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V-SEMI-SLANT SUBMERSIONS

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Abstract. Let F be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). We introduce the notion of the v-semi-slant submersion. And then we obtain some properties on it. In particular, we give some examples for it.

1. Introduction

Let F be a C^∞-submersion from a semi-Riemannian manifold (M, g_M) onto a semi-Riemannian manifold (N, g_N). Then according to the conditions on the map $F : (M, g_M) \mapsto (N, g_N)$, we have the following submersions:

semi-Riemannian submersion and Lorentzian submersion [7], Riemannian submersion ([8], [14]), slant submersion ([5], [19]), almost Hermitian submersion [21], contact-complex submersion [9], quaternionic submersion [10], almost h-slant submersion and h-slant submersion [15], semi-invariant submersion [20], almost h-semi-invariant submersion and h-semi-invariant submersion [16], semi-slant submersions [18], almost h-semi-slant submersions and h-semi-slant submersions [17], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([3], [22]), Kaluza-Klein theory ([2], [11]), Supergravity and superstring theories ([12], [13]), etc.

The paper is organized as follows. In section 2 we remind some notions which is neeed for this paper. In section 3 we give the definition of the v-semi-slant submersion and obtain some interesting properties on it. In section 4 we construct some examples of the v-semi-slant submersion.

2. Preliminaries

Let (M, g_M) and (N, g_N) be Riemannian manifolds, where M, N are $C^\infty-$manifolds and g_M, g_N are Riemannian metrics, and $F : M \mapsto N$ a $C^\infty-$submersion. The map F is said to be *Riemannian submersion* if the differential F_* preserves the lengths of horizontal vectors [10].

Let (M, g_M, J) be an almost Hermitian manifold, where J is an almost complex structure. A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *slant submersion* if the angle $\theta = \theta(X)$ between JX and the space $\ker(F_*)_p$ is constant for any nonzero $X \in T_pM$ and $p \in M$ [19].

We call θ a *slant angle*.

A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *semi-invariant submersion* if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2,\ J(\mathcal{D}_1) = \mathcal{D}_1,\ J(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

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where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$ [19].

A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *semi-slant submersion* if there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We call the angle θ a *semi-slant angle*.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F : (M, g_M) \mapsto (N, g_N)$ a smooth map. The second fundamental form of F is given by

$$(\nabla F_*)(X, Y) := \nabla^F_X F_* Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where ∇^F is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N [4]. Recall that F is said to be *harmonic* if $trace(\nabla F_*) = 0$ and F is called a *totally geodesic* map if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [4].

3. V-Semi-slant submersions

Definition 3.1. Let (M, g_M, J) be an almost Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *v-semi-slant submersion* if there is a distribution $\mathcal{D}_1 \subset (\ker F_*)^\perp$ such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$.

We call the angle θ a *v-semi-slant angle*.

Remark 3.2. Let F be a v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then there is a distribution $\mathcal{D}_1 \subset (\ker F_*)^\perp$ such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$.

If $\mathcal{D}_2 = (\ker F_*)^\perp$, then we call the map F a *v-slant submersion* and the angle θ *v-slant angle* [19]. Otherwise, if $\theta = \frac{\pi}{2}$, then we call the map F a *v-semi-invariant submersion* [20].

Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a v-semi-slant submersion. Then there is a distribution $\mathcal{D}_1 \subset (\ker F_*)^\perp$ such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$ and $p \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$.

Then for $X \in \Gamma((\ker F_*)^\perp)$, we have

$$X = PX + QX,$$

where $PX \in \Gamma(\mathcal{D}_1)$ and $QX \in \Gamma(\mathcal{D}_2)$.

For $X \in \Gamma(\ker F_*)$, we get

$$JX = \phi X + \omega X,$$

where $\phi X \in \Gamma(\ker F_*)$ and $\omega X \in \Gamma((\ker F_*)^\perp)$.

For $Z \in \Gamma((\ker F_*)^\perp)$, we obtain

$$JZ = BZ + CZ,$$

where $BZ \in \Gamma(\ker F_*)$ and $CZ \in \Gamma((\ker F_*)^\perp)$.

For $U \in \Gamma(TM)$, we have

$$U = \mathcal{V}U + \mathcal{H}U,$$

where $\mathcal{V}U \in \Gamma(\ker F_*)$ and $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$.

Then

$$\ker F_* = B\mathcal{D}_2 \oplus \mu,$$

where μ is the orthogonal complement of $B\mathcal{D}_2$ in $\ker F_*$ and is invariant under J. Furthermore,

$$C\mathcal{D}_1 = \mathcal{D}_1, B\mathcal{D}_1 = 0, C\mathcal{D}_2 \subset \mathcal{D}_2, \omega(\ker F_*) = \mathcal{D}_2$$
$$\phi^2 + B\omega = -id, C^2 + \omega B = -id, \omega\phi + C\omega = 0, BC + \phi B = 0.$$

Define the tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$$
$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

for vector fields E, F on M, where ∇ is the Levi-Civita connection of g_M. Define

$$\widehat{\nabla}_X Y := \mathcal{V}\nabla_X Y \quad \text{for } X, Y \in \Gamma(\ker F_*).$$

Then we easily have

Lemma 3.3. *Let (M, g_M, J) be a Kähler manifold and (N, g_N) a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a v-semi-slant submersion. Then we get*

(1)

$$\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y = \phi \widehat{\nabla}_X Y + B\mathcal{T}_X Y$$
$$\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y = \omega \widehat{\nabla}_X Y + C\mathcal{T}_X Y$$

for $X, Y \in \Gamma(\ker F_*)$.

(2)

$$\mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW = \phi \mathcal{A}_Z W + B\mathcal{H}\nabla_Z W$$
$$\mathcal{A}_Z BW + \mathcal{H}\nabla_Z CW = \omega \mathcal{A}_Z W + C\mathcal{H}\nabla_Z W$$

for $Z, W \in \Gamma((\ker F_*)^\perp)$.

(3)

$$\widehat{\nabla}_X BZ + \mathcal{T}_X CZ = \phi \mathcal{T}_X Z + B\mathcal{H}\nabla_X Z$$
$$\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ = \omega \mathcal{T}_X Z + C\mathcal{H}\nabla_X Z$$

for $X \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$.

Theorem 3.4. *Let F be a v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the slant distribution $\mathcal{D}_2$ is integrable if and only if we obtain*

$$\mathcal{A}_X Y = 0 \quad \textit{and} \quad PC(\nabla_X Y - \nabla_Y X) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$

Proof. For $X, Y \in \Gamma(\mathcal{D}_2)$ and $Z \in \Gamma(\mathcal{D}_1)$, assume that $\mathcal{A}_X Y = \frac{1}{2}\mathcal{V}[X, Y] = 0$, we obtain

$$\begin{aligned}
g_M([X, Y], JZ) &= -g_M(J(\nabla_X Y - \nabla_Y X), Z) \\
&= -g_M(B\nabla_X Y + C\nabla_X Y - B\nabla_Y X - C\nabla_Y X, Z) \\
&= -g_M(C(\nabla_X Y - \nabla_Y X), Z).
\end{aligned}$$

Therefore, we have the result. □

Similarly, we get

Theorem 3.5. *Let F be a v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the complex distribution $\mathcal{D}_1$ is integrable if and only if we have*

$$\mathcal{A}_X Y = 0 \quad \textit{and} \quad B(\nabla_X Y - \nabla_Y X) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_1)$.

Lemma 3.6. *Let (M, g_M, J) be a Kähler manifold and (N, g_N) a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a v-semi-slant submersion. Then the complex distribution $\mathcal{D}_1$ is integrable if and only if we get*

$$\mathcal{A}_X Y = 0 \quad \textit{for } X, Y \in \Gamma(\mathcal{D}_1).$$

Proof. For $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\ker F_*)$, assume that $\mathcal{A}_X Y = 0$, we have

$$\begin{aligned}
g_M([X, Y], \omega Z) = g_M([X, Y], JZ) &= -g_M(J(\nabla_X Y - \nabla_Y X), Z) \\
&= -g_M(\mathcal{A}_X JY + \mathcal{H}\nabla_X JY - \mathcal{A}_Y JX - \mathcal{H}\nabla_Y JX, Z) \\
&= -g_M(\mathcal{A}_X JY - \mathcal{A}_Y JX, Z).
\end{aligned}$$

Therefore, the result follows. □

In a similar way, we have

Lemma 3.7. *Let (M, g_M, J) be a Kähler manifold and (N, g_N) a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a v-semi-slant submersion. Then the slant distribution $\mathcal{D}_2$ is integrable if and only if we obtain*

$$\mathcal{A}_X Y = 0 \quad \textit{and} \quad P((\mathcal{A}_X BY - \mathcal{A}_Y BX) + \mathcal{H}(\nabla_X CY - \nabla_Y CX)) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

Proposition 3.8. *Let F be a v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then we obtain*

$$C^2 X = -\cos^2\theta X \quad \textit{for } X \in \Gamma(\mathcal{D}_2),$$

where θ denotes the v-semi-slant angle of $\mathcal{D}_2$.

Proof. Since

$$\cos\theta = \frac{g_M(JX, CX)}{|JX|\cdot|CX|} = \frac{-g_M(X, C^2X)}{|X|\cdot|CX|}$$

and $\cos\theta = \dfrac{|CX|}{|JX|}$, we have

$$\cos^2\theta = -\frac{g_M(X, C^2X)}{|X|^2} \quad \text{for } X \in \Gamma(\mathcal{D}_2).$$

Hence,

$$C^2X = -\cos^2\theta X \quad \text{for } X \in \Gamma(\mathcal{D}_2).$$

□

Remark 3.9. It is easy to see that the converse of Proposition 3.8 is also true.

Assume that the v-semi-slant angle θ is not equal to $\frac{\pi}{2}$ and define an endomorphism $\widehat{J}$ of $(\ker F_*)^\perp$ by

$$\widehat{J} := JP + \frac{1}{\cos\theta}CQ.$$

Then,

$$\widehat{J}^2 = -id \quad \text{on } (\ker F_*)^\perp. \tag{1}$$

From (1), we have

Theorem 3.10. *Let F be a v-semi-slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) with the v-semi-slant angle $\theta \in [0, \frac{\pi}{2})$. Then N is an even-dimensional manifold.*

Proposition 3.11. *Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distribution $\ker F_*$ defines a totally geodesic foliation if and only if*

$$\omega(\widehat{\nabla}_X\phi Y + \mathcal{T}_X\omega Y) + C(\mathcal{T}_X\phi Y + \mathcal{H}\nabla_X\omega Y) = 0 \quad \textit{for } X, Y \in \Gamma(\ker F_*).$$

Proof. For $X, Y \in \Gamma(\ker F_*)$,

$$\begin{aligned}\nabla_X Y &= -J\nabla_X JY = -J(\widehat{\nabla}_X\phi Y + \mathcal{T}_X\phi Y + \mathcal{T}_X\omega Y + \mathcal{H}\nabla_X\omega Y)\\ &= -(\phi\widehat{\nabla}_X\phi Y + \omega\widehat{\nabla}_X\phi Y + B\mathcal{T}_X\phi Y + C\mathcal{T}_X\phi Y + \phi\mathcal{T}_X\omega Y + \omega\mathcal{T}_X\omega Y\\ &\quad + B\mathcal{H}\nabla_X\omega Y + C\mathcal{H}\nabla_X\omega Y).\end{aligned}$$

Thus,

$$\nabla_X Y \in \Gamma(\ker F_*) \Leftrightarrow \omega(\widehat{\nabla}_X\phi Y + \mathcal{T}_X\omega Y) + C(\mathcal{T}_X\phi Y + \mathcal{H}\nabla_X\omega Y) = 0.$$

□

Similarly, we have

Proposition 3.12. *Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation if and only if*

$$\phi(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY) + B(\mathcal{A}_X BY + \mathcal{H}\nabla_X CY) = 0 \quad \textit{for } X, Y \in \Gamma((\ker F_*)^\perp).$$

Proposition 3.13. *Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distribution $\mathcal{D}_1$ defines a totally geodesic foliation if and only if*

$$\phi\mathcal{A}_X JY + B\mathcal{H}\nabla_X JY = 0 \text{ and } Q(\omega\mathcal{A}_X JY + C\mathcal{H}\nabla_X JY) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_1)$.

Proof. For $X, Y \in \Gamma(\mathcal{D}_1)$, we get

$$\begin{aligned}\nabla_X Y &= -J\nabla_X JY = -J(\mathcal{A}_X JY + \mathcal{H}\nabla_X JY) \\ &= -(\phi\mathcal{A}_X JY + \omega\mathcal{A}_X JY + B\mathcal{H}\nabla_X JY + C\mathcal{H}\nabla_X JY).\end{aligned}$$

Hence,

$$\nabla_X Y \in \Gamma(\mathcal{D}_1) \Leftrightarrow \phi\mathcal{A}_X JY + B\mathcal{H}\nabla_X JY = 0 \text{ and } Q(\omega\mathcal{A}_X JY + C\mathcal{H}\nabla_X JY) = 0.$$

□

In a similar way, we obtain

Proposition 3.14. *Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distribution $\mathcal{D}_2$ defines a totally geodesic foliation if and only if*

$$\begin{aligned}&\phi(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY) + B(\mathcal{A}_X BY + \mathcal{H}\nabla_X CY) = 0 \\ &P(\omega(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY) + C(\mathcal{A}_X BY + \mathcal{H}\nabla_X CY)) = 0\end{aligned}$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

Theorem 3.15. *Let F be a v-semi-slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then F is a totally geodesic map if and only if*

$$\begin{aligned}&\omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) = 0 \\ &\omega(\widehat{\nabla}_X BZ + \mathcal{T}_X CZ) + C(\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ) = 0\end{aligned}$$

for $X, Y \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$.

Proof. Since F is a Riemannian submersion, we obtain

$$(\nabla F_*)(Z_1, Z_2) = 0 \quad \text{for } Z_1, Z_2 \in \Gamma((\ker F_*)^\perp).$$

For $X, Y \in \Gamma(\ker F_*)$, we have

$$\begin{aligned}(\nabla F_*)(X, Y) &= -F_*(\nabla_X Y) = F_*(J\nabla_X(\phi Y + \omega Y)) \\ &= F_*(\phi\widehat{\nabla}_X \phi Y + \omega\widehat{\nabla}_X \phi Y + B\mathcal{T}_X \phi Y + C\mathcal{T}_X \phi Y + \phi\mathcal{T}_X \omega Y + \omega\mathcal{T}_X \omega Y \\ &\quad + B\mathcal{H}\nabla_X \omega Y + C\mathcal{H}\nabla_X \omega Y).\end{aligned}$$

Thus,

$$(\nabla F_*)(X, Y) = 0 \Leftrightarrow \omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) = 0.$$

For $X \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$, we get

$$\begin{aligned}(\nabla F_*)(X, Z) &= -F_*(\nabla_X Z) = F_*(J\nabla_X(BZ + CZ)) \\ &= F_*(\phi\widehat{\nabla}_X BZ + \omega\widehat{\nabla}_X BZ + B\mathcal{T}_X BZ + C\mathcal{T}_X BZ + \phi\mathcal{T}_X CZ + \omega\mathcal{T}_X CZ \\ &\quad + B\mathcal{H}\nabla_X CZ + C\mathcal{H}\nabla_X CZ).\end{aligned}$$

Hence,

$$(\nabla F_*)(X,Z)=0 \Leftrightarrow \omega(\widehat{\nabla}_X BZ+\mathcal{T}_X CZ)+C(\mathcal{T}_X BZ+\mathcal{H}\nabla_X CZ)=0.$$

Since $(\nabla F_*)(X,Z)=(\nabla F_*)(Z,X)$, the result follows. □

Let $F:(M,g_M)\mapsto(N,g_N)$ be a Riemannian submersion. The map F is called a Riemannian submersion *with totally umbilical fibers* if

$$\mathcal{T}_X Y=g_M(X,Y)H \quad \text{for } X,Y\in\Gamma(\ker F_*), \tag{2}$$

where H is the mean curvature vector field of the fiber.

Then we obtain

Lemma 3.16. *Let F be a v-semi-slant submersion with totally umbilical fibers from a Kähler manifold (M,g_M,J) onto a Riemannian manifold (N,g_N). Then we have*

$$H\in\Gamma(\mathcal{D}_2).$$

Proof. For $X,Y\in\Gamma(\mu)$ and $W\in\Gamma(\mathcal{D}_1)$, we get

$$\mathcal{T}_X JY+\widehat{\nabla}_X JY=\nabla_X JY=J\nabla_X Y=B\mathcal{T}_X Y+C\mathcal{T}_X Y+\phi\widehat{\nabla}_X Y+\omega\widehat{\nabla}_X Y.$$

Using (2), we easily obtain

$$g_M(X,JY)g_M(H,W)=-g_M(X,Y)g_M(H,JW).$$

Interchanging the role of X and Y, we get

$$g_M(Y,JX)g_M(H,W)=-g_M(Y,X)g_M(H,JW)$$

so that combining the above two equations, we have

$$g_M(X,Y)g_M(H,JW)=0,$$

which means $H\in\Gamma(\mathcal{D}_2)$. □

Corollary 3.17. *Let F be a v-semi-slant submersion with totally umbilical fibers from a Kähler manifold (M,g_M,J) onto a Riemannian manifold (N,g_N) such that $\mathcal{D}_1=(\ker F_*)^\perp$. Then each fiber is minimal.*

Remark 3.18. Let F be a v-semi-slant submersion from a Kähler manifold (M,g_M,J) onto a Riemannian manifold (N,g_N). Then there is a distribution $\mathcal{D}_1\subset(\ker F_*)^\perp$ such that

$$(\ker F_*)^\perp=\mathcal{D}_1\oplus\mathcal{D}_2,\ J(\mathcal{D}_1)=\mathcal{D}_1,$$

and the angle $\theta=\theta(X)$ between JX and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X\in(\mathcal{D}_2)_p$ and $p\in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $(\ker F_*)^\perp$. Furthermore,

$$C\mathcal{D}_2\subset\mathcal{D}_2,\quad B\mathcal{D}_2\subset\ker F_*,\quad \ker F_*=B\mathcal{D}_2\oplus\mu,$$

where μ is the orthogonal complement of $B\mathcal{D}_2$ in $\ker F_*$ and is invariant under J. For the curvature tensor, it is sufficient to calculate the holomorphic sectional curvatures in a Kähler manifold.

Given a plane P being invariant by J in T_pM, $p\in M$, there is an orthonormal basis $\{X,JX\}$ of P. Denote by $K(P)$, $K_*(P)$, and $\widehat{K}(P)$ the sectional curvatures of the plane P in M, N, and the fiber $F^{-1}(F(p))$, respectively, where $K_*(P)$ denotes the sectional curvature of the plane $P_*=<F_*X,F_*JX>$ in N. Let $K(X\wedge Y)$ be the sectional curvature of the plane spanned by the tangent vectors $X,Y\in T_pM$, $p\in M$. Using both Corollary 1 of [14, p.465] and (1.28) of [7, p.13], we obtain the following :

(1) If $P \subset (\mu)_p$, then with some computations we have

$$K(P) = \widehat{K}(P) + |\mathcal{T}_X X|^2 - |\mathcal{T}_X JX|^2 - g_M(\mathcal{T}_X X, J[JX, X]).$$

(2) If $P \subset (\mathcal{D}_2 \oplus B\mathcal{D}_2)_p$ with $X \in (\mathcal{D}_2)_p$, then we get

$$\begin{aligned} K(P) &= \sin^2\theta \cdot K(X \wedge BX) + 2(g_M((\nabla_X \mathcal{A})(X, CX), BX) + g_M(\mathcal{A}_X CX, \mathcal{T}_{BX} X) \\ &\quad - g_M(\mathcal{A}_{CX} X, \mathcal{T}_{BX} X) - g_M(\mathcal{A}_X X, \mathcal{T}_{BX} CX)) + \cos^2\theta \cdot K(X \wedge CX). \end{aligned}$$

(3) If $P \subset (\mathcal{D}_1)_p$, then we obtain

$$K(P) = K_*(P) - 3|\mathcal{V} J \nabla_X X|^2.$$

4. Examples

Example 4.1. Let (M, g_M, J) be an almost Hermitian manifold. Let $\pi : TM \mapsto M$ be the natural projection. Then the map π is a v-semi-slant submersion such that $\mathcal{D}_1 = (\ker \pi_*)^\perp$ [7].

Example 4.2. Let (M, g_M, J) be a $2m$-dimensional almost Hermitian manifold and (N, g_N) a $(2m-1)$-dimensional Riemannian manifold. Let F be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 = ((\ker F_*) \oplus J(\ker F_*))^\perp \quad \text{and} \quad \mathcal{D}_2 = J(\ker F_*)$$

with the v-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 4.3. Define a map $F : \mathbb{R}^6 \mapsto \mathbb{R}^4$ by

$$F(x_1, x_2, \cdots, x_6) = (x_1, x_3 \sin\alpha - x_5 \cos\alpha, x_6, x_2),$$

where $\alpha \in (0, \frac{\pi}{2})$. Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 = < \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} > \text{ and } \mathcal{D}_2 = < \frac{\partial}{\partial x_6}, \sin\alpha \frac{\partial}{\partial x_3} - \cos\alpha \frac{\partial}{\partial x_5} >$$

with the v-semi-slant angle $\theta = \alpha$.

Example 4.4. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$ by

$$F(x_1, x_2, \cdots, x_8) = (x_4, x_3, \frac{x_5 - x_8}{\sqrt{2}}, x_6).$$

Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 = < \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} > \text{ and } \mathcal{D}_2 = < \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_8} >$$

with the v-semi-slant angle $\theta = \frac{\pi}{4}$.

Example 4.5. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^5$ by

$$F(x_1, x_2, \cdots, x_{12}) = (x_2, \frac{x_5 + x_6}{\sqrt{2}}, \frac{x_7 + x_9}{\sqrt{2}}, \frac{x_8 + x_{10}}{\sqrt{2}}, x_1).$$

Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 = < \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}} > \text{ and } \mathcal{D}_2 = < \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} >$$

with the v-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 4.6. Define a map $F : \mathbb{R}^{10} \mapsto \mathbb{R}^6$ by

$$F(x_1, x_2, \cdots, x_{10}) = (\frac{x_3 - x_5}{\sqrt{2}}, x_6, \frac{x_7 + x_9}{\sqrt{2}}, x_8, x_1, x_2).$$

Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} > \text{ and } \mathcal{D}_2 =< \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9} >$$

with the v-semi-slant angle $\theta = \frac{\pi}{4}$.

Example 4.7. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$ by

$$F(x_1, x_2, \cdots, x_8) = (x_1, x_3 \cos\alpha - x_5 \sin\alpha, x_2, x_4 \sin\beta + x_6 \cos\beta),$$

where α and β are constant. Then the map F is a v-semi-slant submersion such that

$$\mathcal{D}_1 =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} > \text{ and } \mathcal{D}_2 =< \cos\alpha \frac{\partial}{\partial x_3} - \sin\alpha \frac{\partial}{\partial x_5}, \sin\beta \frac{\partial}{\partial x_4} + \cos\beta \frac{\partial}{\partial x_6} >$$

with the v-semi-slant angle θ with $\cos\theta = |\sin(\alpha - \beta)|$.

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