Exact spectrum of the Laplacian on a domain in the Sierpinski gasket

HUA QIU

Abstract. For a certain domain Ω in the Sierpinski gasket $\mathcal{SG}$ whose boundary is a line segment, a complete description of the eigenvalues of the Laplacian under the Dirichlet and Neumann boundary conditions is presented. The method developed in this paper is a weak version of the spectral decimation method due to Fukushima and Shima, since for a lot of "bad" eigenvalues the spectral decimation method can not be used directly. We also prove an analogue of Weyl's classical result on the eigenvalue asymptotics of the eigenvalue counting function $\rho^{\Omega}(x)$. The ratio $\rho^{\Omega}(x)/x^{\log 3/\log 5}$ is bounded but non-convergent as $x \to \infty$. Moreover, we explain that the asymptotic expansion of $\rho^{\Omega}(x)$ admits a second term of the order $\log 2/\log 5$, that becomes apparent from the experimental data. This is very analogous to the conjectures of Weyl and Berry.

Keywords. Sierpinski gasket, Laplacian, eigenvalues, spectral decimation, analysis on fractals.

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1 Introduction

The study of the Laplacian on fractals was originated by S. Kusuoka [21] and S. Goldstein [11]. They independently constructed the Laplacian as the generator of a diffusion process on the Sierpinski gasket $\mathcal{SG}$. Later an analytic approach was developed by J. Kigami, who constructed the Laplacian both as a renormalized limit of difference operators and a weak formulation using the theory of Dirichlet forms [15].

We are particularly interested in the eigenvalues of this Laplacian. In the case of the Sierpinski gasket, Physicists R. Rammal and G. Toulouse [27] found that an appropriate choice of a series of eigenvalues of successive difference operators produces an orbit of the dynamical system related to a quadratic polynomial. This is the phenomenon which M. Fukushima and T. Shima [10, 30] described from the mathematical point of view, by saying that $\mathcal{SG}$ admits *spectral decimation* with respect to a quadratic polynomial. Furthermore, they found all the eigenvalues of the Laplacian on the Sierpinski gasket by tracking back the orbits. Later the theory of the Laplacian was developed for nested

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fractals and p.c.f. self-similar sets by T. Lindstrøm [23] and J. Kigami [16] by introducing the notion of *harmonic structure*. Every p.c.f. self-similar set is approximated by an increasing sequence of finite graphs and the harmonic structure determines a sequence of difference operators on the successive graphs, which converges to the Laplacian. Then some generalizations of the spectral decimation to a class of p.c.f. self-similar sets were developed by T. Shima [31], L. Malozemov and A. Teplyaev [24], in which some strong symmetry conditions are supposed to be satisfied to ensure the spectral decimation applies to the corresponding graph sequences. Under such strong symmetry conditions, the spectrum of the Laplacian can also be described in terms of the iteration of a rational function. Recently, the spectra of the Laplacian operators on some other fractals have been analyzed either numerically [1] or using the spectral decimation method [7, 8, 37, 39] by R. S. Strichartz (with co-authors), D. Zhou and A. Teplyaev. In all the references mentioned above, spectral decimation plays a key role in the theoretical study of the spectra of the Laplacian operators.

In this paper, we are mainly concerned with eigenvalue problems for a domain in the Sierpinski gasket. Since analysis on fractals has been made possible by the definition of Laplacian, it is natural to explore the properties of these fractal Laplacians that are natural analogs of results that are known for the usual Laplacian. However, since not much is known about the fractal Laplacian, we can only scratch the surface in attempting the generalization to fractal Laplacian.

For simplicity, here we specifically focus on the Sierpinski gasket $\mathcal{SG}$. Recall that $\mathcal{SG}$ is the attractor of the *iterated function system* $\{F_0, F_1, F_2\}$ where $F_i x = \frac{1}{2}(x + q_i)$ where q_0, q_1, q_2 are the vertices of an equilateral triangle in the plane,

$$\mathcal{SG} = \bigcup_{i=0}^{2} F_i(\mathcal{SG}).$$

Let Δ denote the Laplacian on $\mathcal{SG}$ defined by Kigami. In his theory the boundary of $\mathcal{SG}$ consists of the three points q_0, q_1, q_2 and the space of *harmonic functions* (solutions of $\Delta u = 0$) is three dimensional, with u determined explicitly by its boundary values $u(q_i)$. (Note that this boundary is not a topological boundary.) Thus this theory is more closely related to the theory of linear functions on the unit interval than to harmonic functions on the disk. To get a richer theory we should take an open set Ω in $\mathcal{SG}$ and restrict the Laplacian on $\mathcal{SG}$ to functions defined on Ω. Thus harmonic functions on Ω are the solutions of $\Delta u = 0$.

Here we particularly focus on the certain domain Ω_x which is a triangle obtained by cutting $\mathcal{SG}$ with a horizontal line at any vertical height x ($0 < x \leq 1$ if we suppose that the height of $\mathcal{SG}$ is equal to 1.) below the top vertex q_0. See Fig. 1.1. An important motivation for studying this kind of domains is that they are the simplest examples which

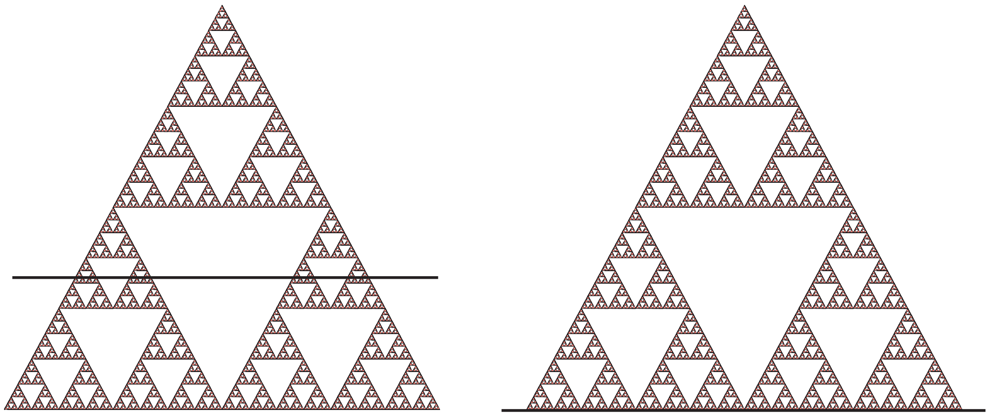

Fig. 1.1. Ω_x and Ω_1.

could serve as a testing ground for questions and conjectures on analysis on more general fractal domains with fractal boundaries. These domains were first introduced by R. S. Strichartz in [32] and later studied by J. Owen and R. S. Strichartz in [25], where they gave an explicit analog of the *Poisson integral formula* to recover a harmonic function u on Ω_x from its boundary values. It is also natural to calculate an explicit *Green's function* for the Laplacian on Ω_x. This was studied by Z. Guo, R. Kogan and R. S. Strichartz in [12] which is completely similar to the construction of the Green's function on $\mathcal{SG}$ given by Kigami in [15, 16]. For some other analytic topics related to this kind of domains, see [13, 14, 19, 20].

In the present paper, we study the spectral properties of the Laplacian on Ω_x which is an open problem posed in [25]. For the simplicity of description, we mainly concentrate our attention to a particular domain Ω_1 (We drop the subscript 1 on Ω in all that follows without causing any confusion.) which is the complement of $q_0 \cup L$, where L is the line segment joining q_1 and q_2 (in this case $\partial\Omega = q_0 \cup L$). We give a complete description of the Dirichlet and Neumann spectra of the Laplaician on Ω.

Unfortunately in our context, for a number of "bad" eigenvalues (whose associated eigenfunctions have supports touching the bottom boundary line L) spectral decimation can not be used directly, which makes things more complicated. However, by choosing a sequence of appropriate graph approximations, we describe a phenomenon on those eigenvalues called *weak spectral decimation* which approximates to spectral decimation when the levels of the successive graphs go to infinity. And we use this weak spectral decimation to replace the role of spectral decimation in the original Fukushima and Shima's work [10]. Actually, similar to the standard case, weak spectral decimation can also produce a "weak" orbit related to the same quadratic polynomial by an appropriate series of eigenvalues of successive difference operators on graph approximations. We can then trace back those "weak" orbits to capture all the "bad" eigenvalues. More precisely, we classify the eigenvalues of Δ on Ω into three types, which we call the *localized eigenvalues*, *primitive eigenvalues* and *miniaturized eigenvalues*. The localized eigenfunctions associated to localized eigenvalues on Ω are just a subspace of the localized eigenfunctions on $\mathcal{SG}$ whose supports are disjoint from L. This kind of eigenvalues can be dealt with in a completely similar way to the $\mathcal{SG}$ case, for which the spectral decimation can apply. The primitive and miniaturized eigenvalues are the so-called "bad" eigenvalues. They are the eigenvalues need to be paid particular attention to.

Recall that in [10], The Weyl asymptotic behavior of the eigenvalue counting function for the $\mathcal{SG}$ case has been studied by Fukushima and Shima. Denote by $\rho(x)$ the number of eigenvalues of Δ(taking the multiplicities into account) not exceeding x. According to their result, there exist positive constant c, C such that $cx^{d_S/2} \leq \rho(x) \leq Cx^{d_S/2}$, for all x large enough, where $d_S = \log 9/\log 5$ is the spectral dimension of $\mathcal{SG}$. In particular,

$$0 < \liminf_{x\to\infty} \rho(x)x^{-d_S/2} < \limsup_{x\to\infty} \rho(x)x^{-d_S/2} < \infty. \tag{1.1}$$

Now what happens to the asymptotic behavior of the eigenvalue counting function on Ω? A natural analogue of (1.1) holds which can be easily proved by first considering the asymptotic behavior of the eigenvalue counting function for each type of eigenvalues separately, then adding up them together.

This is not the entire story. Recall that in the classical case. Suppose D is an arbitrary nonempty bounded open set in $\mathbf{R}^n$ with boundary ∂D, then Weyl's classical asymptotic formula can be extended as follows:

$$\rho(x) = (2\pi)^{-n} c_n |D|_n x^{n/2} + O(x^{(n-1)/2})$$

as $x \to \infty$, where c_n depends only on n. See details in [26, 28, 29]. The above remainder estimate constitutes an important step on the way to H. Weyl's conjecture [38] which states that if ∂D is sufficiently "smooth", then the asymptotic expansion of $\rho(x)$ admits a second term, proportional to $x^{(n-1)/2}$. Extending Weyl's conjecture to the fractal case, M. V. Berry [3, 4] conjectured that if D has a fractal boundary ∂D with Hausdorff dimension (which later was revised into Minkowski dimension in [6, 22]) $d_{\partial D} \in (n-1, n]$, then the order of the second term should be replaced by $d_{\partial D}/2$. See further discussion and a partial resolution of the conjectures of Weyl and Berry in M. L. Lapidus's work [22]. Hence it is natural to ask that is there an analogue result in $\mathcal{SG}$ or Ω setting. For $\mathcal{SG}$ case, using a refinement of the Renewal Theorem, Kigami [18] showed that the remainder is bounded,

$$\rho(x) = g(\log x)x^{d_S/2} + O(1),$$

where g is a periodic function of period $\log 5$. Note that this is consistent with the fact that the boundary of $\mathcal{SG}$ consists of three points, hence has dimension zero. This was refined by Strichartz in [36], where an exact formula was presented with no remainder term at all, provided we restrict attention to almost every x. As for Ω case, We will show that although we are unable to prove, it becomes apparent there is a second term of order $\log 2/\log 5$ in the expansion of the eigenvalue counting function on Ω from observing the experimental data. We note that our work deals with the case when the domain itself is fractal (and hence not open) also. The order of the second term should also has a close connection with the dimension of the boundary $\partial\Omega$ duo to Weyl-Berry conjectures. Moreover, when consider a more general domain Ω_x, we will meet "drums with fractal membrane" with also fractal boundary.

The paper is organized as follows. In Section 2 we briefly introduce some key notions from analysis on fractals and give a precise description of the Dirichlet and Neumann spectra of the Laplacian on $\mathcal{SG}$, which will be used in the rest of the paper.

In Section 3, we first present the structure of the complete Dirichlet spectrum on Ω before going into the technical details. We find an appropriate sequence of graph approximations for the fractal domain Ω, and describe the structures of the corresponding discrete spectra of the successive difference operators on them. Accordingly, for each graph all the graph eigenvalues are also divided into three types, localized, primitive and miniaturized. By using an eigenspace dimensional counting argument, we show that they should make up the whole discrete spectrum. We also briefly describe how to relate the spectra of successive levels and how to pass the graph approximations to the limit by using spectral decimation for localized eigenvalues and weak spectral decimation for other types of eigenvalues. Then we list some conjectures concerning eigenvalue asymptotics (especially the existence of the second term of the expansion of the eigenvalue counting function), gaps in the ratios of eigenvalues and eigenvalue clusters which become apparent from observing the experimental data. At the end of this section, we present the structure of the Neumann spectrum on Ω.

In Section 4, we begin discussion of the discrete Dirichlet primitive eigenvalues. We will divide our discussion into symmetric case and skew-symmetric case. In each case, we will prove that for each level the primitive graph eigenvalues are the total roots of a high degree polynomial. And we describe the weak spectral decimation phenomenon by studying the relation between roots of consecutive polynomials. Moreover, we prove that the complete discrete spectrum is made up of the three types of eigenvalues as expected.

In Section 5, we discuss the Dirichlet primitive eigenvalues on Ω by passing the results of Section 4 on graph approximations to the limit. Since we can only use weak spectral decimation this time, some trivial results in $\mathcal{SG}$ case become nontrivial and need to be proved in this section.

In Section 6, we first prove that the whole Dirichlet spectrum on Ω is made up of the three types of eigenvalues following the basic idea of Fukushima and Shima's work. Then prove an analogue of Weyl's classical result on the eigenvalue asymptotics. The eigenvalue counting function $\rho^{\Omega}(x)$ is shown to be of order $x^{d_S/2}$ as $x \to \infty$ where $d_S = \log 9/\log 5$ is the spectral dimension of $\mathcal{SG}$. Moreover, we also prove that the limit $\rho^{\Omega}(x)/x^{d_S/2}$ is not convergent.

In Section 7, we give a brief discussion on how to deal with the Neumann spectrum. We will find a similar weak spectral decimation for primitive eigenvalues by establishing a relation between primitive symmetric (or skew-symmetric) graph eigenvalues with some high degree polynomials, but the proof is quite different from that in the Dirichlet case.

We will also give a brief discussion on how to extend our method from Ω to Ω_x with $0 < x < 1$ in Section 8. The purpose of this paper is to work out the details for one specific example. We hope this example will provide insights which will inspire future work on a more general theory.

2 Spectral decimation on $\mathcal{SG}$

First we collect some key facts from analysis on $\mathcal{SG}$ that we need to state and prove our results. These come from Kigami's theory of analysis on fractals, and can be found in [15, 16]. An elementary exposition can be found in [33, 35]. The fractal $\mathcal{SG}$ will be realized as the limit of a sequence of graphs $\Gamma_0, \Gamma_1, \cdots$ with vertices $V_0 \subseteq V_1 \subseteq \cdots$. The initial graph Γ_0 is just the complete graph on $V_0 = \{q_0, q_1, q_2\}$, the vertices of an equilateral triangle in the plane, which is considered the boundary of $\mathcal{SG}$. See Fig. 2.1. The entire fractal is the only 0-cell, which has V_0 as its boundary. At stage m of the construction, all the cells of level $m-1$ lie in triangles whose vertices make up V_{m-1}. Each cell of level $m-1$ splits into three cells of level m, adding three new vertices to V_m.

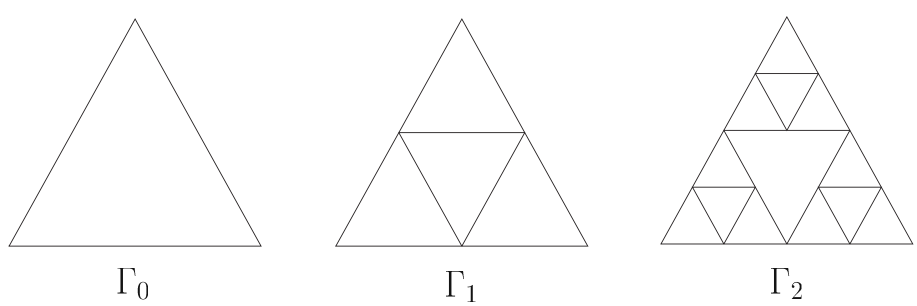

Fig. 2.1. The first 3 graphs, $\Gamma_0, \Gamma_1, \Gamma_2$ in the approximation to the Sierpinski gasket.

We define the unrenormalized energy of a function u on Γ_m by

$$E_m(u) = \sum_{x \sim_m y} (u(x) - u(y))^2.$$

The energy renormalization factor is $r = \frac{3}{5}$, so the renormalized graph energy on Γ_m is

$$\mathcal{E}_m(u) = r^{-m} E_m(u),$$

and we can define the *fractal energy* $\mathcal{E}(u) = \lim_{m \to \infty} \mathcal{E}_m(u)$. We define $dom\mathcal{E}$ as the space of continuous functions with finite energy. Then $\mathcal{E}$ extends by polarization to a bilinear form $\mathcal{E}(u, v)$ which serves as an inner product in this space. The energy $\mathcal{E}$ gives rise to a natural distance on $\mathcal{SG}$ called the *effective resistance metric* on $\mathcal{SG}$, which is defined by

$$d(x, y) = (\min\{\mathcal{E}(u) : u(x) = 0 \text{ and } u(y) = 1\})^{-1} \tag{2.1}$$

for $x, y \in \mathcal{SG}$. It is known that $d(x, y)$ is bounded above and below by constant multiples of $|x - y|^{\log(5/3)/\log 2}$, where $|x - y|$ is the Euclidean distance. Furthermore, the definition (2.1) implies that functions on $dom\mathcal{E}$ are Holder continuous of order $\frac{1}{2}$ in the effective resistance metric.

We let μ denote the standard probability measure on $\mathcal{SG}$ that assigns the measure 3^{-m} to each cell of m level. The standard Laplacian may then be defined using the weak formulation: $u \in dom\Delta$ with $\Delta u = f$ if f is continuous, $u \in dom\mathcal{E}$, and

$$\mathcal{E}(u, v) = -\int f v d\mu \tag{2.2}$$

for all $v \in dom_0\mathcal{E}$, where $dom_0\mathcal{E} = \{v \in \mathcal{E} : v|_{V_0} = 0\}$. There is also a pointwise formula (which is proven to be equivalent in [35]) which, for nonboundary points in $V_* = \bigcup_m V_m$ (not in V_0) computes

$$\Delta u(x) = \frac{3}{2} \lim_{m \to \infty} 5^m \Delta_m u(x),$$

where Δ_m is a discrete Laplacian associated to the graph Γ_m, defined by

$$\Delta_m u(x) = \sum_{y \sim_m x} (u(y) - u(x))$$

for x not on the boundary.

The Laplacian satisfies the scaling property

$$\Delta(u \circ F_i) = \frac{1}{5} (\Delta u) \circ F_i$$

and by iteration

$$\Delta(u \circ F_w) = \frac{1}{5^m} (\Delta u) \circ F_w$$

for $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$.

Although there is no satisfactory analogue of gradient, there is *normal derivative* $\partial_n u(q_i)$ defined at boundary points by

$$\partial_n u(q_i) = \lim_{m\to\infty} \sum_{y\sim_m q_i} r^{-m}(u(q_i) - u(y)),$$

the limit existing for all $u \in dom\Delta$. The definition may be localized to boundary points of cells. For each point $x \in V_m \setminus V_0$, there are two cells containing x as a boundary point, hence two normal derivatives at x. For $u \in dom\Delta$, the normal derivatives at x satisfy the *matching condition* that their sum is zero. The matching conditions allow us to glue together local solutions to $\Delta u = f$.

The above matching condition property follows easily from a local version of the following *Gauss-Green formula*, which is an extension of (2.2) to the case when v doesn't vanish on the boundary:

$$\mathcal{E}(u,v) = -\int (\Delta u)v d\mu + \sum_{V_0} v\partial_n u.$$

The local version of the Gauss-Green formula is

$$\mathcal{E}_A(u,v) = -\int_A (\Delta u)v d\mu + \sum_{\partial A} v\partial_n u$$

where A is any finite union of cells and $\mathcal{E}_A(u,v)$ is the restriction of the energy bilinear form $\mathcal{E}(u,v)$ to A, which can also be defined directly by

$$\mathcal{E}_A(u,v) = \lim_{m\to\infty} \sum_{\substack{x\sim_m y \\ inA}} (u(x) - u(y))(v(x) - v(y)).$$

Now we come to a brief recap of the spectral decimation on $\mathcal{SG}$. Our goal is to find all solutions of the eigenvalue equation

$$-\Delta u = \lambda u \quad \text{on } \mathcal{SG}$$

as limits of solutions of the discrete version

$$-\Delta_m u_m = \lambda_m u_m \quad \text{on } V_m \setminus V_0.$$

In the $\mathcal{SG}$ case, we are lucky that we may take $u_m = u|_{V_m}$, which is necessarily convenient for the spectral decimation. We should emphasize that this is not true for Ω case.

The method of spectral decimation on $\mathcal{SG}$ was invented by Fukushima and Shima [10] to relate eigenfunctions and eigenvalues of the discrete Laplacian Δ_m's on the graph approximation Γ_m's for different values of m to each other and the eigenfunctions and eigenvalues of the fractal Laplacian Δ on $\mathcal{SG}$. In essence, an eigenfunction on Γ_m with eigenvalue λ_m can be extended to an eigenfunction on Γ_{m+1} with eigenvalue λ_{m+1}, where $\lambda_m = f(\lambda_{m+1})$ for an explicit function f defined by $f(x) = x(5-x)$, except for certain specified *forbidden eigenvalues*, and all eigenfunctions on $\mathcal{SG}$ arise as limits of this process starting at some level m_0 which is called the *generation of birth*. This is true regardless of the boundary conditions, but if we specify Dirichlet or Neumann boundary condition we can describe explicitly all eigenspaces and their multiplicities.

Denote the real valued inverse functions of $f(x)$ by $\phi_\pm(x)$. That is

$$\phi_\pm(x) = \frac{5 \pm \sqrt{25-4x}}{2}. \tag{2.3}$$

We describe the procedure briefly here. First, there is a *local extension algorithm* that shows how to uniquely extend an eigenfunction u_m defined on V_m to a function defined on V_{m+1} such that the λ-eigenvalue equations hold on all points of $V_{m+1} \setminus V_m$. For $\mathcal{SG}$, the extension algorithm is: Suppose u_m is an eigenfunction on Γ_m with eigenvalue λ_m. Let $\lambda_{m+1} = \phi_\pm(\lambda_m)$. Consider an m-cell with boundary points x_0, x_1, x_2 and let y_0, y_1, y_2 denote the points in $V_{m+1} \setminus V_m$ in that cell, with y_i opposite x_i. Extend u_m to a function u_{m+1} on V_{m+1} by defining (for simplicity of notation, we drop the subscripts on u)

$$u(y_i) = \frac{(4-\lambda_{m+1})((u(x_{i+1}) + u(x_{i-1}))) + 2u(x_i)}{(2-\lambda_{m+1})(5-\lambda_{m+1})}, \quad i = 0, 1, 2. \tag{2.4}$$

Then we have the following proposition taken from [35].

Proposition 2.1. Suppose $\lambda_{m+1} \neq 2, 5$ or 6, and $\lambda_m = f(\lambda_{m+1})$. *If u_m is a λ_m-eigenfunction of Δ_m and is extended to a function u_{m+1} on V_{m+1} by* (2.4)*, then u_{m+1} is a λ_{m+1}-eigenfunction of Δ_{m+1}, Conversely, if u_{m+1} is a λ_{m+1}-eigenfunction of Δ_{m+1} and is restricted to a function u_m on V_m, then u_m is a λ_m-eigenfunction of Δ_m.*

The forbidden eigenvalues $\{2, 5, 6\}$ are singularities of the spectral decimation function f. It is "forbidden" to decimate to a forbidden eigenvalue. Because forbidden eigenvalues have no predecessor, we speak of forbidden eigenvalues being "born" at a level of approximation m.

Next we want to take the limit as $m \to \infty$. We assume that we have an infinite sequence $\{\lambda_m\}_{m \geq m_0}$ related by $\lambda_{m+1} = \phi_\pm(\lambda_m)$ with all but a finite number of ϕ_-'s. Then we may define

$$\lambda = \frac{3}{2} \lim_{m\to\infty} 5^m \lambda_m.$$

It is easy to see that the limit exists since

$$\phi_-(x) = \frac{1}{5}x + O(x^2) \tag{2.5}$$

as $x \to 0$. Now suppose we start with a λ_{m_0}-eigenfunction u of Δ_{m_0} on V_{m_0}, and extend u to V_* successively using (2.4), assuming that none of λ_m is a forbidden eigenvalue. Since

(2.5) implies $\lambda_m = O(\frac{1}{5^m})$ as $m \to \infty$, it is easy to see that u is uniformly continuous on V_* and so extends to a continuous function on $\mathcal{SG}$. Moreover, it satisfies the λ-eigenvalue equation for Δ.

A proof in [10] guarantees that this spectral decimation produces all possible eigenvalues and eigenfunctions (up to linear combination).

To describe the explicit Dirichlet and Neumann spectra, we have to describe all possible generations of birth and values for λ_{m_0}, and describe the multiplicity of the eigenvalue by giving an explicit basis for the λ_{m_0}-eigenspace of Δ_{m_0}. For each m, we have to add up the dimensions of eigenspaces with generation of birth $m_0 \leq m$, extended to Γ_m in all allowable ways. This total must be $\sharp V_m$ (Neumann) or $\sharp V_m - 3$ (Dirichlet), the dimension of the space on which the symmetric operator Δ_m acts. Now we give a brief description of the structure of the Dirichlet and Neumann spectra on $\mathcal{SG}$ respectively.

Dirichlet spectrum.

We denote by $\mathcal{D}$ the Dirichlet spectrum of Δ on $\mathcal{SG}$ and by $\mathcal{D}_m$ the discrete Dirichlet spectrum of Δ_m on Γ_m for $m \geq 1$. Due to the above discussion, we only need to make clear the spectrum $\mathcal{D}_m$ for each level m. There are two kinds of eigenvalues, *initial* and *continued*. The continued eigenvalues will be those that arise from eigenvalues of $\mathcal{D}_{m-1}$ by the spectral decimation. Those that remain, the initial eigenvalues, must be some of the fobidden eigenvalues by Proposition 2.1.

In [30], it is proved that $\mathcal{D}_1$ consists of two eigenvalues 2 and 5 with multiplicities 1 and 2 respectively, and for $m \geq 2$, the only possible initial eigenvalues in $\mathcal{D}_m$ are the two forbidden eigenvalues 5 and 6 with multiplicities $\frac{3^{m-1}+3}{2}$ and $\frac{3^m-3}{2}$ respectively. Hence we may classify eigenvalues into three series, which we call the 2-series, 5-series, and 6-series, depending on the value of λ_{m_0}. The eigenvalues in the 2-series all have multiplicity 1, while the eigenvalues in the other series all exhibit higher multiplicity. Also, if λ is an eigenvalue in the 5-series or 6-series, then $5^m\lambda$ is also an eigenvalue, corresponding to a generation of birth $m_0 + m$, with the same choice of $\phi_\pm$ relations (suitably reindexed).

Neumann spectrum.

We impose a Neumann condition on the graph Γ_m by imagining that it is embedded in a larger graph by reflecting in each boundary vertex and imposing the λ_m-eigenvalue equation on the even extension of u. This just means that we impose the equation

$$(4 - \lambda_m)u(q_i) = 2u(F_i^m q_{i+1}) + 2u(F_i^m q_{i-1})$$

at q_i for $i = 0, 1, 2$. Then the Neumann λ_m-eigenvalue equations consist of exactly $\sharp V_m$ equations in $\sharp V_m$ unknowns. Similar to the Dirichlet case, we also only need to make clear all the discrete spectra. The result is very similar to the Dirichlet spectrum, with only a few changes. We omit it.

It should be emphasized here that those eigenfunctions which are simultaneously Dirichlet and Neumann play an important role in the spectral analysis of $\mathcal{SG}$. Here we call them *localized eigenfunctions* since all of them have small supports. (Here the definition of localized eigenfunctions is slightly different from that of [2, 18, 35] for the convenience of further discussion for Ω case.) Similar to $\mathcal{D}$, to describe the structure of localized eigenfunctions, we only need to make clear the structure of all initial localized eigenvalues with generation of birth m for each value of m, which consists of 5-series and 6-series eigenvalues. In fact, the multiplicity of a 5-series eigenvalue is $\rho_m(5) = \frac{3^{m-1}-1}{2}$ with an eigenfunction associated to each m-level loop (a m-level circuit around an empty upside-down triangle in the graph Γ_m). The eigenfunction u associated to each loop takes value 0 on all m-level points not lying in that loop. Moreover, the support of u is exactly the union of all m-cells intersecting that loop. The multiplicity of a 6-series eigenvalue is $\rho_m(6) = \frac{3^m-3}{2}$ with an eigenfunction associated to each point x in $V_{m-1} \setminus V_0$. Each such eigenfunction u takes value 0 on all points in V_{m-1} except x. Moreover, u is supported in the union of two $(m-1)$-level cells containing x. The existence of localized eigenfunctions is unprecedented in all of smooth mathematics. However, for a class of p.c.f. self-similar sets, including $\mathcal{SG}$, localized eigenfunctions dominate global eigenfunctions. See more details in [18].

3 The structures of Dirichlet and Neumann spectra on Ω

To give the reader an intuitive perception of the structure of the spectrum of Δ on Ω in advance, in this section we describe all Dirichlet or Neumann eigenvalues and eigenfunctions and their multiplicities on Ω without proof. We will go to the technical details in the following sections.

3.1 Dirichlet spectrum

We begin with the Dirichlet case. Let $\mathcal{S}$ denote the Dirichlet spectrum of Δ on Ω. We will consider three kinds of eigenfunctions, *primitive*, *localized* and *miniaturized*.

The *localized eigenfunctions* are just a subspace of the localized eigenfunctions on $\mathcal{SG}$ whose supports are disjoint from L (the line segment joining q_1 and q_2). We let $\mathcal{L}$ denote the eigenvalues associated to them. These have generation of birth $m_0 \geq 3$ (the ones with $m_0 = 2$ all have supports intersecting L) and $\lambda_{m_0} = 5$ or 6.

Comparing to the $\mathcal{SG}$ case, instead of the eigenfunctions associated to the 2-series eigenvalues, there is a type of global eigenfunctions, which we will call *primitive eigen-*

functions, which are sorted into symmetric and skew-symmetric parts according to the reflection symmetry fixing q_0. We let $\mathcal{P}^+$ and $\mathcal{P}^-$ denote the symmetric and skew-symmetric primitive eigenvalues associated to them respectively. And let $\mathcal{P}$ denote all of this kind of eigenvalues. They have multiplicity one. (An explanation of this will come later.) In fact, we call an eigenfunction u a symmetric primitive eigenfunction if it is symmetric under the reflection symmetry fixing q_0 and also local symmetric in each cell $F_w(\mathcal{SG})$ under the reflection symmetry fixing $F_w q_0$ with word w taking symbols only from $\{1, 2\}$. For the skew-symmetric case, instead, we require u to be skew-symmetric under the reflection symmetry fixing q_0, but still local symmetric in small cells. Fig. 3.1. gives a symbolic picture of the symmetries, indicated by dotted lines. The primitive eigenfunction u (either the symmetric or skew-symmetric case) is unique and determined by the values denoted by $(b_0, b_1, b_2, \cdots)$ of u on vertex points $(q_0, F_1 q_0, F_1^2 q_0, \cdots)$ by using the eigenfunction extension algorithm described in (2.4). Due to the Dirichlet boundary condition, we always have $b_0 = 0$ and $\lim_{m\to} b_m = 0$. We call $(q_0, F_1 q_0, F_1^2 q_0, \cdots)$ a *skeleton* of Ω since it plays a critical role in the study of primitive eigenfunctions. Another equivalent definition of the primitive eigenfunctions will be presented later by considering first *primitive graph eigenfunctions* then passing the approximation to the limit.

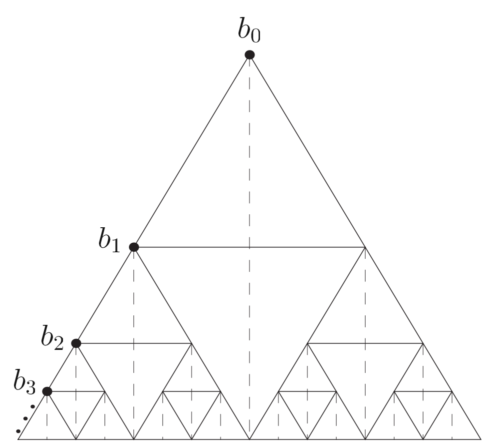

Fig. 3.1. The first 4 level symmetries and the skeleton of Ω.

There is another type of eigenfunctions that we call *miniaturized eigenfunctions*. For each $\lambda \in \mathcal{P}^-$ there is a family of eigenfunctions with eigenvalue $5^k\lambda$ and multiplicity 2^k for $k = 1, 2, 3, \cdots$. To get such an eigenfunction, just take the λ-eigenfunction u, contract it k times, place it in any one of the 2^k bottom cells of level k, and take value 0 elsewhere. See Fig 3.2. The reason we can do this is that on the boundary point q_0 the matching condition of u holds automatically. Let $\mathcal{M}$ denote all the eigenvalues associated to them.

In section 6 we will prove that all eigenfunctions of Δ on Ω fall into one of these three

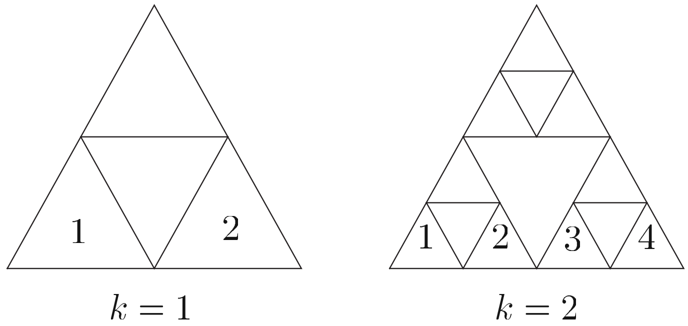

Fig. 3.2. The first 2 level miniaturized eigenfunctions.

types, and there are no coincidences of eigenvalues among different types. That is

$$\mathcal{S} = \mathcal{L} \cup \mathcal{P} \cup \mathcal{M} \text{ (disjoint union).}$$

For the purpose of studying the structure of the spectrum $\mathcal{S}$ of Δ, the first thing we should consider is to describe the spectra of Δ_m's on the associated graph approximations. Similar to the $\mathcal{SG}$ case, the fractal domain Ω can be realized as the limit of a sequence of graphs Ω_m. More precisely, $\forall m \geq 1$, let V_m^{Ω} be a subset of V_m with all vertices lying along L removed. Let Ω_m be the subgraph of Γ_m restricted to V_m^{Ω}. Denote by $\partial\Omega_m$ the boundary of the finite graph Ω_m. It is easy to find that $V_m^{\Omega} \setminus \partial\Omega_m$ and $\partial\Omega_m$ approximate to Ω and $\partial\Omega$ as m goes to infinity respectively. See Fig. 3.3. We denote by $\mathcal{S}_m$ the discrete Dirichlet spectrum of Δ_m on Ω_m for $m \geq 2$. On Ω_m the Dirichlet λ_m-eigenvalue equations consist of exactly $\sharp(V_m^{\Omega} \setminus \partial\Omega_m)$ equations in $\sharp(V_m^{\Omega} \setminus \partial\Omega_m)$ unknowns. We start from m=2 since there is no λ_1-eigenvalue equation. For simplicity, let $a_m = \sharp(V_m^{\Omega} \setminus \partial\Omega_m)$. It is easy to check that $a_2 = 5$, $a_3 = 24$, and more generally,

$$a_m = \frac{3^{m+1} - 1}{2} - 2^{m+1},$$

noticing that $a_m = a_{m-1} + 3^m + 2^{m-1} - 3 \cdot 2^{m-1}$, where $3^m = \sharp(V_m \setminus V_{m-1})$, 2^{m-1} is the number of points lying on the bottom boundary of Ω_{m-1}, and $3 \cdot 2^{m-1}$ is the number of points in $V_m \setminus V_{m-1}$ lying on L or $\partial\Omega_m$.

Due to different types of eigenvalues of Δ, we should consider the associated different types of graph eigenvalues of Δ_m. We now describe how to define $\mathcal{L}_m$, $\mathcal{P}_m$ and $\mathcal{M}_m$ respectively. In fact, by the spectral decimation recipe, each localized eigenfunction u of Δ whose generation of birth $m_0 \leq m$ can be restricted to Ω_m to get a graph eigenfunction u_m of Δ_m with the Dirichlet boundary condition on $\partial\Omega_m$ holding automatically. We call all this kind of graph eigenfunctions m-level localized graph eigenfunctions and all the associated eigenvalues are denoted by $\mathcal{L}_m$. However, we can not imitate this process to get the m-level primitive graph eigenfunctions since the Dirichlet boundary condition

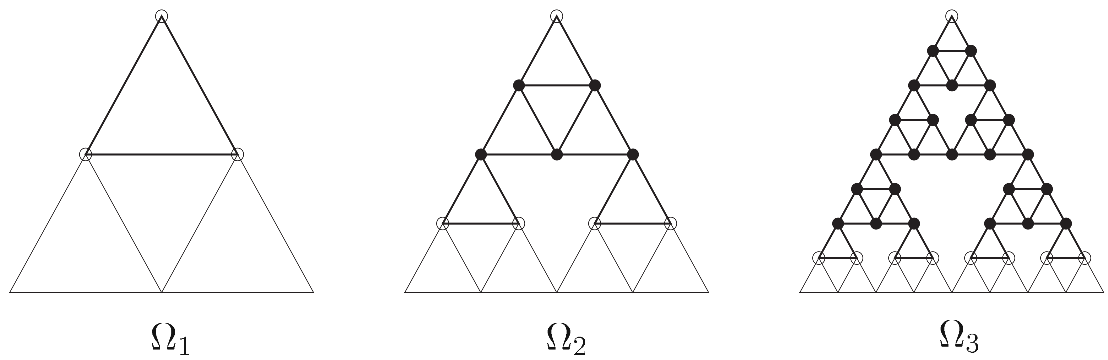

Fig. 3.3. The first 3 graphs, $\Omega_1, \Omega_2, \Omega_3$ in the approximation to Ω with inside points and boundary points represented by dots and circles respectively.

would be destroyed if we do the similar restriction. But we can define m-level primitive graph eigenfunctions on Ω_m directly using the following way. We call an eigenfunction u_m of Δ_m a m-level symmetric primitive graph eigenfunction if it is symmetric under the reflection symmetry fixing q_0 and also local symmetric in $F_w(\mathcal{SG}) \cap V_m^{\Omega}$ under the reflection symmetry fixing $F_w q_0$ with word w taking symbols only from $\{1, 2\}$. Denote by $\mathcal{P}_m^+$ all the eigenvalues associated to them. $\mathcal{P}_m^-$ can be defined in a similar way. Let $\mathcal{P}_m$ denote all of them. The primitive graph eigenfunction u_m (either the symmetric or skew-symmetric case) is unique and determined by the values denoted by $(b_0, b_1, b_2, \cdots, b_m)$ of u_m on vertex points $(q_0, F_1 q_0, F_1^2 q_0, \cdots, F_1^m q_0)$ by using the eigenfunction extension algorithm described in (2.4). Due to the Dirichlet boundary condition, we always have $b_0 = b_m = 0$. We call $(q_0, F_1 q_0, F_1^2 q_0, \cdots, F_1^m q_0)$ a *skeleton* of Ω_m. It also plays a critical role in the study of primitive graph eigenfunctions. Miniaturized graph eigenfunctions on Ω_m can be defined in a similar way by using miniaturization of skew-symmetric primitive graph eigenfunctions whose level strictly less than m. Denote by $\mathcal{M}_m$ all the associated eigenvalues.

It is not difficult to describe all the localized graph eigenvalues in $\mathcal{L}_m$, since they are almost the same as the $\mathcal{SG}$ case. There are two kinds of eigenvalues in $\mathcal{L}_m$, initial and continued. The initial eigenvalues are 5 and 6. For the 6-eigenfunctions of Δ_m on Ω_m, comparing to the 6-eigenfunctions of Δ_m on Γ_m, the only difference is those eigenfunctions whose support intersecting the boundary $\partial\Omega_m$ should be removed. A similar analysis shows that they are indexed by points in $V_{m-1}^{\Omega} \setminus \partial\Omega_{m-1}$. Hence the multiplicity of 6 is $\rho_m^{\Omega}(6) = a_{m-1} = \frac{3^m-1}{2} - 2^m$. Similarly, the 5-eigenfunctions of Δ_m on Ω_m are indexed by m-level loops except those loops touching $\partial\Omega_m$. Hence the multiplicity $\rho_m^{\Omega}(5) = \rho_m(5) - (1 + 2 + 2^2 + \cdots + 2^{m-2}) = \frac{3^{m-1}+1}{2} - 2^{m-1}$. So that is the story for initial eigenvalues. The continued eigenvalues will be those that arise from eigenvalues of $\mathcal{L}_{m-1}$ by the spectral decimation. Note that every eigenvalue λ_{m-1} of Δ_{m-1} bifurcates into two choices of λ_m of Δ_m by (2.3), except $\lambda_{m-1} = 6$, which just yields the single choice $\lambda_m = 3$ since the other is a forbidden eigenvalue 2. We know that $\rho^{\Omega}_{m-1}(6)$ of $\mathcal{L}_{m-1}$ correspond to eigenvalue 6 of Δ_{m-1}, while the remaining $\sharp\mathcal{L}_{m-1} - \rho^{\Omega}_{m-1}(6)$ of them correspond to other eigenvalues, leading to a space of continued eigenfunctions of dimension $2 \cdot (\sharp\mathcal{L}_{m-1} - \rho^{\Omega}_{m-1}(6)) + \rho^{\Omega}_{m-1}(6) = 2 \cdot \sharp\mathcal{L}_{m-1} - \frac{3^{m-1}-1}{2} + 2^{m-1}$. If we add to this $\rho^{\Omega}_{m}(6) = \frac{3^m-1}{2} - 2^m$ and $\rho^{\Omega}_{m}(5) = \frac{3^{m-1}+1}{2} - 2^{m-1}$, we should obtain $\sharp\mathcal{L}_m$. Hence we have

$$\begin{aligned}\sharp\mathcal{L}_m &= 2 \cdot \sharp\mathcal{L}_{m-1} - \frac{3^{m-1}-1}{2} + 2^{m-1} + \frac{3^m-1}{2} - 2^m + \frac{3^{m-1}+1}{2} - 2^{m-1} \\ &= 2 \cdot \sharp\mathcal{L}_{m-1} + \frac{3^m+1}{2} - 2^m.\end{aligned}$$

Combining this with $\sharp\mathcal{L}_2 = 0$, we can easily get

$$\sharp\mathcal{L}_m = \frac{3^{m+1}-1}{2} - (m-2)\cdot 2^m - 26 \cdot 2^{m-3} \quad \text{for } m \geq 2.$$

As for primitive graph eigenvalues $\mathcal{P}_m$, things become more complicated. We consider $\mathcal{P}^+_m$ and $\mathcal{P}^-_m$ respectively. We will show in the next section the spectral decimation recipe for this type of eigenvalues can not be used directly. In fact there is even not an analytic relation between elements in $\mathcal{P}^+_m$ (or $\mathcal{P}^-_m$) and elements in $\mathcal{P}^+_{m+1}$ (or $\mathcal{P}^-_{m+1}$). A rough but intuitive explanation of why does this "bad" thing happen is that the Dirichlet boundary condition will be destroyed when we use the eigenfunction extension algorithm (2.4) to extend a λ_m-eigenfunction u_m from Ω_m to Ω_{m+1} or restrict a λ_{m+1}-eigenfunction u_{m+1} from Ω_{m+1} to Ω_m. However, a weak but useful relation between $\mathcal{P}^+_m$ (or $\mathcal{P}^-_m$) and $\mathcal{P}^+_{m+1}$ (or $\mathcal{P}^-_{m+1}$) will be found in the next section, which will take the place of spectral decimation in the further discussion. We will prove that: For each $m \geq 2$, $\mathcal{P}^+_m$ consists of $r_m = 2^m + 2^{m-2} - 2$ distinct eigenvalues with multiplicity 1, between 0 and 6 strictly, denoted by $\lambda_{m,1}, \lambda_{m,2}, \cdots, \lambda_{m,r_m}$ in increasing order. Moreover, $r_{m+1} = 2r_m + 2$ and

$$\begin{aligned}0 &< \lambda_{m+1,1} < \phi_-(\lambda_{m,1}), \\ \phi_-(\lambda_{m,k-1}) &< \lambda_{m+1,k} < \phi_-(\lambda_{m,k}), \quad \forall 2 \leq k \leq r_m, \\ \phi_-(\lambda_{m,r_m}) &< \lambda_{m+1,r_m+1} < \phi_+(\lambda_{m,r_m}), \\ \phi_+(\lambda_{m,2r_m+2-k}) &< \lambda_{m+1,k} < \phi_+(\lambda_{m,2r_m+1-k}), \quad \forall r_m + 2 \leq k \leq 2r_m, \\ \phi_+(\lambda_{m,1}) &< \lambda_{m+1,2r_m+1} < 5, \\ 5 &< \lambda_{m+1,2r_m+2} < 6.\end{aligned}$$

Similar property holds for $\mathcal{P}^-_m$ with r_m replaced by $s_m = 2^m - 2$. In order to study the relation between $\mathcal{P}^+_m$ (or $\mathcal{P}^-_m$) and $\mathcal{P}^+_{m+1}$ (or $\mathcal{P}^-_{m+1}$), we introduce the following notations. In symmetric case, let $\widetilde{\phi}_-(\lambda_{m,1})$ denote the $(m+1)$-level eigenvalue between 0 and $\phi_-(\lambda_{m,1})$.

Let $\widetilde{\phi}_-(\lambda_{m,k})$ denote the $(m+1)$-level eigenvalue between $\phi_-(\lambda_{m,k-1})$ and $\phi_-(\lambda_{m,k})$ for each $2 \leq k \leq r_m$. Let $\widetilde{\phi}_+(\lambda_{m,k})$ denote the $(m+1)$-level eigenvalue between $\phi_+(\lambda_{m,k})$ and $\phi_+(\lambda_{m,k-1})$ for each $2 \leq k \leq r_m$. Let $\widetilde{\phi}_+(\lambda_{m,1})$ denote the $(m+1)$-level eigenvalue between $\phi_+(\lambda_{m,1})$ and 5. Call this kind of $(m+1)$-level eigenvalues continued eigenvalues. There are another two $(m+1)$-level eigenvalues: one is between $\phi_-(\lambda_{m,r_m})$ and $\phi_+(\lambda_{m,r_m})$, the other is between 5 and 6. Call these two $(m+1)$-level eigenvalues initial eigenvalues with generation of birth $m+1$. For the 2 level, all $r_2 = 3$ primitive symmetric eigenvalues $\lambda_{2,1}, \lambda_{2,2}$ and $\lambda_{2,3}$ are called initial eigenvalues with generation of birth 2. We define the similar notations for skew-symmetric case in an obvious way with r_m replaced by s_m. From this point of view, the continued primitive eigenvalues in $\mathcal{P}^+_{m+1}$ (or $\mathcal{P}^-_{m+1}$) will be those arise from eigenvalues in $\mathcal{P}^+_m$ (or $\mathcal{P}^-_m$) by a $\widetilde{\phi}_\pm$ bifurcation similar (but never equal) to $\phi_\pm$ bifurcation. We call this phenomenon *weak spectral decimation*, which will be proved playing a critical role in the study of the structure of primitive eigenvalues on Ω in stead of spectral decimation. We should emphasize here that $\widetilde{\phi}_\pm$ is not a real function relation. (It is just a notation for simplicity.) See the following diagram for the relation between $\mathcal{P}^+_m$ and $\mathcal{P}^+_{m+1}$. The skew-symmetric case is similar.

$$\begin{array}{cccccccccc} & \lambda_{m,1} & & & \lambda_{m,2} & & \cdots & & \lambda_{m,r_m} & & & \\ \swarrow & & \searrow & \swarrow & & \searrow & \cdots & \swarrow & & \searrow & & \\ \widetilde{\phi}_-(\lambda_{m,1}) & & \widetilde{\phi}_+(\lambda_{m,1}) & \widetilde{\phi}_-(\lambda_{m,2}) & & \widetilde{\phi}_+(\lambda_{m,2}) & \cdots & \widetilde{\phi}_-(\lambda_{m,2}) & & \widetilde{\phi}_+(\lambda_{m,2}) & \lambda_{m+1,r_m+1} & \lambda_{m+1,r_{m+1}} \end{array}$$

The structure of $\mathcal{M}_m$ depends on the structure of all $\mathcal{P}^-_k$'s with $k < m$ by the definition of $\mathcal{M}_m$. In fact, it is easy to check that

$$\sharp\mathcal{M}_m = \sum_{k=2}^{m-1} 2^{m-k} \sharp\mathcal{P}^-_k = \sum_{k=2}^{m-1} 2^{m-k}(2^k - 2) = (m-3) \cdot 2^m + 4 \quad \text{for } m \geq 2.$$

It will be proved that different types of eigenfunctions of Δ_m on Ω_m are linearly independent in the next section. Moreover, it is easy to check $\sharp\mathcal{L}_m$, $\sharp\mathcal{P}_m$ and $\sharp\mathcal{M}_m$ add up to $\sharp(V^\Omega_m \setminus \partial\Omega_m)$ since

$$\sharp\mathcal{L}_m + \sharp\mathcal{P}_m + \sharp\mathcal{M}_m = \frac{3^{m+1}-1}{2} - (m-2)\cdot 2^m - 26\cdot 2^{m-3} + r_m + s_m + (m-3)\cdot 2^m + 4 = a_m. \quad (3.1)$$

Hence we have the complete spectrum $\mathcal{S}_m$ of Δ_m. It consists of the only above mentioned three types of eigenvalues. In Table 3.1, we list the eigenspace dimensions of all different types of eigenvalues in $\mathcal{S}_m$ for level $m = 2, 3, 4, 5$.

level	$\sharp\mathcal{L}_m$	$\sharp\mathcal{P}^+_m$	$\sharp\mathcal{P}^-_m$	$\sharp\mathcal{M}_m$	$\sharp\mathcal{S}_m$
m	$\frac{3^{m+1}-1}{2} - (m-2)\cdot 2^m - 26\cdot 2^{m-3}$	$2^m + 2^{m-2} - 2$	$2^m - 2$	$(m-3)\cdot 2^m + 4$	$\frac{3^{m+1}-1}{2} - 2^{m+1}$
2	0	3	2	0	5
3	6	8	6	4	24
4	37	18	14	20	89
5	164	38	30	68	300

Table 3.1. Eigenspace dimensions of different types of eigenvalues in $\mathcal{S}_m$.

Next we want to take the limit as $m \to \infty$. For $\mathcal{L}$ case, we assume that we have an infinite sequence of localized graph eigenvalues $\{\lambda_m\}_{m \geq m_0}$ related by $\phi_\pm$ relations, with all but a finite number of ϕ_-'s. Then we define

$$\lambda = \frac{3}{2} \lim_{m \to \infty} 5^m \lambda_m.$$

By successively using the eigenfunction extension algorithm (2.4) from a λ_{m_0}-eigenfunction u of Δ_{m_0} on Ω_{m_0}, one can extend u to a localized eigenfunction of Δ on Ω associated to λ. This method generates all the localized eigenfunctions $\mathcal{L}$ similar to the $\mathcal{SG}$ case. For $\mathcal{P}^+$ case, we also assume that we have an infinite sequence of $\mathcal{P}^+$ type graph eigenvalues $\{\lambda_m\}_{m \geq m_0}$ related by $\widetilde{\phi}_\pm$ relations, with all but a finite number of $\widetilde{\phi}_-$'s. Then we define

$$\lambda = \frac{3}{2} \lim_{m \to \infty} 5^m \lambda_m.$$

We will also show the existence of the limit λ. As shown before, now we can not use the eigenfunction extension algorithm. However, we can still prove that λ is a $\mathcal{P}^+$ type eigenvalue of Δ on Ω by a nonconstructive method. Furthermore, all $\mathcal{P}^+$ type eigenvalues come in this way. This will be done in Section 5 by using the weak spectral decimation. The $\mathcal{P}^-$ and $\mathcal{M}$ cases are completely similar to the $\mathcal{P}^+$ case. Hence we get the complete spectrum $\mathcal{S}$ of Δ on Ω.

3.2 Spectral asymptotics, ratio gaps and clusters

In Tables 3.2, 3.3, 3.4 and 3.5 we present the eigenvalues and their multiplicities in $\mathcal{S}_m$ for level $m = 2, 3, 4, 5$, where we use $\lambda^+_{m,k}$, $\lambda^-_{m,k}$, $\lambda_{m,k}$ to denote the k'th $\mathcal{P}^+$, $\mathcal{P}^-$, $\mathcal{L}$ type eigenvalues respectively, and use $\mathcal{M}_m(\lambda^-_{m',k})$ to denote the miniaturized eigenvalue generated from $\lambda^-_{m',k}$.

The following conjectures list some interesting phenomena we observed from the tables.

Conjecture 3.1. *Let $\rho_m(x)$ denote the eigenvalue counting function of $\mathcal{S}_m$, i.e., $\rho_m(x) = \sharp\{\lambda_m \in \mathcal{S}_m : \lambda_m \leq x\}$. Then $\rho_m(\phi_-^{(m-k)}(5)) = 3^k - 2^k$ for $k < m$.*

Remark. *Here $3^k - 2^k$ is the difference between a_k and a_{k-1}.*

This conjecture suggests that the bottom $3^k - 2^k$ eigenvalues of the Dirichlet spectrum of Ω should be generated from the bottom $3^k - 2^k$ eigenvalues in $\mathcal{S}_m$ and the largest of these eigenvalues should be $\lim_{n\to\infty} \frac{3}{2} 5^n \phi_-^{(n-k)}(5) = c5^k$ for the appropriate choice of c. If we define the Dirichlet eigenvalue counting function

$$\rho^\Omega(x) = \sharp\{\lambda \in \mathcal{S} : \lambda \leq x\},$$

then we have $\rho^\Omega(c5^k) = 3^k - 2^k$. This suggests an asymptotic growth rate $\rho^\Omega(x) \sim x^{\log 3/\log 5}$ as $x \to \infty$. In analogy with the Weyl asymptotic law in $\mathcal{SG}$ case, noticing high multiplicity of $\phi^{(m-k)}(5)$ (which is $\frac{3^{k-1}+1}{2} - 2^{k-1}$) in $\mathcal{S}_m$, using similar arguments we can get that the ratio $\rho^\Omega(x)/x^{\log 3/\log 5}$ is bounded above and bounded away from zero, but non-convergent as $x \to \infty$. (This Weyl asymptotic law can be still proved without using Conjecture 3.1, by first considering the asymptotic law of the eigenvalue counting function for each type of eigenvalues separately, then adding up them together. See Section 6.) Moreover, of course,

$$\rho^\Omega(x) = x^{\log 3/\log 5} - x^{\log 2/\log 5}$$

along the sequence $x = c5^k$. Hence, in analogy with the $\mathcal{SG}$ case, we hopefully believe the following more precise conjecture.

Conjecture 3.2. *There exist two periodic functions $g_1(t)$ and $g_2(t)$ of period $\log 5$, which are bounded above, bounded away from zero, and necessarily discontinuous at the value $\log c$, such that*

$$\rho^\Omega(x) = g_1(\log x)x^{\log 3/\log 5} + g_2(\log x)x^{\log 2/\log 5} + o(x^{\log 2/\log 5}). \tag{3.2}$$

Here besides the leading term $g_1(\log x)x^{\log 3/\log 5}$ in Weyl's formula, the asymptotic second term of the eigenvalue counting function appears. This is very analogous to the conjectures of Weyl and Berry.

Conjecture 3.3. *There exist gaps in the ratios of eigenvalues from the Dirichlet spectrum $\mathcal{S}$ of Δ. That is, we can find infinitely many pairs of consecutive eigenvalues λ, λ' with $\frac{\lambda'}{\lambda} \geq c$ for some constant $c > 1$.*

Remark. *In fact, in the discrete spectrum $\mathcal{S}_m$, one can observe that gap appears above each $\phi_-^{(m-k)}(5)$ for $k < m$. Moreover, there are also smaller gaps below miniaturized eigenvalues.*

In [5] it was shown that on $\mathcal{SG}$ there exist gaps in the ratios of eigenvalues. The existence of gaps is an interesting phenomenon in itself, but it also has important applications to analysis on fractals. See details in [5], [18], [36]. Thus it is of great interest to know whether similar phenomenon exists for fractals other than $\mathcal{SG}$. In fact [39] shows that this is the case for Vicsek set. Also [8] investigates this question for a variant of the $\mathcal{SG}$ type fractal.

Conjecture 3.4. *In the spectrum $\mathcal{S}_m$, between consecutive 5 and 6 type localized eigenvalues, there is exactly one $\mathcal{P}^+$ and one $\mathcal{P}^-$ type eigenvalue (except the case that the two consecutive eigenvalues are $\phi_-(5)$ and $\phi_+(6) = 3$, where there is nothing in between).*

Conjecture 3.5. *In the spectrum $\mathcal{S}_m$, the number of distinct eigenvalues between $5-\varepsilon$ and 5 goes to ∞ as $m \to \infty$ for any $\varepsilon > 0$.*

Remark. *This means in $\mathcal{S}$ there exist eigenvalue clusters, that is, arbitrarily many distinct eigenvalues in an arbitrarily small interval.*

We say the spectrum $\mathcal{S}$ exhibits spectral clustering. Clustering does not occur on the $\mathcal{SG}$ case. Experimental evidence suggests that it does occur on the pentagasket [1] and on the Julia sets [9]. It is proved that in [7] it also does occur on Vicsek set.

$m=2$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type	$m=2$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type
1	$\lambda_{2,1}^{+}$=1.064568	39.92	1	$\mathcal{P}^+$	4	$\lambda_{2,3}^{+}$=5.472834	205.23	1	$\mathcal{P}^+$
2	$\lambda_{2,1}^{-}$=3.381966	126.82	1	$\mathcal{P}^-$	5	$\lambda_{2,2}^{-}$=5.618034	210.68	1	$\mathcal{P}^-$
3	$\lambda_{2,2}^{+}$=4.462598	167.35	1	$\mathcal{P}^+$					

Table 3.2. The 2-level eigenvalues in $\mathcal{S}_2$ in increasing order.

$m=3$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type	$m=3$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type
1	$\lambda_{3,1}^{+}$=0.187518	35.16	1	$\mathcal{P}^+$	11	$\lambda_{3,4}^{-}$=3.902230	731.67	1	$\mathcal{P}^-$
2	$\lambda_{3,1}^{-}$=0.558733	104.76	1	$\mathcal{P}^-$	12	$\lambda_{3,6}^{+}$=4.517231	846.98	1	$\mathcal{P}^+$
3	$\lambda_{3,2}^{+}$=0.805532	151.04	1	$\mathcal{P}^+$	13	$\lambda_{3,5}^{-}$=4.803115	900.58	1	$\mathcal{P}^-$
4	$\lambda_{3,2}^{-}$=1.247636	233.93	1	$\mathcal{P}^-$	14	$\lambda_{3,7}^{+}$=4.946726	927.51	1	$\mathcal{P}^+$
5	$\lambda_{3,3}^{+}$=1.329287	249.24	1	$\mathcal{P}^+$	15	$\lambda_{3,1}$=5	937.50	1	$\mathcal{L}$
6	$\lambda_{3,3}^{-}$=3.059152	573.59	1	$\mathcal{P}^-$	16	$\lambda_{3,8}^{+}$=5.424059	1017.01	1	$\mathcal{P}^+$
7	$\lambda_{3,4}^{+}$=3.075910	576.73	1	$\mathcal{P}^+$	17	$\lambda_{3,6}^{-}$=5.429135	1017.96	1	$\mathcal{P}^-$
8,9	$\mathcal{M}_3(\lambda_{2,1}^{-})$=3.381966	634.12	2	$\mathcal{M}$	18,19	$\mathcal{M}_3(\lambda_{2,2}^{-})$=5.618034	1053.38	2	$\mathcal{M}$
10	$\lambda_{3,5}^{+}$=3.713736	696.33	1	$\mathcal{P}^+$	20–24	$\lambda_{3,2}$=6	1125.00	5	$\mathcal{L}$

Table 3.3. The 3-level eigenvalues in $\mathcal{S}_3$ in increasing order.

$m=4$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type	$m=4$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type
1	$\lambda_{4,1}^{+}$=0.035755	33.52	1	$\mathcal{P}^+$	34	$\lambda_{4,10}^{+}$=3.631877	3404.88	1	$\mathcal{P}^+$
2	$\lambda_{4,1}^{-}$=0.100554	94.27	1	$\mathcal{P}^-$	35	$\lambda_{4,8}^{-}$=3.656967	3428.41	1	$\mathcal{P}^-$
3	$\lambda_{4,2}^{+}$=0.146945	137.76	1	$\mathcal{P}^+$	36	$\lambda_{4,11}^{+}$=3.760496	3525.46	1	$\mathcal{P}^+$
4	$\lambda_{4,2}^{-}$=0.249495	233.90	1	$\mathcal{P}^-$	37,38	$\mathcal{M}_4(\lambda_{3,4}^{-})$=3.902230	3658.34	2	$\mathcal{M}$
5	$\lambda_{4,3}^{+}$=0.277423	260.08	1	$\mathcal{P}^+$	39	$\lambda_{4,9}^{-}$=3.982762	3733.84	1	$\mathcal{P}^-$
6,7	$\mathcal{M}_4(\lambda_{3,1}^{-})$=0.558733	523.81	2	$\mathcal{M}$	40	$\lambda_{4,12}^{+}$=4.074531	3819.87	1	$\mathcal{P}^+$
8	$\lambda_{4,4}^{+}$=0.645454	605.11	1	$\mathcal{P}^+$	41	$\lambda_{4,13}^{+}$=4.223191	3959.24	1	$\mathcal{P}^+$
9	$\lambda_{4,3}^{-}$=0.652593	611.81	1	$\mathcal{P}^-$	42	$\lambda_{4,10}^{-}$=4.241362	3976.28	1	$\mathcal{P}^-$
10	$\lambda_{4,4}^{-}$=0.843591	790.87	1	$\mathcal{P}^-$	43	$\lambda_{4,11}^{-}$=4.573615	4287.76	1	$\mathcal{P}^-$
11	$\lambda_{4,5}^{+}$=0.857718	804.11	1	$\mathcal{P}^+$	44	$\lambda_{4,14}^{+}$=4.586787	4300.11	1	$\mathcal{P}^+$
12	$\lambda_{4,6}^{+}$=0.965805	905.44	1	$\mathcal{P}^+$	45	$\lambda_{4,15}^{+}$=4.735683	4439.70	1	$\mathcal{P}^+$
13	$\lambda_{4,5}^{-}$=1.065699	999.09	1	$\mathcal{P}^-$	46	$\lambda_{4,12}^{-}$=4.793032	4493.47	1	$\mathcal{P}^-$
14,15	$\mathcal{M}_4(\lambda_{3,2}^{-})$=1.247636	1169.66	2	$\mathcal{M}$	47,48	$\mathcal{M}_4(\lambda_{3,5}^{-})$=4.803115	4502.92	2	$\mathcal{M}$
16	$\lambda_{4,7}^{+}$=1.263652	1184.67	1	$\mathcal{P}^+$	49	$\lambda_{4,16}^{+}$=4.926848	4618.92	1	$\mathcal{P}^+$
17	$\lambda_{4,6}^{-}$=1.358256	1273.37	1	$\mathcal{P}^-$	50	$\lambda_{4,13}^{-}$=4.979948	4668.70	1	$\mathcal{P}^-$
18	$\lambda_{4,8}^{+}$=1.372367	1286.59	1	$\mathcal{P}^+$	51	$\lambda_{4,17}^{+}$=4.993259	4681.18	1	$\mathcal{P}^+$
19	$\lambda_{4,1}$=1.381966	1295.59	1	$\mathcal{L}$	52–57	$\lambda_{4,4}$=5	4687.50	6	$\mathcal{L}$
20–24	$\lambda_{4,2}$=3	2812.50	5	$\mathcal{L}$	58	$\lambda_{4,18}^{+}$=5.423778	5084.79	1	$\mathcal{P}^+$
25,26	$\mathcal{M}_4(\lambda_{3,3}^{-})$=3.059152	2867.96	2	$\mathcal{M}$	59	$\lambda_{4,14}^{-}$=5.423779	5084.79	1	$\mathcal{P}^-$
27	$\lambda_{4,7}^{-}$=3.078348	2885.95	1	$\mathcal{P}^-$	60,61	$\mathcal{M}_4(\lambda_{3,6}^{-})$=5.429135	5089.81	2	$\mathcal{M}$
28	$\lambda_{4,9}^{+}$=3.078431	2886.03	1	$\mathcal{P}^+$	62–65	$\mathcal{M}_4(\lambda_{2,2}^{-})$=5.618034	5266.91	4	$\mathcal{M}$
29–32	$\mathcal{M}_4(\lambda_{2,1}^{-})$=3.381966	3170.59	4	$\mathcal{M}$	66–89	$\lambda_{4,5}$=6	5625.00	24	$\mathcal{L}$
33	$\lambda_{4,3}$=3.618034	3391.91	1	$\mathcal{L}$					

Table 3.4. The 4-level eigenvalues in $\mathcal{S}_4$ in increasing order.

$m=5$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type	$m=5$	eigenvalue λ_m	$\frac{3}{2}5^m\lambda_m$	multi	type
1	$\lambda_{5,1}^{+}$=0.007039	33.00	1	$\mathcal{P}^+$	112	$\lambda_{5,20}^{+}$=3.620288	16970.1	1	$\mathcal{P}^+$
2	$\lambda_{5,1}^{-}$=0.019385	90.87	1	$\mathcal{P}^-$	113	$\lambda_{5,16}^{-}$=3.623927	16987.2	1	$\mathcal{P}^-$
3	$\lambda_{5,2}^{+}$=0.028430	133.27	1	$\mathcal{P}^+$	114	$\lambda_{5,21}^{+}$=3.644882	17085.4	1	$\mathcal{P}^+$
4	$\lambda_{5,2}^{-}$=0.049571	232.36	1	$\mathcal{P}^-$	115,116	$\mathcal{M}_5(\lambda_{4,8}^{-})$=3.656967	17142.0	2	$\mathcal{M}$
5	$\lambda_{5,3}^{+}$=0.055860	261.84	1	$\mathcal{P}^+$	117	$\lambda_{5,17}^{-}$=3.694772	17319.2	1	$\mathcal{P}^-$
6,7	$\mathcal{M}_5(\lambda_{4,1}^{-})$=0.100554	471.35	2	$\mathcal{M}$	118	$\lambda_{5,22}^{+}$=3.720985	17442.1	1	$\mathcal{P}^+$
8	$\lambda_{5,4}^{+}$=0.123515	578.98	1	$\mathcal{P}^+$	119	$\lambda_{5,23}^{+}$=3.749413	17575.4	1	$\mathcal{P}^+$
9	$\lambda_{5,3}^{-}$=0.125398	587.80	1	$\mathcal{P}^-$	120	$\lambda_{5,18}^{-}$=3.753145	17592.9	1	$\mathcal{P}^-$
10	$\lambda_{5,4}^{-}$=0.166319	779.62	1	$\mathcal{P}^-$	121–124	$\mathcal{M}_5(\lambda_{3,4}^{-})$=3.902230	18291.7	4	$\mathcal{M}$
11	$\lambda_{5,5}^{+}$=0.170850	800.86	1	$\mathcal{P}^+$	125	$\lambda_{5,19}^{-}$=3.908588	18321.5	1	$\mathcal{P}^-$
12	$\lambda_{5,6}^{+}$=0.196017	918.83	1	$\mathcal{P}^+$	126	$\lambda_{5,24}^{+}$=3.912510	18339.9	1	$\mathcal{P}^+$
13	$\lambda_{5,5}^{-}$=0.217665	1020.30	1	$\mathcal{P}^-$	127	$\lambda_{5,25}^{+}$=3.971467	18616.3	1	$\mathcal{P}^+$
14,15	$\mathcal{M}_5(\lambda_{4,2}^{-})$=0.249495	1169.51	2	$\mathcal{M}$	128,129	$\mathcal{M}_5(\lambda_{4,9}^{-})$=3.982762	18669.2	2	$\mathcal{M}$
16	$\lambda_{5,7}^{+}$=0.264441	1239.57	1	$\mathcal{P}^+$	130	$\lambda_{5,20}^{-}$=3.997137	18736.6	1	$\mathcal{P}^-$
17	$\lambda_{5,6}^{-}$=0.286684	1343.83	1	$\mathcal{P}^-$	131	$\lambda_{5,26}^{+}$=4.069518	19075.9	1	$\mathcal{P}^+$
18	$\lambda_{5,8}^{+}$=0.290993	1364.03	1	$\mathcal{P}^+$	132	$\lambda_{5,21}^{-}$=4.103862	19236.9	1	$\mathcal{P}^-$
19	$\lambda_{5,1}$=0.293638	1376.43	1	$\mathcal{L}$	133	$\lambda_{5,27}^{+}$=4.116582	19296.5	1	$\mathcal{P}^+$
20–23	$\mathcal{M}_5(\lambda_{3,1}^{-})$=0.558733	2619.06	4	$\mathcal{M}$	134	$\lambda_{5,7}$=4.122334	19323.4	1	$\mathcal{L}$
24	$\lambda_{5,9}^{+}$=0.644676	3021.92	1	$\mathcal{P}^+$	135	$\lambda_{5,28}^{+}$=4.219041	19776.8	1	$\mathcal{P}^+$
25	$\lambda_{5,7}^{-}$=0.644693	3022.00	1	$\mathcal{P}^-$	136	$\lambda_{5,22}^{-}$=4.219295	19777.9	1	$\mathcal{P}^-$
26,27	$\mathcal{M}_5(\lambda_{4,3}^{-})$=0.652593	3059.03	2	$\mathcal{M}$	137,138	$\mathcal{M}_5(\lambda_{4,10}^{-})$=4.241362	19881.4	2	$\mathcal{M}$
28–32	$\lambda_{5,2}$=0.697224	3268.24	5	$\mathcal{L}$	139–143	$\lambda_{5,8}$=4.302776	20169.3	5	$\mathcal{L}$
33,34	$\mathcal{M}_5(\lambda_{4,4}^{-})$=0.843591	3954.33	2	$\mathcal{M}$	144,145	$\mathcal{M}_5(\lambda_{4,11}^{-})$=4.573615	21438.8	2	$\mathcal{M}$
35	$\lambda_{5,8}^{-}$=0.864034	4050.16	1	$\mathcal{P}^-$	146	$\lambda_{5,23}^{-}$=4.588806	21510.0	1	$\mathcal{P}^-$
36	$\lambda_{5,10}^{+}$=0.866936	4063.76	1	$\mathcal{P}^+$	147	$\lambda_{5,29}^{+}$=4.588882	21510.4	1	$\mathcal{P}^+$
37	$\lambda_{5,3}$=0.877666	4114.06	1	$\mathcal{L}$	148	$\lambda_{5,9}$=4.706362	22061.1	1	$\mathcal{L}$
38	$\lambda_{5,11}^{+}$=0.890579	4174.59	1	$\mathcal{P}^+$	149	$\lambda_{5,30}^{+}$=4.710126	22078.7	1	$\mathcal{P}^+$
39	$\lambda_{5,9}^{-}$=0.921042	4317.38	1	$\mathcal{P}^-$	150	$\lambda_{5,24}^{-}$=4.717827	22114.8	1	$\mathcal{P}^-$
40	$\lambda_{5,12}^{+}$=0.951360	4459.50	1	$\mathcal{P}^+$	151	$\lambda_{5,31}^{+}$=4.742035	22228.3	1	$\mathcal{P}^+$
41	$\lambda_{5,10}^{-}$=1.013289	4749.79	1	$\mathcal{P}^-$	152	$\lambda_{5,25}^{-}$=4.791572	22460.5	1	$\mathcal{P}^-$
42	$\lambda_{5,13}^{+}$=1.031636	4835.79	1	$\mathcal{P}^+$	153,154	$\mathcal{M}_5(\lambda_{4,12}^{-})$=4.793032	22467.3	2	$\mathcal{M}$
43,44	$\mathcal{M}_5(\lambda_{4,5}^{-})$=1.065699	4995.46	2	$\mathcal{M}$	155–158	$\mathcal{M}_5(\lambda_{3,5}^{-})$=4.803115	22514.6	4	$\mathcal{M}$
45	$\lambda_{5,14}^{+}$=1.095777	5136.45	1	$\mathcal{P}^+$	159	$\lambda_{5,32}^{+}$=4.809185	22543.1	1	$\mathcal{P}^+$
46	$\lambda_{5,11}^{-}$=1.097686	5145.40	1	$\mathcal{P}^-$	160	$\lambda_{5,33}^{+}$=4.844770	22709.9	1	$\mathcal{P}^+$
47–50	$\mathcal{M}_5(\lambda_{3,2}^{-})$=1.247636	5848.29	4	$\mathcal{M}$	161	$\lambda_{5,26}^{-}$=4.847489	22722.6	1	$\mathcal{P}^-$
51	$\lambda_{5,15}^{+}$=1.259109	5902.07	1	$\mathcal{P}^+$	162	$\lambda_{5,27}^{-}$=4.932207	23119.7	1	$\mathcal{P}^-$
52	$\lambda_{5,12}^{-}$=1.260744	5909.74	1	$\mathcal{P}^-$	163	$\lambda_{5,34}^{+}$=4.934639	23131.1	1	$\mathcal{P}^+$
53	$\lambda_{5,16}^{+}$=1.291565	6054.21	1	$\mathcal{P}^+$	164	$\lambda_{5,35}^{+}$=4.950036	23203.3	1	$\mathcal{P}^+$
54	$\lambda_{5,13}^{-}$=1.314754	6162.91	1	$\mathcal{P}^-$	165	$\lambda_{5,28}^{-}$=4.963126	23264.7	1	$\mathcal{P}^-$
55	$\lambda_{5,17}^{+}$=1.358055	6365.88	1	$\mathcal{P}^+$	166,167	$\mathcal{M}_5(\lambda_{4,13}^{-})$=4.979948	23343.5	2	$\mathcal{M}$
56,57	$\mathcal{M}_5(\lambda_{4,6}^{-})$=1.358256	6366.83	2	$\mathcal{M}$	168	$\lambda_{5,36}^{+}$=4.987488	23378.9	1	$\mathcal{P}^+$
58	$\lambda_{5,14}^{-}$=1.377582	6457.42	1	$\mathcal{P}^-$	169	$\lambda_{5,29}^{-}$=4.997193	23424.3	1	$\mathcal{P}^-$
59	$\lambda_{5,18}^{+}$=1.380161	6469.50	1	$\mathcal{P}^+$	170	$\lambda_{5,37}^{+}$=4.998947	23432.6	1	$\mathcal{P}^+$
60–65	$\lambda_{5,4}$=1.381966	6477.97	6	$\mathcal{L}$	171–195	$\lambda_{5,10}$=5	23437.5	25	$\mathcal{L}$
66–89	$\lambda_{5,5}$=3	14063.0	24	$\mathcal{L}$	196	$\lambda_{5,38}^{+}$=5.423778	25424.0	1	$\mathcal{P}^+$
90–93	$\mathcal{M}_5(\lambda_{3,3}^{-})$=3.059152	14339.8	4	$\mathcal{M}$	197	$\lambda_{5,30}^{-}$=5.423778	25424.0	1	$\mathcal{P}^-$
94,95	$\mathcal{M}_5(\lambda_{4,7}^{-})$=3.078348	14429.8	2	$\mathcal{M}$	198,199	$\mathcal{M}_5(\lambda_{4,14}^{-})$=5.423779	25424.0	2	$\mathcal{M}$
96	$\lambda_{5,19}^{+}$=3.078432	14430.2	1	$\mathcal{P}^+$	200–203	$\mathcal{M}_5(\lambda_{3,6}^{-})$=5.429135	25449.1	4	$\mathcal{M}$
97	$\lambda_{5,15}^{-}$=3.078432	14430.2	1	$\mathcal{P}^-$	204–211	$\mathcal{M}_5(\lambda_{2,2}^{-})$=5.618034	26334.5	8	$\mathcal{M}$
98–105	$\mathcal{M}_5(\lambda_{2,1}^{-})$=3.381966	15853.0	8	$\mathcal{M}$	212–300	$\lambda_{5,11}$=6	28125.0	89	$\mathcal{L}$
106–111	$\lambda_{5,6}$=3.618034	16959.5	6	$\mathcal{L}$					

Table 3.5. The 5-level eigenvalues in $\mathcal{S}_5$ in increasing order.

3.3 Neumann spectrum

Next we give a brief discussion of the Neumann spectrum of Δ. Similar to $\mathcal{SG}$ case, we want to impose a Neumann condition on the graph Ω_m by extending functions from Ω_m by even reflection, and imposing the pointwise eigenvalue equation at the boundary points in $\partial\Omega_m$, which now have 4 neighbors. Then the Neumann λ_m-eigenvalue equations consist of exactly $\sharp V_m^\Omega$ equations in $\sharp V_m^\Omega$ unknowns. It is even convenient to allow $m = 1$, in which case there are three equations associated to the boundary $\partial\Omega_1$ and no others. In particular, on Ω_1 we find eigenvalues $\lambda_1 = 0$ corresponding to the constant function, and $\lambda_1 = 6$ corresponding to the two dimensional space of functions satisfying $u(q_0) + u(F_1 q_0) + u(F_2 q_0) = 0$ which can be split into an one dimensional symmetric space and an one dimensional skew-symmetric space under the reflection symmetry fixing q_0. For simplicity, let $b_m = \sharp V_m^\Omega$. It is easy to check that $b_1 = 3$, $b_2 = 10$, and more generally,

$$b_m = a_m + \sharp\partial\Omega_m = \frac{3^{m+1}-1}{2} - 2^{m+1} + 2^m + 1 = \frac{3^{m+1}+1}{2} - 2^m.$$

We denote $\mathcal{S}^N$ the Neumann spectrum of Δ on Ω and $\mathcal{S}_m^N$ the Neumann spectrum of Δ_m on Ω_m respectively. $\mathcal{S}^N$ still consists of three types of eigenvalues, localized, primitive and miniaturized, denoted by $\mathcal{L}^N$, $\mathcal{P}^N$ and $\mathcal{M}^N$ respectively. And correspondingly, $\mathcal{S}_m^N$ consists of three types of eigenvalues, denoted by $\mathcal{L}_m^N$, $\mathcal{P}_m^N$ and $\mathcal{M}_m^N$ respectively. Moreover, $\mathcal{P}^N(\mathcal{P}_m^N)$ can also be split into symmetric part $\mathcal{P}^{+,N}(\mathcal{P}_m^{+,N})$ and skew-symmetric part $\mathcal{P}^{-,N}(\mathcal{P}_m^{-,N})$ in the same sense as the Dirichlet case.

The structure of localized (graph) eigenvalues is very similar to the Dirichlet case, with only a few changes: The 6-series has multiplicity increasing by 1, namely the eigenfunction associated to q_0, while the 5-series is unchanged. Hence $\rho_m^{\Omega,N}(6) = \rho_m^{\Omega}(6) + 1 = \frac{3^m+1}{2} - 2^m$ and $\rho_m^{\Omega,N}(5) = \rho_m^{\Omega}(5) = \frac{3^{m-1}+1}{2} - 2^{m-1}$, $\forall m \geq 1$, where $\rho_m^{\Omega,N}(6)$ and $\rho_m^{\Omega,N}(5)$ denote the multiplicities of the m-level initial eigenvalues 6 and 5 respectively. A similar discussion shows that

$$\sharp\mathcal{L}_m^N = 2\cdot\sharp\mathcal{L}_{m-1}^N - \rho_{m-1}^{\Omega,N}(6) + \rho_m^{\Omega,N}(6) + \rho_m^{\Omega,N}(5).$$

Hence we have

$$\sharp\mathcal{L}_m^N = 2\cdot\sharp\mathcal{L}_{m-1}^N + \frac{3^m+1}{2} - 2^m,$$

which yields that

$$\sharp\mathcal{L}_m^N = \frac{3^{m+1}-1}{2} - 2^{m+1} - (m-1)\cdot 2^m \quad \text{for } m \geq 1,$$

since $\sharp\mathcal{L}_1^N = 0$.

The structure of primitive (graph) eigenvalues $\mathcal{P}^N(\mathcal{P}_m^N)$ is also similar to the Dirichlet case. We consider the symmetric and skew-symmetric case respectively. In symmetric case, we will prove that: For each $m \geq 1$, $\mathcal{P}_m^{+,N}$ consists of 2^m distinct eigenvalues with multiplicity 1, between 0 and 6 with 0, 6 included, denoted by $\lambda_{m,1} = 0, \lambda_{m,2}, \cdots, \lambda_{m,2^m} = 6$ in increasing order. Moreover,

$$\begin{aligned} \phi_-(\lambda_{m,k-1}) &< \lambda_{m+1,k} < \phi_-(\lambda_{m,k}), \quad \forall 2 \leq k \leq 2^m, \\ \phi_+(\lambda_{m,2^{m+1}-k+1}) &< \lambda_{m+1,k} < \phi_+(\lambda_{m,2^{m+1}-k}), \quad \forall 2^m + 1 \leq k \leq 2^{m+1} - 1. \end{aligned}$$

Similarly, there is also a weak spectral decimation which relates $\mathcal{P}_m^{+,N}$ and $\mathcal{P}_{m+1}^{+,N}$ by introducing the following notations. Let $\widetilde{\phi}_-(\lambda_{m,k})$ denote the $(m+1)$-level eigenvalue between $\phi_-(\lambda_{m,k-1})$ and $\phi_-(\lambda_{m,k})$ for each $2 \leq k \leq 2^m$. Let $\widetilde{\phi}_+(\lambda_{m,k})$ denote the $(m+1)$-level eigenvalue between $\phi_+(\lambda_{m,k})$ and $\phi_+(\lambda_{m,k-1})$ for each $2 \leq k \leq 2^m$. Call this kind of $(m+1)$-level eigenvalues continued eigenvalues. Hence the continued primitive eigenvalues in $\mathcal{P}_{m+1}^{+,N}$ will be those arise from eigenvalues of $\mathcal{P}_m^{+,N} \setminus \{0\}$ by a $\widetilde{\phi}_\pm$ bifurcation similar (but never equal) to $\phi_\pm$ bifurcation. There are another two $(m+1)$-level eigenvalues: one is 0, called *zero eigenvalue*, the other is 6, called initial eigenvalue with generation of birth $m+1$. See the following diagram of eigenvalues in $\mathcal{P}_m^{+,N}$.

$$\begin{array}{llllllllll}
\mathcal{P}_1^{+,N}: & 0 & & & & 6 & & & & \\
 & \downarrow & & & \swarrow & & \searrow & & & \\
\mathcal{P}_2^{+,N}: & 0 & & \widetilde{\phi}_-(6) & & & & \widetilde{\phi}_+(6) & & 6 \\
 & \downarrow & \swarrow & & \searrow & \swarrow & & \searrow & \swarrow & \searrow \\
\mathcal{P}_3^{+,N}: & 0 & \widetilde{\phi}_-\widetilde{\phi}_-(6) & & \widetilde{\phi}_+\widetilde{\phi}_-(6) & \widetilde{\phi}_-\widetilde{\phi}_+(6) & & \widetilde{\phi}_+\widetilde{\phi}_+(6) & \widetilde{\phi}_-(6) & \widetilde{\phi}_+(6) \quad 6 \\
\vdots & \vdots & & & & \vdots & & & &
\end{array}$$

For skew-symmetric case, the only difference is that there is no zero eigenvalue in $\mathcal{P}_m^{-,N}$. $\mathcal{P}_m^{-,N}$ consists of $2^m - 1$ distinct eigenvalues between 0 and 6, including 6, where 6 is an initial eigenvalue with generation of birth m and the others are continued eigenvalues arise from previous level eigenvalues by a similar weak bifurcation. We have now the following decimation diagram of eigenvalues in $\mathcal{P}_m^{-,N}$.

$$\begin{array}{lllllllll}
\mathcal{P}_1^{-,N}: & & & & 6 & & & & \\
 & & & \swarrow & & \searrow & & & \\
\mathcal{P}_2^{-,N}: & & \widetilde{\phi}_-(6) & & & & \widetilde{\phi}_+(6) & & 6 \\
 & \swarrow & & \searrow & \swarrow & & \searrow & \swarrow & \searrow \\
\mathcal{P}_3^{-,N}: & \widetilde{\phi}_-\widetilde{\phi}_-(6) & & \widetilde{\phi}_+\widetilde{\phi}_-(6) & \widetilde{\phi}_-\widetilde{\phi}_+(6) & & \widetilde{\phi}_+\widetilde{\phi}_+(6) & \widetilde{\phi}_-(6) & \widetilde{\phi}_+(6) \quad 6 \\
\vdots & \vdots & & & & \vdots & & &
\end{array}$$

As for miniaturized eigenvalues, the structure of $\mathcal{M}_m^N$ depends on the structure of all $\mathcal{P}_k^{-,N}$'s with $k < m$ in a completely similar way as the Dirichlet case. In fact, it is easy to check that

$$\sharp \mathcal{M}_m^N = \sum_{k=1}^{m-1} 2^{m-k} \sharp \mathcal{P}_k^{-,N} = \sum_{k=1}^{m-1} 2^{m-k}(2^k - 1) = (m-2) \cdot 2^m + 2 \quad \text{for } m \geq 1.$$

It is easy to check $\sharp\mathcal{L}_m^N$, $\sharp\mathcal{P}_m^N$ and $\sharp\mathcal{M}_m^N$ add up to $\sharp V_m^\Omega$, since

$$\sharp\mathcal{L}_m^N+\sharp\mathcal{P}_m^N+\sharp\mathcal{M}_m^N=\frac{3^{m+1}-1}{2}-2^{m+1}-(m-1)\cdot 2^m+2^m+2^m-1+(m-2)\cdot 2^m+2=b_m.$$

Hence we have the complete Neumann spectrum of Δ_m. Then a completely similar discussion leads to the Neumann spectrum $\mathcal{S}^N$ of Δ on Ω. In Table 3.6, we list the eigenspace dimensions of all different types of eigenvalues in $\mathcal{S}_m^N$ for level $m = 1, 2, 3, 4, 5$.

level	$\sharp\mathcal{L}_m^N$	$\sharp\mathcal{P}_m^{+,N}$	$\sharp\mathcal{P}_m^{-,N}$	$\sharp\mathcal{M}_m^N$	$\sharp\mathcal{S}_m^N$
m	$\frac{3^{m+1}-1}{2}-2^{m+1}-(m-1)\cdot 2^m$	2^m	2^m-1	$(m-2)\cdot 2^m+2$	$\frac{3^{m+1}+1}{2}-2^m$
1	0	2	1	0	3
2	1	4	3	2	10
3	8	8	7	10	33
4	41	16	15	34	106
5	172	32	31	98	333

Table 3.6. Eigenspace dimensions of different types of eigenvalues in $\mathcal{S}_m^N$.

All the unproved details except the conjectures will be proved in the following sections.

4 Primitive graph Dirichlet eigenvalues of Δ_m

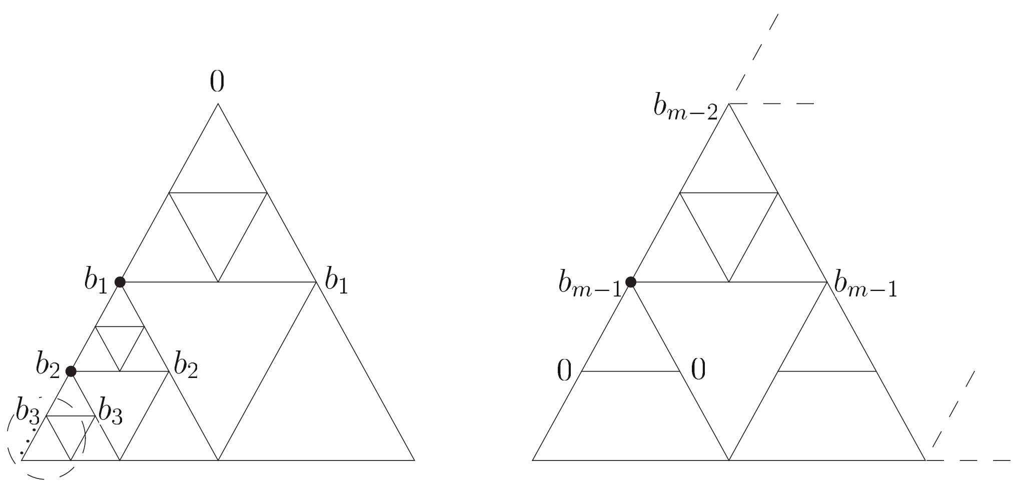

Fig. 4.1. The values of the λ_m-eigenfunction u_m on the skeleton of Ω_m with $\lambda_m \in \mathcal{P}_m^+$.

In this section, we work with m-level graph approximation Ω_m, $m = 2, 3, 4\cdots$. Denote by $\mathcal{P}_m$ the totality of primitive eigenvalues of the discrete Laplacian Δ_m. In the following we use $f^{(n)}$ to denote the n'th iteration of f, $n \geq 1$. We let $f^{(0)}(x) = x$. If $w = f^{(n)}(x)$, w is called a *successor* of x of order n with respect to f, and x is called a *predecessor* of w of order n with respect to f.

We begin with $\mathcal{P}_m^+$, the symmetric eigenvalues in $\mathcal{P}_m$. Let u_m be a λ_m-eigenfunction of Δ_m with $\lambda_m \in \mathcal{P}_m^+$. Denote by $(b_0, b_1, b_2, \cdots, b_m)$ the values of u_m on the skeleton $(q_0, F_1 q_0, F_1^2 q_0, \cdots, F_1^m q_0)$ of Ω_m where $b_0 = b_m = 0$ by the Dirichlet boundary condition. See Fig. 4.1. Write $\lambda_i^{(m)}$ the successor of λ_m of order $(m-i)$ with $2 \le i \le m$ for simplicity. Assume that none of $\lambda_i^{(m)}$'s is equal to 2 or 5. (Later we will show this assumption automatically holds for any $\lambda_m \in \mathcal{P}_m^+$.) The eigenfunction extension algorithm (2.4) gives the value of u_m on the four $(i+1)$-level neighbors of $F_1^i q_0$ for each $1 \le i \le m-1$, shown in Fig. 4.2. Hence the $\lambda_{i+1}^{(m)}$-eigenvalue equation at the vertex $F_1^i q_0$ gives

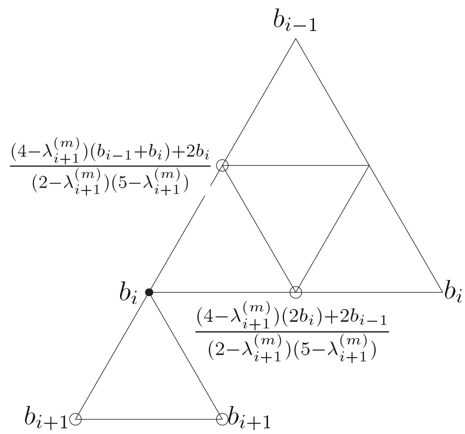

Fig. 4.2. Values of u_m on neighbors of $F_1^i q_0$.

$$(4-\lambda_{i+1}^{(m)})b_i = 2b_{i+1} + \frac{(14-3\lambda_{i+1}^{(m)})b_i + (6-\lambda_{i+1}^{(m)})b_{i-1}}{(2-\lambda_{i+1}^{(m)})(5-\lambda_{i+1}^{(m)})}, \quad \forall 1 \le i \le m-1, \tag{4.1}$$

which can be modified into

$$l(\lambda_{i+1}^{(m)})b_{i-1} + s(\lambda_{i+1}^{(m)})b_i + r(\lambda_{i+1}^{(m)})b_{i+1} = 0, \quad \forall 1 \le i \le m-1, \tag{4.2}$$

with $l(x) = x - 6$, $s(x) = (2-x)(4-x)(5-x) - (14-3x)$ and $r(x) = -2(2-x)(5-x)$. Still from the eigenfunction extension algorithm, u_m is uniquely determined by $(b_1, b_2, \cdots, b_{m-1})$. Here $(b_1, b_2, \cdots, b_{m-1})$ can be viewed as a non-zero vector solution of either of the above two systems of equations consisting of $m-1$ equations in $m-1$ unknowns. Hence the determinants of them should both be equal to 0. For simplicity, we are interested in the second determinant although comparing to the first one, it brings the possibility that $\lambda_i^{(m)}$ $(2 \le i \le m)$ could be 2 or 5, which should be removed.

The determinant associated to system (4.2) is a tridiagonal determinant,

$$\begin{vmatrix} s(\lambda_2^{(m)}) & r(\lambda_2^{(m)}) & & & \\ l(\lambda_3^{(m)}) & s(\lambda_3^{(m)}) & r(\lambda_3^{(m)}) & & \\ & \ddots & \ddots & \ddots & \\ & & l(\lambda_{m-1}^{(m)}) & s(\lambda_{m-1}^{(m)}) & r(\lambda_{m-1}^{(m)}) \\ & & & l(\lambda_m^{(m)}) & s(\lambda_m^{(m)}) \end{vmatrix}.$$

Hence λ_m should be a solution of the following equation

$$q_m(x) \triangleq \begin{vmatrix} s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & & & \\ l(f^{(m-3)}(x)) & s(f^{(m-3)}(x)) & r(f^{(m-3)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(f(x)) & s(f(x)) & r(f(x)) \\ & & & l(x) & s(x) \end{vmatrix} = 0. \qquad (4.3)$$

Conversely, if λ_m is a root of the polynomial $q_m(x)$ and none of $f^{(i)}(\lambda_m)$'s with $0 \leq i \leq m-2$ is equal to 2 or 5, then $\lambda_m \in \mathcal{P}_m^+$. Hence we are particular interested in all the root x's of the polynomial $q_m(x)$ excluding those satisfying $f^{(i)}(x) = 2$ or 5 for some $0 \leq i \leq m-2$. We list some useful facts about the polynomial $q_m(x)$.

Proposition 4.1. *Let $m \geq 2$, then*

(1) $q_m(0) > 0$;

(2) $q_m(5) > 0$;

(3) $q_m(6) < 0$;

(4) $q_m(\phi_-^{(m-1)}(5)) < 0$;

(5) $q_m(\phi_-^{(m-1)}(2)) > 0$;

(6) $q_{m+2}(\phi_-^{(m-1)}(3)) < 0$ and $q_3(3) > 0$.

Proof. (1) We will prove a stronger result,

$$q_{m+1}(0) > 20q_m(0) > 0 \qquad (4.4)$$

for $m \geq 2$. This can be proved by induction. It is easy to check that $q_2(0) = 26 > 0$ and $q_3(0) = 556 > 20q_2(0)$ by a direct computation. If we assume $q_m(0) > 20q_{m-1}(0) > 0$, then the expansion along the first row of $q_{m+1}(0)$ yields that

$$q_{m+1}(0) = 26q_m(0) - 6 \cdot 20q_{m-1}(0) > 26q_m(0) - 6q_m(0) = 20q_m(0) > 0.$$

(2) It is easy to compute that $q_2(5) = 1 > 0$ and $q_3(5) = 6 > 0$. For $m \geq 4$, $q_m(5) = q_{m-1}(0) - 20q_{m-2}(0) > 0$ by using (4.4).

(3) It is easy to compute that $q_2(6) = -4 < 0$, $q_3(6) = -3392 \leq q_2(6) < 0$ and

$$q_m(6) = s(f^{(m-2)}(6)) \cdot q_{m-1}(6) - r(f^{(m-2)}(6)) \cdot l(f^{(m-3)}(6)) \cdot q_{m-2}(6)$$

for $m \geq 4$ by the expansion along the first row of $q_m(6)$.

Consider a polynomial defined by $g_1(x) = s(f(x)) - r(f(x))l(x)$, it is easy to check that $g_1(x) \geq 1$ whenever $x \leq -6$. In fact, we can write $g_1(x) = (2 - f(x))(5 - f(x))(4 - f(x)+2(x-6))-(14-3f(x))$ by substituting the expressions for $s(f(x))$, $r(f(x))$ and $l(x)$. Noticing that $4-f(x)+2(x-6) = x^2-3x-8 \geq 46$ and $f(x) < 0$ whenever $x \leq -6$, we have $g_1(x) \geq 46(2-f(x))(5-f(x))-(14-3f(x)) = 46(f(x))^2-319f(x)+446$. Moreover, since $f(x) \leq -66$ whenever $x \leq -6$, we finally have $g_1(x) \geq 46(-66)^2 - 319 \cdot (-66) + 446 \geq 1$.

Then we can prove $q_m(6) \leq q_{m-1}(6) < 0$ by induction. Suppose $q_{m-1}(6) \leq q_{m-2}(6) < 0$. (This is true for $m = 4$.) Write $q_m(6) = aq_{m-1}(6) + bq_{m-2}(6)$ with $a = s(f^{(m-2)}(6))$ and $b = -r(f^{(m-2)}(6)) \cdot l(f^{(m-3)}(6))$. Noticing that $m \geq 4$, we have $f^{(m-3)}(6) \leq -6$ and $f^{(m-2)}(6) < 0$. Hence $a + b = g_1(f^{(m-3)}(6)) \geq 1$ and $b < 0$. So by the induction assumption, we have

$$q_m(6) \leq aq_{m-1}(6) + bq_{m-1}(6) = (a + b)q_{m-1}(6) \leq q_{m-1}(6) < 0.$$

Hence we always have $q_m(6) < 0$ for $m \geq 2$.

(4) For simplicity, denote $\alpha_m = q_m(\phi_-^{(m-1)}(5))$. By direct computation, we have $\alpha_2 = -4 < 0$ and $\alpha_3 \approx -92.10 < 0$. We will prove a stronger result, $\alpha_{m+1} \leq 10\alpha_m < 0$, $\forall m \geq 2$. It holds for $m = 2$. In order to use the induction, we assume $\alpha_{m+1} \leq 10\alpha_m < 0$. An expansion of α_{m+2} along the last row yields that

$$\alpha_{m+2} = s(\phi_-^{(m+1)}(5))\alpha_{m+1} - r(\phi_-^{(m)}(5))l(\phi_-^{(m+1)}(5))\alpha_m.$$

Since $2 - \phi_-^{(m)}(5) > 0$, $5 - \phi_-^{(m)}(5) > 0$ and $\phi_-^{(m+1)}(5) - 6 < 0$, we have

$$\begin{aligned}
\alpha_{m+2} &= s(\phi_-^{(m+1)}(5))\alpha_{m+1} - \frac{1}{10}r(\phi_-^{(m)}(5))l(\phi_-^{(m+1)}(5)) \cdot (10\alpha_m) \\
&\leq s(\phi_-^{(m+1)}(5))\alpha_{m+1} - \frac{1}{10}r(\phi_-^{(m)}(5))l(\phi_-^{(m+1)}(5))\alpha_{m+1} \\
&= [s(\phi_-^{(m+1)}(5)) - \frac{1}{10}r(\phi_-^{(m)}(5))l(\phi_-^{(m+1)}(5))]\alpha_{m+1}.
\end{aligned}$$

Consider a polynomial

$$g_2(x) = s(x) - \frac{1}{10}r(f(x))l(x) = 14 + 9x - \frac{172}{5}x^2 + \frac{87}{5}x^3 - \frac{16}{5}x^4 + \frac{1}{5}x^5.$$

It is easy to compute that

$$g_2'(x) = 9 - \frac{344}{5}x + \frac{261}{5}x^2 - \frac{64}{5}x^3 + x^4 \geq 9 - \frac{344}{5}(\phi_-^{(3)}(5)) - \frac{64}{5}(\phi_-^{(3)}(5))^3 \approx 4.91 > 0$$

whenever $0 \leq x \leq \phi_-^{(3)}(5)$. Hence $g_2(x)$ is monotone increasing in the interval $[0, \phi_-^{(3)}(5)]$.

Since $0 < \phi_-^{(m+1)}(5) \leq \phi_-^{(3)}(5)$, we have $g_2(\phi_-^{(m+1)}(5)) \geq g_2(0) \geq 10$. Hence $\alpha_{m+2} \leq g_2(\phi_-^{(m+1)}(5))\alpha_{m+1} \leq 10\alpha_{m+1} < 0$.

The proofs of (5) and (6) are similar to that of (4). $\square$

Now we discuss the possibility of the roots of $q_m(x)$ satisfying $f^{(i)}(x) = 2$ or 5 for some $0 \le i \le m-2$. The following well-known basic algebra lemma should be useful.

Lemma 4.1. *Let g, h be two polynomials whose coefficients all belong to $\mathbf{Q}$ (the field of rational numbers), i.e., $g, h \in \mathbf{Q}[x]$. If g is irreducible in $\mathbf{Q}[x]$ and g, h have a common root in $\mathbf{R}$, then g divides h in $\mathbf{Q}[x]$, i.e., all real roots of g belong to those of h.*

Lemma 4.2. *Let x be a predecessor of 2 of order i with $0 \le i \le m-3$. Then $q_m(x) = 0$. Let x be a predecessor of 2 of order $m-2$. Then $q_m(x) \ne 0$.*

Proof. Firstly, let $m \ge 3$ and x be a predecessor of 2 of order i with $0 \le i \le m-3$. Then $f^{(i)}(x) = 2$ and $f^{(i+1)}(x) = 6$. Substituting them into (4.3), noticing $s(2) = r(6) = -8$, $s(6) = l(2) = -4$ and $r(2) = l(6) = 0$, we get

$$\begin{aligned} q_m(x) &= \begin{vmatrix} \ddots & \ddots & \ddots & & & \\ & l(f^{(i+1)}(x)) & s(f^{(i+1)}(x)) & r(f^{(i+1)}(x)) & & \\ & & l(f^{(i)}(x)) & s(f^{(i)}(x)) & r(f^{(i)}(x)) & \\ & & & \ddots & \ddots & \ddots \end{vmatrix} \\ &= \begin{vmatrix} \ddots & \ddots & \ddots & & & \\ & 0 & -4 & -8 & & \\ & & -4 & -8 & 0 & \\ & & & \ddots & \ddots & \ddots \end{vmatrix} = 0. \end{aligned}$$

Secondly, let x be a predecessor of 2 of order $m-2$. Then $f^{(m-2)}(x) = 2$. If $m = 2$, then $x = 2$. It is easy to check that $x = 2$ is not a root of $q_2(x)$. If $m \ge 3$, suppose x is a root of $q_m(x)$, then using the basic algebraic lemma Lemma 4.1, all roots of $f^{(m-2)}(x) - 2$ are roots of $q_m(x)$. Noticing that $\phi_-^{(m-2)}(2)$ is also a root of $f^{(m-2)}(x) - 2$, we have $q_m(\phi_-^{(m-2)}(2)) = 0$. But

$$\begin{aligned} q_m(\phi_-^{(m-2)}(2)) &= \begin{vmatrix} s(2) & r(2) & & \\ l(\phi_-(2)) & s(\phi_-(2)) & r(\phi_-(2)) & \\ & \ddots & \ddots & \ddots \\ & & l(\phi_-^{(m-2)}(2)) & s(\phi_-^{(m-2)}(2)) \end{vmatrix} \\ &= s(2) \cdot \begin{vmatrix} s(\phi_-(2)) & r(\phi_-(2)) & \\ \ddots & \ddots & \ddots \\ & l(\phi_-^{(m-2)}(2)) & s(\phi_-^{(m-2)}(2)) \end{vmatrix} \\ &= (-8) \cdot q_{m-1}(\phi_-^{(m-2)}(2)), \end{aligned}$$

since $r(2) = 0$. By using Proposition 4.1(5), we get $q_m(\phi_-^{(m-2)}(2)) < 0$ which contradicts to $q_m(\phi_-^{(m-2)}(2)) = 0$. Hence $q_m(x) \ne 0$. $\square$

Lemma 4.3. *Let x be a predecessor of 5 of order i with $0 \le i \le m-2$. Then $q_m(x) \neq 0$.*

Proof. Let x be a predecessor of 5 of order i with $0 \le i \le m-2$. Then $f^{(i)}(x) = 5$. Hence if x is a root of $q_m(x)$, then using Lemma 4.1, all roots of $f^{(i)}(x) - 5$ are roots of $q_m(x)$. Noticing that $\phi_-^{(i)}(5)$ is also a root of $f^{(i)}(x) - 5$, we have $q_m(\phi_-^{(i)}(5)) = 0$. But $q_m(\phi_-^{(0)}(5)) = q_m(5) > 0$ by Proposition 4.1(2). More generally for $0 < i \le m-2$,

$$\begin{aligned} q_m(\phi_-^{(i)}(5)) &= \begin{vmatrix} s(f^{(m-2-i)}(5)) & r(f^{(m-2-i)}(5)) & & & & \\ \ddots & \ddots & \ddots & & & \\ & l(5) & s(5) & r(5) & & \\ & & l(\phi_-(5)) & s(\phi_-(5)) & r(\phi_-(5)) & \\ & & & \ddots & \ddots & \ddots \\ & & & & l(\phi_-^{(i)}(5)) & s(\phi_-^{(i)}(5)) \end{vmatrix} \\ &= \begin{vmatrix} s(f^{(m-2-i)}(5)) & r(f^{(m-2-i)}(5)) & & & & \\ \ddots & \ddots & \ddots & & & \\ & l(5) & s(5) & 0 & & \\ & & l(\phi_-(5)) & s(\phi_-(5)) & r(\phi_-(5)) & \\ & & & \ddots & \ddots & \ddots \\ & & & & l(\phi_-^{(i)}(5)) & s(\phi_-^{(i)}(5)) \end{vmatrix}, \end{aligned}$$

since $r(5) = 0$. Thus

$$\begin{aligned} q_m(\phi_-^{(i)}(5)) &= \begin{vmatrix} s(f^{(m-2-i)}(5)) & r(f^{(m-2-i)}(5)) & & \\ \ddots & \ddots & \ddots & \\ & & l(5) & s(5) \end{vmatrix} \\ &\quad \cdot \begin{vmatrix} s(\phi_-(5)) & r(\phi_-(5)) & & \\ \ddots & \ddots & \ddots & \\ & & l(\phi_-^{(i)}(5)) & s(\phi_-^{(i)}(5)) \end{vmatrix} \\ &= q_{m-i}(5) \cdot q_{i+1}(\phi_-^{(i)}(5)) < 0 \end{aligned}$$

by the 2'nd and 4'th statements in Proposition 4.1. Hence $\forall 0 \le i \le m-2$, we have proved $q_m(\phi_-^{(i)}(5)) \neq 0$ which yields a contradiction to $q_m(\phi_-^{(i)}(5)) = 0$. So $q_m(x) \neq 0$. □

From Lemma 4.2 and Lemma 4.3, for $m \ge 3$, the total unwanted roots of $q_m(x)$ are those predecessors of 2 of order i with $0 \le i \le m-3$. $q_2(x)$ does not have any unwanted root. Hence to exclude them out, we define

$$p_m(x) = \frac{q_m(x)}{(x-2)(f(x)-2)\cdots(f^{(m-3)}(x)-2)}, \qquad \text{for } m \ge 3,$$

and

$$p_2(x) = q_2(x) = s(x).$$

$p_m(x)$ is still a polynomial from Lemma 4.2, although it looks like a rational function. Now we can say if λ_m is a root of the polynomial $p_m(x)$, then $\lambda_m \in \mathcal{P}_m^+$. Note that the degree of the polynomial $q_m(x)$ is $3 + 3 \cdot 2 + \cdots + 3 \cdot 2^{m-2} = 3(2^{m-1} - 1)$ and the number of all the unwanted roots of $q_m(x)$ is $1 + 2 + \cdots + 2^{m-3} = 2^{m-2} - 1$ for $m \geq 3$ and 0 for $m = 2$. Hence it is easy to check that the degree of $p_m(x)$ is $r_m = 2^m + 2^{m-2} - 2$. The following is a list of some useful facts about the polynomial $p_m(x)$.

Proposition 4.2. *(1)* $(-1)^m p_m(0) > 0$, $\forall m \geq 2$;

(2) $p_2(5) > 0$ *and* $(-1)^{m-1} p_m(5) > 0$, $\forall m \geq 3$;

(3) $p_2(6) < 0$ *and* $(-1)^m p_m(6) > 0$, $\forall m \geq 3$.

Proof. It can be checked by a direct computation when $m = 2$. When $m \geq 3$, noticing that by the definition of $p_m(x)$,

$$p_m(0) = \frac{q_m(0)}{(-2)^{m-2}}, \quad p_m(5) = \frac{q_m(5)}{3 \cdot (-2)^{m-3}}$$

and

$$p_m(6) = \frac{q_m(6)}{(6-2)(f(6)-2)\cdots(f^{(m-3)}(6)-2)}.$$

Using Proposition 4.1(1)-(3), we get the desired result. □

We now present a more precise result about the distribution of the roots of $p_m(x)$ and show an useful relation between roots of two consecutive polynomials.

Lemma 4.4. *For each* $m \geq 2$, $p_m(x)$ *has* r_m *distinct real roots satisfying*

$$0 < \lambda_{m,1} < \lambda_{m,2} < \cdots < \lambda_{m,r_m-1} < 5 < \lambda_{m,r_m} < 6.$$

Moreover, $(-1)^{m+k-1} p_{m+1}(\phi_-(\lambda_{m,k})) > 0$ *and* $(-1)^{m+k} p_{m+1}(\phi_+(\lambda_{m,k})) > 0$, $\forall 1 \leq k \leq r_m$.

Proof. We prove it by using the induction on m.

When $m = 2$, $p_2(x) = s(x)$ has 3 distinct roots: $\lambda_{2,1} \approx 1.0646$, $\lambda_{2,2} \approx 4.4626$ and $\lambda_{2,3} \approx 5.4728$ by a direct computation.

Let λ be one of $\lambda_{2,k}$'s, then $p_2(\lambda) = 0$, i.e., $s(\lambda) = 0$, and $p_3(\phi_-(\lambda)) = \frac{q_3(\phi_-(\lambda))}{\phi_-(\lambda)-2} = \frac{2(\phi_-(\lambda)-6)(2-\lambda)(5-\lambda)}{\phi_-(\lambda)-2}$ by using $s(\lambda) = 0$. Since $0 < \lambda < 6$, we have $\phi_-(\lambda) - 2 < 0$ and $\phi_-(\lambda) - 6 < 0$. Hence $p_3(\phi_-(\lambda)) \sim (2-\lambda)(5-\lambda)$ where "$\sim$" means both sides of "$\sim$" have the same signs. Similarly, $p_3(\phi_+(\lambda)) = \frac{2(\phi_+(\lambda)-6)(2-\lambda)(5-\lambda)}{\phi_+(\lambda)-2}$ and $p_3(\phi_+(\lambda)) \sim -(2-\lambda)(5-\lambda)$.

Hence $0 < \lambda_{2,1} < 2$ yields that $p_3(\phi_-(\lambda_{2,1})) > 0$ and $p_3(\phi_+(\lambda_{2,1})) < 0$; $2 < \lambda_{2,2} < 5$ yields that $p_3(\phi_-(\lambda_{2,2})) < 0$ and $p_3(\phi_+(\lambda_{2,2})) > 0$; $\lambda_{2,1} > 5$ yields that $p_3(\phi_-(\lambda_{2,3})) > 0$ and $p_3(\phi_+(\lambda_{2,3})) < 0$. So our lemma holds for $m = 2$.

We now assume our lemma holds for m, and prove it for $m + 1$.

Noticing that from Proposition 4.2, we have $p_{m+1}(0) \sim (-1)^{m-1}$, $p_{m+1}(5) \sim (-1)^m$ and $p_{m+1}(6) \sim (-1)^{m-1}$. Hence if we write

$$0, \phi_-(\lambda_{m,1}), \phi_-(\lambda_{m,2}), \cdots, \phi_-(\lambda_{m,r_m}), \phi_+(\lambda_{m,r_m}), \cdots, \phi_+(\lambda_{m,2}), \phi_+(\lambda_{m,1}), 5, 6 \qquad (4.5)$$

in increasing order, then the values of p_{m+1} on them have alternating signs by the induction assumption. Hence there exist at least $2r_m + 2 = r_{m+1}$ distinct roots of $p_{m+1}(x)$, with each located strictly between each two consecutive points in (4.5). Moreover, these are the totality of the roots of $p_{m+1}(x)$ since the degree of $p_{m+1}(x)$ is also r_{m+1}. Hence we can write them in increasing order:

$$0 < \lambda_{m+1,1} < \lambda_{m+1,2} < \cdots < \lambda_{m+1,r_{m+1}-1} < 5 < \lambda_{m+1,r_{m+1}} < 6.$$

Now we study the signs of $p_{m+2}(\phi_\pm(\lambda_{m+1,k}))$'s. Let λ be one of $\lambda_{m+1,k}$'s, then $p_{m+1}(\lambda) = 0$. Moreover,

$$\begin{aligned} p_{m+2}(\phi_-(\lambda)) &= \frac{q_{m+2}(\phi_-(\lambda))}{(\phi_-(\lambda)-2)(\lambda-2)\cdots(f^{(m-2)}(\lambda)-2)} \\ &= \frac{s(\phi_-(\lambda))q_{m+1}(\lambda) + 2(\phi_-(\lambda)-6)(2-\lambda)(5-\lambda)q_m(f(\lambda))}{(\phi_-(\lambda)-2)(\lambda-2)\cdots(f^{(m-2)}(\lambda)-2)} \end{aligned}$$

by using the expansion of $q_{m+2}(\phi_-(\lambda))$ along the last row. Since $p_{m+1}(\lambda) = 0$, we have $q_{m+1}(\lambda) = 0$. Hence

$$p_{m+2}(\phi_-(\lambda)) = \frac{2(\phi_-(\lambda)-6)(2-\lambda)(5-\lambda)q_m(f(\lambda))}{(\phi_-(\lambda)-2)(\lambda-2)\cdots(f^{(m-2)}(\lambda)-2)} = \frac{-2(\phi_-(\lambda)-6)(5-\lambda)p_m(f(\lambda))}{\phi_-(\lambda)-2}.$$

Since $0 < \lambda < 6$, we have $\phi_-(\lambda) - 2 < 0$ and $\phi_-(\lambda) - 6 < 0$, hence

$$p_{m+2}(\phi_-(\lambda)) \sim (\lambda - 5)p_m(f(\lambda)).$$

Similarly,

$$p_{m+2}(\phi_+(\lambda)) = \frac{-2(\phi_+(\lambda)-6)(5-\lambda)p_m(f(\lambda))}{\phi_+(\lambda)-2}$$

and

$$p_{m+2}(\phi_+(\lambda)) \sim (5-\lambda)p_m(f(\lambda)).$$

When $\lambda = \lambda_{m+1,1}$, we have $0 < \lambda < \phi_-(\lambda_{m,1})$, hence $0 < f(\lambda) < \lambda_{m,1}$. Noticing that $\lambda_{m,1}$ is the least root of $p_m(x)$ and $\lambda_{m,1} > 0$ by the induction assumption, we have $p_m(f(\lambda)) \sim p_m(0) \sim (-1)^m$. Hence $p_{m+2}(\phi_-(\lambda_{m+1,1})) \sim (-1)^{m+1}$ and $p_{m+2}(\phi_+(\lambda_{m+1,1})) \sim (-1)^m$ since $\lambda_{m+1,1} < 5$.

When $\lambda = \lambda_{m+1,k}$ with $2 \le k \le r_m$, we have $\phi_-(\lambda_{m,k-1}) < \lambda < \phi_-(\lambda_{m,k})$, hence $\lambda_{m,k-1} < f(\lambda) < \lambda_{m,k}$. Noticing that $p_m(\lambda_{m,k-1}) = 0$ and $p_m(0) \sim (-1)^m$, we have

$p_m(f(\lambda)) \sim (-1)^{m+k+1}$ by using the induction assumption. Hence $p_{m+2}(\phi_-(\lambda_{m+1,k})) \sim (-1)^{m+k}$ and $p_{m+2}(\phi_+(\lambda_{m+1,k})) \sim (-1)^{m+k+1}$ since $\lambda_{m+1,k} < 5$.

When $\lambda = \lambda_{m+1,r_m+1}$, we have $\phi_-(\lambda_{m,r_m}) < \lambda < \phi_+(\lambda_{m,r_m})$, hence $f(\lambda) > \lambda_{m,r_m}$. Noticing that λ_{m,r_m} is the last root of $p_m(x)$ and $p_m(0) \sim (-1)^m$, we have $p_m(f(\lambda)) \sim (-1)^{m+r_m}$ by using the induction assumption. Hence $p_{m+2}(\phi_-(\lambda_{m+1,r_m+1})) \sim (-1)^{m+1+r_m}$ and $p_{m+2}(\phi_+(\lambda_{m+1,r_m+1})) \sim (-1)^{m+r_m}$ since $\lambda_{m+1,r_m+1} < 5$.

When $\lambda = \lambda_{m+1,k}$ with $r_m+2 \le k \le 2r_m$, we have $\phi_+(\lambda_{m,r_{m+1}-k}) < \lambda < \phi_+(\lambda_{m,r_{m+1}-k-1})$, hence $\lambda_{m,r_{m+1}-k-1} < f(\lambda) < \lambda_{m,r_{m+1}-k}$. Noticing that $p_m(\lambda_{m,r_{m+1}-k-1}) = 0$ and $p_m(0) \sim (-1)^m$, we have $p_m(f(\lambda)) \sim (-1)^{m+r_{m+1}-k-1} \sim (-1)^{m+k-1}$ by using the induction assumption. Hence $p_{m+2}(\phi_-(\lambda_{m+1,k})) \sim (-1)^{m+k}$ and $p_{m+2}(\phi_+(\lambda_{m+1,k})) \sim (-1)^{m+k-1}$ since $\lambda_{m+1,k} < 5$.

When $\lambda = \lambda_{m+1,2r_m+1}$, we have $\phi_+(\lambda_{m,1}) < \lambda < 5$, hence $f(\lambda) < \lambda_{m,1}$. So we have $p_m(f(\lambda)) \sim p_m(0) \sim (-1)^m$. Hence $p_{m+2}(\phi_-(\lambda_{m+1,2r_m+1})) \sim (-1)^{m+1}$ and $p_{m+2}(\phi_+(\lambda_{m+1,2r_m+1})) \sim (-1)^m$ since $\lambda_{m+1,2r_m+1} < 5$.

When $\lambda = \lambda_{m+1,2r_m+2}$, we have $5 < \lambda < 6$, hence $f(\lambda) < 0$. So we have $p_m(f(\lambda)) \sim p_m(0) \sim (-1)^m$. But now $\lambda > 5$, hence $p_{m+2}(\phi_-(\lambda_{m+1,2r_m+2})) \sim (-1)^m$ and $p_{m+2}(\phi_+(\lambda_{m+1,2r_m+2})) \sim (-1)^{m-1}$.

Hence we have proved $(-1)^{m+1+k-1}p_{m+2}(\phi_-(\lambda_{m+1,k})) > 0$ and $(-1)^{m+1+k}p_{m+2}(\phi_+(\lambda_{m+1,k})) > 0$, $\forall 1 \le k \le r_{m+1}$. So our lemma holds for $m+1$. □

Thus by Lemma 4.4, in particular the proof of Lemma 4.4 and the fact that each root of $p_m(x)$ belongs to $\mathcal{P}_m^+$, we have the following result:

Lemma 4.5. *For each $m \ge 2$, $\mathcal{P}_m^+$ consists of at least r_m distinct eigenvalues satisfying*

$$0 < \lambda_{m,1} < \lambda_{m,2} < \cdots < \lambda_{m,r_m-1} < 5 < \lambda_{m,r_m} < 6. \tag{4.6}$$

Moreover,

$$\begin{aligned} 0 &< \lambda_{m+1,1} < \phi_-(\lambda_{m,1}), \\ \phi_-(\lambda_{m,k-1}) &< \lambda_{m+1,k} < \phi_-(\lambda_{m,k}), \quad \forall 2 \le k \le r_m, \\ \phi_-(\lambda_{m,r_m}) &< \lambda_{m+1,r_m+1} < \phi_+(\lambda_{m,r_m}), \\ \phi_+(\lambda_{m,2r_m+2-k}) &< \lambda_{m+1,k} < \phi_+(\lambda_{m,2r_m+1-k}), \quad \forall r_m+2 \le k \le 2r_m, \\ \phi_+(\lambda_{m,1}) &< \lambda_{m+1,2r_m+1} < 5, \\ 5 &< \lambda_{m+1,2r_m+2} < 6. \end{aligned} \tag{4.7}$$

Remark. *The third inequality in* (4.7) *can be refined into* $2 < \lambda_{m+1,r_m+1} < \phi_+(\lambda_{m,r_m})$. *See details in Theorem A in Appendix.*

Moreover, we have

Lemma 4.6. *Let λ_m be a root of $p_m(x)$, u_m a primitive λ_m-eigenfunction on Ω_m, and $(b_0, b_1, \cdots, b_m)$ the values of u_m on the skeleton of Ω_m. Then $b_1 \neq 0$ and $b_{m-1} \neq 0$.*

Proof. Without loss of generality, assume $m \geq 3$. We still use $\lambda_i^{(m)}$ to denote the successor of λ_m of order $(m-i)$ with $2 \leq i \leq m$. From the definition of $p_m(x)$, $\lambda_i^{(m)} \neq 2$ or 5, for each $2 \leq i \leq m$. From the discussion in the beginning of this section, the vector $(b_1, b_2, \cdots, b_{m-1})$ can be viewed as a non-zero vector solution of system (4.2) of equations.

Suppose $b_{m-1} = 0$. Then $(b_1, b_2, \cdots, b_{m-2})$ can be viewed as a non-zero vector solution of the system of equations consisting of the first $(m-2)$ equations of (4.2) in $(m-2)$ unknowns. Hence the determinant of this system $q_{m-1}(\lambda_{m-1}^{(m)})$ should be equal to 0. Thus $\lambda_{m-1}^{(m)}$ is a root of $p_{m-1}(x)$ since its all successors $\lambda_2^{(m)}, \cdots, \lambda_{m-1}^{(m)}$ do not take value from $\{2, 5\}$ obviously. Then Lemma 4.4 says that neither of $\phi_\pm(\lambda_{m-1}^{(m)})$ should be a root of $p_m(x)$. This contradicts to $p_m(\lambda_m) = 0$ since λ_m is equal to either of $\phi_\pm(\lambda_{m-1}^{(m)})$. Hence $b_{m-1} \neq 0$.

On the other hand, if $b_1 = 0$, then by substituting it into (4.2), noticing that none of $\lambda_i^{(m)}$'s is equal to 2 or 5, we can get $b_2 = 0, \cdots, b_{m-1} = 0$ successively, which contradicts to $b_{m-1} \neq 0$. Hence $b_1 \neq 0$. □

Next we give a brief discussion of the skew-symmetric case. It is very similar to the symmetric case. Let u_m be a λ_m-eigenfunction of Δ_m with $\lambda_m \in \mathcal{P}_m^-$. Denote by $(b_0, b_1, b_2, \cdots, b_m)$ the values of u_m on the skeleton of Ω_m where $b_0 = b_m = 0$ by the Dirichlet boundary condition. Write $\lambda_i^{(m)}$ the successor of λ_m of order $(m-i)$ with $2 \leq i \leq m$. Comparing to the symmetric case, the eigenvalue equations at the vertex $F_1^i q_0$'s are unchanged except the one at $F_1 q_0$, since now the values of u_m on the four 2-level neighbors of $F_1 q_0$ are modified as shown in Fig. 4.3. Hence we still have the same

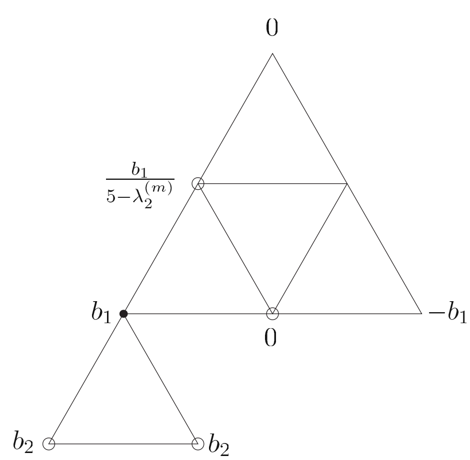

Fig. 4.3. Values of u_m on neighbors of $F_1 q_0$.

modified eigenvalue equation

$$l(\lambda_{i+1}^{(m)})b_{i-1} + s(\lambda_{i+1}^{(m)})b_i + r(\lambda_{i+1}^{(m)})b_{i+1} = 0, \quad \forall 2 \le i \le m-1,$$

while the first equation in (4.2) is replaced by

$$\widetilde{s}(\lambda_2^{(m)})b_1 + \widetilde{r}(\lambda_2^{(m)})b_2 = 0,$$

with $\widetilde{s}(x) = (4-x)(5-x)-1$ and $\widetilde{r}(x) = -2(5-x)$. Now we assume $\lambda_2^{(m)} \neq 5$ and none of $\lambda_i^{(m)}$'s is equal to 2 or 5 for $3 \le i \le m$. (Later we will show this assumption automatically holds for any $\lambda_m \in \mathcal{P}_m^-$.) Then by the eigenfunction extension algorithm, u_m is unique and determined by its values on the skeleton of Ω_m. Using similar discussion, λ_m should be a solution of the following equation

$$\widetilde{q}_m(x) \triangleq \begin{vmatrix} \widetilde{s}(f^{(m-2)}(x)) & \widetilde{r}(f^{(m-2)}(x)) & & & \\ l(f^{(m-3)}(x)) & s(f^{(m-3)}(x)) & r(f^{(m-3)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(f(x)) & s(f(x)) & r(f(x)) \\ & & & l(x) & s(x) \end{vmatrix} = 0, \qquad (4.8)$$

instead of $q_m(x) = 0$ in the symmetric case. Hence if λ_m is a root of $\widetilde{q}_m(x)$, $f^{(m-2)}(\lambda_m) \neq 5$ and none of $f^{(i)}(\lambda_m)$'s is equal to 2 or 5 for $0 \le i \le m-3$, then $\lambda_m \in \mathcal{P}_m^-$. Similar to Proposition 4.1, we have

Proposition 4.3. *(1)* $\widetilde{q}_m(0) > 0$, $\forall m \ge 2$;

(2) $\widetilde{q}_2(5) < 0$ *and* $\widetilde{q}_m(5) > 0$, $\forall m \ge 3$;

(3) $\widetilde{q}_2(6) > 0$ *and* $\widetilde{q}_m(6) < 0$, $\forall m \ge 3$.

Proof. The first two statements follow from a very similar argument in the proof of Proposition 4.1. We only need to prove the third one.

It is easy to check that $\widetilde{q}_2(6) = 1 > 0$ and $\widetilde{q}_3(6) = -436 < 0$ by a direct computation. For $m \ge 4$, an expansion along the first row yields that

$$\widetilde{q}_m(6) = \widetilde{s}(f^{(m-2)}(6))q_{m-1}(6) + 2(5 - f^{(m-2)}(6))(f^{(m-3)}(6) - 6)q_{m-2}(6).$$

Recall that in the proof of Proposition 4.1(3), we have proved that $q_{m-1}(6) \le q_{m-2}(6) < 0$. Hence

$$\widetilde{q}_m(6) \le (\widetilde{s}(f^{(m-2)}(6)) + 2(5 - f^{(m-2)}(6))(f^{(m-3)}(6) - 6))q_{m-1}(6),$$

noticing that $f^{(m-2)}(6) < f^{(m-3)}(6) \le -6$. An easy calculus shows that $\widetilde{s}(f^{(m-2)}(6)) + 2(5 - f^{(m-2)}(6))(f^{(m-3)}(6) - 6) \ge 1$, hence $\widetilde{q}_m(6) \le q_{m-1}(6) < 0$. □

Similar to the symmetric case, the following two lemmas focus on the possibility of the roots of $\widetilde{q}_m(x)$ satisfying $f^{(m-2)}(x) = 5$, or $f^{(i)}(x) = 2$ or 5 for some $0 \le i \le m-3$.

Lemma 4.7. *Let x be a predecessor of 2 of order i with $0 \le i \le m-3$. Then $\widetilde{q}_m(x) = 0$.*

Proof. If $0 \le i < m-3$, the proof is the same as that of Lemma 4.2. So we only need to check the $i = m-3$ case. In this case, $f^{(m-3)}(x) = 2$ and $f^{(m-2)}(x) = 6$. Substituting them into (4.8), noticing $s(2) = -8$, $l(2) = -4$, $r(2) = 0$, $\widetilde{s}(6) = 1$ and $\widetilde{r}(6) = 2$, we get

$$\widetilde{q}_m(x) = \begin{vmatrix} \widetilde{s}(f^{(m-2)}(x)) & \widetilde{r}(f^{(m-2)}(x)) & & \\ l(f^{(m-3)}(x)) & s(f^{(m-3)}(x)) & r(f^{(m-3)}(x)) & \\ & \ddots & \ddots & \ddots \end{vmatrix} = \begin{vmatrix} 1 & 2 & & \\ -4 & -8 & 0 & \\ & \ddots & \ddots & \ddots \end{vmatrix} = 0. \quad \square$$

Lemma 4.8. *Let x be a predecessor of 5 of order i with $0 \le i \le m-2$. Then $\widetilde{q}_m(x) \neq 0$.*

Proof. Similar to the proof of Lemma 4.3, we only need to prove $\widetilde{q}_m(\phi_-^{(i)}(5)) \neq 0$. A similar argument yields that $\widetilde{q}_m(\phi_-^{(0)}(5)) = \widetilde{q}_m(5)$ and $\widetilde{q}_m(\phi_-^{(i)}(5)) = \widetilde{q}_{m-i}(5) \cdot q_{i+1}(\phi_-^{(i)}(5))$ for $0 < i \le m-2$. Combined with Proposition 4.1(4) and Proposition 4.3(2), it follows the desired result. $\square$

Hence the total unwanted roots of $\widetilde{q}_m(x)$ consist of those predecessors of 2 of order i with $0 \le i \le m-3$ for $m \ge 3$ and $\widetilde{q}_2(x)$ does not have any unwanted root. This is exactly the same as the symmetric case. To exclude them out, we define

$$\widetilde{p}_m(x) = \frac{\widetilde{q}_m(x)}{(x-2)(f(x)-2)\cdots(f^{(m-3)}(x)-2)}, \qquad \text{for } m \ge 3,$$

and

$$\widetilde{p}_2(x) = \widetilde{q}_2(x) = \widetilde{s}(x).$$

These polynomials play a very similar role to $p_m(x)$'s in the symmetric case. It is easy to check that the degree of $\widetilde{p}_m(x)$ is $s_m = 2^m - 2$, since the degree of the polynomial $\widetilde{q}_m(x)$ is $3 + 3 \cdot 2 + \cdots + 3 \cdot 2^{m-3} + 2 \cdot 2^{m-2} = 3(2^{m-2} - 1) + 2^{m-1}$, and the number of all the unwanted roots of $\widetilde{q}_m(x)$ is $1 + 2 + \cdots + 2^{m-3} = 2^{m-2} - 1$ for $m \ge 3$ and 0 for $m = 2$. The following is a list of some facts about $\widetilde{p}_m(x)$ similar to Proposition 4.2, which can be easily get from Proposition 4.3.

Proposition 4.4. *Let $m \ge 2$, then*

(1) $(-1)^m \widetilde{p}_m(0) > 0$*;*

(2) $(-1)^{m-1} \widetilde{p}_m(5) > 0$*;*

(3) $(-1)^m \widetilde{p}_m(6) > 0$.

Then following a similar argument, the results in Lemma 4.5 and Lemma 4.6 still hold with $\mathcal{P}_m^+$, $p_m(x)$ and r_m replaced by $\mathcal{P}_m^-$, $\widetilde{p}_m(x)$ and s_m respectively.

Hence we have found r_m distinct eigenvalues in $\mathcal{P}_m^+$ and s_m distinct eigenvalues in $\mathcal{P}_m^-$. We will show these eigenvalues are the totality of $\mathcal{P}_m$. To prove this, the following lemma is needed.

Lemma 4.9. *Let $\mathcal{P}_m^{+,*}$ and $\mathcal{P}_m^{-,*}$ be the sets of total roots of $p_m(x)$ and $\widetilde{p}_m(x)$ respectively. Let $\mathcal{M}_m^*$ be the set of miniaturized eigenvalues generated by $\mathcal{P}_k^{-,*}$ with $2 \leq k < m$. Let $\mathcal{L}_m$ denote the set of m-level localized eigenvalues. Then all eigenfunctions associated to these eigenvalues are linearly independent.*

Proof. Without loss of generality, assume $m \geq 3$. It is easy to check that for each m-level localized eigenfunction $u_m^{\mathcal{L}}$, it must be 0 on $\partial\Omega_{m-1}$. Lemma 4.6 says that each m-level primitive symmetric λ_m-eigenfunction $u_m^{\mathcal{P},+}$ with $\lambda_m \in \mathcal{P}_m^{+,*}$ must be a non-zero constant on $\partial\Omega_{m-1} \setminus \{q_0\}$ and be a non-zero constant on $\partial\Omega_1 \setminus \{q_0\}$. The skew-symmetric analog of Lemma 4.6 says that each m-level primitive skew-symmetric λ_m-eigenfunction $u_m^{\mathcal{P},-}$ with $\lambda_m \in \mathcal{P}_m^{-,*}$ must be a non-zero constant on each symmetric part of $\partial\Omega_{m-1} \setminus \{q_0\}$ under the symmetry fixing q_0, and take non-zero value on $F_1 q_0$ and $F_2 q_0$ only different in signs. From the construction of the miniaturized eigenfunctions, for each m-level miniaturized eigenfunction $u_m^{\mathcal{M}}$ with eigenvalue in $\mathcal{M}_m^*$, $u_m^{\mathcal{M}}$ must take non-zero value on a subset of $\partial\Omega_{m-1} \setminus \{q_0\}$ and be 0 on $\partial\Omega_1$. These observations implies the linearly independence of eigenfunctions among different types. □

Hence we have

Lemma 4.10. *For each $m \geq 2$, $\mathcal{P}_m^{+,*} = \mathcal{P}_m^+$ and $\mathcal{P}_m^{-,*} = \mathcal{P}_m^-$.*

Proof. It follows directly from Lemma 4.5 and its skew-symmetric analog, Lemma 4.9 and the eigenspace dimensional counting formula (3.1). □

By this lemma, it is easy to see that the assumptions we made before on symmetric and skew-symmetric m-level primitive eigenvalues hold automatically.

Next we will prove each primitive eigenvalue $\lambda \in \mathcal{P}_m$ has multiplicity 1. For this purpose, we need the following lemma.

Lemma 4.11. *For each $m \geq 2$, $\mathcal{P}_m^+ \cap \mathcal{P}_{m+1}^+ = \emptyset$.*

Proof. For $m = 2$, it can be checked by a direct computation. In order to use the induction, we assume that $\mathcal{P}_m^+ \cap \mathcal{P}_{m+1}^+ = \emptyset$ and will prove $\mathcal{P}_{m+1}^+ \cap \mathcal{P}_{m+2}^+ = \emptyset$.

Suppose there is a $\lambda \in \mathcal{P}_{m+1}^+ \cap \mathcal{P}_{m+2}^+$. Then $p_{m+1}(\lambda) = p_{m+2}(\lambda) = 0$ (hence $q_{m+1}(\lambda) = q_{m+2}(\lambda) = 0$). Moreover, none of $f^{(i)}(\lambda)$ $(0 \leq i \leq m)$ is equal to 2 or 5.

The expansion along the first row of $q_{m+2}(\lambda)$ gives

$$q_{m+2}(\lambda) = s(f^{(m)}(\lambda))q_{m+1}(\lambda) - r(f^{(m)}(\lambda))l(f^{(m-1)}(\lambda))q_m(\lambda).$$

Noticing $q_{m+1}(\lambda) = q_{m+2}(\lambda) = 0$, we have

$$r(f^{(m)}(\lambda))l(f^{(m-1)}(\lambda))q_m(\lambda) = -2(2 - f^{(m)}(\lambda))(5 - f^{(m)}(\lambda))(f^{(m-1)}(\lambda) - 6)q_m(\lambda) = 0.$$

Hence $q_m(\lambda) = 0$ or $f^{(m-1)}(\lambda) = 6$, since $f^{(m)}(\lambda) \neq 2$ or 5.

If $q_m(\lambda) = 0$, then $\lambda \in \mathcal{P}_m^+$, hence $\mathcal{P}_m^+ \cap \mathcal{P}_{m+1}^+ \neq \emptyset$. This contradicts to our induction assumption.

Hence we have $f^{(m-1)}(\lambda) = 6$, i.e., $f^{(m-2)}(\lambda) = 3$. Noticing that λ is also a root of $p_{m+1}(x)$, Lemma 4.1 says that $\phi_-^{(m-2)}(3)$ is a root of $p_{m+1}(x)$. Hence $p_{m+1}(\phi_-^{(m-2)}(3)) = 0$, which contradict to Proposition 4.1(6). Hence such λ can not exist. So we get the desired result. □

Then we can prove:

Lemma 4.12. *For each $m \geq 2$, $\mathcal{P}_m^+ \cap \mathcal{P}_m^- = \emptyset$.*

Proof. For $m = 2$ or 3, it can be checked by a direct computation. Let $m \geq 4$. Suppose there is an eigenvalue $\lambda_m \in \mathcal{P}_m^+ \cap \mathcal{P}_m^-$. Then by Lemma 4.10, $p_m(\lambda_m) = \widetilde{p}_m(\lambda_m) = 0$. For each $2 \leq i \leq m$, denote by $\lambda_i^{(m)}$ the successor of λ_m of order $(m-i)$. Obviously we have $q_m(\lambda_m) = \widetilde{q}_m(\lambda_m) = 0$ and $\lambda_i^{(m)} \neq 2$ or 5 for $2 \leq i \leq m$. Furthermore, by Lemma 4.11, we have $p_{m-1}(\lambda_m) \neq 0$, hence $q_{m-1}(\lambda_m) \neq 0$.

Using the expansions of $q_m(\lambda_m)$ and $\widetilde{q}_m(\lambda_m)$ along their first rows respectively, we have

$$s(\lambda_2^{(m)})q_{m-1}(\lambda_m) - r(\lambda_2^{(m)})l(\lambda_3^{(m)})q_{m-2}(\lambda_m) = 0$$

and

$$\widetilde{s}(\lambda_2^{(m)})q_{m-1}(\lambda_m) - \widetilde{r}(\lambda_2^{(m)})l(\lambda_3^{(m)})q_{m-2}(\lambda_m) = 0.$$

Hence, the vector $(q_{m-1}(\lambda_m), q_{m-2}(\lambda_m))$ can be viewed as a non-zero solution of the system of linear equations,

$$\begin{cases} s(\lambda_2^{(m)})x - r(\lambda_2^{(m)})l(\lambda_3^{(m)})y = 0 \\ \widetilde{s}(\lambda_2^{(m)})x - \widetilde{r}(\lambda_2^{(m)})l(\lambda_3^{(m)})y = 0. \end{cases}$$

Thus

$$\begin{vmatrix} s(\lambda_2^{(m)}) & 2(2-\lambda_2^{(m)})(5-\lambda_2^{(m)})(\lambda_3^{(m)}-6) \\ \widetilde{s}(\lambda_2^{(m)}) & 2(5-\lambda_2^{(m)})(\lambda_3^{(m)}-6) \end{vmatrix} = 0.$$

Since $\lambda_2^{(m)} \neq 5$, we have $\lambda_3^{(m)} = 6$ or $s(\lambda_2^{(m)}) = (2-\lambda_2^{(m)})\widetilde{s}(\lambda_2^{(m)})$. By Substituting the expressions for $s(x)$ and $\widetilde{s}(x)$, we get $\lambda_2^{(m)} = 6$ or $\lambda_3^{(m)} = 6$. Hence we have $\lambda_3^{(m)} = 3$ or $\lambda_4^{(m)} = 3$, i.e., $f^{(m-3)}(\lambda_m) = 3$, or $f^{(m-4)}(\lambda_m) = 3$.

Noticing that λ_m is a root of $q_m(x)$, by using Lemma 4.1, we can see that either $\phi_-^{(m-3)}(3)$ or $\phi_-^{(m-4)}(3)$ is a root of $q_m(x)$, i.e., $q_m(\phi_-^{(m-3)}(3)) = 0$ or $q_m(\phi_-^{(m-4)}(3)) = 0$. An expansion of $q_m(\phi_-^{(m-4)}(3))$ along the first row yields that

$$q_m(\phi_-^{(m-4)}(3)) = s(f^{(2)}(3))q_{m-1}(\phi_-^{(m-4)}(3)) = 848q_{m-1}(\phi_-^{(m-4)}(3))$$

since $l(f(3)) = 0$. Hence we have either $q_m(\phi_-^{(m-3)}(3)) = 0$ or $q_{m-1}(\phi_-^{(m-4)}(3)) = 0$. By Proposition 4.1(6), this is impossible. Hence such λ_m can not exist. So $\mathcal{P}_m^+ \cap \mathcal{P}_m^- = \emptyset$. □

We summarize what we have proved:

Theorem 4.1. *For each $m \geq 2$, $\mathcal{P}_m^+$ consists of r_m distinct eigenvalues satisfying*

$$0 < \lambda_{m,1} < \lambda_{m,2} < \cdots < \lambda_{m,r_m-1} < 5 < \lambda_{m,r_m} < 6.$$

A relation between $\mathcal{P}_m^+$ and $\mathcal{P}_{m+1}^+$ is shown in (4.7). Similar properties hold for $\mathcal{P}_m^-$ with r_m replaced by s_m. Each $\lambda_m \in \mathcal{P}_m$ has multiplicity 1. Moreover, the spectrum $\mathcal{S}_m$ of Δ_m on Ω_m satisfies

$$\mathcal{S}_m = \mathcal{L}_m \cup \mathcal{P}_m^+ \cup \mathcal{P}_m^- \cup \mathcal{M}_m,$$

where the union is disjoint.

5 Primitive Dirichlet eigenvalues of Δ

Having found the primitive Dirichlet eigenvalues and eigenfunctions for Δ_m, it is natural to believe that the primitive Dirichlet eigenvalues of Δ can be obtained in the limit as m goes to infinity. This is true for the spectrum in $\mathcal{SG}$ case, benefiting from the spectral decimation method and the eigenfunction extension algorithm. Our goal in this section is to extend this recipe to Ω case by instead using the weak spectral decimation introduced in Section 3. Comparing to the $\mathcal{SG}$ case, our method is more based on estimates. We will focus on the symmetric case, since the skew-symmetric case can be got by using a similar discussion. We use the $\widetilde{\phi}_\pm$ notations introduced in Section 3. Recall that if α_m, β_m are two consecutive eigenvalues in $\mathcal{P}_m^+$ with $\alpha_m < \beta_m$, then we always have

$$\phi_-(\alpha_m) < \widetilde{\phi}_-(\beta_m) < \phi_-(\beta_m) \text{ and } \phi_+(\beta_m) < \widetilde{\phi}_+(\beta_m) < \phi_+(\alpha_m), \tag{5.1}$$

and if β_m is the least eigenvalue in $\mathcal{P}_m^+$, then instead we have

$$0 < \widetilde{\phi}_-(\beta_m) < \phi_-(\beta_m) \text{ and } \phi_+(\beta_m) < \widetilde{\phi}_+(\beta_m) < 5.$$

Let $m_0 \geq 2$, λ_{m_0} be a m_0-level primitive symmetric eigenvalue, $\{\lambda_m\}_{m\geq m_0}$ be an infinite sequence related by $\lambda_{m+1} = \widetilde{\phi}_-(\lambda_m)$ or $\widetilde{\phi}_+(\lambda_m)$, $\forall m \geq m_0$, assuming that there are only a finite number of $\widetilde{\phi}_+$ relations. Call the minimum value m_1, such that $\forall m \geq m_1$, $\lambda_{m+1} = \widetilde{\phi}_-(\lambda_m)$, the *generation of fixation* of the sequence $\{\lambda_m\}_{m\geq m_0}$. In all that follows in this section, we always use $\{\lambda_m\}_{m\geq m_0}$ as such a sequence without specifical declaration.

The first fact about this sequence is:

Lemma 5.1. $\lim_{m\to\infty} 5^m\lambda_m$ *exists.*

Proof. Without loss of generality, assume $\lambda_{m_1} < 5$, otherwise, we could choose $\widetilde{m}_1 = m_1 + 1$ and use $\widetilde{m}_1$ to replace m_1 in the following proof.

Let $m \geq m_1$, then $\frac{\lambda_{m+1}}{\lambda_m} = \frac{\widetilde{\phi}_-(\lambda_m)}{\lambda_m} \leq \frac{\phi_-(\lambda_m)}{\lambda_m} = \frac{\phi_-(\lambda_m)}{\phi_-(\lambda_m)(5-\phi_-(\lambda_m))} = \frac{1}{5-\phi_-(\lambda_m)}$. Since $0 < \lambda_m < 5$, we have $0 < \phi_-(\lambda_m) < 2$, hence $\frac{1}{5-\phi_-(\lambda_m)} < \frac{1}{3}$. Thus $\sum_{m\geq m_1} \lambda_m < \infty$.

Furthermore, $\frac{5^{m+1}\lambda_{m+1}}{5^m\lambda_m} = 5\frac{\lambda_{m+1}}{\lambda_m} \leq \frac{5}{5-\phi_-(\lambda_m)} = 1+\frac{\phi_-(\lambda_m)}{5-\phi_-(\lambda_m)}$. Noticing that $\sum_{m\geq m_1} \frac{\phi_-(\lambda_m)}{5-\phi_-(\lambda_m)} \leq \frac{1}{3}\sum_{m\geq m_1} \phi_-(\lambda_m) \leq \frac{1}{3}\sum_{m\geq m_1} \lambda_m < \infty$ since $\phi_-'(x) < 1$ whenever $0 < x < 5$, we get that $\Pi_{m\geq m_1} \frac{5^{m+1}\lambda_{m+1}}{5^m\lambda_m}$ converges. Hence $\lim_{m\to\infty} 5^m\lambda_m$ exists. □

The following is an estimate of the difference between $\widetilde{\phi}_-(\lambda_m)$ and $\phi_-(\lambda_m)$ for λ_m in the sequence $\{\lambda_m\}_{m\geq m_0}$.

Proposition 5.1.

$$\sum_{m\geq m_1} 5^m(\widetilde{\phi}_-(\lambda_m) - \phi_-(\lambda_m)) < \infty.$$

In particular, $\lim_{m\to\infty} 5^m(\widetilde{\phi}_-(\lambda_m) - \phi_-(\lambda_m)) = 0$.

Proof. Without loss of generality, assume $\lambda_{m_1} < 5$. From Lemma 5.1, we have $\sum_{m\geq m_1}(5^{m+1}\lambda_{m+1} - 5^m\lambda_m) < \infty$. Hence

$$\begin{aligned} & \sum_{m\geq m_1} 5^m(5\lambda_{m+1} - \lambda_m) \\ = & \sum_{m\geq m_1} 5^m(5\widetilde{\phi}_-(\lambda_m) - \phi_-(\lambda_m)(5 - \phi_-(\lambda_m))) \\ = & \sum_{m\geq m_1} (5^{m+1}(\widetilde{\phi}_-(\lambda_m) - \phi_-(\lambda_m)) + 5^m(\phi_-(\lambda_m))^2) < \infty. \end{aligned} \tag{5.2}$$

Since $0 < \phi_-'(x) < 1$ whenever $0 < x < 5$, we have $5^m(\phi_-(\lambda_m))^2 \leq 5^m\lambda_m^2$. Still from Lemma 5.1, we have $\lambda_m = O(\frac{1}{5^m})$, hence $5^m(\phi_-(\lambda_m))^2 \leq \frac{c}{5^m}$ for some constant c. Thus $\sum_{m\geq m_1} 5^m(\phi_-(\lambda_m))^2 < \infty$. Combining this with (5.2), we get $\sum_{m\geq m_1} 5^m(\widetilde{\phi}_-(\lambda_m) - \phi_-(\lambda_m)) < \infty$. □

To reveal some further properties of the limit $\lim_{m\to\infty} 5^m\lambda_m$, the following lemma is required, which is a generalization of formula (5.1).

Lemma 5.2. *Let $m \geq 2$. α_m, β_m be two consecutive eigenvalues in $\mathcal{P}_m^+$ with $\alpha_m < \beta_m$. Then $\forall l \in \mathbf{N}$,*

$$\phi_-^{(l)}(\alpha_m) < \widetilde{\phi}_-^{(l)}(\beta_m). \tag{5.3}$$

Proof. First we need to prove the following relation.

$$p_{m+l}(\phi_-^{(l)}(\alpha_m)) \sim (-1)^{l-1}p_{m+1}(\phi_-(\alpha_m)), \quad \forall l \in \mathbf{N}. \tag{5.4}$$

In fact, when $l \geq 3$, using the Laplace theorem to expand the determinant $q_{m+l}(\phi_-^{(l)}(\alpha_m))$ according to the last $(l-1)$ rows, we have

$$q_{m+l}(\phi_-^{(l)}(\alpha_m)) = q_l(\phi_-^{(l)}(\alpha_m))q_{m+1}(\phi_-(\alpha_m)) - l(\phi_-^{(2)}(\alpha_m))q_{l-1}(\phi_-^{(l)}(\alpha_m))r(\phi_-(\alpha_m))q_m(\alpha_m).$$

Since $q_m(\alpha_m) = 0$, we have

$$q_{m+l}(\phi_-^{(l)}(\alpha_m)) = q_l(\phi_-^{(l)}(\alpha_m))q_{m+1}(\phi_-(\alpha_m)).$$

This equality also holds for $l = 2$ by instead using an expansion along the last row of $q_{m+2}(\phi_-^{(2)}(\alpha_m))$. Hence for each $l \geq 2$, we always have $q_{m+l}(\phi_-^{(l)}(\alpha_m)) = q_l(\phi_-^{(l)}(\alpha_m))q_{m+1}(\phi_-(\alpha_m))$.

Then from Lemma B in Appendix, we have $q_l(\phi_-^{(l)}(\alpha_m)) > 0$, hence $q_{m+l}(\phi_-^{(l)}(\alpha_m)) \sim q_{m+1}(\phi_-(\alpha_m))$. By the relation between $p_{m+l}(x)$ and $q_{m+l}(x)$, we can easily get (5.4).

Now we prove (5.3). When $l = 1$, (5.3) follows from (5.1) directly. In order to use the induction, assuming (5.3) holds for l, we turn to prove

$$\phi_-^{(l+1)}(\alpha_m) < \widetilde{\phi}_-^{(l+1)}(\beta_m).$$

Suppose α_m and β_m are the k'th and $(k+1)$'th eigenvalues in $\mathcal{P}_m^+$ respectively. Recall that in Lemma 4.4, we have proved that $p_{m+1}(\phi_-(\alpha_m)) \sim (-1)^{m+k-1}$. Combining this with (5.4), we have

$$p_{m+l+1}(\phi_-^{(l+1)}(\alpha_m)) \sim (-1)^{m+k+l-1}. \tag{5.5}$$

On the other hand, if we denote $\alpha_{m+l} = \widetilde{\phi}_-^{(l)}(\alpha_m)$ and $\beta_{m+l} = \widetilde{\phi}_-^{(l)}(\beta_m)$, then it is easy to see that α_{m+l} and β_{m+l} are the k'th and $(k+1)$'th eigenvalues in $\mathcal{P}_{m+l}^+$ respectively. Lemma 4.4 says that

$$p_{m+l+1}(\phi_-(\alpha_{m+l})) \sim (-1)^{m+l+k-1} \tag{5.6}$$

and

$$p_{m+l+1}(\phi_-(\beta_{m+l})) \sim (-1)^{m+l+k}. \tag{5.7}$$

Furthermore, if we denote $\beta_{m+l+1} = \widetilde{\phi}_-^{(l+1)}(\beta_m)$, then β_{m+l+1} is the only root of $p_{m+l+1}(x)$ located between $\phi_-(\alpha_{m+l})$ and $\phi_-(\beta_{m+l})$, i.e.,

$$\phi_-(\alpha_{m+l}) < \beta_{m+l+1} < \phi_-(\beta_{m+l}). \tag{5.8}$$

Noticing that from the induction assumption, we have $\phi_-^{(l+1)}(\alpha_m) < \phi_-(\beta_{m+l})$ since $\beta_{m+l} = \widetilde{\phi}_-^{(l)}(\beta_m)$. Moreover, (5.5) and (5.7) say that there exists at least one root of $p_{m+l+1}(x)$, denoted by β^*_{m+l+1}, between $\phi_-^{(l+1)}(\alpha_m)$ and $\phi_-(\beta_{m+l})$, i.e.,

$$\phi_-^{(l+1)}(\alpha_m) < \beta^*_{m+l+1} < \phi_-(\beta_{m+l}). \tag{5.9}$$

Since $\phi_-(\alpha_{m+l}) = \phi_-(\widetilde{\phi}_-^{(l)}(\alpha_m)) < \phi_-^{(l+1)}(\alpha_m)$, we have

$$\phi_-(\alpha_{m+l}) < \phi_-^{(l+1)}(\alpha_m) < \beta^*_{m+l+1} < \phi_-(\beta_{m+l}).$$

Combing this with (5.8), from the uniqueness of β_{m+l+1}, we have $\beta_{m+l+1} = \beta^*_{m+l+1}$. Hence substituting it into (5.9), we finally get $\phi_-^{(l+1)}(\alpha_m) < \beta_{m+l+1}$, i.e., $\phi_-^{(l+1)}(\alpha_m) < \widetilde{\phi}_-^{(l+1)}(\beta_m)$, which is the desired result. □

The following is an application of Lemma 5.2.

Lemma 5.3. *Let $m_1 \geq 2$, $\alpha_{m_1}, \beta_{m_1}$ be two consecutive eigenvalues in $\mathcal{P}_{m_1}^+$ with $\alpha_{m_1} < \beta_{m_1}$. $\{\alpha_m\}_{m \geq m_1}$ is an infinite sequence related by $\alpha_{m+1} = \widetilde{\phi}_-(\alpha_m), \forall m \geq m_1$;*

$\{\beta_m\}_{m\geq m_1}$ *is an infinite sequence related by* $\beta_{m+1} = \widetilde{\phi}_-(\beta_m), \forall m \geq m_1$. *Then* $\forall m \geq m_1$, $\alpha_m < \beta_m$. *Moreover,*

$$\lim_{m\to\infty} 5^m \alpha_m < \lim_{m\to\infty} 5^m \beta_m.$$

Remark. *In* $\mathcal{SG}$ *case, this is a direct result since* $\phi_-(x)$ *is a definite strictly increasing continuous function.*

Proof of Lemma 5.3. Let $m > m_1$. Since $\alpha_m = \widetilde{\phi}_-^{(m-m_1)}(\alpha_{m_1})$ and $\beta_m = \widetilde{\phi}_-^{(m-m_1)}(\beta_{m_1})$, we have

$$\alpha_m < \phi_-^{(m-m_1)}(\alpha_{m_1}) < \widetilde{\phi}_-^{(m-m_1)}(\beta_{m_1}) = \beta_m \tag{5.10}$$

by Lemma 5.2. Hence $\forall m > m_1$, $\alpha_m < \beta_m$.

Now we prove $\lim_{m\to\infty} 5^m \alpha_m < \lim_{m\to\infty} 5^m \beta_m$.

Let $m > m_1$. Then from (5.10), we have

$$\alpha_m < \phi_-^{(m-m_1-1)}(\widetilde{\phi}_-(\alpha_{m_1})) < \phi_-^{(m-m_1)}(\alpha_{m_1}) < \beta_m.$$

Hence $\beta_m - \alpha_m > \phi_-^{(m-m_1-1)}(\phi_-(\alpha_{m_1})) - \phi_-^{(m-m_1-1)}(\widetilde{\phi}_-(\alpha_{m_1}))$. Since $\phi_-'(x) \geq \frac{1}{5}$ whenever $0 < x < 5$, and $0 < \widetilde{\phi}_-(\alpha_{m_1}) < \phi_-(\alpha_{m_1}) < 5$, we have

$$\beta_m - \alpha_m > \frac{1}{5^{m-m_1-1}}(\phi_-(\alpha_{m_1}) - \widetilde{\phi}_-(\alpha_{m_1})).$$

Hence $5^m(\beta_m - \alpha_m) > 5^{m_1+1}(\phi_-(\alpha_{m_1}) - \widetilde{\phi}_-(\alpha_{m_1}))$ which yields that

$$\lim_{m\to\infty} 5^m(\beta_m - \alpha_m) \geq 5^{m_1+1}(\phi_-(\alpha_{m_1}) - \widetilde{\phi}_-(\alpha_{m_1})) > 0.$$

Thus $\lim_{m\to\infty} 5^m \alpha_m < \lim_{m\to\infty} 5^m \beta_m$. □

Lemma 5.4. $\lim_{m\to\infty} 5^m \lambda_m > 0$.

Remark. *In* $\mathcal{SG}$ *case, this is a direct result, since* $\{5^m \lambda_m\}_{m\geq m_1}$ *is then a monotone increasing sequence.*

Proof of Lemma 5.4. Without loss of generality, we assume that λ_{m_1} is the least eigenvalue in $\mathcal{P}_{m_1}^+$, since Lemma 5.3 says that it suffices to prove for this special case. Then $\forall m \geq m_1$, λ_m is also the least eigenvalue in $\mathcal{P}_m^+$. Note that Lemma B in Appendix says that $\forall m \geq m_1$, we have $\lambda_m \geq \phi_-^{(m)}(6)$. Hence

$$\lim_{m\to\infty} 5^m \lambda_m \geq \lim_{m\to\infty} 5^m \phi_-^{(m)}(6) > 0,$$

where the existence and positivity of the second limit are already shown in $\mathcal{SG}$ case. See [10]. □

Now we define

$$\lambda = \frac{3}{2} \lim_{m\to\infty} 5^m \lambda_m.$$

We will prove λ is an primitive eigenvalue of Δ on the fractal Ω.

Note that $\forall m \geq m_0$, $\lambda_m \in \mathcal{P}_m^+$, i.e., λ_m is a root of both $p_m(x)$ and $q_m(x)$ by Lemma 4.5 and Theorem 4.1. As in Section 4, denote by $\lambda_i^{(m)}$ the successor of λ_m of order $(m-i)$ with $2 \leq i \leq m$ for simplicity. Lemma 4.6 and Theorem 4.1 say that the system (4.2) of equations has 1-dimensional solutions $(b_1, b_2, \cdots, b_{m-1})$ with $b_1 \neq 0$ and $b_{m-1} \neq 0$. We normalize the solution by requiring $b_1 = 1$, and write it as $(b_1^{(m)}, b_2^{(m)}, \cdots, b_{m-1}^{(m)})$ with $b_1^{(m)} = 1$ to specify its relation to λ_m. We always denote $b_0^{(m)} = 0$ for convenience. As described in Section 4, from $(b_1^{(m)}, b_2^{(m)}, \cdots, b_{m-1}^{(m)})$ one can recover the unique (up to a constant) λ_m-eigenfunction u_m on Ω_m (noticing that $\lambda_i^{(m)} \neq 2$ or 5, $\forall 2 \leq i \leq m$). Hence

$$\begin{cases} -\Delta_m u_m = \lambda_m u_m \text{ on } \Omega_m, \\ u_m|_{\partial\Omega_m} = 0. \end{cases}$$

For each $m \geq m_0$, we start with the λ_m-eigenfunction u_m on Ω_m, and extend u_m to Ω by successively using the eigenfunction extension algorithm corresponding to the revised eigenvalue sequence $\{\lambda_m, \phi_-(\lambda_m), \phi_-^{(2)}(\lambda_m), \cdots\}$ (starting from λ_m, but continued with the standard spectral decimation eigenvalues) to get a primitive eigenfunction (possessing the symmetry in each cell $F_w(\mathcal{SG})$ under the reflection symmetry fixing $F_w q_0$ with word w taking symbols only from $\{1, 2\}$) on Ω. We still denote u_m for this function. Obviously, u_m may not satisfy the Dirichlet boundary condition. $\forall i > m$, we use $\lambda_i^{(m)} = \phi_-^{(i-m)}(\lambda_m)$ to denote the i-level revised eigenvalue. Hence for each $m \geq m_0$, u_m is an eigenfunction associated to the eigenvalue sequence $\{\lambda_i^{(m)}\}_{i\geq 2}$, where $\lambda_i^{(m)} = f^{(m-i)}(\lambda_m)$, $\forall 2 \leq i \leq m$, and $\lambda_i^{(m)} = \phi_-^{(i-m)}(\lambda_m)$, $\forall i > m$. We use $b_i^{(m)}$ ($\forall i \geq m$) to denote the value of u_m at vertex $F_1^i q_0$. Hence $\{b_i^{(m)}\}_{i\geq 0}$ are the values of u_m on the skeleton of Ω which conversely determine u_m on Ω. We have the following relationship between $\{\lambda_i^{(m)}\}_{i\geq 2}$ and $\{b_i^{(m)}\}_{i\geq 0}$.

$$(4 - \lambda_{i+1}^{(m)}) b_i^{(m)} = 2 b_{i+1}^{(m)} + \frac{(14 - 3\lambda_{i+1}^{(m)}) b_i^{(m)} + (6 - \lambda_{i+1}^{(m)}) b_{i-1}^{(m)}}{(2 - \lambda_{i+1}^{(m)})(5 - \lambda_{i+1}^{(m)})}, \quad \forall i \geq 1, \tag{5.11}$$

which follows from the eigenvalue equation at the vertex $F_1^i q_0$. Note that when $1 \leq i \leq m-1$, these are exactly the equations in (4.1). Moreover, u_m on Ω satisfies that

$$\begin{cases} -\Delta u_m = 5^m \Phi(\lambda_m) u_m \text{ on } \Omega, \\ u_m(q_0) = 0, \\ u_m|_L = \lim_{i\to\infty} b_i^{(m)} < \infty, \end{cases}$$

where $\Phi(z)$ is a function defined by $\Phi(z) = \frac{3}{2}\lim_{k\to\infty} 5^k \phi_-^{(k)}(z)$. The existence of the limit $\lim_{i\to\infty} b_i^{(m)}$ will be given later.

It is easy to find that $5^m \Phi(\lambda_m) \to \lambda$ as m goes to infinity. For each $m \geq m_1$, let $v_m = \frac{u_m}{\|u_m\|_\infty}$. We will prove that $\{v_m\}_{m\geq m_1}$ contains a subsequence which converges uniformly to a continuous function v on Ω and v is a Dirichlet eigenfunction associated to λ. We need the following lemmas.

Lemma 5.5. *There exists a constant $C_1 > 0$ depending only on m_1, such that $\forall i \in \mathbf{N}$, $\forall p \in \mathbf{N}$, we have $|b_{i+p}^{(m)} - b_i^{(m)}| \leq C_1(\frac{3}{10})^i \|u_m\|_\infty$ uniformly on $m \geq m_1$.*

Proof. Without loss of generality, assume $i > m_1$ and λ_{m_1} is not the largest eigenvalue in $\mathcal{P}_{m_1}^+$. Denote by γ_{m_1} the next eigenvalue of λ_{m_1} in $\mathcal{P}_{m_1}^+$. Let $\{\gamma_m\}_{m \geq m_1}$ be the infinite sequence staring from γ_{m_1} related by $\gamma_{m+1} = \widetilde{\phi}_-(\gamma_m)$, $\forall m \geq m_1$. We now show

$$\lambda_{i+1}^{(m)} < \gamma_{i+1} < \phi_-(2), \quad \forall m \geq m_1. \tag{5.12}$$

In fact if $m \geq i+1$, then

$$\lambda_{i+1}^{(m)} = f^{(m-i-1)}(\lambda_m) = f^{(m-i-1)}(\widetilde{\phi}_-^{(m-i-1)}(\lambda_{i+1})) \leq f^{(m-i-1)}(\phi_-^{(m-i-1)}(\lambda_{i+1})) = \lambda_{i+1} < \gamma_{i+1}.$$

If $m < i+1$, then $\lambda_{i+1}^{(m)} = \phi_-^{(i+1-m)}(\lambda_m) < \widetilde{\phi}_-^{(i+1-m)}(\gamma_m) = \gamma_{i+1}$ by using Lemma 5.2. The right inequality of (5.12) is obvious. Hence (5.12) always holds.

On the other hand, notice that from (5.11),

$$\begin{aligned} b_{i+1}^{(m)} - b_i^{(m)} &= \frac{s(\lambda_{i+1}^{(m)})b_i^{(m)} - (6-\lambda_{i+1}^{(m)})b_{i-1}^{(m)}}{2(2-\lambda_{i+1}^{(m)})(5-\lambda_{i+1}^{(m)})} - b_i^{(m)} \\ &= \frac{(6-\lambda_{i+1}^{(m)})(b_i^{(m)} - b_{i-1}^{(m)}) - (20\lambda_{i+1}^{(m)} - 9(\lambda_{i+1}^{(m)})^2 + (\lambda_{i+1}^{(m)})^3)b_i^{(m)}}{2(2-\lambda_{i+1}^{(m)})(5-\lambda_{i+1}^{(m)})}. \end{aligned}$$

Hence

$$|b_{i+1}^{(m)} - b_i^{(m)}| \leq \frac{|6-\lambda_{i+1}^{(m)}|}{2|2-\lambda_{i+1}^{(m)}| \cdot |5-\lambda_{i+1}^{(m)}|}|b_i^{(m)} - b_{i-1}^{(m)}| + \frac{|20 - 9\lambda_{i+1}^{(m)} + (\lambda_{i+1}^{(m)})^2|}{2|2-\lambda_{i+1}^{(m)}| \cdot |5-\lambda_{i+1}^{(m)}|}|\lambda_{i+1}^{(m)}| \cdot |b_i^{(m)}|.$$

In the remaining proof, we use c to denote different constants.

By (5.12), we have

$$|b_{i+1}^{(m)} - b_i^{(m)}| \leq \frac{3}{(2-\gamma_{i+1})(5-\gamma_{i+1})}|b_i^{(m)} - b_{i-1}^{(m)}| + c\gamma_{i+1}|b_i^{(m)}|.$$

Noticing that $\gamma_i = O(\frac{1}{5^i})$ and $|b_i^{(m)}| \leq \|u_m\|_\infty$, we get

$$|b_{i+1}^{(m)} - b_i^{(m)}| \leq (\frac{3}{10} + \frac{c}{5^i})|b_i^{(m)} - b_{i-1}^{(m)}| + \frac{c}{5^i}\|u_m\|_\infty.$$

Hence

$$|b_{i+1}^{(m)} - b_i^{(m)}| \leq \frac{3}{10}|b_i^{(m)} - b_{i-1}^{(m)}| + \frac{c}{5^i}\|u_m\|_\infty.$$

Similarly we have the estimates

$$|b_i^{(m)} - b_{i-1}^{(m)}| \leq \frac{3}{10}|b_{i-1}^{(m)} - b_{i-2}^{(m)}| + \frac{c}{5^{i-1}}\|u_m\|_\infty$$

till

$$|b^{(m)}_{m_1+2} - b^{(m)}_{m_1+1}| \le \frac{3}{10}|b^{(m)}_{m_1+1} - b^{(m)}_{m_1}| + \frac{c}{5^{m_1+1}}\|u_m\|_\infty.$$

A routine argument shows that

$$|b^{(m)}_{i+1} - b^{(m)}_{i}| \le (\frac{3}{10})^{i-m_1}|b^{(m)}_{m_1+1} - b^{(m)}_{m_1}| + (\frac{3}{10})^{i-m_1-1}\frac{c}{5^{m_1+1}}\|u_m\|_\infty.$$

Hence we have proved that

$$|b^{(m)}_{i+1} - b^{(m)}_{i}| \le c(\frac{3}{10})^{i}\|u_m\|_\infty$$

where c depends only on m_1.

Similarly, we have

$$|b^{(m)}_{i+2} - b^{(m)}_{i+1}| \le c(\frac{3}{10})^{i+1}\|u_m\|_\infty,$$

till

$$|b^{(m)}_{i+p} - b^{(m)}_{i+p-1}| \le c(\frac{3}{10})^{i+p-1}\|u_m\|_\infty.$$

By adding up the above estimates, we finally get $|b^{(m)}_{i+p} - b^{(m)}_{i}| \le C_1(\frac{3}{10})^{i}\|u_m\|_\infty$.□

Lemma 5.6. *For each $m \ge m_1$, $\lim_{i\to\infty} b^{(m)}_i$ exists. Moreover, there exists a constant $C_2 > 0$ depending only on m_1, such that $|\lim_{i\to\infty} b^{(m)}_i| \le C_2(\frac{3}{10})^m\|u_m\|_\infty$ uniformly on $m \ge m_1$.*

Proof. For each $m \ge m_1$, Lemma 5.5 says that each sequence $\{b^{(m)}_i\}_{i\ge1}$ is a Cauchy sequence, hence $\lim_{i\to\infty} b^{(m)}_i$ exists.

Taking $i = m$, $p = 1$ in Lemma 5.5, noticing that $b^{(m)}_m = 0$, we get that $|b^{(m)}_{m+1}| \le C_1(\frac{3}{10})^m\|u_m\|_\infty$.

On the other hand, $\forall i > m + 1$, notice that $|b^{(m)}_i| \le |b^{(m)}_i - b^{(m)}_{m+1}| + |b^{(m)}_{m+1}|$. By using Lemma 5.5 again, we have

$$|b^{(m)}_i| \le C_1(\frac{3}{10})^{m+1}\|u_m\|_\infty + C_1(\frac{3}{10})^{m}\|u_m\|_\infty = C_2(\frac{3}{10})^{m}\|u_m\|_\infty.$$

Letting $i \to \infty$, we get the desired result. □

In the following context, let θ_m denote the limit $\lim_{i\to\infty} b^{(m)}_i/\|u_m\|_\infty$. Lemma 5.6 guarantees the existence of this limit, and furthermore, $|\theta_m| \le C_2(\frac{3}{10})^m$. Let $v_m = \frac{u_m}{\|u_m\|_\infty}$, $\forall m \ge m_1$. Then v_m on Ω satisfies that

$$\begin{cases} -\Delta v_m = 5^m\Phi(\lambda_m)v_m \text{ on } \Omega, \\ v_m(q_0) = 0, \\ v_m|_L = \theta_m. \end{cases}$$

Lemma 5.7. *$\{\partial_n v_m(q_0)\}_{m\ge m_1}$ is uniformly bounded, i.e., there exist a constant $C_3 > 0$ depending only on m_1, such that $|\partial_n v_m(q_0)| \le C_3$.*

Proof. Let $m \geq m_1$. Choosing a harmonic function h such that $h(q_0) = 1$, $h(F_1 q_0) = h(F_2 q_0) = 0$, the local Gauss-Green formula on $F_0(\mathcal{SG})$ says that

$$\mathcal{E}_{F_0(\mathcal{SG})}(v_m, h) = -\int_{F_0(\mathcal{SG})} (\Delta v_m) h d\mu + \sum_{\partial F_0(\mathcal{SG})} h \partial_n v_m.$$

Hence $|\partial_n v_m(q_0)| \leq |\mathcal{E}_{F_0(\mathcal{SG})}(v_m, h)| + |\int_{F_0(\mathcal{SG})} (\Delta v_m) h d\mu|$.

Since h is harmonic on $F_0(\mathcal{SG})$, we have $\mathcal{E}_{F_0(\mathcal{SG})}(v_m, h) = \frac{5}{3}\mathcal{E}(v_m \circ F_0, h \circ F_0) = \frac{5}{3}\mathcal{E}_0(v_m \circ F_0, h \circ F_0)$. Noticing that $h(q_0) = 1$, $h(F_1 q_0) = h(F_2 q_0) = 0$, we get $|\mathcal{E}_{F_0(\mathcal{SG})}(v_m, h)| \leq c_1$, since $\|v_m\|_\infty = 1$.

On the other hand, since $\Delta v_m = -5^m \Phi(\lambda_m) v_m$, we have $|\int_{F_0(\mathcal{SG})} (\Delta v_m) h d\mu| \leq 5^m \Phi(\lambda_m) \|v_m\|_\infty \cdot \|h\|_\infty \mu(F_0(\mathcal{SG})) \leq c_2$, since $5^m \Phi(\lambda_m) \to \lambda$.

Hence $|\partial_n v_m(q_0)| \leq c_1 + c_2 \triangleq C_3$. □

Lemma 5.8. *$\{\mathcal{E}(v_m)\}_{m \geq m_1}$ is uniformly bounded, i.e., there exists a constant $C_4 > 0$ depending only on m_1, such that $\mathcal{E}(v_m) \leq C_4$.*

Proof. $\forall n \geq m_1$, let K_n be the part of Ω above $\partial\Omega_n \setminus \{q_0\}$. We first prove $\{\mathcal{E}_{K_n}(v_m)\}_{m \geq m_1}$ is uniformly bounded and the upper bound is independent on n.

Fix $n \geq m_1$, $m \geq m_1$. The Gauss-Green formula says that $\int_{K_n} \Delta v_m d\mu = \sum_{\partial K_n} \partial_n v_m$. From the symmetry property of v_m, $\partial_n v_m$ takes same value along $\partial K_n \setminus \{q_0\}$. Hence we get

$$-5^m \Phi(\lambda_m) \int_{K_n} v_m d\mu = \partial_n v_m(q_0) + 2^n \partial_n v_m(F_1^n(q_0)). \tag{5.13}$$

On the other hand, the Gauss-Green formula also says that

$$\begin{aligned} \mathcal{E}_{K_n}(v_m) &= -\int_{K_n} (\Delta v_m) v_m d\mu + \sum_{\partial K_n} v_m \partial_n v_m \\ &= 5^m \Phi(\lambda_m) \int_{K_n} v_m^2 d\mu + 2^n v_m(F_1^n q_0) \partial_n v_m(F_1^n q_0), \end{aligned}$$

since $v_m(q_0) = 0$. Combined with (5.13), it follows

$$\mathcal{E}_{K_n}(v_m) = 5^m \Phi(\lambda_m) \int_{K_n} v_m^2 d\mu + v_m(F_1^n q_0)(-5^m \Phi(\lambda_m) \int_{K_n} v_m d\mu - \partial_n v_m(q_0)).$$

Since $5^m \Phi(\lambda_m) \to \lambda$, there exists a constant $c > 0$, such that $5^m \Phi(\lambda_m) \leq c$. Hence

$$\mathcal{E}_{K_n}(v_m) \leq c\|v_m\|_\infty^2 + \|v_m\|_\infty (c\|v_m\|_\infty + |\partial_n v_m(q_0)|).$$

Using Lemma 5.7, we get $\mathcal{E}_{K_n}(v_m) \leq 2c + C_3 \triangleq C_4$. Since the above inequality is independent on n, we then get the desired result by passing n to infinity. □

Now we come to the main result of this section.

Theorem 5.1. *There is a subsequence of $\{v_m\}_{m\geq m_1}$, converging uniformly to a continuous function v on Ω. Furthermore, v is a Dirichlet eigenfunction associated to $\lambda = \frac{3}{2}\lim_{m\to\infty} 5^m\lambda_m$.*

Proof. For each $m \geq m_1$, since $v_m \in dom\mathcal{E}$, we have

$$|v_m(x) - v_m(y)| \leq \mathcal{E}(v_m)^{1/2} d(x,y)^{1/2}, \quad \forall x, y \in \Omega,$$

where $d(\cdot,\cdot)$ is the effective resistance metric on Ω. Hence by Lemma 5.8,

$$|v_m(x) - v_m(y)| \leq C_4^{1/2} d(x,y)^{1/2}, \quad \forall x, y \in \Omega$$

holds uniformly on $m \geq m_1$. Thus $\{v_m\}_{m\geq m_1}$ is equicontinuous. Moreover, notice that $\{v_m\}_{m\geq m_1}$ is also uniformly bounded. Then using the Arzelà-Ascoli theorem, there exists a subsequence $\{v_{m_k}\}$ of $\{v_m\}$ which converges uniformly to a continuous function v on Ω.

Let $G_\Omega(x,y)$ denote the Green's function associated to Ω. See the explicit expression for $G_\Omega(x,y)$ in [12]. Then $\forall k$, we have

$$v_{m_k}(x) = \int_\Omega G_\Omega(x,y) 5^{m_k}\Phi(\lambda_{m_k}) v_{m_k}(y) d\mu(y) + h_{m_k}(x), \tag{5.14}$$

where h_{m_k} is a harmonic function on Ω taking the same boundary values as v_{m_k}, i.e., $h_{m_k}(q_0) = 0$ and $h_{m_k}|_L = \theta_{m_k}$. If $k \to \infty$, then $\theta_{m_k} \to 0$ and hence h_{m_k} goes to 0 uniformly on Ω by the maximum principle. Hence by letting $k \to \infty$ on both side of (5.14), we get

$$v(x) = \int_\Omega G_\Omega(x,y)(\lambda v(y)) d\mu(y).$$

Thus we finally get

$$\begin{cases} -\Delta v = \lambda v \text{ in } \Omega, \\ v|_\Omega = 0, \end{cases}$$

i.e., v is a Dirichlet eigenfunction associated to λ. □

Thus for each sequence $\{\lambda_m\}_{m\geq m_0}$, we have proved that $\lambda = \frac{3}{2}\lim_{m\to} 5^m\lambda_m$ is a primitive Dirichlet symmetric eigenvalue of Δ. We denote by $\mathcal{P}^+$ the totality of all this kind of eigenvalues. Lemma 5.2 and Lemma 5.3 guarantee that all these eigenvalues are distinct and they are all greater than 0. The skew-symmetric case is similar. We denote by $\mathcal{P}^-$ the set of skew-symmetric eigenvalues generated in this way. Let $\mathcal{P} = \mathcal{P}^+ \cup \mathcal{P}^-$ denote all the primitive Dirichlet eigenvalues of Δ.

6 Complete Dirichlet spectrum of Δ

It is clear that the weak spectral decimation recipe constructs many primitive eigenvalues (hence also many miniaturized eigenvalues) of Δ. Recall that the standard spectral decimation recipe also constructs many localized eigenvalues of Δ. It is natural to ask do these recipes construct all of the spectrum of Δ? In this section, we will answer this question.

Till now, for each $m \geq 2$, we have proved that the spectrum $\mathcal{S}_m$ of the discrete Laplacian Δ_m consists of $\mathcal{L}_m$, $\mathcal{P}_m$ and $\mathcal{M}_m$ the three types of eigenvalues. After passing the approximation to the limit, we have proved that there are at least three types of eigenvalues $\mathcal{L}$, $\mathcal{P}$ and $\mathcal{M}$ in the spectrum $\mathcal{S}$ of Δ, i.e., $\mathcal{S} \supset \mathcal{L} \cup \mathcal{P} \cup \mathcal{M}$. We call all of the above three types of eigenvalues *raw eigenvalues*. By the *raw multiplicity* of the raw eigenvalue λ, we mean the multiplicity of the associated eigenvalue λ_{m_0} of Δ_{m_0}, where m_0 is the generation of birth. Since linear independent eigenfunctions of Δ_{m_0} belonging to λ_{m_0} give rise to linearly independent eigenfunctions of Δ, and the fact that all primitive graph eigenvalues have only raw multiplicity 1, the raw multiplicity of λ is not greater than the true multiplicity of λ.

Denote by $\mathcal{S}'$ the collection of raw eigenvalues of Δ, then $\mathcal{S}' = \mathcal{L} \cup \mathcal{P} \cup \mathcal{M}$ and $\mathcal{S}' \subset \mathcal{S}$. Hence we need to prove $\mathcal{S}' = \mathcal{S}$ and the raw multiplicity of each element of $\mathcal{S}'$ coincides with its true multiplicity.

Comparing to the proof of the same problem in the $\mathcal{SG}$ case, we only need to prove the following result. See details in [10]. Recall that $a_m = \sharp(V_m^\Omega \setminus \partial\Omega_m) = \frac{3^{m+1}-1}{2} - 2^{m+1}$.

Theorem 6.1. *Let $0 < \kappa_1 \leq \kappa_2 \leq \cdots$ be the rearrangement of elements of $\mathcal{S}'$ each repeated according to its raw multiplicity. Let $\{\kappa_{m,i}\}_{1\leq i\leq a_m}$ be the m-level graph eigenvalues of Δ_m on Ω_m including multiplicities. Then*

$$\lim_{m\to\infty} \sum_{1\leq i\leq a_m} \frac{1}{\frac{3}{2}5^m \kappa_{m,i}} = \sum_{i=1}^{\infty} \frac{1}{\kappa_i} < \infty.$$

In order to prove this theorem, we first list some notations and lemmas. It is more convenient to consider the following slightly different classification of all the raw eigenvalues of Δ,

$$\mathcal{S}' = \mathcal{L} \cup \mathcal{P}^+ \cup \widetilde{\mathcal{P}}^-$$

where $\widetilde{\mathcal{P}}^- = \mathcal{P}^- \cup \mathcal{M}$, since miniaturized eigenvalues have the same generation mechanism as the skew-symmetric primitive eigenvalues. In the following, we always use α, β, γ to denote $\mathcal{L}$, $\mathcal{P}^+$, $\widetilde{\mathcal{P}}^-$ type eigenvalues respectively. Accordingly, $\forall m \geq 2$, all the m-level graph eigenvalues are classified into the three types $\mathcal{L}_m$, $\mathcal{P}_m^+$ and $\widetilde{\mathcal{P}}_m^-$, where $\widetilde{\mathcal{P}}_m^- = \mathcal{P}_m^- \cup \mathcal{M}_m$, and we always use $\alpha_m, \beta_m, \gamma_m$ to denote them respectively. For simplicity, we denote $A_m = \sharp\mathcal{L}_m$, $B_m = \sharp\mathcal{P}_m^+$ and $C_m = \sharp\widetilde{\mathcal{P}}_m^-$. Of course, $a_m = A_m + B_m + C_m$. Moreover, recall that $\rho_m^\Omega(5)$ and $\rho_m^\Omega(6)$ are the multiplicities of m-level initial eigenvalue 5 and 6 respectively. See the exact values of them in Section 3.

Lemma 6.1. $\mathcal{L} = \bigcup_{k=3}^\infty \mathcal{L}^k$ *(disjoint union) where* $\mathcal{L}^k \subset [5^k\Phi(3), 5^k\Phi(5)]$.

Proof. $\forall \alpha \in \mathcal{L}$, let $\{\alpha_m\}_{m\geq m_0}$ be the corresponding sequence of eigenvalues with a generation of fixation m_1. Then $\alpha = \frac{3}{2}\lim_{m\to\infty} 5^m\alpha_m = 5^{m_1}\Phi(\alpha_{m_1})$.

If α_{m_1} is an initial eigenvalue, then α_{m_1} can only be equal to 5. If α_{m_1} is a continued eigenvalue, then $\alpha_{m_1} = \phi_+(\alpha_{m_1-1})$, which yields that $3 \leq \alpha_{m_1} \leq 5$. Hence we always have $3 \leq \alpha_{m_1} \leq 5$.

Noticing that each localized eigenvalue has generation of birth at least 3, denote by $\mathcal{L}^k$ the set of eigenvalues with $m_1 = k$, $k = 3, 4, \cdots$. Then $\mathcal{L} = \bigcup_{k=3}^{\infty} \mathcal{L}^k$ and $\mathcal{L}^k \subset [5^k\Phi(3), 5^k\Phi(5)]$. Since $\phi_-(5) < 3$, we have $\Phi(5) < 5\Phi(3)$. Hence $\mathcal{L} = \bigcup_{k=3}^{\infty} \mathcal{L}^k$ is a disjoint union. □

Lemma 6.2. $\mathcal{P}^+ = \bigcup_{k=2}^{\infty} \mathcal{P}^{+,k}$ *(disjoint union) where* $\mathcal{P}^{+,2} \subset (0, 5^2\Phi(6)]$ *and* $\mathcal{P}^{+,k} \subset [5^k\Phi(\phi_-(3)), 5^k\Phi(6)]$ *for* $k \geq 3$.

Proof. $\forall \beta \in \mathcal{P}^+$, let $\{\beta_m\}_{m\geq m_0}$ be the corresponding sequence of eigenvalues with a generation of fixation m_1. Then $\beta = \frac{3}{2}\lim_{m\to\infty} 5^m\beta_m = 5^{m_1}\lim_{n\to\infty}\frac{3}{2}5^n\widetilde{\phi}_-^{(n)}(\beta_{m_1})$.

If β_{m_1} is a continued eigenvalue (hence $m_1 \geq 3$), then we must have $\beta_{m_1} = \widetilde{\phi}_+(\beta_{m_1-1})$, which obviously yields that $\beta_{m_1} > \widetilde{\phi}_-(\beta^*_{m_1-1})$ where $\beta^*_{m_1-1}$ denotes the largest eigenvalue in $\mathcal{P}^+_{m_1-1}$. If β_{m_1} is an initial eigenvalue with $m_1 \geq 3$, then obviously $\beta_{m_1} > \widetilde{\phi}_-(\beta^*_{m_1-1})$. Hence we always have $\beta_{m_1} > \widetilde{\phi}_-(\beta^*_{m_1-1})$ if $m_1 \geq 3$.

Moreover, When $m_1 > 3$, if we denote $\beta^{**}_{m_1-1}$ the largest eigenvalue in $\mathcal{P}^+_{m_1-1}$ except for $\beta^*_{m_1-1}$, then we have $\widetilde{\phi}_-(\beta^*_{m_1-1}) > \phi_-(\beta^{**}_{m_1-1})$. It is easy to check that $\beta^{**}_{m_1-1} > \phi_+(\beta^*_{m_1-2}) > 3$ since $m_1 > 3$. Thus $\beta_{m_1} > \widetilde{\phi}_-(\beta^*_{m_1-1}) > \phi_-(3)$. When $m_1 = 3$, it can be checked directly that $\beta_3 > \widetilde{\phi}_-(\beta^*_2) \approx 1.33 > \phi_-(3)$. Hence we always have $\beta_{m_1} > \widetilde{\phi}_-(\beta^*_{m_1-1}) > \phi_-(3)$ if $m_1 \geq 3$. By Lemma 5.2, we have $\widetilde{\phi}_-^{(n)}(\beta_{m_1}) > \phi_-^{(n)}(\widetilde{\phi}_-(\beta^*_{m_1-1})), \forall n \in \mathbf{N}$. Hence if $m_1 \geq 3$, we have

$$\begin{aligned} \beta &= 5^{m_1}\lim_{n\to\infty}\frac{3}{2}5^n\widetilde{\phi}_-^{(n)}(\beta_{m_1}) \\ &\geq 5^{m_1}\lim_{n\to\infty}\frac{3}{2}5^n\phi_-^{(n)}(\widetilde{\phi}_-(\beta^*_{m_1-1})) \\ &\geq 5^{m_1}\lim_{n\to\infty}\frac{3}{2}5^n\phi_-^{(n)}(\phi_-(3)) \\ &= 5^{m_1}\Phi(\phi_-(3)). \end{aligned}$$

On the other hand, when $m_1 \geq 2$, we always have

$$\beta = 5^{m_1}\lim_{n\to\infty}\frac{3}{2}5^n\widetilde{\phi}_-^{(n)}(\beta_{m_1}) \leq 5^{m_1}\lim_{n\to\infty}\frac{3}{2}5^n\phi_-^{(n)}(6) = 5^{m_1}\Phi(6).$$

Denote by $\mathcal{P}^{+,k}$ the set of eigenvalues with $m_1 = k$, $k = 2, 3, \cdots$. Then $\mathcal{P}^+ = \bigcup_{k=2}^{\infty} \mathcal{P}^{+,k}$ where $\mathcal{P}^{+,2} \subset (0, 5^2\Phi(6)]$ and $\mathcal{P}^{+,k} \subset [5^k\Phi(\phi_-(3)), 5^k\Phi(6)]$ for $k \geq 3$.

Next we need to prove $\mathcal{P}^+ = \bigcup_{k=2}^{\infty} \mathcal{P}^{+,k}$ is a disjoint union. $\forall 2 \leq k < k'$, take an element β in $\mathcal{P}^{+,k}$, β' in $\mathcal{P}^{+,k'}$ respectively. Then $\beta = 5^k\lim_{n\to\infty}\frac{3}{2}5^n\widetilde{\phi}_-^{(n)}(\beta_k)$ for some eigenvalue β_k in $\mathcal{P}^+_k$, and $\beta' = 5^{k'}\lim_{n\to\infty}\frac{3}{2}5^n\widetilde{\phi}_-^{(n)}(\beta'_{k'})$ for some eigenvalue $\beta'_{k'}$ in $\mathcal{P}^+_{k'}$.

Note that $\widetilde{\phi}_-^{(k'-k)}(\beta_k)$ and $\beta'_{k'}$ both belong to $\mathcal{P}^+_{k'}$. Since k' is the generation of fixation of β', we can easily get $\widetilde{\phi}_-^{(k'-k)}(\beta_k) < \beta'_{k'}$. Then by using Lemma 5.3, we have $\beta < \beta'$.

From the arbitrariness of β, β' and k, k', we finally get that $\mathcal{P}^+ = \bigcup_{k=2}^{\infty} \mathcal{P}^{+,k}$ is a disjoint union. □

Lemma 6.3. *Let $0 < \alpha_1 \leq \alpha_2 \leq \cdots$ be the rearrangement of elements of $\mathcal{L}$ each repeated according to its raw multiplicity. Let $\{\alpha_{m,i}\}_{1\leq i\leq A_m}$ be the m-level localized eigenvalues of Δ_m on Ω_m including multiplicities. Then*

$$\lim_{m\to\infty} \sum_{1\leq i\leq A_m} \frac{1}{\frac{3}{2}5^m\alpha_{m,i}} = \sum_{i=1}^{\infty} \frac{1}{\alpha_i},$$

providing $\sum_{i=1}^{\infty} \frac{1}{\alpha_i} < \infty$.

Proof. Noticing that $\lim_{m\to\infty} \frac{\rho_m^\Omega(6)}{5^m} = 0$, it suffices to show that

$$\sum_{\substack{1\leq i\leq A_m \\ \alpha_{m,i}\neq 6}} \frac{1}{\frac{3}{2}5^m\alpha_{m,i}} - \sum_{i=1}^{A_m-\rho_m^\Omega(6)} \frac{1}{\alpha_i}, \tag{6.1}$$

converges to 0 as m goes to infinity.

$\forall m \geq 2$, denote $D_m = A_m - \rho_m^\Omega(6)$. By Lemma 6.1, $\{\alpha_1, \alpha_2, \cdots, \alpha_{D_m}\}$ is an arrangement of elements of $\bigcup_{k=3}^m \mathcal{L}^k$ each being repeated according to its raw multiplicity. The first sum of (6.1) has also D_m terms, which can be rearranged so that

$$\lim_{n\to\infty} \frac{3}{2}5^{m+n}\phi_-^{(n)}(\alpha_{m,i}) = \alpha_i, \quad 1 \leq i \leq D_m.$$

Hence by using Lemma 6.1, (6.1) is equal to $\sum_{k=3}^m \sum_{\alpha_i\in\mathcal{L}^k} (\frac{1}{\frac{3}{2}5^m\alpha_{m,i}} - \frac{1}{\alpha_i})$. If $\alpha_i \in \mathcal{L}^k$ $(k = 3, \cdots, m)$, then $\alpha_i = 5^k\Phi(\theta)$ for some $\theta \in [3, 5]$ and accordingly the corresponding $\alpha_{m,i}$ is of the form $\alpha_{m,i} = \phi_-^{(m-k)}(\theta)$. Hence

$$0 < \frac{1}{\frac{3}{2}5^m\alpha_{m,i}} - \frac{1}{\alpha_i} = \frac{1}{5^k}\left(\frac{1}{\frac{3}{2}5^{m-k}\phi_-^{(m-k)}(\theta)} - \frac{1}{\Phi(\theta)}\right).$$

Since $\frac{1}{\frac{3}{2}5^n\phi_-^{(n)}(x)}$ converges to $\frac{1}{\Phi(x)}$ uniformly on $[3, 5]$ as n goes to infinity, $\forall\varepsilon > 0$, the last expression is dominated by $\frac{\varepsilon}{5^k}$ whenever $m - k$ is greater than some number N. When $m - k \leq N$, the same expression is dominated by $\frac{1}{5^m R}$ for $R = \frac{3}{2}\inf_{3\leq x\leq 5}\phi_-^{(N)}(x)$. The number of α_i's in $\mathcal{L}^k$ is less than $A_{k-1} + \rho_k^\Omega(5)$, so (6.1) is dominated by

$$\sum_{k=3}^{m-N-1} \frac{A_{k-1} + \rho_k^\Omega(5)}{5^k}\varepsilon + \sum_{k=m-N}^{m} \frac{A_{k-1} + \rho_k^\Omega(5)}{5^m R} \leq c_1\varepsilon + c_2\left(\frac{3}{5}\right)^m\frac{1}{R}$$

for some constants $c_1, c_2 > 0$. Then let m be large enough, (6.1) can be dominated by $(c_1 + c_2)\varepsilon$. Hence we have proved $\sum_{\substack{1\leq i\leq A_m \\ \alpha_{m,i}\neq 6}} \frac{1}{\frac{3}{2}5^m\alpha_{m,i}} - \sum_{i=1}^{D_m} \frac{1}{\alpha_i}$ converges to 0 as m goes to infinity. □

Lemma 6.4. *Let $0 < \beta_1 < \beta_2 < \cdots$ be the elements of $\mathcal{P}^+$ in increasing order. Let $\{\beta_{m,i}\}_{1 \le i \le B_m}$ be the m-level primitive symmetric eigenvalues of Δ_m on Ω_m. Then*

$$\lim_{m\to\infty} \sum_{1\le i\le B_m} \frac{1}{\frac{3}{2}5^m \beta_{m,i}} = \sum_{i=1}^{\infty} \frac{1}{\beta_i},$$

providing $\sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty$.

Proof. It suffices to prove that

$$\sum_{1\le i\le B_m} \frac{1}{\frac{3}{2}5^m \beta_{m,i}} - \sum_{i=1}^{B_m} \frac{1}{\beta_i}, \tag{6.2}$$

converges to 0 as m goes to infinity.

By Lemma 6.2, $\{\beta_1, \beta_2, \cdots, \beta_{B_m}\}$ is an arrangement of elements of $\bigcup_{k=2}^m \mathcal{P}^{+,k}$. The first sum of (6.2) can be rearranged so that

$$\lim_{n\to\infty} \frac{3}{2} 5^{m+n} \widetilde{\phi}_-^{(n)}(\beta_{m,i}) = \beta_i, \quad 1 \le i \le B_m.$$

Hence by using Lemma 6.2, (6.2) is equal to $\sum_{k=2}^m \sum_{\beta_i \in \mathcal{P}^{+,k}} (\frac{1}{\frac{3}{2}5^m \beta_{m,i}} - \frac{1}{\beta_i})$. The $k = 2$ term converges to 0 as m goes to infinity since $\sharp \mathcal{P}^{+,2} = B_2 = 3$.

Hence we only need to prove

$$\sum_{k=3}^m \sum_{\beta_i \in \mathcal{P}^{+,k}} |\frac{1}{\frac{3}{2}5^m \beta_{m,i}} - \frac{1}{\beta_i}|. \tag{6.3}$$

converges to 0 as m goes to infinity.

If $\beta_i \in \mathcal{P}^{+,k}$ $(k = 3, \cdots, m)$, then $\beta_i = 5^k \lim_{n\to\infty} \frac{3}{2} 5^n \widetilde{\phi}_-^{(n)}(\theta)$ for some $\theta \in \mathcal{P}_k^+$ and accordingly the corresponding $\beta_{m,i}$ is of the form $\beta_{m,i} = \widetilde{\phi}_-^{(m-k)}(\theta)$. Hence

$$|\frac{1}{\frac{3}{2}5^m \beta_{m,i}} - \frac{1}{\beta_i}| = \frac{1}{5^k} |\frac{1}{\frac{3}{2}5^{m-k} \widetilde{\phi}_-^{(m-k)}(\theta)} - \frac{1}{\lim_{n\to\infty} \frac{3}{2}5^n \widetilde{\phi}_-^{(n)}(\theta)}|. \tag{6.4}$$

From the proof of Lemma 6.2, we have

$$\frac{3}{2}5^n \phi_-^{(n)}(\phi_-(3)) < \frac{3}{2}5^n \widetilde{\phi}_-^{(n)}(\theta) < \frac{3}{2}5^n \phi_-^{(n)}(6).$$

Then by the proof of Lemma 5.1, $\forall \varepsilon > 0$, the right side of formula (6.4) is dominated by $\frac{1}{5^k}\varepsilon$ whenever $m - k$ is greater than some number N. When $m - k \le N$, $\frac{1}{\frac{3}{2}5^m \widetilde{\phi}_-^{(m-k)}(\theta)}$ is dominated by $\frac{1}{5^m R}$ for $R = \frac{3}{2}\phi_-^{(N+1)}(3)$. The number of β_i's in $\mathcal{P}^{+,k}$ is controlled by B_k, so the sum (6.3) is dominated by

$$\begin{aligned} & \sum_{k=3}^{m-N-1} \frac{B_k}{5^k}\varepsilon + \sum_{k=m-N}^{m} \frac{B_k}{5^m R} + \sum_{k=m-N}^{m} \sum_{\beta_i \in \mathcal{P}^{+,k}} \frac{1}{\beta_i} \\ \le \quad & c_1\varepsilon + c_2(\frac{2}{5})^m \frac{1}{R} + \sum_{k=m-N}^{m} \sum_{\beta_i \in \mathcal{P}^{+,k}} \frac{1}{\beta_i}. \end{aligned} \tag{6.5}$$

Noticing that $\sum_{i=1}^{\infty}\frac{1}{\beta_i}<\infty$, the last term goes to 0 as m goes to infinity. Hence for large m, (6.5) is less than $(c_1+c_2+1)\varepsilon$. Thus we have proved $\sum_{1\leq i\leq B_m}\frac{1}{\frac{3}{2}5^m\beta_{m,i}}-\sum_{i=1}^{B_m}\frac{1}{\beta_i}$ converges to 0 as m goes to infinity. □

Lemma 6.5. *Let $0<\gamma_1\leq\gamma_2\leq\cdots$ be the elements of $\widetilde{\mathcal{P}}^-$ in increasing order repeated according to their raw multiplicities. Let $\{\gamma_{m,i}\}_{1\leq i\leq C_m}$ be the m-level $\widetilde{\mathcal{P}}^-$ type eigenvalues of Δ_m on Ω_m including multiplicities. Then*

$$\lim_{m\to\infty}\sum_{1\leq i\leq C_m}\frac{1}{\frac{3}{2}5^m\gamma_{m,i}}=\sum_{i=1}^{\infty}\frac{1}{\gamma_i},$$

providing $\sum_{i=1}^{\infty}\frac{1}{\gamma_i}<\infty$.

The proof is similar to those of Lemma 6.3 and Lemma 6.4.

Proof of Theorem 6.1. Let $0<\widetilde{\kappa}_1\leq\widetilde{\kappa}_2\leq\cdots$ be the rearrangement of elements of $\mathcal{S}$ each repeated according to its true multiplicity. Let $\widetilde{v}_1,\widetilde{v}_2,\cdots$ be the associated eigenfunctions. Let $G_\Omega(x,y)$ be the Green's function for Ω. Then $G_\Omega(x,y)$ can be expanded as a uniformly convergence series

$$G_\Omega(x,y)=\sum_{i=1}^{\infty}\frac{\widetilde{v}_i(x)\widetilde{v}_i(y)}{\widetilde{\kappa}_i},\quad \forall x,y\in\Omega.$$

Since $\mathcal{S}'\subset\mathcal{S}$ and the raw multiplicity is not greater than the true one, we get that $\sum_{i=1}^{\infty}\frac{1}{\kappa_i}<\infty$. Hence $\sum_{i=1}^{\infty}\frac{1}{\alpha_i}<\infty$, $\sum_{i=1}^{\infty}\frac{1}{\beta_i}<\infty$, and $\sum_{i=1}^{\infty}\frac{1}{\gamma_i}<\infty$. The by adding up the results in Lemma 6.3, Lemma 6.4 and Lemma 6.5, we have

$$\lim_{m\to\infty}\sum_{1\leq i\leq a_m}\frac{1}{\frac{3}{2}5^m\kappa_{m,i}}=\sum_{i=1}^{\infty}\frac{1}{\kappa_i}<\infty.\square$$

Based on Theorem 6.1, using similar argument in [10], we finally get $\mathcal{S}'=\mathcal{S}$ and the raw multiplicity of each element of $\mathcal{S}'$ coincides with its true multiplicity. Thus we have

Theorem 6.2. *$\mathcal{S}=\mathcal{L}\cup\mathcal{P}\cup\mathcal{M}$ where the union is disjoint.*

Hence we have constructed the complete Dirichlet spectrum of Δ on Ω.

Next we describe the Weyl's eigenvalue asymptotics on Ω. We will find a formula analogous to (1.1). We define the Dirichlet eigenvalue counting function

$$\rho^{\Omega}(x)=\sharp\{\lambda\in\mathcal{S}:\lambda\leq x\},$$

repeated according to multiplicities. Then we have:

Theorem 6.3. *There exist positive constant c, C such that $cx^{d_S/2}\leq\rho^{\Omega}(x)\leq Cx^{d_S/2}$, for all x large enough, where $d_S=\log 9/\log 5$ is the spectral dimension of $\mathcal{SG}$. Moreover,*

$$0<\liminf_{x\to\infty}\rho^{\Omega}(x)x^{-d_S/2}<\limsup_{x\to\infty}\rho^{\Omega}(x)x^{-d_S/2}<\infty.$$

Proof. We divide $\rho^{\Omega}(x)$ into four parts $\rho^{\mathcal{L}}(x)$, $\rho^{\mathcal{P}^+}(x)$, $\rho^{\mathcal{P}^-}(x)$ and $\rho^{\mathcal{M}}(x)$ corresponding to different types of eigenvalues. The exact definitions are: $\rho^{\mathcal{L}}(x) = \sharp\{\lambda \in \mathcal{L} : \lambda \leq x\}$, $\rho^{\mathcal{P}^+}(x) = \sharp\{\lambda \in \mathcal{P}^+ : \lambda \leq x\}$, $\rho^{\mathcal{P}^-}(x) = \sharp\{\lambda \in \mathcal{P}^- : \lambda \leq x\}$ and $\rho^{\mathcal{M}}(x) = \sharp\{\lambda \in \mathcal{M} : \lambda \leq x\}$. Obviously,

$$\rho^{\Omega}(x) = \rho^{\mathcal{L}}(x) + \rho^{\mathcal{P}^+}(x) + \rho^{\mathcal{P}^-}(x) + \rho^{\mathcal{M}}(x).$$

For $\rho^{\mathcal{L}}(x)$, it is same as the $\mathcal{SG}$ case, hence there exist positive constant c', C' such that $c'x^{d_S/2} \leq \rho^{\mathcal{L}}(x) \leq C'x^{d_S/2}$, for all x large enough, and furthermore

$$0 < \liminf_{x\to\infty} \rho^{\mathcal{L}}(x)x^{-d_S/2} < \limsup_{x\to\infty} \rho^{\mathcal{L}}(x)x^{-d_S/2} < \infty.$$

See details in [10].

Next we consider $\rho^{\mathcal{P}^+}(x)$. Denote β_m^* the largest eigenvalue in $\mathcal{P}_m^+$, and $\beta^{(m)}$ the eigenvalue in $\mathcal{P}^+$ corresponding to the sequence $\{\widetilde{\phi}_-^{(n)}(\beta_m^*)\}_{n\geq 0}$, i.e., $\beta^{(m)} = \lim_{n\to\infty} \frac{3}{2}5^{n+m}\widetilde{\phi}_-^{(n)}(\beta_m^*)$. By using Lemma 5.2, it is easy to check that

$$c_1 5^m = \lim_{n\to\infty} \frac{3}{2}5^{n+m}\phi_-^{(n)}(2) \leq \lim_{n\to\infty} \frac{3}{2}5^{n+m}\phi_-^{(n)}(\beta_m^{**}) \leq \beta^{(m)} \leq \lim_{n\to\infty} \frac{3}{2}5^{n+m}\phi_-^{(n)}(6) = c_2 5^m, \tag{6.6}$$

for appropriate constants $c_1, c_2 > 0$, where β_m^{**} denote the largest eigenvalue in $\mathcal{P}_m^+$ except β_m^*.

Notice that the bottom r_m eigenvalues in $\mathcal{P}^+$ are generated from eigenvalues in $\mathcal{P}_m^+$ by extending these eigenvalues by choosing $\widetilde{\phi}_-$ relation for all $m' > m$. Hence we get

$$\rho^{\mathcal{P}^+}(\beta^{(m)}) = r_m, \quad \forall m \geq 2.$$

Using (6.6), we get $\rho^{\mathcal{P}^+}(c_1 5^m) \leq r_m$, and $\rho^{\mathcal{P}^+}(c_2 5^m) \geq r_m$.

Denote by k_0 the least number such that $5^{k_0}c_1 \geq c_2$. $\forall x \geq 25c_2$, choose a number m such that $c_2 5^m \leq x < c_2 5^{m+1}$. Then $c_2 5^m \leq x < c_1 5^{m+k_0+1}$. Hence

$$c_3 x^{\log 2/\log 5} \leq r_m \leq \rho^{\mathcal{P}^+}(c_2 5^m) \leq \rho^{\mathcal{P}^+}(x) \leq \rho^{\mathcal{P}^+}(c_1 5^{m+k_0+1}) \leq r_{m+k_0+1} \leq c_4 x^{\log 2/\log 5},$$

for appropriate constants $c_3, c_4 > 0$. Thus we have proved that for x large enough,

$$c_3 x^{\log 2/\log 5} \leq \rho^{\mathcal{P}^+}(x) \leq c_4 x^{\log 2/\log 5}.$$

Similar argument yields that for x large enough,

$$c_5 x^{\log 2/\log 5} \leq \rho^{\mathcal{P}^-}(x) \leq c_6 x^{\log 2/\log 5},$$

for appropriate constants $c_5, c_6 > 0$.

Now we consider $\rho^{\mathcal{M}}(x)$. Notice that for each $\lambda' \in \{\lambda \in \mathcal{M} : \lambda \leq x\}$, there exists a $k \geq 1$, such that λ' has multiplicity 2^k in $\mathcal{M}$, and $\frac{1}{5^k}\lambda' \in \{\lambda \in \mathcal{P}^- : \lambda \leq \frac{x}{5^k}\}$. Hence

$$\rho^{\mathcal{M}}(x) \leq \sum_k 2^k \rho^{\mathcal{P}^-}(\frac{x}{5^k}).$$

Denote λ_* the least eigenvalue in $\mathcal{P}^-$. Then

$$\rho^{\mathcal{M}}(x) \leq \sum_{k=1}^{[\log(x/\lambda_*)/\log 5]} 2^k \rho^{\mathcal{P}^-}(\frac{x}{5^k}) \leq c_6 \cdot \sum_{k=1}^{[\log(x/\lambda_*)/\log 5]} 2^k (\frac{x}{5^k})^{\log 2/\log 5} \leq c_7 (\log x) x^{\log 2/\log 5},$$

for an appropriate constant $c_7 > 0$.

Taking the above estimates into account, we finally get the desired result. □

7 The Neumann case

In this section, we give a brief discussion on the Neumann spectrum of Δ. It suffices to make clear all the discrete Neumann spectra of Δ_m's. As indicated in Section 3, we want to impose a Neumann condition on the graph Ω_m by imagining that it is embedded in a larger graph by reflecting in each boundary vertex and imposing the λ_m-eigenvalue equation on the even extension of u_m. It is convenient to allow $m = 1$, in which case there are only three boundary points in Ω_1 and no others. Denote by $\mathcal{P}_m^N$ the totality of primitive Neumann eigenvalues of the discrete Laplacian Δ_m. Due to the eigenspace dimensional counting argument in Section 3, this time we need to find out 2^m symmetric primitive eigenvalues and $2^m - 1$ skew-symmetric primitive eigenvalues.

We focus our discussion on $\mathcal{P}_m^{+,N}$, the symmetric case, and describe a similar weak spectral decimation which relates $\mathcal{P}_m^{+,N}$ with $\mathcal{P}_{m+1}^{+,N}$. Let u_m be a λ_m-eigenfunction of Δ_m with $\lambda_m \in \mathcal{P}_m^{+,N}$. Still denote by $(b_0, b_1, \cdots, b_m)$ the values of u_m on the skeleton of Ω_m. Write $\lambda_i^{(m)}$ the successor of λ_m of order $(m - i)$ with $1 \leq i \leq m$. (This time we begin with $\lambda_1^{(m)}$.) Assume that none of $\lambda_i^{(m)}$'s is equal to 2 or 5 for $2 \leq i \leq m$. Then u_m is uniquely determined by $(b_0, b_1, \cdots, b_m)$. In addition to the eigenvalue equations at the vertex $F_1q_0, F_1^2q_0, \cdots, F_1^{m-1}q_0$ as described in Section 4, we impose the equations

$$(4 - \lambda_1^{(m)})b_0 = 4b_1 \tag{7.1}$$

at q_0 and

$$(4 - \lambda_m)b_m = 2b_{m-1} + 2b_m \tag{7.2}$$

at $F_1^m q_0$ according to the Neumann boundary condition. Hence $(b_0, b_1, \cdots, b_m)$ can be viewed as a non-zero vector solution of a system of equations consisting of $m+1$ equations in $m+1$ unknowns, whose determinant is

$$\begin{vmatrix} 4-\lambda_1^{(m)} & -4 & & & \\ l(\lambda_2^{(m)}) & s(\lambda_2^{(m)}) & r(\lambda_2^{(m)}) & & \\ & \ddots & \ddots & \ddots & \\ & & l(\lambda_m^{(m)}) & s(\lambda_m^{(m)}) & r(\lambda_m^{(m)}) \\ & & & -2 & 2-\lambda_m^{(m)} \end{vmatrix}.$$

Hence λ_m should be a solution of the following equation

$$q_m^N(x) \triangleq \begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(x) & s(x) & r(x) \\ & & & -2 & 2-x \end{vmatrix} = 0. \qquad (7.3)$$

Thus if λ_m is a root of $q_m^N(x)$ and none of $f^{(i)}(\lambda_m)$'s with $0 \le i \le m-2$ is equal to 2 or 5, then $\lambda_m \in \mathcal{P}_m^{+,N}$. We should mention here that when $m \ge 2$, comparing to $q_m(x)$ in the Dirichlet case, $q_m^N(x)$ is a $(m+1)\times(m+1)$ tridiagonal determinant, containing $q_m(x)$ in the center as a $(m-1)\times(m-1)$ minor, i.e., we have

$$q_m^N(x) = \begin{vmatrix} 4-f^{(m-1)}(x) & -4 & 0 & \cdots & 0 & 0 \\ l(f^{(m-2)}(x)) & & & & & 0 \\ \vdots & & q_m(x) & & & \vdots \\ 0 & & & & & r(x) \\ 0 & 0 & \cdots & 0 & -2 & 2-x \end{vmatrix}.$$

The degree of $q_m^N(x)$ is $3(2^{m-1}-1)+2^{m-1}+1 = 2^{m+1}-2$, since the degree of $q_m(x)$ is $3(2^{m-1}-1)$. The analysis on $q_m^N(x)$ is more complicated than that on $q_m(x)$ since for $q_m(x)$ we can always use the expansion of $q_m(x)$ along the first or last row to get a relation with two polynomials in same type but with smaller degree.

The following lemma is a slight modification of the form of $q_m^N(x)$ from a $(m+1)\times(m+1)$ determinant to a $m\times m$ determinant.

Lemma 7.1. *Let $m \ge 2$. Then*

$$q_m^N(x) = (2-x)(x-6) \begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(f(x)) & s(f(x)) & r(f(x)) \\ & & & 1 & f(x)-1 \end{vmatrix}.$$

Proof. Substituting the expression for $r(x)$ into (7.3), we get

$$
\begin{aligned}
q_m^N(x) &= (2-x)\begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(x) & s(x) & -2(5-x) \\ & & & -2 & 1 \end{vmatrix} \\
&= (2-x)\begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & & \\ & \ddots & \ddots & \ddots & \\ & & l(x) & s(x)-4(5-x) & -2(5-x) \\ & & & 0 & 1 \end{vmatrix} \\
&= (2-x)\begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & \\ & \ddots & \ddots & \ddots \\ & & l(x) & s(x)-4(5-x) \end{vmatrix}.
\end{aligned}
$$

Noticing that $s(x) - 4(5-x) = (x-6)(f(x)-1)$ and $l(x) = x-6$, we get the desired result. □

The following lemma focuses on the possibility of the roots of $q_m^N(x)$ satisfying $f^{(i)}(x) = 2$ or 5 for some $0 \le i \le m-2$.

Lemma 7.2. *Let $m \ge 2$, and x be a predecessor of 2 or 5 of order i with $0 \le i \le m-2$. Then $q_m^N(x) = 0$.*

Proof. Firstly, let x be a predecessor of 2 of order i with $0 \le i \le m-2$. Then $f^{(i)}(x) = 2$ and $f^{(i+1)}(x) = 6$. If $0 \le i < m-2$, the proof is the same as that of Lemma 4.2. So we only need to check the $i = m-2$ case. In this case we have

$$
q_m^N(x) = \begin{vmatrix} 4-f^{(m-1)}(x) & -4 & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & \\ & \ddots & \ddots & \ddots \end{vmatrix} = \begin{vmatrix} -2 & -4 & \\ -4 & -8 & 0 \\ & \ddots & \ddots & \ddots \end{vmatrix} = 0.
$$

Secondly, let x be a predecessor of 5 of order i with $0 \le i \le m-2$. Then $f^{(i)}(x) = 5$

and $f^{(i+1)}(x) = \cdots f^{(m-1)}(x) = 0$. Hence we have

$$q_m^N(x) = \begin{vmatrix} 4 & -4 & & & & & \\ l(0) & s(0) & r(0) & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & l(0) & s(0) & r(0) & & \\ & & & l(5) & s(5) & r(5) & \\ & & & & \ddots & \ddots & \ddots \end{vmatrix} = \begin{vmatrix} 4 & -4 & & & & & \\ -6 & 26 & -20 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -6 & 26 & -20 & & \\ & & & -1 & 1 & 0 & \\ & & & & \ddots & \ddots & \ddots \end{vmatrix} = 0.$$

Thus we always have $q_m^N(x) = 0$. □

This lemma means that for $m \geq 2$, all the predecessors of 2 or 5 of order i with $0 \leq i \leq m-2$ are unwanted roots of $q_m^N(x)$. To exclude them out, we define

$$p_m^N(x) = \frac{q_m^N(x)}{(x-2)(x-5)\cdots(f^{(m-2)}(x)-2)(f^{(m-2)}(x)-5)}, \qquad \text{for } m \geq 2,$$

and

$$p_1^N(x) = q_1^N(x).$$

Now we can say if λ_m is a root of the polynomial $p_m^N(x)$, then $\lambda_m \in \mathcal{P}_m^{+,N}$. It is easy to check that the degree of $p_m^N(x)$ is 2^m, since the degree of $q_m^N(x)$ is $2^{m+1}-2$ and the number of all the unwanted roots of $q_m^N(x)$ is $2(1+2+\cdots 2^{m-2}) = 2^m - 2$ for $m \geq 2$ and 0 for $m = 1$. The following is an easy observation on $p_m^N(x)$.

Lemma 7.3. *For $m \geq 1$, $p_m^N(x)$ always has roots 0 and 6.*

Proof. We only need to check $q_m^N(0) = q_m^N(6) = 0$. It is easy to see that

$$q_m^N(0) = \begin{vmatrix} 4 & -4 & & & \\ l(0) & s(0) & r(0) & & \\ & \ddots & \ddots & \ddots & \\ & & l(0) & s(0) & r(0) \\ & & & -2 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -4 & & & \\ -6 & 26 & -20 & & \\ & \ddots & \ddots & \ddots & \\ & & -6 & 26 & -20 \\ & & & -2 & 2 \end{vmatrix} = 0.$$

$q_m^N(6) = 0$ follows from Lemma 7.1 for $m \geq 2$, and from direct computation for $m = 1$.
□

In order to study the distribution of roots of $p_m^N(x)$, we now introduce a type of auxiliary polynomials $l_m(x)$ associated to $p_m^N(x)$. First, $\forall m \geq 1$, let $\widetilde{l}_m(x)$ denote the $m \times m$ minor located in the upper left corner of $q_m^N(x)$, i.e., $\widetilde{l}_1(x) = 4 - x$ and for $m \geq 2$,

$$\widetilde{l}_m(x) = \begin{vmatrix} 4 - f^{(m-1)}(x) & -4 & & \\ l(f^{(m-2)}(x)) & s(f^{(m-2)}(x)) & r(f^{(m-2)}(x)) & \\ & \ddots & \ddots & \ddots \\ & & l(x) & s(x) \end{vmatrix}.$$

Note that the $(m-1)\times(m-1)$ minor located in the bottom right corner of $\widetilde{l}_m(x)$ is $q_m(x)$. The degree of $\widetilde{l}_m(x)$ is $2^{m+1}-3$ since it is reduced by 1 comparing to the degree of $q_m^N(x)$. With similar argument in the proof of Lemma 7.2, all the predecessors of 2 or 5 of order i with $0\le i\le m-2$ are roots of $\widetilde{l}_m(x)$. To exclude them out, we define

$$l_m(x)=\frac{\widetilde{l}_m(x)}{(x-2)(x-5)\cdots(f^{(m-2)}(x)-2)(f^{(m-2)}(x)-5)},\qquad \text{for } m\ge 2,$$

and

$$l_1(x)=\widetilde{l}_1(x).$$

It is easy to check that the degree of $l_m(x)$ is 2^m-1, since the degree of $\widetilde{l}_m(x)$ is $2^{m+1}-3$ and the number of all the unwanted roots of $\widetilde{l}_m(x)$ is $2(1+2+\cdots 2^{m-2})=2^m-2$ for $m\ge 2$ and 0 for $m=1$.

Based on the property

$$\widetilde{l}_m(x)=s(x)\widetilde{l}_{m-1}(f(x))-r(f(x))l(x)\widetilde{l}_{m-2}(f^{(2)}(x)),$$

$l_m(x)$ can be analyzed in a similar way that of $p_m(x)$ or $\widetilde{p}_m(x)$ in the Dirichlet case. We then have:

Lemma 7.4. *$l_m(0)>0$ and $l_m(6)<0$, $\forall m\ge 1$.*

Proof. $l_m(0)>0$ follows from a similar argument in the proof of Proposition 4.1 and Proposition 4.2.

To prove $l_m(6)<0$, we only need to prove $\widetilde{l}_m(6)<0$ by the definition of $l_m(x)$. It can be checked that $\widetilde{l}_1(6)=-2<0$ and $\widetilde{l}_2(6)=-40<0$ by an easy computation. For $m\ge 3$, an expansion of $\widetilde{l}_m(6)$ along the first row yields that

$$\widetilde{l}_m(6)=(4-f^{(m-1)}(6))q_m(6)+4(f^{(m-2)}(6)-6)q_{m-1}(6).$$

Recall that in the proof of Proposition 4.1(3), we have proved that $q_m(6)\le q_{m-1}(6)<0$, $\forall m\ge 3$. Hence

$$\widetilde{l}_m(6)\le(4-f^{(m-1)}(6)+4f^{(m-2)}(6)-24)q_m(6)=(f^{(m-2)}(6)-5)(f^{(m-2)}(6)+4)q_m(6)<0,$$

noticing that $f^{(m-2)}(6)\le -6$ whenever $m\ge 3$. □

Lemma 7.5. *For each $m\ge 1$, $l_m(x)$ has 2^m-1 distinct real roots between 0 and 6 satisfying*

$$0<\beta_{m,1}<\beta_{m,2}<\cdots<\beta_{m,2^m-1}<6.$$

Moreover,

$$\begin{aligned}
0 &< \beta_{m+1,1} < \phi_-(\beta_{m,1}),\\
\phi_-(\beta_{m,k-1}) &< \beta_{m+1,k} < \phi_-(\beta_{m,k}), \quad \forall 2 \le k \le 2^m - 1,\\
\phi_-(\beta_{m,2^m-1}) &< \beta_{m+1,2^m} < \phi_+(\beta_{m,2^m-1}),\\
\phi_+(\beta_{m,2^{m+1}-k}) &< \beta_{m+1,k} < \phi_+(\beta_{m,2^{m+1}-k-1}), \quad \forall 2^m + 1 \le k \le 2^{m+1} - 2,\\
\phi_+(\beta_{m,1}) &< \beta_{m+1,2^{m+1}-1} < 6.
\end{aligned}$$

Proof. It follows from a similar argument in the proof of Lemma 4.4. □

The following lemma shows a relation between $p_m^N(x)$'s and $l_m(x)$'s.

Lemma 7.6. *Let $m \ge 2$. Then $p_m^N(x) = (2-x)l_m(x) - 4l_{m-1}(f(x))$.*

Proof. This is easy to get since we have

$$q_m^N(x) = (2-x)\widetilde{l}_m(x) - 4(2-x)(5-x)\widetilde{l}_{m-1}(f(x)), \quad \forall m \ge 2,$$

using the expansion along the last row of $q_m^N(x)$. □

Now we consider the distribution of roots of $p_m^N(x)$.

Lemma 7.7. *For each $m \ge 1$, $p_m^N(x)$ has 2^m distinct roots between 0 and 6 (including 0 and 6). Moreover, $p_m^N(0+) < 0$ and $p_m^N(6-) < 0$.*

Proof. When $m = 1$, it naturally holds.

Let $m \ge 2$. From Lemma 7.5, $l_m(x)$ has $2^m - 1$ distinct real roots between 0 and 6 satisfying

$$0 < \beta_{m,1} < \beta_{m,2} < \cdots < \beta_{m,2^m-1} < 6.$$

For each $1 \le k \le 2^m - 1$, using Lemma 7.6, we have

$$p_m^N(\beta_{m,k}) = (2-\beta_{m,k})l_m(\beta_{m,k}) - 4l_{m-1}(f(\beta_{m,k})) = -4l_{m-1}(f(\beta_{m,k})).$$

When $k = 1$, by Lemma 7.5, $0 < \beta_{m,1} < \phi_-(\beta_{m-1,1})$, hence $0 < f(\beta_{m,1}) < \beta_{m-1,1}$. Combined with $l_{m-1}(0) > 0$ from Lemma 7.4, it follows $l_{m-1}(f(\beta_{m,1})) > 0$, hence $p_m^N(\beta_{m,1}) < 0$.

When $2 \le k \le 2^{m-1} - 1$, following from Lemma 7.5, we have $\phi_-(\beta_{m-1,k-1}) < \beta_{m,k} < \phi_-(\beta_{m-1,k})$, hence $\beta_{m-1,k-1} < f(\beta_{m,k}) < \beta_{m-1,k}$. Combined with $l_{m-1}(0) > 0$, it follows $l_{m-1}(f(\beta_{m,k})) \sim (-1)^{k-1}$, hence $p_m^N(\beta_{m,k}) \sim (-1)^k$.

When $k = 2^{m-1}$, following from Lemma 7.5, we have $\phi_-(\beta_{m-1,2^{m-1}-1}) < \beta_{m,2^{m-1}} < \phi_+(\beta_{m-1,2^{m-1}-1})$, hence $f(\beta_{m,2^{m-1}}) > \beta_{m-1,2^{m-1}-1}$. Combined with $l_{m-1}(0) > 0$, it follows $l_{m-1}(f(\beta_{m,2^{m-1}})) < 0$, hence $p_m^N(\beta_{m,2^{m-1}}) > 0$.

When $2^{m-1} + 1 \le k \le 2^m - 2$, following from Lemma 7.5, we have $\phi_+(\beta_{m-1,2^m-k}) < \beta_{m,k} < \phi_+(\beta_{m-1,2^m-k-1})$, hence $\beta_{m-1,2^m-k-1} < f(\beta_{m,k}) < \beta_{m-1,2^m-k}$. Combined with $l_{m-1}(0) > 0$, it follows $l_{m-1}(f(\beta_{m,k})) \sim (-1)^{k-1}$, hence $p_m^N(\beta_{m,k}) \sim (-1)^k$.

When $k = 2^m - 1$, following from Lemma 7.5, we have $\phi_+(\beta_{m-1,1}) < \beta_{m,2^m-1} < 6$, hence $f(\beta_{m,2^m-1}) < \beta_{m-1,1}$. Combined with $l_{m-1}(0) > 0$, it follows $l_{m-1}(f(\beta_{m,2^m-1})) > 0$, hence $p_m^N(\beta_{m,2^m-1}) < 0$.

Hence we have proved $p_m^N(\beta_{m,k}) \sim (-1)^k, \forall 1 \le k \le 2^m - 1$. So there exist at least $2^m - 2$ roots of $p_m^N(x)$, each located strictly between each two consecutive $\beta_{m,k}$'s. Moreover, Lemma 7.3 says that 0 and 6 are also roots of $p_m^N(x)$. Thus we have found 2^m distinct roots of $p_m^N(x)$. Since the order of $p_m^N(x)$ is also 2^m, these are the totality of roots of $p_m^N(x)$.

Furthermore, from the fact that $p_m^N(\beta_{m,1}) < 0$ and $p_m^N(\beta_{m,2^m-1}) < 0$, we have $p_m^N(0+) < 0$ and $p_m^N(6-) < 0$. □

In all that follows, we denote

$$\lambda_{m,1} = 0 < \lambda_{m,2} < \cdots < \lambda_{m,2^m-1} < \lambda_{m,2^m} = 6$$

the 2^m distinct roots of $p_m^N(x)$ in increasing order, $\forall m \ge 1$. In order to study the relation of roots of two consecutive $p_m^N(x)$'s, we prove the following two lemmas:

Lemma 7.8. *Let $m \ge 1$ and $2 \le k \le 2^m$, then*

$$p_{m+1}^N(\phi_-(\lambda_{m,k})) = -\frac{\lambda_{m,k}}{2}\frac{\phi_-(\lambda_{m,k}) - 6}{\phi_-(\lambda_{m,k}) - 5} \cdot l_m(\lambda_{m,k}),$$

and

$$p_{m+1}^N(\phi_+(\lambda_{m,k})) = -\frac{\lambda_{m,k}}{2}\frac{\phi_+(\lambda_{m,k}) - 6}{\phi_+(\lambda_{m,k}) - 5} \cdot l_m(\lambda_{m,k}).$$

Proof. For simplicity we only prove the first equality. The second will follow from a similar argument. It is easy to see that $\lambda_{m,k}$ is also a root of $q_m^N(x)$ and none of $f^{(i)}(\lambda_{m,k})$'s $(0 \le i \le m-2)$ is equal to 2 or 5.

By Lemma 7.1, $q_{m+1}^N(\phi_-(\lambda_{m,k})) = (2 - \phi_-(\lambda_{m,k}))(\phi_-(\lambda_{m,k}) - 6) \cdot A$ where

$$A = \begin{vmatrix} 4 - f^{(m-1)}(\lambda_{m,k}) & -4 & & & \\ l(f^{(m-2)}(\lambda_{m,k})) & s(f^{(m-2)}(\lambda_{m,k})) & r(f^{(m-2)}(\lambda_{m,k})) & & \\ & \ddots & \ddots & \ddots & \\ & & l(\lambda_{m,k}) & s(\lambda_{m,k}) & r(\lambda_{m,k}) \\ & & & 1 & \lambda_{m,k} - 1 \end{vmatrix}.$$

Noticing that from $q_m^N(\lambda_{m,k}) = 0$, we have

$$\begin{vmatrix} 4 - f^{(m-1)}(\lambda_{m,k}) & -4 & & & \\ l(f^{(m-2)}(\lambda_{m,k})) & s(f^{(m-2)}(\lambda_{m,k})) & r(f^{(m-2)}(\lambda_{m,k})) & & \\ & \ddots & \ddots & \ddots & \\ & & l(\lambda_{m,k}) & s(\lambda_{m,k}) & r(\lambda_{m,k}) \\ & & & -1 & 1 - \lambda_{m,k}/2 \end{vmatrix} = 0.$$

The summation of the above two determinants yields that $A = \frac{\lambda_{m,k}}{2}\widetilde{l}_m(\lambda_{m,k})$. Hence

$$q^N_{m+1}(\phi_-(\lambda_{m,k})) = \frac{\lambda_{m,k}}{2}(2-\phi_-(\lambda_{m,k}))(\phi_-(\lambda_{m,k})-6)\cdot \widetilde{l}_m(\lambda_{m,k}),$$

which yields the desired result. □

Lemma 7.9. *Let $m \geq 1$. Then $(-1)^{k-1}l_m(\lambda_{m,k}) > 0, \quad \forall 1 \leq k \leq 2^m$.*

Proof. Let $\beta_{m,1}, \beta_{m,2}, \cdots \beta_{m,2^m-1}$ denote the 2^m-1 distinct roots of $l_m(x)$ in increasing order as described in Lemma 7.5. Then by the proof of Lemma 7.7, we have

$$\lambda_{m,1} = 0 < \beta_{m,1} < \lambda_{m,2} < \beta_{m,2} < \cdots < \lambda_{m,2^m-1} < \beta_{m,2^m-1} < \lambda_{m,2^m} = 6.$$

Combined with the fact $l_m(\lambda_{m,1}) = l_m(0) > 0$ by Lemma 7.4, it follows the desired result. □

Now we can prove the following Neumann analog of Lemma 4.5,

Lemma 7.10. *For each $m \geq 1$, $\mathcal{P}^{+,N}_m$ consists of at least 2^m distinct eigenvalues satisfying*

$$\lambda_{m,1} = 0 < \lambda_{m,2} < \cdots < \lambda_{m,2^m-1} < \lambda_{m,2^m} = 6.$$

Moreover,

$$\begin{aligned} \phi_-(\lambda_{m,k-1}) &< \lambda_{m+1,k} < \phi_-(\lambda_{m,k}), \quad \forall 2 \leq k \leq 2^m, \\ \phi_+(\lambda_{m,2^{m+1}-k+1}) &< \lambda_{m+1,k} < \phi_+(\lambda_{m,2^{m+1}-k}), \quad \forall 2^m+1 \leq k \leq 2^{m+1}-2, \\ \phi_+(\lambda_{m,2}) &< \lambda_{m+1,2^{m+1}-1} < 6. \end{aligned} \tag{7.4}$$

Proof. Noticing that each root of $p^N_m(x)$ belongs to $\mathcal{P}^{+,N}_m$, we only need to prove the results for the roots of $p^N_m(x)$. The first statement follows from Lemma 7.7. We now prove the second statement. From Lemma 7.8 and Lemma 7.9, we have

$$p^N_{m+1}(\phi_-(\lambda_{m,k})) = -\frac{\lambda_{m,k}}{2}\frac{\phi_-(\lambda_{m,k})-5}{\phi_-(\lambda_{m,k})-6}\cdot l_m(\lambda_{m,k}) \sim -l_m(\lambda_{m,k}) \sim (-1)^k, \quad \forall 2 \leq k \leq 2^m,$$

and similarly,

$$p^N_{m+1}(\phi_+(\lambda_{m,k})) = -\frac{\lambda_{m,k}}{2}\frac{\phi_+(\lambda_{m,k})-5}{\phi_+(\lambda_{m,k})-6}\cdot l_m(\lambda_{m,k}) \sim -l_m(\lambda_{m,k}) \sim (-1)^k, \quad \forall 2 \leq k \leq 2^m.$$

Following the above facts and Lemma 7.7, we can list the signs of the values of $p^N_{m+1}(x)$ at different point x in the following table.

$x:$	0	$0+$	$\phi_-(\lambda_{m,2})$	$\phi_-(\lambda_{m,3})$	$\cdots$	$\phi_-(\lambda_{m,2^m})$	$\phi_+(\lambda_{m,2^m})$	$\cdots$	$\phi_+(\lambda_{m,3})$	$\phi_+(\lambda_{m,2})$	$6-$	6
$p^N_{m+1}(x):$	0	$-$	$+$	$-$	$\cdots$	$+$	$+$	$\cdots$	$-$	$+$	$-$	0

Hence there exist at least 2^{m+1} distinct roots of $p^N_{m+1}(x)$ satisfying (7.4). Moreover, these are the totality of the roots of $p^N_{m+1}(x)$ since the degree of $p^N_{m+1}(x)$ is also 2^{m+1}. Hence we get the desired distribution of roots of $p^N_{m+1}(x)$. □

The estimate $\phi_+(\lambda_{m,2}) < \lambda_{m+1,2^{m+1}-1} < 6$ in Lemma 7.10 can be refined into

$$\phi_+(\lambda_{m,2}) < \lambda_{m+1,2^{m+1}-1} < 5 \tag{7.5}$$

by using the following lemma.

Lemma 7.11. *For $m \geq 2$, let $\lambda_{m,1} = 0, \lambda_{m,2}, \cdots, \lambda_{m,2^m-1}, \lambda_{m,2^m} = 6$ be the 2^m distinct roots of $p^N_m(x)$ in increasing order. Then*

$$\lambda_{m,k} + \lambda_{m,2^m-k+1} = 5, \quad \forall 2 \leq k \leq 2^m - 1.$$

Proof. From Lemma 7.1, it is easy to see that if $q^N_m(x) = 0$ and $x \neq 2$ or 6, then $q^N_m(5 - x) = 0$. Obviously, each $\lambda_{m,k}$ $(2 \leq k \leq 2^m - 1)$ satisfies this property. □

Now we come to the main result of this section:

Theorem 7.1. *For each $m \geq 1$, $\mathcal{P}^{+,N}_m$ consists of at least 2^m distinct eigenvalues satisfying*

$$\lambda_{m,1} = 0 < \lambda_{m,2} < \cdots < \lambda_{m,2^m-1} < 5 < \lambda_{m,2^m} = 6.$$

Moreover,

$$\begin{aligned} \phi_-(\lambda_{m,k-1}) &< \lambda_{m+1,k} < \phi_-(\lambda_{m,k}), \quad \forall 2 \leq k \leq 2^m, \\ \phi_+(\lambda_{m,2^{m+1}-k+1}) &< \lambda_{m+1,k} < \phi_+(\lambda_{m,2^{m+1}-k}), \quad \forall 2^m + 1 \leq k \leq 2^{m+1} - 1. \end{aligned} \tag{7.6}$$

Proof. It follows from Lemma 7.10 and Lemma 7.11. □

The following is a Neumann analog of Lemma 4.6.

Lemma 7.12. *Let λ_m be a root of $p^N_m(x)$, u_m a primitive λ_m-eigenfunction on Ω_m, and $(b_0, b_1, \cdots, b_m)$ the values of u_m on the skeleton of Ω_m. Then $b_0 \neq 0$ and $b_m \neq 0$.*

Proof. Without loss of generality, assume $m \geq 3$. We still use $\lambda^{(m)}_i$ to denote the successor of λ_m of order $(m-i)$ with $1 \leq i \leq m$. From the definition of $p^N_m(x)$, $\lambda^{(m)}_i \neq 2$ or 5, for each $2 \leq i \leq m$. Vector $(b_0, b_1, \cdots, b_m)$ can be viewed as a non-zero vector solution of system (4.2) of equations and in addition the two Neumann boundary eigenvalue equations (7.1) and (7.2).

Suppose $b_m = 0$. Then from (7.2), $b_{m-1} = 0$. It is easy to check that the determinant of the remaining equations in $m - 1$ unknowns $(b_0, b_1, \cdots, b_{m-2})$ is $\widetilde{l}_{m-1}(\lambda^{(m)}_{m-1})$. Since $(b_0, b_1, \cdots, b_{m-2})$ should be a non-zero vector, we have $\widetilde{l}_{m-1}(\lambda^{(m)}_{m-1}) = 0$, hence $l_{m-1}(\lambda^{(m)}_{m-1}) = 0$. Noticing that from Lemma 7.6, we have $p^N_m(\lambda_m) = (2 - \lambda_m) l_m(\lambda_m) -$

$4l_{m-1}(\lambda_{m-1}^{(m)})$. Hence we get that $l_m(\lambda_m) = 0$ since both $l_{m-1}(\lambda_{m-1}^{(m)})$ and $p_m^N(\lambda_m)$ are equal to 0. But this is impossible, since Lemma 7.5 says that if $l_{m-1}(\lambda_{m-1}^{(m)}) = 0$ then $l_m(\lambda_m)$ could not equal to 0. Hence $b_m \neq 0$.

On the other hand, if $b_0 = 0$, then by substituting it into the system, noticing that none of $\lambda_i^{(m)}$'s is equal to 2 or 5, we can get $b_1 = 0, \cdots, b_m = 0$ successively, which contradicts to $b_m \neq 0$. Hence $b_0 \neq 0$. □

This is the whole story of the symmetric case. The skew-symmetric case is slightly different but very similar. The result is shown in Section 3, but the proof is omitted.

With similar argument in the Dirichlet case, we finally get

Theorem 7.2. *For each $m \geq 1$, $\mathcal{P}_m^{+,N}$ consists of 2^m distinct eigenvalues satisfying*

$$\lambda_{m,1} = 0 < \lambda_{m,2} < \cdots < \lambda_{m,2^m-1} < 5 < \lambda_{m,2^m} = 6.$$

A relation between $\mathcal{P}_m^{+,N}$ and $\mathcal{P}_{m+1}^{+,N}$ is shown in (7.6). *Similar properties hold for $\mathcal{P}_m^{-,N}$ with 2^m replaced by $2^m - 1$, and $\lambda_{m,1} > 0$ in that case. Each $\lambda_m \in \mathcal{P}_m^N$ has multiplicity* 1. *Moreover, the Neumann spectrum $\mathcal{S}_m^N$ of Δ_m on Ω_m satisfies*

$$\mathcal{S}_m^N = \mathcal{L}_m^N \cup \mathcal{P}_m^{+,N} \cup \mathcal{P}_m^{-,N} \cup \mathcal{M}_m^N,$$

where the union is disjoint.

We should mention that Lemma 7.12 and its skew-symmetric analog show that there is no primitive eigenfunction (or miniaturized eigenfunction) that is simultaneously Dirichlet and Neumann ($D - N$). Hence the only possible $D - N$ eigenfunctions are localized eigenfunctions. This is same as the $\mathcal{SG}$ case.

8 Further discussion

In this section, we discuss to what extent our method can be extended to other domains in $\mathcal{SG}$. In particular, we will focus on Ω_x $(0 < x < 1)$. It seems that we can analyze the spectrum of Δ on Ω_x case by case following the similar recipe for the Ω_1 case. However, it is hardly to develop a general method which is suitable for all cases, although we believe that we are clear about the structures of the spectra. We let L_x denote the bottom boundary of Ω_x. Thus L_x will be a Cantor set for generic x, and a union of intervals if x is a dyadic rational. We may assume without loss of generality that $\frac{1}{2} < x < 1$, for if not we may first solve the problem for Ω_{2x}, and then simply dilate the solution to Ω_x.

For simplicity, we only discuss the Dirichlet spectrum of Δ. Obviously, it will suffice to describe the discrete Dirichlet spectra of Δ_m's for all m. Hence the first problem is how to define the graph approximations. Similar to Ω_1, the fractal domain Ω_x can be realized as the limit of a sequence of graphs $\Omega_{x,m}$. More precisely, $\forall m \geq 1$, let $V_m^{\Omega_x}$

be a subset of V_m with all vertices lying along or under L_x removed. Let $\Omega_{x,m}$ be the subgraph of Γ_m restricted to $V_m^{\Omega_x}$. Denote by $\partial\Omega_{x,m}$ the boundary of the finite graph $\Omega_{x,m}$. It is easy to find that $V_m^{\Omega_x} \setminus \partial\Omega_{x,m}$, $\partial\Omega_{x,m}$ approximate to Ω_x and $\partial\Omega_x$ as m goes to infinity respectively. See Fig. 8.1 and Fig. 8.2 for Ω_x and $\Omega_{x,m}$ where $x = 3/4$. On $\Omega_{x,m}$ the Dirichlet λ_m-eigenvalue equations consists of exactly $\sharp(V_m^{\Omega_x} \setminus \partial\Omega_{x,m})$ equations in $\sharp(V_m^{\Omega_x} \setminus \partial\Omega_{x,m})$ unknowns. We denote by $\mathcal{S}_m(x)$ the spectrum of Δ_m on $\Omega_{x,m}$ for each $m \geq 1$. Accordingly, $\mathcal{S}_m(x)$ should consists of (at least) three types of eigenvalues, denoted by $\mathcal{L}_m(x)$, $\mathcal{P}_m(x)$ and $\mathcal{M}_m(x)$ respectively. $\mathcal{P}_m(x)$ can also be split into symmetric part $\mathcal{P}_m^+(x)$ and skew-symmetric part $\mathcal{P}_m^-(x)$. We omit the precise definitions since they are obvious. To ensure that there is no other eigenvalue in $\mathcal{S}_m(x)$, the following eigenspace dimensional counting formula is hoped to be held,

$$\sharp(V_m^{\Omega_x} \setminus \partial\Omega_{x,m}) = \sharp\mathcal{L}_m(x) + \sharp\mathcal{P}_m(x) + \sharp\mathcal{M}_m(x).$$

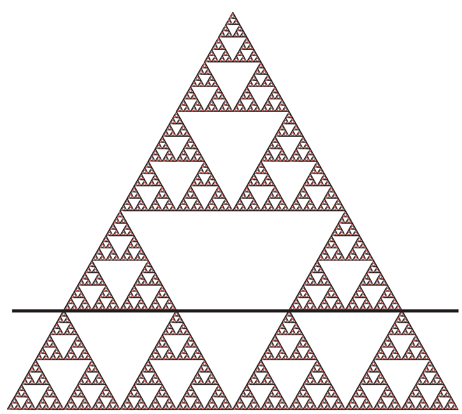

Fig. 8.1. $\Omega_{3/4}$.

We now focus on a particular example $\Omega_{3/4}$ to illustrate how to extend the recipe for Ω_1. We should be particular interested in the primitive eigenvalues. We begin with $\mathcal{P}_m^+(3/4)$, the symmetric case. It is convenient to define the skeleton of $\Omega_m(3/4)$ by $(q_0, F_1q_0, F_{10}F_1q_0, \cdots, F_{10}F_1^{m-2}q_0)$ for $m \geq 3$ and (q_0, F_1q_0) for $m = 1$ or 2. Let u_m be a λ_m-eigenfunction of Δ_m with $\lambda_m \in \mathcal{P}_m^+(3/4)$. Denote by $(b_0, b_1, b_2, \cdots, b_m)$ the values of u_m on the skeleton of $\Omega_{3/4,m}$ where $b_0 = b_m = 0$ by the Dirichlet boundary condition. It is easy to observe that when $i \geq 2$, the eigenvalue equation at the vertex $F_{10}F_1^{i-1}q_0$ is exactly same as that of Ω_1 case with suitably reindexed. Hence the generation mechanism of primitive symmetric eigenvalues is quite similar to the Ω_1 case. Based on this observation, one can easily find that $\sharp\mathcal{P}_m^+(3/4) = 2^m - 2$ for $m \geq 2$ by still using the weak spectral decimation method. A similar argument yields that $\sharp\mathcal{P}_m^-(3/4) = 2^m - 2^{m-2} - 2$ for $m \geq 2$.

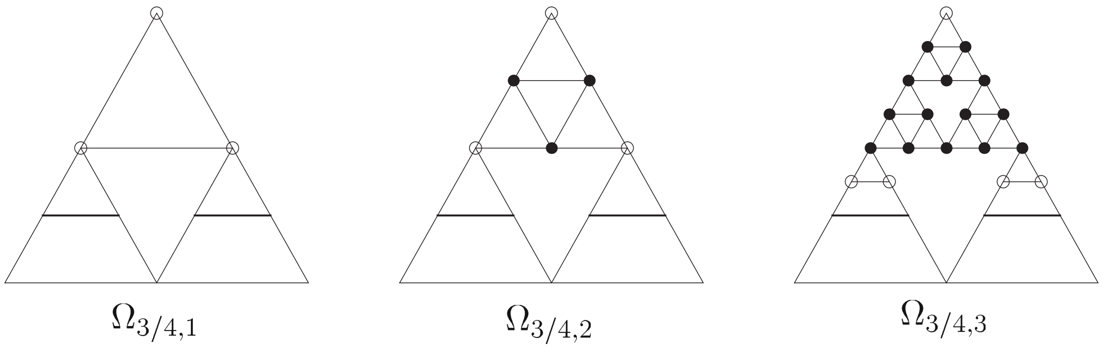

Fig. 8.2. The first 3 graphs, $\Omega_{3/4,1}, \Omega_{3/4,2}, \Omega_{3/4,3}$ in the approximation to $\Omega_{3/4}$ with inside points and boundary points represented by dots and circles respectively.

To verify the eigenspace dimensional counting formula, we only look at the first 4 levels of approximations since the continued process is similar.

When $m = 1$, the result is trivial since there is no inside point in $\Omega_{3/4,1}$. Hence $\sharp\mathcal{S}_1(3/4) = 0 = \sharp V_1^{\Omega_{3/4}} \setminus \partial\Omega_{3/4,1}$.

When $m = 2$, it is easy to check that there are only primitive eigenvalues. Hence $\sharp\mathcal{S}_2(3/4) = \sharp\mathcal{P}_2^+(3/4) + \sharp\mathcal{P}_2^-(3/4) = 2 + 1 = \sharp V_2^{\Omega_{3/4}} \setminus \partial\Omega_{3/4,2}$.

When $m = 3$, it is easy to check that there are 4 initial localized eigenvalues, i.e., 5 with multiplicity 1 and 6 with multiplicity 3; there are 6 primitive symmetric eigenvalues and 4 primitive skew-symmetric eigenvalues; there is no miniaturized eigenvalues. Hence $\sharp\mathcal{S}_3(3/4) = \sharp\mathcal{L}_3(3/4) + \sharp\mathcal{P}_3^+(3/4) + \sharp\mathcal{P}_3^-(3/4) = 4 + 6 + 4 = \sharp V_3^{\Omega_{3/4}} \setminus \partial\Omega_{3/4,3}$.

When $m = 4$, it is easy to check that besides $1 \cdot 2 + 3 \cdot 1 = 5$ continued localized eigenvalues, there are 18 initial localized eigenvalues, i.e., 5 with multiplicity 4 and 6 with multiplicity 14. Hence $\sharp\mathcal{L}_4(3/4) = 5 + 18 = 23$. There are 14 primitive symmetric eigenvalues and 10 primitive skew-symmetric eigenvalues. Hence $\sharp\mathcal{P}_4(3/4) = 14+10 = 24$. Moreover, there are some miniaturized eigenvalues which come from the miniaturizations of eigenvalues in $\mathcal{P}_2^-(1)$. Hence $\sharp\mathcal{M}_4(3/4) = 2 \cdot \mathcal{P}_2^-(1) = 2 \cdot 2 = 4$. Thus $\sharp\mathcal{S}_4(3/4) = 23 + 24 + 4 = \sharp V_4^{\Omega_{3/4}} \setminus \partial\Omega_{3/4,4}$.

It is easy to verify the general formula for general m. We will not attempt to list the details here. However, a more important fact should be pointed out is that for $\Omega_{3/4}$ case, the miniaturized eigenvalues in $\mathcal{M}_m(3/4)$ are generated not from those in $\mathcal{P}_k^-(3/4)$ but from those in $\mathcal{P}_k^-(1)$ for $k \leq m-2$. This means to study $\mathcal{S}_m(3/4)$, one should first make clear $\mathcal{S}_m(1)$. Things will be more complicated for general Ω_x.

Next we briefly present another observation. Still consider a domain Ω_x with a series of graph approximations $\{\Omega_{x,m}\}$. Notice that there are only two possible patterns when passing from the m-level graph approximation to its next level. One is that the boundary $\partial\Omega_{x,m+1}$ remains unchanged, i.e., $\partial\Omega_{x,m+1} = \partial\Omega_{x,m}$, the other is that $\partial\Omega_{x,m} \setminus \{q_0\}$ becomes a collection of inside points of $\Omega_{x,m+1}$, i.e., each point in $\partial\Omega_{x,m} \setminus \{q_0\}$ is connected with two new $(m+1)$-level points in $\partial\Omega_{x,m+1}$. In fact, for the $\mathcal{SG}$ case, when passing from one level to the next level, the boundaries of graphs are always V_0, keeping unchanged. This is also the reason why spectral decimation can work for 2-series eigenvalues (which should be considered as the primitive eigenvalues in $\mathcal{SG}$ case). As for the Ω_1 case, when passing from one level to the next level, the boundaries always change. Due to this phenomenon, the spectral decimation recipe should be replaced by the weak spectral decimation recipe for primitive or miniaturized eigenvalues since their supports always touch the boundaries. For general Ω_x $(0 < x < 1)$, these two possible patterns can both exist. It is natural to expect that under the first pattern, the two levels of primitive eigenvalues are related by the spectral decimation (it is obviously true.), while under the second pattern, they are related by a weak spectral decimation instead. Thus we expect:

Conjecture 8.1. *For a domain Ω_x $(0 < x < 1)$ with a series of graph approximations $\{\Omega_{x,m}\}$, if the boundaries change when passing from m-level to $(m+1)$-level, then there is a weak spectral decimation relating the two levels of primitive symmetric (or skew-symmetric) eigenvalues.*

9 Appendix

Theorem A. *For each $m \geq 2$, let $\lambda_{m,1}, \lambda_{m,2}, \cdots, \lambda_{m,r_m}$ be the r_m distinct eigenvalues in $\mathcal{P}_m^+$ in increasing order. Then $\lambda_{m+1,r_m+1} > 2$.*

To prove this theorem, we need the following lemma:

Lemma A. $p_2(2) < 0$, $p_3(2) > 0$ and $(-1)^m p_m(2) > 0$, $\forall m \geq 4$.

Proof. It is easy to check that $p_2(2) = -8 < 0$ and $p_3(2) = 68 > 0$.

Let $m \geq 4$. Then

$$\begin{aligned} p_m(x) &= \frac{q_m(x)}{(x-2)(f(x)-2)\cdots(f^{(m-3)}(x)-2)} \\ &= \frac{s(f^{(m-2)}(x))q_{m-1}(x) - l(f^{(m-3)}(x))r(f^{(m-2)}(x))q_{m-2}(x)}{(x-2)(f(x)-2)\cdots(f^{(m-3)}(x)-2)} \\ &= \frac{s(f^{(m-2)}(x))}{f^{(m-3)}(x)-2}p_{m-1}(x) + \frac{-l(f^{(m-3)}(x))r(f^{(m-2)}(x))}{(f^{(m-4)}(x)-2)(f^{(m-3)}(x)-2)}p_{m-2}(x). \end{aligned}$$

Noticing that $l(f^{(m-3)}(x)) = f^{(m-3)}(x) - 6 = (f^{(m-4)}(x) - 2)(3 - f^{(m-4)}(x))$ and choosing $x = 2$, we have

$$p_m(2) = \frac{s(f^{(m-2)}(2))p_{m-1}(2) + 2(2 - f^{(m-2)}(2))(5 - f^{(m-2)}(2))(3 - f^{(m-4)}(2))p_{m-2}(2)}{f^{(m-3)}(2) - 2}. \tag{9.1}$$

We will prove the following stronger result than that stated in Lemma A.

$$p_m(2) \sim (-1)^m \text{ and } p_{m+1}(2) + p_m(2) \sim (-1)^{m+1}, \forall m \geq 4. \tag{9.2}$$

Using (9.1), it is easy to check that $p_4(2) = 14064 > 0$ and $p_5(2) = -593514756 < 0$ by a direct computation. Hence (9.2) holds for $m = 4$. In order to use the induction, we assume (9.2) holds for m and will prove it for $m+1$.

First, it is easy to get that $p_{m+1}(2) \sim (-1)^{m+1}$, since otherwise $p_{m+1}(2) + p_m(2) \sim (-1)^m$, which contradicts to the induction assumption. Hence we only need to prove $p_{m+2}(2) + p_{m+1}(2) \sim (-1)^m$.

Note that from (9.1),

$$\begin{aligned}
& p_{m+2}(2) + p_{m+1}(2) \\
= & \frac{(s(f^{(m)}(2)) + f^{(m-1)}(2) - 2)p_{m+1}(2) + 2(2 - f^{(m)}(2))(5 - f^{(m)}(2))(3 - f^{(m-2)}(2))p_m(2)}{f^{(m-1)}(2) - 2} \\
= & a_m p_{m+1}(2) + b_m(p_{m+1}(2) + p_m(2)),
\end{aligned}$$

where

$$a_m = \frac{s(f^{(m)}(2)) + f^{(m-1)}(2) - 2 - 2(2 - f^{(m)}(2))(5 - f^{(m)}(2))(3 - f^{(m-2)}(2))}{f^{(m-1)}(2) - 2}$$

and

$$b_m = \frac{2(2 - f^{(m)}(2))(5 - f^{(m)}(2))(3 - f^{(m-2)}(2))}{f^{(m-1)}(2) - 2}.$$

It is easy to check that $b_m < 0$, since $f^{(m)}(2) < f^{(m-1)}(2) < f^{(m-2)}(2) < 0$ noticing that $f^{(2)}(2) = -6$ and $m \geq 4$. We will prove that $a_m < 0$ also. In fact, the numerator of a_m is $s(\gamma) + f(\beta) - 2 - 2(2-\gamma)(5-\gamma)(3-\beta)$, where $\gamma = f^{(m)}(2)$ and $\beta = f^{(m-2)}(2)$ for simplicity. By using $\gamma < f(\beta) < \beta \leq -6$, it is easy to get

$$\begin{aligned}
s(\gamma) + f(\beta) - 2 &= (2-\gamma)(4-\gamma)(5-\gamma) - 14 + 3\gamma + f(\beta) - 2 \\
&> (2-\gamma)(4-\gamma)(5-\gamma) - 16 + 4\gamma \\
&> (2-\gamma)(4-\gamma)(5-\gamma) - (2-\gamma)(5-\gamma) \\
&= (3-\gamma)(2-\gamma)(5-\gamma),
\end{aligned}$$

and

$$3 - \gamma > 3 - f(\beta) = 3 - \beta(5-\beta) > 3 - 5\beta > 2(3-\beta).$$

Hence we have $s(\gamma) + f(\beta) - 2 > 2(2-\gamma)(5-\gamma)(3-\beta)$. Thus the numerator of a_m is positive. Since the denominator of a_m is obviously negative, we get $a_m < 0$.

Hence since $p_{m+1}(2) \sim (-1)^{m+1}$ we have proved before, and $p_{m+1}(2)+p_m(2) \sim (-1)^{m+1}$ by the induction assumption, we finally get $p_{m+2}(2) + p_{m+1}(2) \sim (-1)^m$. □

Proof of Theorem A. Recall that in Lemma 4.4, we have proved that $p_{m+1}(\phi_-(\lambda_{m,r_m})) \sim (-1)^{m+r_m-1}$ and $p_{m+1}(\phi_+(\lambda_{m,r_m})) \sim (-1)^{m+r_m}$. Furthermore, λ_{m+1,r_m+1} is the only root of $p_{m+1}(x)$ between $\phi_-(\lambda_{m,r_m})$ and $\phi_+(\lambda_{m,r_m})$.

When $m = 2$, we have $p_3(\phi_-(\lambda_{2,r_2})) > 0$ and $p_3(\phi_+(\lambda_{2,r_2})) < 0$ since r_2 is odd. By Lemma A, we have $p_3(2) > 0$. Since λ_{3,r_2+1} is the only root between $\phi_-(\lambda_{2,r_2})$ and $\phi_+(\lambda_{2,r_2})$, we get $\lambda_{3,r_2+1} > 2$.

When $m \geq 3$, we have $p_{m+1}(\phi_-(\lambda_{m,r_m})) \sim (-1)^{m-1}$ and $p_{m+1}(\phi_+(\lambda_{m,r_m})) \sim (-1)^m$ since r_m is always even. Still by Lemma A, we have $p_{m+1}(2) \sim (-1)^{m-1}$. Since λ_{m+1,r_m+1} is the only root between $\phi_-(\lambda_{m,r_m})$ and $\phi_+(\lambda_{m,r_m})$, we get $\lambda_{m+1,r_m+1} > 2$. $\square$

Remark. *This theorem says that when $m \geq 3$, the first m-level initial eigenvalue is always greater than 2.*

Lemma B. *Let $m \geq 2$. Then $q_m(x) > 0$ whenever $0 < x < \phi_-^{(m)}(6)$.*

Proof. Define $\theta_m(z) = q_m(\phi_-^{(m)}(z))$ on $0 < z < 6$, $\forall m \geq 2$.

When $m = 2$, $\theta_2(z) = q_2(\phi_-^{(2)}(z))$. Noticing that $q_2(x) = s(x)$ and $q_2'(x) = -3x^2 + 22x - 35$, an easy calculus shows that $q_2(x)$ is monotone decreasing when $0 < x < \phi_-^{(2)}(6)$. Hence $\forall 0 < x < \phi_-^{(2)}(6)$, we have $q_2(0) = 26 > q_2(x) > q_2(\phi_-^{(2)}(6)) \approx 12.68$. Thus

$$26 > \theta_2(z) > 12.68, \forall 0 < z < 6. \tag{9.3}$$

When $m = 3$, $\theta_3(z) = q_3(\phi_-^{(3)}(z)) = s(\phi_-^{(3)}(z))\theta_2(z) - l(\phi_-^{(3)}(z))r(\phi_-^{(2)}(z))$ on $0 < z < 6$. Noticing that $s(\phi_-^{(3)}(z)) = q_2(\phi_-^{(3)}(z))$ and $q_2(x)$ is monotone decreasing when $0 < x < \phi_-^{(2)}(6)$, we have

$$s(0) = 26 > s(\phi_-^{(3)}(z)) > s(\phi_-^{(3)}(6)) \approx 22.96.$$

The monotone property of $-l(\phi_-^{(3)}(z))r(\phi_-^{(2)}(z))$ on $0 < z < 6$ implies that

$$-84.21 > -l(\phi_-^{(3)}(z))r(\phi_-^{(2)}(z)) > -120.$$

Hence by using (9.3), we get

$$26 \cdot 26 - 84.21 = 591.80 > \theta_3(z) > 22.96 \cdot 12.68 - 120 = 171.16, \forall 0 < z < 6.$$

Hence $\theta_3(z) \geq 6\theta_2(z) > 0$ on $0 < z < 6$.

We now use induction to prove:

$$\theta_{m+1}(z) \geq 6\theta_m(z) > 0 \text{ on } 0 < z < 6, \quad \forall m \geq 2. \tag{9.4}$$

Of course, it holds for $m = 2$. To use the induction, Assuming $\theta_{m+1}(z) \geq 6\theta_m(z) > 0$ on $0 < z < 6$, we will prove $\theta_{m+2}(z) \geq 6\theta_{m+1}(z) > 0$ on $0 < z < 6$.

Consider a polynomial $g(x) = s(x) - \frac{1}{6}l(x)r(f(x)) = 6 + \frac{115}{3}x - \frac{194}{3}x^2 + \frac{89}{3}x^3 - \frac{16}{3}x^4 + \frac{1}{3}x^5$. It is easy to compute that

$$g'(x) = \frac{115}{3} - \frac{388}{3}x + 89x^2 - \frac{64}{3}x^3 + \frac{5}{3}x^4 \geq \frac{115}{3} - \frac{388}{3}\phi_-^{(4)}(6) - \frac{64}{3}(\phi_-^{(4)}(6))^3 \approx 36.02 > 0$$

on $0 < x < \phi_-^{(4)}(6)$. Hence $g(x)$ is a monotone increasing function on the interval $[0, \phi_-^{(4)}(6)]$. So $g(x) \geq g(0) = 6$ on $0 < x < \phi_-^{(4)}(6)$.

By using an expansion along the last row of $\theta_{m+2}(z) = q_{m+2}(\phi_-^{(m+2)}(z))$, we have

$$\theta_{m+2}(z) = s(\phi_-^{(m+2)}(z))\theta_{m+1}(z) - \frac{1}{6}l(\phi_-^{(m+2)}(z))r(\phi_-^{(m+1)}(z)) \cdot 6\theta_m(z).$$

By the induction assumption and the fact that $\phi_-^{(m+2)}(z) < \phi_-^{(m+1)}(z) < 2$, we have

$$\begin{aligned}\theta_{m+2}(z) &\geq s(\phi_-^{(m+2)}(z))\theta_{m+1}(z) - \frac{1}{6}l(\phi_-^{(m+2)}(z))r(\phi_-^{(m+1)}(z))\theta_{m+1}(z) \\ &= g(\phi_-^{m+2}(z))\theta_{m+1}(z).\end{aligned}$$

Since $0 < \phi_-^{(m+2)}(z) < \phi_-^{(4)}(6)$ on $0 < z < 6$ when $m \geq 2$, we have $g(\phi_-^{(m+2)}(z)) \geq 6$. Hence

$$\theta_{m+2}(z) \geq 6\theta_{m+1}(z) > 0.$$

Hence we have proved (9.4) holds for $m+1$. From (9.4), we get the desired result. □

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(Hua Qiu) DEPARTMENT OF MATHEMATICS, NANJING UNIVERISITY, NANJING, 210093, CHINA

E-mail address: huatony@gmail.com