0) \cup N_{\delta_j}(\Gamma_j, x_0)$, for some $\delta_j > 0$. The next step in the proof of Proposition 5 will consist of showing that the curves $\Gamma_1, \dots, \Gamma_r$ are disjoint. Then we will be able to extend the initial ball $B(x_0, R_0)$ to a new admissible domain $\Omega = B(x_0, R_0) \cup (\cup_{j=1}^r N_\delta(\Gamma_j, x_0))$, for some $\delta \leq \min\{\delta_j : j = 1, \dots, r\}$. The function associated to such Ω will have boundary values $+\infty$ or $-\infty$ along each of the Γ_j (depending on the sign of its geodesic curvature). Therefore, for a further extension of Ω we will work in $\cap_{j=1}^r N_\infty(\Gamma_j, x_0)$, i.e., we will extend Ω in the direction of $\partial\Omega \cap \partial B(x_0, R_0)$. As this process will be iterated, we describe a general situation for the extension procedure.

Definition 8. Let $\mathcal{C}$ be a finite family of curves in $\mathbb{H}^2$, each one with constant geodesic curvature $2H$ or $-2H$. We will say that $\mathcal{C}$ is *in general position* for some radius $R > 0$ when the following three conditions are satisfied:

a) Each $\Gamma \in \mathcal{C}$ intersects the closed ball $\overline{B}(x_0, R)$ and $x_0 \notin \Gamma$.
b) $\partial\Omega_{\mathcal{C}} \cap \Gamma \neq \emptyset$ for every $\Gamma \in \mathcal{C}$, where

$$\Omega_{\mathcal{C}} = B(x_0, R) \cap (\cap_{\Gamma \in \mathcal{C}} N_\infty(\Gamma, x_0)). \tag{3.2}$$

c) No intersection point of curves in $\mathcal{C}$ lies on $\overline{B}(x_0, R)$.

(We recall that $N_\infty(\Gamma, x_0)$ is the open connected component of $\mathbb{H}^2 \setminus \Gamma$ containing x_0.) It is clear that the family of curves $\{\Gamma_1, \dots, \Gamma_r\}$ in the discussion above is in general position for the radius R_0. This condition will be preserved under the successive steps for enlarging the admissible domain. The next lemma gives some information about a family of curves in general position.

Lemma 9. *Let $\mathcal{C}$ be a finite family of curves in general position for a radius $R > 0$. Suppose that $\Omega_{\mathcal{C}}$, defined by 3.2, is an admissible domain and that, for any $\Gamma \in \mathcal{C}$ there exists a vertical H-graph v over $N_\delta(\Gamma, x_0)$, for some $\delta > 0$, with*

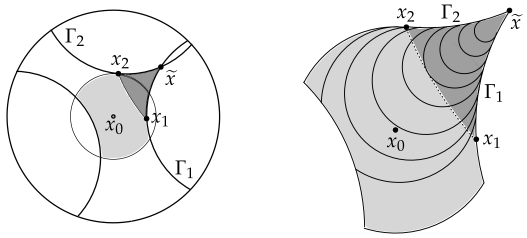

FIGURE 2. Left: Three curves in general position, two of which intersect at $\widetilde{x} \notin \overline{B}(x_0, R)$. Right: $\Omega_{\mathcal{C}} \cup T$ have been magnified and some of the curves Λ_t which cover T up to $\widetilde{x}$ are shown.

constant infinite boundary values along Γ*, which coincides with the function* u *associated to* $\Omega_{\mathcal{C}}$ *on* $\Omega_{\mathcal{C}} \cap N_\delta(\Gamma, x_0)$*. Then:*

a) Any two curves in $\mathcal{C}$ *are disjoint.*
b) There exists $\delta' > 0$ *so that* $\Omega_{\mathcal{C}} \cup (\cup_{\Gamma \in \mathcal{C}} N_{\delta'}(\Gamma, x_0))$ *is admissible.*

Proof of Lemma 9. Reasoning by contradiction, let us suppose that there exists a point $\widetilde{x} \in \Gamma_1 \cap \Gamma_2$ for some $\Gamma_1, \Gamma_2 \in \mathcal{C}$. Condition c) in Definition 8 tells us that $\widetilde{x} \notin \overline{B}(x_0, R)$. We can take a continuous family $\{D_t\}_{t \in [0,\ell)}$ of geodesic balls in $\mathbb{H}^2$ satisfying the following three conditions:

(1) $D_0 \subset B(x_0, R)$.
(2) ∂D_t is tangent to Γ_1 and Γ_2, for any $t \in [0, \ell)$.
(3) The radius of D_t strictly decreases with respect to t, and D_t converges to the point $\widetilde{x}$ when $t \to \ell$.

Let us call, respectively, x_1 and x_2 the points in Γ_1 and Γ_2 which are closest to x_0. Denote by T an open triangle with vertices $\widetilde{x}$, x_1 and x_2, which has two sides lying on Γ_1 and Γ_2 and the third one is a curve interior to $\Omega_{\mathcal{C}}$ joining x_1 and x_2. Let us denote by $\Lambda_t \subset \partial D_t$ the intersection of $\Omega_{\mathcal{C}} \cup T$ with the longest of the two curves in which ∂D_t is divided by Γ_1 and Γ_2. Then $\Omega_{\mathcal{C}} \cup T = \Omega_{\mathcal{C}} \cup (\cup_{[0,\ell)} \Lambda_t)$ (see figure 2).

Let $r \in [0, \ell]$ be the supreme of the values of $t \in [0, \ell)$ for which the function u associated to $\Omega_{\mathcal{C}}$ can be extended as an H-graph on $\Omega_{\mathcal{C}} \cup (\cup_{s \in [0,t]} \Lambda_s)$. Note that $r > 0$ since the graph can be extended to $\Omega_{\mathcal{C}} \cup N_\varepsilon(\Gamma_i, x_0)$, for $i \in \{1, 2\}$ and some $\varepsilon > 0$, and the curves Λ_s lie on the union of these two open sets for small values of s. Moreover, it must be $r = \ell$: Otherwise there would exist a point in Λ_r such that u cannot be extended to any neighborhood of it as an H-graph. But Λ_r has constant geodesic curvature smaller than $-1 < -2H$ with respect to the normal pointing to $\Omega_{\mathcal{C}} \cup (\cup_{s \in [0,r]} \Lambda_s)$, contradicting Lemma 7.

Hence we have proved that $\Omega_1 = \Omega_{\mathcal{C}} \cup T$ is admissible, and the extension u_1 of u over Ω_1 has non-bounded values in the part of $\partial\Omega_1$ lying on $\Gamma_1 \cup \Gamma_2$. Nonetheless, the graph u can also be extended to u_2 defined over

$\Omega_2 = \Omega_{\mathcal{C}} \cup N_\varepsilon(\Gamma_1, x_0)$. Thus $u_1 = u_2$ in $\Omega_1 \cap \Omega_2$, by uniqueness of the analytic prolongation. This is a contradiction because u_2 has bounded values in $\Gamma_2 \cap N_\varepsilon(\Gamma_1, x_0) \subset \partial(\Omega_1 \cap \Omega_2)$ whereas u_1 does not. Such a contradiction proves a). Item b) also follows from the previous argument. □

Let us now prove that, apart from the curves along which the graph takes infinite boundary values, the number of points which do not admit an extension to any neighborhood of them is finite at any step of the extension procedure.

Lemma 10. *Let $\mathcal{C}$ be a finite family of curves in general position for some radius $R > 0$ and suppose that $\Omega_{\mathcal{C}}$ is admissible. Then the set $A(R)$ of points in $\partial\Omega_{\mathcal{C}} \cap \partial B(x_0, R)$ such that the function u associated to $\Omega_{\mathcal{C}}$ cannot be extended to any neighborhood of them as an H-graph, is finite.*

Proof of Lemma 10. Since $\mathcal{C}$ is finite, $\partial\Omega_{\mathcal{C}} \cap \partial B(x_0, R)$ consists of finitely many regular arcs. Their endpoints belong to $A(R)$, but they are finitely many and, by Lemma 7, it is known that the graph can be extended as an H-graph to a neighborhood in $\mathbb{H}^2$ of any point in a neighborhood in $\partial\Omega_{\mathcal{C}} \cap \partial B(x_0, R)$ of any of them. Thus we will not consider such endpoints.

Lemma 7 also says that the rest of points in $A(R)$ are isolated. On the other hand, it is easy to check that $A(R)$ is closed: We suppose there exists a sequence $\{x_n\}$ in $A(R)$ converging to $x_\infty \notin A(R)$. Such a point x_∞ must be interior to one of the arcs in $\partial\Omega_{\mathcal{C}} \cap \partial B(x_0, R)$ because of the discussion above. Then u admits an extension to a neighborhood of x_∞ in $\mathbb{H}^2$. But x_n lies in such a neighborhood for n large enough, a contradiction.

Hence $A(R)$ is closed, consists of isolated points and is contained in the compact set $\partial B(x_0, R)$, so it is finite. □

We now have all the ingredients to prove the desired result.

Proof of Proposition 5. Repeating the argument leading to equation (3.1) and provided that $R_0 < +\infty$ (otherwise we would be done), we can consider the maximal admissible ball $B(x_0, R_0)$. Lemma 10 guarantees the existence of a finite collection of points in $\partial B(x_0, R_0)$ such that $B(x_0, R_0)$ cannot be extended in an admissible way to any neighborhood of any of them. Hence, there exists a finite family $\mathcal{C}_0$ of complete curves with constant geodesic curvature $2H$ or $-2H$, each one tangent to $\partial B(x_0, R_0)$ at one of those points, under the conditions of Lemma 7. The family $\mathcal{C}_0$ is in general position. Then Lemma 9 says that the curves in $\mathcal{C}_0$ are disjoint and that there exists $\delta > 0$ such that the domain $\Omega' = \Omega_{\mathcal{C}_0} \cup (\cup_{\Gamma \in \mathcal{C}_0} N_\delta(\Gamma, x_0))$ is admissible.

Moreover, the initial graph can be extended to a neighborhood of any point interior to $\partial\Omega' \cap \partial B(x_0, R_0)$, so we can define the supremum of $R > R_0$ satisfying that $B(x_0, R) \cap (\cap_{\Gamma \in \mathcal{C}_0} N_\infty(\Gamma, x_0))$ is admissible, called R_1. If $R_1 < +\infty$, then let us consider the admissible domain $\Omega_1 = B(x_0, R_1) \cap (\cap_{\Gamma \in \mathcal{C}_0} N_\infty(\Gamma, x_0))$. By definition of R_1, there will be points in the interior of $\partial\Omega_1 \cap \partial B(x_0, R_1)$ for which Ω_1 cannot be extended in an admissible way to any neighborhood of any of them. By Lemma 10, these points are finitely many, and we can apply Lemma 7 to obtain a new family $\mathcal{C}_1 \supset \mathcal{C}_0$ in general position for the radius R_1. Note that $\mathcal{C}_1 - \mathcal{C}_0 \neq \emptyset$.

This procedure can be iterated to get a strictly increasing sequence (possibly finite) or radii $\{R_n\}$ in such a way that, for any n, there exists $\mathcal{C}_n$, a family of curves in general position for R_n, with $\mathcal{C}_{n-1} \subsetneq \mathcal{C}_n$. Besides, the arguments above show that the domain

$$\Omega_{\mathcal{C}_n} = B(x_0, R_n) \cap \left(\bigcap_{\Gamma \in \mathcal{C}_n} N_\infty(\Gamma, x_0) \right),$$

is admissible, $\Omega_{\mathcal{C}_{n-1}} \subset \Omega_{\mathcal{C}_n}$ and the associate function u_n extends u_{n-1}. Note that there exists $\delta_n > 0$ such that $\Omega_{\mathcal{C}_n}$ can be extended to the admissible domain $\Omega_{\mathcal{C}_n} \cup (\cup_{\Gamma \in \mathcal{C}_n} N_{\delta_n}(\Gamma, x_0))$, by item b) of Lemma 9.

In this situation, there are two possibilities: Either $R_{n_0} = +\infty$ for some $n_0 \geq 0$ (and we would be done since Σ would be a complete vertical graph), or the sequence $\{R_n\}$ has infinitely many terms. Suppose we are in the latter case. Then we claim that $\lim\{R_n\} = +\infty$. In order to prove this, we observe that the curves in $\mathcal{C}_n$ are all disjoint by Lemma 9. If $R_\infty = \lim\{R_n\} < +\infty$, we would have an infinite family of disjoint curves (infinite, as each $\mathcal{C}_n$ strictly contains $\mathcal{C}_{n-1}$), each one with constant geodesic curvature $2H$ or $-2H$ and intersecting the compact ball $\overline{B}(x_0, R_\infty)$. But this situation is impossible by condition b) in Definition 8 (it tells us that, when we fix one of such curves, Γ, the rest of them must lie in one of the two components of $\mathbb{H}^2 \smallsetminus \Gamma$).

Given $n \in \mathbb{N}$, the open set $O_n = F_{u_n}(\Omega_{\mathcal{C}_n}) \subset \Sigma$ satisfies that $\pi(\partial O_n) \subset \partial B(x_0, R_n)$, since u_n has boundary values $\pm\infty$ on the curves of $\mathcal{C}_n$. Then we get that the length of any piecewise regular curve $\alpha : [a, b] \to \Sigma$ with $\alpha(a) = p$ and $\alpha(b) \in \partial O_n$ is bigger than R_n, as the projected curve $\pi \circ \alpha$ in $\mathbb{H}^2$ is shorter than α. Thus, the distance in Σ from p to ∂O_n is at least R_n, so $B_\Sigma(p, R_n) \subset F_u(\Omega_{\mathcal{C}_n})$ and we are done since R_n is arbitrarily large. □

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