CLASSIFICATION OF 2-FANO MANIFOLDS WITH HIGH INDEX

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Dedicated to Joe Harris.

ABSTRACT. In this paper we classify n-dimensional Fano manifolds with index $\geq n - 2$ and positive second Chern character.

CONTENTS

1. Introduction

A Fano manifold is a smooth complex projective variety X having ample anticanonical class, $-K_X > 0$. This simple condition has far reaching geometric implications. For instance, any Fano manifold X is *rationally connected*, i.e., there are rational curves connecting any two points of X ([Cam92] and [KMM92a]).

The Fano condition $-K_X > 0$ also plays a distinguished role in arithmetic geometry. In the landmark paper [GHS03], Graber, Harris and Starr showed that proper families of rationally connected varieties over curves always admit sections. This generalizes Tsen's theorem in the case of function fields of curves.

Theorem (Tsen's Teorem)**.** *Let K be a field of transcendence degree r over an algebraically closed field k. Let $\mathcal{X} \subset \mathbf{P}_K^n$ be a hypersurface of degree d. If $d^r \leq n$, then $\mathcal{X}$ has a K-point.*

For hypersurfaces of degree d in $\mathbf{P}^n$, being Fano or rationally connected is equivalent to the numerical condition $d \leq n$. So, for $r = 1$, [GHS03] replaces the condition of $\mathcal{X}$ being a hypersurface of degree $d \leq n$ with the condition of $\mathcal{X}$ being rationally connected. It turns out that rationally connected varieties form the largest class of varieties for which such statement holds true when $r = 1$ (see [GHMS02] for the precise statement).

Since then, there has been quite some effort towards finding suitable geometric conditions on $\mathcal{X}$ that generalize Tsen's theorem for function fields of higher dimensional varieties. In [deJHS08], de Jong and Starr considered a possible notion of *rationally simply connectedness*. They established a version of Tsen's theorem for function fields of surfaces, replacing the condition of $\mathcal{X}$ being a hypersurface of degree d, $d^2 \leq n$, with the condition of $\mathcal{X}$ being rationally simply connected (see [deJHS08, Corollary 1.1] for a precise statement). Several attempts have been made to define the appropriate notion of rationally simply connectedness. Roughly speaking, one would like to ask that a suitable irreducible component of the space of rational curves through two general points of X is itself rationally connected. However, in order to make the definition applicable, one is led to introduce some technical hypothesis, which makes this condition difficult to verify in practice. It is then desirable to have natural geometric conditions that imply rationally simply connectedness. In this context, *2-Fano manifolds* were introduced by de Jong and Starr in [deJS06] and [deJS07]. In order to define these, we introduce some notation. Given a smooth projective variety X and a positive integer k, we denote by $N_k(X)$ the $\mathbf{R}$-vector space of k-cycles on X modulo numerical equivalence, and by $\overline{\mathrm{NE}}_k(X)$ the closed convex cone in $N_k(X)$ generated by classes of effective k-cycles. Recall that the second Chern character of X is

$$\mathrm{ch}_2(X) = \frac{c_1(X)^2}{2} - c_2(X),$$

where $c_i(X) = c_i(T_X)$. We say that a manifold X is *2-Fano* (respectively *weakly 2-Fano*) if it is Fano and $\mathrm{ch}_2(X) \cdot \alpha > 0$ (respectively $\mathrm{ch}_2(X) \cdot \alpha \geq 0$) for every $\alpha \in \overline{\mathrm{NE}}_2(X) \setminus \{0\}$.

Questions 1.1. Do 2-Fano manifolds satisfy some version of rationally simply connectedness? Is this a good condition to impose on the general member of fibrations over surfaces in order to prove existence of rational sections (modulo the vanishing of Brauer obstruction)?

Motivated by these questions, in [AC12], we investigated and classified certain spaces of rational curves on 2-Fano manifolds, and gave evidence for a positive answer to Questions 1.1. In that work, we announced the following threefold classification.

Theorem 1.2. *The only 2-Fano threefolds are* $\mathbf{P}^3$ *and the smooth quadric hypersurface* $Q^3 \subset \mathbf{P}^4$.

In this paper we write down a complete proof of Theorem 1.2. In fact, Theorem 1.2 will follow from a more general classification. Recall that the index i_X of a Fano manifold X is the largest integer dividing $-K_X$ in $\mathrm{Pic}(X)$. Our main result is the following.

Theorem 1.3. *Let* X *be a 2-Fano manifold of dimension* $n \geq 3$ *and index* $i_X \geq n - 2$. *Then* X *is isomorphic to one of the following.*
   - $\mathbf{P}^n$.
   - *Complete intersections in projective spaces:*
       - *Quadric hypersurfaces* $Q^n \subset \mathbf{P}^{n+1}$ *with* $n > 2$;
       - *Complete intersections of quadrics* $X_{2 \cdot 2} \subset \mathbf{P}^{n+2}$ *with* $n > 5$;
       - *Cubic hypersurfaces* $X_3 \subset \mathbf{P}^{n+1}$ *with* $n > 7$;
       - *Quartic hypersurfaces in* $\mathbf{P}^{n+1}$ *with* $n > 15$;
       - *Complete intersections* $X_{2 \cdot 3} \subset \mathbf{P}^{n+2}$ *with* $n > 11$;
       - *Complete intersections* $X_{2 \cdot 2 \cdot 2} \subset \mathbf{P}^{n+3}$ *with* $n > 9$.
   - *Complete interesections in weighted projective spaces:*
       - *Degree 4 hypersurfaces in* $\mathbf{P}(2, 1, \ldots, 1)$ *with* $n > 11$;
       - *Degree 6 hypersurfaces in* $\mathbf{P}(3, 2, 1, \ldots, 1)$ *with* $n > 23$;
       - *Degree 6 hypersurfaces in* $\mathbf{P}(3, 1, \ldots, 1)$ *with* $n > 26$;
       - *Complete intersections of two quadrics in* $\mathbf{P}(2, 1, \ldots, 1)$ *with* $n > 14$.
   - $G(2, 5)$.
   - $OG_+(5, 10)$ *and its linear sections of codimension* $c < 4$.
   - $SG(3, 6)$.
   - G_2/P_2.

Here $OG_+(5, 10)$ denotes a connected component of the 10-dimensional orthogonal Grassmannian $OG(5, 10)$ in the half-spinor embedding (see Section 6.2), $SG(3, 6)$ is a 6-dimensional symplectic Grassmannian (see Section 6.3), and G_2/P_2 is a 5-dimensional homogeneous variety for a group of type G_2 (see Section 6.4).

In order to prove Theorem 1.3, we will go through the classification of Fano manifolds of dimension $n \geq 3$ and index $i_X \geq n-2$, and check positivity of the second Chern character for each of them. In the course of the proof, we also determine which of these manifolds are weakly 2-Fano.

The paper is organized as follows. In Section 2 we revise the classification of Fano manifolds of high index. In Section 3, we check the 2-Fano condition for the simplest ones: (weighted) projective spaces and complete intersections on them, and Grassmannians. Most of the others can be described as double covers, blow-ups or projective bundles over simple ones. So in Section 4 we compute Chern characters for these constructions. In Section 5, we revise some results from [AC12], which describe certain families of rational curves on 2-Fano manifolds. These results are then used in Section 6 to check the 2-Fano condition for certain Fano manifolds described as complete intersections on homogeneous spaces. After all these computations, we are ready to prove Theorem 1.3. In Section 7, we address n-dimensional Fano manifolds with index $i_X \geq n - 2$, except Fano threefolds and fourfolds with Picard number ≥ 2. These are treated in Sections 8 and 9 respectively.

We remark that toric 2-Fano manifolds have been addressed in [Nob11], [Sat11] and [Nob12]. The only known examples are projective spaces.

Notation. Given a vector bundle $\mathcal{E}$ on a variety X, we denote by $\mathbf{P}_X(\mathcal{E})$, or simply $\mathbf{P}(\mathcal{E})$, the projective bundle of one-dimensional quotients of the fibers of $\mathcal{E}$, i.e., $\mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}\mathcal{E})$.

We denote by $G(k, n)$ the Grassmannian of k-dimensional subspaces of an n-dimensional vector space V, and we always assume that $2 \leq k \leq \frac{n}{2}$. We write

$$0 \to \mathcal{S} \to \mathcal{O} \otimes V \to \mathcal{Q} \to 0$$

for the universal sequence on $G(k, n)$. For subvarieties X of $G(k, n)$, we denote by the same symbols $\sigma_{a_1, \ldots, a_k}$ the restrictions to X of the corresponding Schubert cycles.

2. Classification of Fano manifolds

In this section we discuss the classification of Fano manifolds. A modern survey on this subject can be found in [IP99].

Notation. When X is an n-dimensional Fano manifold with $\rho(X) = 1$, we denote by L the ample generator of $\mathrm{Pic}(X)$, and define the degree of X as $d_X := c_1(L)^n$.

For a fixed positive integer n, Fano n-folds form a bounded family (see [KMM92b]). For $n \leq 3$, Fano n-folds are completely classified. The classification of Fano surfaces, also known as *Del Pezzo surfaces*, is a classical

result. They are $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, and the blow-up S_{9-n} of $\mathbf{P}^2$ at n points in general position, $1 \leq n \leq 8$. It is easy to check that among those only $\mathbf{P}^2$ is 2-Fano, and among the others only $S_8 = \mathbf{F}_1$ and $\mathbf{P}^1 \times \mathbf{P}^1$ are weakly 2-Fano (see 4.3.1).

The classification of Fano threefolds of Picard number $\rho = 1$ was established by Iskovskikh in [Isk77] and [Isk78]. There are 17 deformation types of these. The classification of Fano threefolds of Picard number $\rho \geq 2$ was established by Mori and Mukai in [MM81] and [MM03]. There are 88 deformation types of those. We will revise this list in Section 8.

In higher dimensions, there is no complete classification. On the other hand, one can get results in this direction if one fixes some invariants of the Fano manifold. For instance, we have the following result by Wiśniewski.

Theorem 2.1 ([Wiś91])**.** *Let X be an n-dimensional Fano manifold with index $i_X \geq \frac{n+1}{2}$. Then X satisfies one of the following conditions:*

- $\rho(X) = 1$;
- $X \cong \mathbf{P}^{\frac{n}{2}} \times \mathbf{P}^{\frac{n}{2}}$ *(n even)*;
- $X \cong \mathbf{P}^{\frac{n-1}{2}} \times Q^{\frac{n+1}{2}}$ *(n odd)*;
- $X \cong \mathbf{P}(T_{\mathbf{P}^{\frac{n+1}{2}}})$ *(n odd); or*
- $X \cong \mathbf{P}_{\mathbf{P}^{\frac{n+1}{2}}}(\mathcal{O}(1) \oplus \mathcal{O}^{\frac{n-1}{2}})$ *(n odd).*

Fano manifolds of dimension n and index $i_X \geq n - 2$ have been classified. A classical result of Kobayachi-Ochiai's asserts that $i_X \leq n+1$, and equality holds if and only if $X \simeq \mathbf{P}^n$. Moreover, $i_X = n$ if and only if X is a quadric hypersurface $Q^n \subset \mathbf{P}^{n+1}$ ([KO73]). Fano manifolds with index $i_X = n - 1$ are called *Del Pezzo* manifolds. They were classified by Fujita in [Fuj82a] and [Fuj82b]:

Theorem 2.2. *Let X be an n-dimensional Fano manifold with index $i_X = n - 1$, $n \geq 3$.*

(1) *Suppose that $\rho(X) = 1$. Then $1 \leq d_X \leq 5$. Moreover, for each $1 \leq d \leq 4$ and $n \geq 3$, and for $d = 5$ and $3 \leq n \leq 6$, there exists a unique deformation class of n-dimensional Fano manifolds Y_d with $\rho(X) = 1$, $i_X = n - 1$ and $d_X = d$. They have the following description:*

  (i) *Y_5 is a linear section of the Grassmannian $G(2,5) \subset \mathbf{P}^9$ (embedded via the Plücker embedding).*

  (ii) *$Y_4 = Q \cap Q' \subset \mathbf{P}^{n+2}$ is an intersection of two quadrics in $\mathbf{P}^{n+2}$.*

  (iii) *$Y_3 \subset \mathbf{P}^{n+1}$ is a cubic hypersurface.*

  (iv) *$Y_2 \to \mathbf{P}^n$ is a double cover branched along a quartic $B \subset \mathbf{P}^n$ (alternatively, Y_2 is a hypersurface of degree 4 in the weighted projective space $\mathbf{P}(2, 1, \ldots, 1)$).*

  (v) *Y_1 is a hypersurface of degree 6 in the weighted projective space $\mathbf{P}(3, 2, 1, \ldots, 1)$.*

(2) *Suppose that $\rho(X) > 1$. Then X is isomorphic to one of the following:*

- $\mathbf{P}^2 \times \mathbf{P}^2$ *(n = 4);*
- $\mathbf{P}(T_{\mathbf{P}^2})$ *(n = 3);*
- $\mathbf{P}\big(\mathcal{O}_{\mathbf{P}^2}(1) \oplus \mathcal{O}_{\mathbf{P}^2}\big)$ *(n = 3); or*
- $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ *(n = 3).*

An n-dimensional Fano manifold X with index $i_X = n - 2$ is called a *Mukai* manifold. The classification of such manifolds was first announced in [Muk89] (see also [IP99] and references therein). First we note that, by Theorem 2.1, if $n \geq 5$, then n-dimensional Mukai manifolds have Picard number $\rho = 1$, except in the cases of $\mathbf{P}^3 \times \mathbf{P}^3$, $\mathbf{P}^2 \times Q^3$, $\mathbf{P}(T_{\mathbf{P}^3})$ and $\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1) \oplus \mathcal{O})$.

So we start by considering n-dimensional Mukai manifolds X with $\rho(X) = 1$. In this case there is an integer $g = g_X$, called the *genus* of X, such that $d_X = c_1(L)^n = 2g - 2$. The linear system $|L|$ determines a morphism

$$\phi_{|L|} : X \to \mathbf{P}^{g+n-2},$$

which is an embedding if $g \geq 4$ (see [IP99, Theorem 5.2.1]).

Theorem 2.3. *Let X be an n-dimensional Mukai manifold with $\rho(X) = 1$. Then X has genus $g \leq 12$ $(g \neq 11)$ and we have the following descriptions.*

(1) *If $g = 12$, then $n = 3$ and X is the zero locus of a global section of the vector bundle $\wedge^2 \mathcal{S}^* \oplus \wedge^2 \mathcal{S}^* \oplus \wedge^2 \mathcal{S}^*$ on the Grassmannian $G(3,7)$.*

(2) *If $6 \leq g \leq 10$, then X is a linear section of a variety*

$$\Sigma_{2g-2}^{n(g)} \subset \mathbf{P}^{g+n(g)-2}$$

of dimension $n(g)$ and degree $2g - 2$, which can be described as follows:

(g=6) *$\Sigma_{10}^6 \subset \mathbf{P}^{10}$ is a quadric section of the cone over the Grassmannian $G(2,5) \subset \mathbf{P}^9$ in the Plücker embedding;*

(g=7) *$\Sigma_{12}^{10} = OG_+(5,10) \subset \mathbf{P}^{15}$ is a connected component of the orthogonal Grassmannian $OG(5,10)$ in the half-spinor embedding;*

(g=8) *$\Sigma_{14}^8 = G(2,6) \subset \mathbf{P}^{14}$ is the Grassmannian $G(2,6)$ in the Plücker embedding;*

(g=9) *$\Sigma_{16}^6 = SG(3,6) \subset \mathbf{P}^{13}$ is the symplectic Grassmannian $SG(3,6)$ in the Plücker embedding;*

(g=10) *$\Sigma_{18}^5 = (G_2/P_2) \subset \mathbf{P}^{13}$ is a 5-dimensional homogeneous variety for a group of type G_2.*

(3) *If $g \leq 5$, and the map $\phi_{|L|}$ is an embedding, then X is a complete intersection as follows:*

(g=3) *$X_4 \subset \mathbf{P}^{n+1}$ a quartic hypersurface;*

(g=4) *$X_{2\cdot3} \subset \mathbf{P}^{n+2}$ a complete intersection of a quadric and a cubic;*

(g=5) *$X_{2\cdot2\cdot2} \subset \mathbf{P}^{n+3}$ a complete intersection of three quadrics.*

(4) *If $g \leq 3$, and the map $\phi_{|L|}$ is not an embedding, then:*

(g=2) $\phi_{|L|} : X \to \mathbf{P}^n$ *is a double cover branched along a sextic (alternatively, X is a degree* 6 *hypersurface in the weighted projective space* $\mathbf{P}(3, 1, \dots, 1)$*);*

(g=3) $\phi_{|L|} : X \to Q^n \subset \mathbf{P}^{n+1}$ *is a double cover branched along the intersection of Q with a quartic hypersurface (alternatively, X is a complete intersection of two quadric hypersurfaces in the weighted projective space* $\mathbf{P}(2, 1, \dots, 1)$*).*

We will go through the classification of Mukai manifolds with Picard number $\rho \geq 2$ and dimension $n \in \{3, 4\}$ in Sections 8 and 9.

3. First Examples

In this section we compute the first Chern characters for the simplest examples of Fano manifolds with high index: (weighted) projective spaces and complete intersection on them, and Grassmannians.

3.1. **Projective spaces.** Set $h := c_1(\mathcal{O}_{\mathbf{P}^n}(1))$. Then

$$\mathrm{ch}(\mathbf{P}^n) = n + \sum_{k=1}^{n} \frac{n+1}{k!} h^k.$$

In particular, $\mathbf{P}^n$ is 2-Fano, with second Chern character given by:

$$\mathrm{ch}_2(\mathbf{P}^n) = \frac{n+1}{2} h^2.$$

3.2. **Weighted projective spaces.** Let $\mathbf{P} = \mathbf{P}(a_0, \dots, a_n)$ be a weighted projective space, with $\gcd(a_0, \dots, a_n) = 1$. Denote by H the effective generator of the class group $\mathrm{Cl}(\mathbf{P}) \cong \mathbf{Z}$. Recall that H is an ample $\mathbf{Q}$-Cartier divisor. From the Euler sequence, on the smooth locus of $\mathbf{P}$, we have:

$$\mathrm{ch}(\mathbf{P}) = n + \sum_{k=1}^{n} \frac{a_0^k + \dots + a_n^k}{k!} c_1(H)^k.$$

3.3. **Zero loci of sections of vector bundles.** Several Fano manifolds with $\rho(X) = 1$ and high index are described as the zero locus $X = Z(s) \subseteq Y$ of a global section s of a vector bundle $\mathcal{E}$ on a simpler variety Y. So we investigate the 2-Fano condition in this case.

Lemma 3.1. *Let Y be a smooth projective variety, and $\mathcal{E}$ a vector bundle on Y. Let s be a global section of $\mathcal{E}$, and X its zero locus $Z(s)$. Assume that X is smooth of dimension $\dim(Y) - rk(\mathcal{E})$. Then*

$$ch_i(X) = \big(ch_i(Y) - ch_i(\mathcal{E})\big)_{|X}.$$

Proof. Since the normal bundle $N_{X|Y}$ is $\mathcal{E}_{|X}$, the lemma follows from the normal bundle sequence. $\square$

Special cases of these are complete intersections. If Y is a smooth projective variety, and X is a smooth complete intersection of divisors $D_1, \ldots, D_c$ in X, then Lemma 3.1 becomes:

$$(3.1) \qquad \mathrm{ch}_k(X) = \big(ch_k(Y) - \frac{1}{k!}\sum D_i^k\big)_{|X}.$$

3.3.1. Complete intersections in $\mathbf{P}^n$. Let X be a smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_c$. Then by (3.1):

$$\mathrm{ch}_k(X) = \frac{1}{k!}\big((n+1) - \sum d_i^k\big)h_{|X}^k.$$

It follows that

  (i) X is 2-Fano if an only if $\sum d_i^2 \leq n$.
  (ii) X is weakly 2-Fano if an only if $\sum d_i^2 \leq n+1$.

3.3.2. Complete intersections in weighted projective spaces. We use the same notation as in 3.2. Let $\mathbf{P}$ denote the weighted projective space $\mathbf{P}(a_0, \ldots, a_n)$, with $\gcd(a_0, \ldots, a_n) = 1$, and let X be a smooth complete intersection of hypersurfaces with classes $d_1 H, \ldots, d_c H$ in $\mathbf{P}$. Assume X is smooth, and contained in the smooth locus of $\mathbf{P}$. Then the Chern character of X is given by

$$\mathrm{ch}(X) = (n-c) + \sum_{k=1}^{n} \frac{a_0^k + \ldots + a_n^k - \sum d_i^k}{k!} c_1(H_{|X})^k.$$

It follows that

  (i) X is 2-Fano if an only if $\sum d_i^2 < \sum a_i^2$.
  (ii) X is weakly 2-Fano if an only if $\sum d_i^2 \leq \sum a_i^2$.

3.4. Grassmannians. Let $\mathcal{S}^*$ denote the dual of the universal rank k vector bundle $\mathcal{S}$ on $G(k,n)$. The Chern classes of $\mathcal{S}^*$ are given by:

$$c_i(\mathcal{S}^*) = \sigma_{1,\ldots,1}, \quad (i \geq 1)$$

where $\sigma_{a_1,\ldots,a_k}$ denotes the usual Schubert cycle on $G(k,n)$. Recall that σ_1 is the class of a hyperplane via the Plücker embedding and generates $\mathrm{Pic}(G(k,n))$. Since the tangent bundle of $G(k,n)$ is given by

$$T_{G(k,n)} = \mathcal{S}^* \otimes \mathcal{Q},$$

the Chern character of $G(k,n)$ can be calculated from

$$\mathrm{ch}(G(k,n)) = \mathrm{ch}(\mathcal{S}^*)\mathrm{ch}(\mathcal{Q}) = \mathrm{ch}(\mathcal{S}^*)\big(n - \mathrm{ch}(\mathcal{S})\big).$$

The Chern character of $\mathcal{S}^*$ is given by

$$\mathrm{ch}(\mathcal{S}^*) = k + \sigma_1 + \frac{1}{2}(\sigma_2 - \sigma_{1,1}) + \frac{1}{6}(\sigma_3 - \sigma_{2,1} + \sigma_{1,1,1}) + \ldots.$$

As computed in [deJS06, 2.2]),

$$\mathrm{ch}(G(k,n)) = k(n-k) + n\sigma_1 + \big(\frac{n+2-2k}{2}\sigma_2 - \frac{n-2-2k}{2}\sigma_{1,1}\big) +$$

$$+\frac{n-2k}{6}\big(\sigma_3 - \sigma_{2,1} + \sigma_{1,1,1}\big) + \dots .$$

The cone $\overline{\mathrm{NE}}_2(G(k,n))$ is generated by the dual Schubert cycles σ_2^* and $\sigma_{1,1}^*$. It follows that $G(k,n)$ is 2-Fano if and only if $n = 2k$ or $2k + 1$. Moreover, $G(k,n)$ is weakly 2-Fano if and only if $n = 2k$, $2k + 1$ or $2k + 2$.

Remark 3.2. Complete intersections and, more generally, zero loci of vector bundles in Grassmannians will be addressed in Section 6. We will need the following formulas, obtained by standard Chern class computations:

$$\mathrm{ch}(\wedge^2(\mathcal{S}^*)) = \binom{k}{2} + (k-1)\sigma_1 + \big(\frac{k-1}{2}\sigma_2 - \frac{k-3}{2}\sigma_{1,1}\big) +$$

$$+\big(\frac{k-1}{6}\sigma_3 - \frac{k-4}{6}\sigma_{2,1} + \frac{k-7}{6}\sigma_{1,1,1}\big) + \dots ,$$

$$\mathrm{ch}(\mathrm{Sym}^2(\mathcal{S}^*)) = \binom{k+1}{2} + (k+1)\sigma_1 + \big(\frac{k+3}{2}\sigma_2 - \frac{k+1}{2}\sigma_{1,1}\big) +$$

$$+\big(\frac{k+7}{6}\sigma_3 - \frac{k+4}{6}\sigma_{2,1} + \frac{k+1}{6}\sigma_{1,1,1}\big) + \dots .$$

4. Chern class computations

From the classification of Fano manifolds with high index, we see that many of those with $\rho = 1$ are described as double covers, while most of the ones with $\rho > 1$ are obtained from simpler ones by blow-ups and taking projective bundles. So in this section we compute Chern characters for these constructions.

4.1. Double covers.

Lemma 4.1. *Let $f : X \to Y$ be a finite map of degree 2 between smooth projective varieties X and Y. Let $R \subset X$ denote the ramification divisor, and $B = f(R) \subset Y$ the branch divisor. Then $f^{-1}(B) = 2R$ and there is an exact sequence:*

$$0 \to T_X \to f^*T_Y \to \mathcal{O}(2R)_{|R} \to 0.$$

The first and second Chern characters are related by

$$c_1(X) = f^*\big(c_1(Y) - \frac{1}{2}B\big),$$

$$ch_2(X) = f^*\big(ch_2(Y) - \frac{3}{8}B^2\big).$$

Proof. This follows from the exact sequence:

$$0 \to f^*\Omega_Y \to \Omega_X \to \mathcal{O}(-R)_{|R} \to 0.$$

$\square$

Corollary 4.2. *Let $f : X \to Y$ be a finite map of degree 2 between smooth projective varieties X and Y. Let $B \subset Y$ be the branch divisor. Then:*

(i) X is Fano if and only if $-K_Y - \frac{1}{2}B$ is an ample divisor. In particular, if X is Fano and B is nef, then Y is Fano.

(ii) X is 2-Fano (respectively weakly 2-Fano) if and only if X is Fano and
$$ch_2(Y) - \frac{3}{8}B^2 > 0 \quad (\text{respectively } \geq 0).$$

If X is weakly 2-Fano and B is ample, then Y is 2-Fano.

4.2. **Projective Bundles.** The following two lemmas appear in [deJS06]. (Note that in [deJS06] the notation $\mathbf{P}(E)$ stands for $\mathrm{Proj}(\mathrm{Sym}E^*)$.)

Lemma 4.3. [deJS06, Lemma 3.1] Let X be a smooth irreducible projective variety and let $\mathcal{E}$ be a rank r vector bundle on X. Denote by $\pi : \mathbf{P}(\mathcal{E}) \to X$ the natural projection and let $\xi = c_1(\mathcal{O}_\pi(1))$. Then
$$c_1(\mathbf{P}(\mathcal{E})) = \pi^*\big(c_1(X) + c_1(\mathcal{E}^*)\big) + r\xi,$$
$$ch_2(\mathbf{P}(\mathcal{E})) = \pi^*\big(ch_2(X) + ch_2(\mathcal{E}^*))\big) + \pi^* c_1(\mathcal{E}^*) \cdot \xi + \frac{r}{2}\xi^2.$$

Lemma 4.4. [deJS06, Prop 3.3] Let X be a smooth irreducible projective variety and let $\mathcal{E}$ be a rank 2 vector bundle on X. Denote by $\pi : \mathbf{P}(\mathcal{E}) \to X$ the natural projection and let $\xi = c_1(\mathcal{O}_\pi(1))$. Then
$$c_1(\mathbf{P}(\mathcal{E})) = 2\xi + \pi^*\big(c_1(X) - c_1(\mathcal{E})\big),$$
$$ch_2(\mathbf{P}(\mathcal{E})) = \pi^*\big(ch_2(X) + \frac{1}{2}(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}))\big).$$

Therefore, $ch_2(\mathbf{P}(\mathcal{E})) \geq 0$ if and only if

(4.1) $$ch_2(X) + \frac{1}{2}(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E})) \geq 0.$$

If $\dim(X) > 0$, then $\mathbf{P}(\mathcal{E})$ is not 2-Fano. $\mathbf{P}(\mathcal{E})$ is weakly 2-Fano if it is Fano and condition (4.1) holds.

Corollary 4.5. Let X be a smooth irreducible projective variety and let L be a line bundle on X. The projective bundle $\mathbf{P}_X(\mathcal{O} \oplus L)$ is not 2-Fano and it is weakly 2-Fano if and only if it is Fano and we have:

(4.2) $$ch_2(X) + \frac{1}{2}c_1(L)^2 \geq 0.$$

In particular, (4.2) holds if X is weakly 2-Fano and L is nef. For example:

(i) $\mathbf{P}_{\mathbf{P}^n}(\mathcal{O} \oplus \mathcal{O}(a))$ is weakly 2-Fano if and only if $|a| \leq n$.

(ii) $\mathbf{P}_{\mathbf{P}^n \times \mathbf{P}^m}(\mathcal{O} \oplus \mathcal{O}(a,b))$ is weakly 2-Fano if and only if $|a| \leq n$, $|b| \leq m$, and $ab \geq 0$.

Proof. The result is an immediate consequence of Lemma 4.4. For examples (i) and (ii) we use that $\mathbf{P}_{\mathbf{P}^n}(\mathcal{O} \oplus \mathcal{O}(a))$ is Fano if and only if $|a| \leq n$, and similarly, $\mathbf{P}_{\mathbf{P}^n \times \mathbf{P}^m}(\mathcal{O} \oplus \mathcal{O}(a,b))$ is Fano if and only if $|a| \leq n$ and $|b| \leq m$. $\square$

Example 4.6. Consider the Fano manifold $X = \mathbf{P}(T_{\mathbf{P}^n})$, $n \geq 2$.

If $n = 2$, then X is not 2-Fano, but weakly 2-Fano by Lemma 4.4 since

$$\mathrm{ch}_2(\mathbf{P}^2) + \frac{1}{2}\big(c_1(\mathbf{P}^2)^2 - 4c_2(\mathbf{P}^2)\big) = c_1(\mathbf{P}^2)^2 - 3c_2(\mathbf{P}^2) = 0.$$

Suppose $n \geq 3$. Denote by $\pi : X \to \mathbf{P}^n$ the natural morphism, and let $\ell \subset \mathbf{P}^n$ be a line. Consider the surface S in $\pi^{-1}(\ell)$, ruled over ℓ, corresponding to the surjection

$$T_{\mathbf{P}^n}|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1} \twoheadrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1).$$

Using the formula for ch_2 from Lemma 4.3, one gets that $\mathrm{ch}_2(X) \cdot S = -1$. Hence X is not weakly 2-Fano.

Lemma 4.7. *A product $X \times Y$ of smooth projective irreducible varieties is not 2-Fano. It is weakly 2-Fano if and only if both X and Y are weakly 2-Fano.*

Proof. This follows from the projection formula and the formula

$$\mathrm{ch}_2(X \times Y) = \pi_1^*\mathrm{ch}_2(X) \oplus \pi_2^*\mathrm{ch}_2(Y),$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the two projections. For any two curves $B \subset X$ and $C \subset Y$, if we let $S = B \times C$, then $X \times Y$ is not 2-Fano since $\mathrm{ch}_2(X \times Y) \cdot S = 0$. $\qquad\square$

4.2.1. **Complete intersections in in products of projective spaces.** Let Y be a smooth divisor of type $(a_1, \ldots, a_r)$ in $\mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_r}$, and set $h_i := c_1(\pi_i^*\mathcal{O}(1))$. By a direct computation using the normal bundle sequence, we have:

$$\mathrm{ch}_2(Y) = \frac{1}{2}\sum_{i=1}^{r}(n_i + 1 - a_i^2)\big(h_i^2\big)_{|Y} - \sum_{i<j}(a_i a_j)\big(h_i \cdot h_j\big)_{|Y}.$$

We compute some examples of intersection numbers $\mathrm{ch}_2(Y) \cdot S$ with

$$S = h_{1|Y}^{c_1} \cdot \ldots \cdot h_{r|Y}^{c_r}, \quad \sum c_i = \sum n_i - 3 \quad (c_i \geq 0).$$

Note that if γ_1, γ_2 are cycles on a manifold X and $Y \subset X$ is a submanifold, then the intersection of the restrictions to Y can be computed by

$$\gamma_{1|Y} \cdot \gamma_{2|Y} = \gamma_1 \cdot \gamma_2 \cdot Y.$$

Example 4.8. If Y is a divisor of type (a, b) on $\mathbf{P}^n \times \mathbf{P}^m$ $(a, b > 0)$, then

$$\mathrm{ch}_2(Y) = \frac{1}{2}(n + 1 - a^2)h_{1|Y}^2 + \frac{1}{2}(n + 1 - b^2)h_{2|Y}^2 - ab(h_1 \cdot h_2)_{|Y}.$$

Since $h_1^n \cdot h_2^m = 1$, it follows that

$$\mathrm{ch}_2(Y) \cdot h_{1|Y}^{n-2} \cdot h_{2|Y}^{m-1} = \frac{b}{2}(n + 1 - 3a^2).$$

In particular, Y is not weakly 2-Fano if either $3a^2 > n+1$ or $3b^2 > m+1$.

Example 4.9. If Y is a divisor of type (a, b, c) on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$, then

$$\mathrm{ch}_2(Y) = \frac{1}{2}(3 - c^2)h_{3|Y}^2 - ab(h_1 \cdot h_2)_{|Y} - ac(h_1 \cdot h_3)_{|Y} - bc(h_2 \cdot h_3)_{|Y}.$$

Since $h_1^2 = h_2^2 = 0$ and $h_1 \cdot h_2 \cdot h_3^2 = 1$, it follows that

$$\mathrm{ch}_2(Y) \cdot h_{1|Y} = \frac{3}{2}(1 - bc^2), \quad \mathrm{ch}_2(Y) \cdot h_{2|Y} = \frac{3}{2}(1 - ac^2),$$

$$\mathrm{ch}_2(Y) \cdot h_{3|Y} = -3abc.$$

In particular, if $a, b, c > 0$ then Y is not weakly 2-Fano.

Example 4.10. If Y is a divisor of type $(a_1, \ldots, a_r)$ on $(\mathbf{P}^1)^r$, then

$$\mathrm{ch}_2(Y) = -\sum_{i<j} a_i a_j (h_i \cdot h_j)_{|Y}.$$

Since $h_i^2 = 0$ and $h_1 \cdot \ldots \cdot h_r = 1$, it follows that

$$\mathrm{ch}_2(Y) \cdot h_{1|Y} \cdot h_{2|Y} \ldots \cdot h_{r-3|Y} = -3a_{r-2}a_{r-1}a_r.$$

In particular, if $a_i > 0$ for all $i = 1, \ldots r$, then Y is not weakly 2-Fano.

Example 4.11. Let Y be a complete intersection in $\mathbf{P}^n \times \mathbf{P}^m$ of a divisor D_1 of type (a_1, b_1) and a divisor D_2 of type (a_2, b_2). Then

$$\mathrm{ch}_2(Y) = \frac{1}{2}(n + 1 - a_1{}^2 - a_2{}^2)h_{1|Y}^2 + \frac{1}{2}(m + 1 - b_1{}^2 - b_2{}^2)h_{2|Y}^2 -$$

$$-(a_1 b_1 + a_2 b_2)(h_1 \cdot h_2)_{|Y}.$$

It follows that

$$\mathrm{ch}_2(Y) \cdot h_{1|Y}^{n-2} \cdot h_{2|Y}^{m-2} = -(a_1 b_1 + a_2 b_2)(a_1 b_2 + a_2 b_1)$$

In particular, Y is not weakly 2-Fano if $a_i, b_i > 0$.

4.3. **Blow-ups.** The following Lemma appeared first in [deJS06]. See also [Nob12] for a detailed computation.

Lemma 4.12. [deJS06, Lemma 4.1] *Let X be a smooth projective variety and let $i : Y \hookrightarrow X$ be a smooth irreducible subvariety of codimension $c \geq 2$. Let $f : \tilde{X} \to X$ be the blow-up of X along Y and let E be the exceptional divisor. Denote by $j : E \hookrightarrow \tilde{X}$ the natural inclusion map and let $\pi = f_{|E} : E \to Y$. Let N be the normal bundle of Y in X. The Chern characters of $\tilde{X}$ are given by the following formulas:*

$$c_1(\tilde{X}) = f^* c_1(X) - (c - 1)[E],$$

$$ch_2(\tilde{X}) = f^* ch_2(X) + \frac{c+1}{2}[E]^2 - j_* \pi^* c_1(N).$$

4.3.1. **Del Pezzo surfaces.** Let S_d $(1 \leq d \leq 9)$ denote a Del Pezzo surface of degree d, i.e., S_d is the blow-up of $\mathbf{P}^2$ at $9 - d$ points in general position. By Lemma 4.12, we have

$$(4.3) \qquad \mathrm{ch}_2(S_d) = \mathrm{ch}_2(\mathbf{P}^2) - \frac{3}{2}(9 - d) = \frac{3}{2}(d - 8).$$

It follows that the only 2-Fano Del Pezzo surface is $\mathbf{P}^2$, while $S_8 = \mathbf{F}_1$ and $\mathbf{P}^1 \times \mathbf{P}^1$ (Lemma 4.7) are the only other weakly 2-Fano surfaces.

4.3.2. **The case of threefolds.** We compute several intersection numbers of $\mathrm{ch}_2(\tilde{X})$ with surfaces in the case when $\tilde{X}$ is a blow-up of a threefold, first along a smooth curve (Lemma 4.13), and then along points (Lemma 4.14).

Lemma 4.13. *Let X be a smooth projective variety of dimension 3 and let C be a smooth irreducible curve in X. Let $\tilde{X}$ be the blow-up of X along C, E the exceptional divisor, and N the normal bundle of C in X. Then $E^3 = -\deg(N)$ and we have*

$$ch_2(\tilde{X}) \cdot E = -\frac{1}{2}\deg(N).$$

Let T be a smooth surface in X and let $\tilde{T}$ be its proper transform in $\tilde{X}$.

(i) *If $T \cap C$ is a 0-dimensional reduced scheme of length r, then*

$$ch_2(\tilde{X}) \cdot \tilde{T} = ch_2(X) \cdot T - \frac{3r}{2}.$$

(ii) *If $C \subset T$ and $(C^2)_T$ denotes the self-intersection of C on T, then*

$$ch_2(\tilde{X}) \cdot \tilde{T} = ch_2(X) \cdot T + \frac{3}{2}(C^2)_T - \deg(N).$$

Proof. By Lemma 4.12, we have:

$$\mathrm{ch}_2(\tilde{X}) \cdot E = \frac{3}{2}E^3 - \left(j_*\pi^*c_1(N)\right) \cdot E.$$

Denote

$$\xi = c_1(\mathcal{O}_E(1)).$$

Since $E \cong \mathbf{P}_C(N^*)$, by [Ful98, Rmk. 3.2.4]) we have

$$\xi^2 + \pi^*c_1(N)\xi + \pi^*c_2(N) = 0.$$

(Note that in [Ful98], $\mathbf{P}(E) = \mathrm{Proj}(\mathrm{Sym}(E^*))$.)
    It follows that $\xi^2 = -\deg(N)$ and hence,

$$E^3 = \xi^2 = -\deg(N).$$

If α is a cycle on E and D is a divisor on X, then

$$(4.4) \qquad j_*\alpha \cdot D = (j^*D \cdot \alpha)_E,$$

where $(,)_E$ denotes the intersection on E. Applying (4.4) for $D = E$ and $\alpha = \pi^*c_1(N)$, it follows that

$$\left(j_*\pi^*c_1(N)\right) \cdot E = -\left(\xi \cdot \pi^*c_1(N)\right)_E = -\deg(N),$$

$$\mathrm{ch}_2(T_{\tilde{X}}) \cdot E = -\frac{3}{2} \deg(N) + \deg(N) = -\frac{1}{2} \deg(N).$$

For Cases (i) and (ii), by Lemma 4.12, we have:

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(X) \cdot T + \frac{3}{2} E^2 \cdot \tilde{T} - \left(j_* \pi^* c_1(N)\right) \cdot \tilde{T}.$$

Consider now Case (i). Then $\tilde{T}$ is the blow-up of T along the r points in $C \cap T$. Then $E \cap \tilde{T}$ is the union of the r exceptional divisors of the blow-up $\tilde{T} \to T$. Since $\tilde{T}_{|E}$ consists of fibers of $\pi : E \to C$, it follows using (4.4) that

$$\left(j_* \pi^* c_1(N)\right) \cdot \tilde{T} = \tilde{T}_{|E} \cdot \pi^* c_1(N) = 0.$$

As $E^2 \cdot \tilde{T} = \left(E^2_{|\tilde{T}}\right)_{\tilde{T}} = -r$, the result follows.

Consider now Case (i). Then $\tilde{T} \cong T$ and $E \cap \tilde{T}$ is a section of $\pi : E \to C$. By (4.4) it follows that

$$\left(j_* \pi^* c_1(N)\right) \cdot \tilde{T} = \tilde{T}_{|E} \cdot \pi^* c_1(N) = \deg(N).$$

Since $E^2 \cdot \tilde{T} = (C^2)_T$, the result follows. $\qquad\square$

Lemma 4.14. *Let X be a smooth projective variety of dimension 3 and let $\tilde{X}$ be the blow-up of X along a point q in X, with exceptional divisor E. Then $E^3 = 1$ and we have:*

$$ch_2(\tilde{X}) \cdot E = 2.$$

Let T be a surface in X and let $\tilde{T}$ be its proper transform in $\tilde{X}$. If $m \geq 0$ is the multiplicity of T at q, then we have:

$$ch_2(\tilde{X}) \cdot \tilde{T} = ch_2(X) \cdot T - 2m.$$

Proof. Since $E_{|E}$ is the tautological line bundle $\mathcal{O}_E(-1)$ on $E \cong \mathbf{P}^2$, it follows that $E^3 = \left(\mathcal{O}_E(-1)^2\right)_E = 1$. By Lemma 4.12, we have:

$$\mathrm{ch}_2(\tilde{X}) \cdot E = 2E^3 = 2.$$

If T is a surface that contains q with multiplicity m, then

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(\tilde{X}) \cdot (f^* T - mE) = \mathrm{ch}_2(X) \cdot T - 2mE^3 = \mathrm{ch}_2(X) \cdot T - 2m.$$

$\qquad\square$

Corollary 4.15. *Let X be a smooth projective variety of dimension 3 and $\tilde{X}$ be the blow-up of X along disjoint smooth curves $C_1, \ldots, C_k$ and l distinct points. If T is a smooth surface in X containing $0 \leq s \leq l$ of the blown-up points, and intersecting $\cup_i C_i$ along a zero-dimensional reduced scheme of length r, then we have:*

$$ch_2(\tilde{X}) \cdot \tilde{T} = ch_2(X) \cdot T - \frac{3r}{2} - 2s.$$

In particular, $\tilde{X}$ is not weakly 2-Fano if

$$ch_2(X) \cdot T < \frac{3r}{2} + 2s.$$

Proof. This is clear from Lemma 4.13 and Lemma 4.14. □

The results in 4.3.2 (for example Corollary 4.15) give ways to check that some blow-ups of threefolds are not weakly 2-Fano. Here we list a few more.

Corollary 4.16. *(Under the assumptions and notations of Lemma 4.13.) Suppose that either $g(C) > 0$ and $-K_X \cdot C > 0$, or $C \cong \mathbf{P}^1$ and $-K_X \cdot C > 2$. Then*

$$ch_2(\tilde{X}) \cdot E < 0.$$

In particular, if X is Fano and either $g(C) > 0$, or X has index $i_X \geq 3$, then $ch_2(\tilde{X})$ is not nef.
If $C \cong \mathbf{P}^1$ and $-K_X \cdot C = 2$, then

$$ch_2(\tilde{X}) \cdot E = 0.$$

Proof. The result follows immediately from Lemma 4.13 since

$$\deg(N) = \deg(T_{X|C}) - \deg(T_C) = -K_X \cdot C - 2g(C) - 2.$$

□

Lemma 4.17. *Let X be a smooth projective threefold. Assume X has a divisor T such that T is semiample and*

$$ch_2(X) \cdot T < 0.$$

Then any blow-up of X along points and smooth curves is not weakly 2-Fano.

Proof. By replacing T with a multiple, we may assume that $|T|$ is a base-point free linear system. In this case we can find a surface T that avoids any of the blown-up points and intersects each of the blown-up curves in a reduced 0-dimensional scheme of length $r \geq 0$. By Lemma 4.15, we have:

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(X) \cdot T - \frac{3r}{2} < 0.$$

In particular, $\tilde{X}$ is not weakly 2-Fano. □

Corollary 4.18. *Let X be a smooth projective threefold with $\rho = 1$. If X is not weakly 2-Fano, then any blow-up of X along points and smooth curves is not weakly 2-Fano.*

Proof. If X is not weakly 2-Fano, it follows that there exists a surface $T \subset X$ such that $\mathrm{ch}_2(X) \cdot T < 0$. Since T is an effective divisor and X has Picard number 1, it follows that T is a semimaple divisor and the result follows from Lemma 4.17. □

Corollary 4.19. *Let $f : X \to Y$ be a finite map of degree 2 between smooth projective threefolds with ample branch divisor B. Moreover, assume Y has a divisor T such that T is semiample and*

$$ch_2(X) \cdot T \leq 0.$$

Then any blow-up of X along points and smooth curves is not weakly 2-Fano.

Proof. By replacing T with a multiple, we may assume that $|T|$ is base-point free. Note that $|f^*T|$ is also base-point free. By Lemma 4.1, we have:

$$\text{ch}_2(X) \cdot f^*(T) = \left(\text{ch}_2(Y) - \frac{3}{8}B^2\right) \cdot T < 0.$$

The result now follows from Lemma 4.17. $\square$

5. FAMILIES OF RATIONAL CURVES ON 2-FANO MANIFOLDS

In this section we revise some results from [AC12], to which we refer for details and further references.

Let X be a smooth complex projective uniruled variety, and $x \in X$ a general point. There is a scheme $\text{RatCurves}^n(X, x)$ parametrizing rational curves on X passing through x, and it always contains a smooth and proper irreducible component H_x. For instance, one can take H_x to be an irreducible component of $\text{RatCurves}^n(X, x)$ parametrizing rational curves through x having minimal degree with respect to some fixed ample line bundle on X. We denote by $\pi_x : U_x \to H_x$ and $\text{ev}_x : U_x \to X$ the usual universal family morphisms, and set $d := \dim(H_x)$. Since ev_x is proper and π_x is a $\mathbf{P}^1$-bundle, we have a linear map

$$T_1 = \text{ev}_{x*}\pi_x^* : N_1(H_x) \to N_2(X),$$

which maps $\overline{\text{NE}}_1(H_x) \setminus \{0\}$ into $\overline{\text{NE}}_2(X) \setminus \{0\}$.

The variety H_x comes with a natural polarization L_x, which can be defined as follows. There is a finite morphism $\tau_x : H_x \to \mathbf{P}(T_x X^*)$ that sends a point parametrizing a curve smooth at x to its tangent direction at x. We then set $L_x := \tau_x^* \mathcal{O}(1)$.

The pair (H_x, L_x) is called a *polarized minimal family of rational curves through x*, and reflects much of the geometry of X. It is well understood for homogeneous spaces and complete intersections on them (see [Hwa01]). In [AC12], we computed all the Chern classes of the variety H_x in terms of the Chern classes of X and $c_1(L_x)$. For instance,

$$(5.1) \qquad c_1(H_x) = \pi_{x*}\text{ev}_x^*\big(\text{ch}_2(X)\big) + \frac{d}{2}c_1(L_x).$$

In particular, if X is 2-Fano (respectively weakly 2-Fano), then $-2K_{H_x} - dL_x$ is ample (respectively nef). This necessary condition is also sufficient provided that $T_1\big(\overline{\text{NE}}_1(H_x)\big) = \overline{\text{NE}}_2(X)$.

Example 5.1. We consider the special case of the Grassmannian $G(k, n)$. The variety H_x of lines in $G(k, n)$ that pass through a general point $x = [W]$ can be identified with $\mathbf{P}(W) \times \mathbf{P}(V/W)^* \cong \mathbf{P}^{k-1} \times \mathbf{P}^{n-k-1}$, and the map $\tau_x : \mathbf{P}^{k-1} \times \mathbf{P}^{n-k-1} \to \mathbf{P}(T_x X^*)$ is the Segre embedding. So the polarization L_x corresponds to a divisor of type $(1, 1)$. We denote by π_1 and π_2 the projections from $\mathbf{P}^{k-1} \times \mathbf{P}^{n-k-1}$. The map

$$T_1 : \overline{\text{NE}}_1(H_x) \to \overline{\text{NE}}_2(G(k, n))$$

sends classes of lines in the fibers of π_1 and π_2, to the dual cycles σ_2^* and $\sigma_{1,1}^*$, respectively.

Now let $X = H_1 \cap \ldots \cap H_c \subseteq G(k,n)$ be a smooth complete intersection of hyperplane sections $H_1, \ldots, H_c$ under the Plücker embedding, with $c \leq n-2$. We may assume that $x \in X$ is a general point, and consider the variety of lines in X, $Z_x \subset H_x$. Notice that Z_x is a complete intersection of c divisors D_i of type $(1,1)$ in H_x:

$$Z_x = D_1 \cap \ldots \cap D_c \subset H_x \cong \mathbf{P}^{k-1} \times \mathbf{P}^{n-k-1}.$$

Claim.

  (i) If $c \leq k-1$, then Z_x contains a line from a fiber of π_2. In particular, X contains a surface with class $\sigma_{1,1}^*$.
  (ii) If $c < n-k-1$, then Z_x contains a line from a fiber of π_1. In particular, X contains a surface with class σ_2^*.

In particular, if $c \leq k-1$ and $n > 2k$, then the natural map

$$u_2 : \overline{\mathrm{NE}}_2(X) \to \overline{\mathrm{NE}}_2(G(k,n))$$

is surjective.

Proof. Let $x_0, \ldots, x_{k-1}$ (respectively $y_0, \ldots, y_{n-k-1}$) denote the coordinates on $\mathbf{P}^{k-1}$ (respectively on $\mathbf{P}^{n-k-1}$). Each divisor D_i has an equation of type:

$$x_0 F_0^{(i)} + \ldots + x_{k-1} F_{k-1}^{(i)} = 0,$$

where $F_j^{(i)}$ are linear forms in (y_i). Clearly, if $c < k-1$, then Z_x contains a line from any fiber of π_2. By the same argument, if $c < n-k-1$, then Z_x contains a line from any fiber of π_1, and this proves (ii).

Note that if $c = k-1$ and $k \leq n-k-1$, then the locus in $\mathbf{P}^{n-k-1}$ where the k minors of size $(k-1) \times (k-1)$ of the matrix of linear forms $\left(F_j^{(i)}\right)$ vanish is non-empty. This proves (i) and the claim follows. $\qquad\square$

Note that inequalities (i) and (ii) are optimal. Indeed, consider the case when $X = H_1 \cap H_2 \subset G(2,5)$. Let x_0, x_1 (respectively y_0, y_1, y_2) denote the coordinates on $\mathbf{P}^1$ (respectively on $\mathbf{P}^2$). The variety Z_x is a complete intersection in $\mathbf{P}^1 \times \mathbf{P}^2$ of two divisors of type $(1,1)$:

$$D_1 : x_0 F_0 + x_1 F_1 = 0,$$

$$D_2 : x_0 G_0 + x_1 G_1 = 0,$$

where F_i, G_i are linear forms in (y_i). Thus Z_x is isomorphic to the smooth conic $F_0 G_1 - F_1 G_0 = 0$ in $\mathbf{P}^2$, and $Z_x \subset \mathbf{P}^1 \times \mathbf{P}^2$ is a curve of type $(2,2)$. It follows that $T_1 : \overline{\mathrm{NE}}_1(Z_x) \to \overline{\mathrm{NE}}_2(X)$ maps the fundamental class of Z_x to $\sigma_2^* + \sigma_{1,1}^*$.

6. Complete intersections on homogeneous spaces

6.1. Complete intersections in Grassmannians.

We apply the results in Section 3.3 to the case when the ambient space is a Grassmannian.

Proposition 6.1. *Consider a smooth complete intersection*

$$X = (d_1 H) \cap \ldots \cap (d_c H) \subseteq G(k, n) \quad (2 \leq k \leq \frac{n}{2}, 1 \leq c),$$

where $H = \sigma_1$ is the class of a hyperplane class via the Plücker embedding. The Chern character of X is given by:

$$ch(X) = \big(k(n-k) - c\big) + (n - \sum d_i)\sigma_1 +$$

$$\big(\frac{n + 2 - 2k - \sum d_i^2}{2}\sigma_2 - \frac{n - 2 - 2k + \sum d_i^2}{2}\sigma_{1,1}\big) +$$

$$+ \big(\frac{n - 2k - \sum d_i^3}{6}(\sigma_3 + \sigma_{1,1,1}) - \frac{n - 2k + \sum d_i^3}{6}\sigma_{2,1}\big) + \ldots.$$

Then X is Fano if and only if $\sum d_i < n$. Moreover, X is not weakly 2-Fano if

$$\sum d_i^2 \geq n - 2k + 2,$$

with the exception of the case when $n = 2k, c = 2, d_1 = d_2 = 1$, in which case X is weakly 2-Fano (see also Proposition 6.2).

Proof of Proposition 6.1. The formula for ch(X) follows from the formula for ch$(G(k,n))$ computed in 3.4 and the formula for complete intersections from 3.3. The criterion for X to be Fano follows immediately.

To prove the last statement, we may assume X is Fano. If $\sum d_i^2 \geq n - 2k + 2$ then ch$_2(X) = a\sigma_2 + b\sigma_{1,1}$, with $a, b \leq 0$. Note that $b = 0$ if and only if $n = 2k$ and $\sum d_i^2 = 2$, i.e., $c = 2$ and $d_1 = d_2 = 1$. In this case, ch$_2(X) = 0$. But if $b < 0$, then either $a < 0$, in which case ch$_2(X) \cdot S < 0$ for any surface $S \subset X$, or $a = 0$ and we have:

$$\sum d_i^2 = n - 2k + 2, \quad \text{ch}_2(X) = -(n - 2k)\sigma_{1,1|X} \quad (n > 2k).$$

Since $\sigma_{1,1} \cdot \sigma_1^{\dim G(k,n)-2} > 0$ and $\sigma_{1|X}$ is ample, in the latter case X is not weakly 2-Fano for

$$(6.1) \qquad\qquad \text{ch}_2(X) \cdot \sigma_1^{\dim(X)-2} < 0.$$

$\square$

Proposition 6.2. *Consider a smooth complete intersection*

$$X = H_1 \cap \ldots \cap H_c \subseteq G(k, n) \quad (2 \leq k \leq \frac{n}{2}, 1 \leq c < n),$$

of hyperplane sections $H_1, \ldots, H_c$ under the Plücker embedding. Then

$$ch_2(X) = \frac{n + 2 - 2k - c}{2}\sigma_2 - \frac{n - 2 - 2k + c}{2}\sigma_{1,1}.$$

Moreover:

(i)   If $c \geq n - 2k + 2$ (with $c \neq 2$ if $n = 2k$) then X is not weakly 2-Fano.
(ii) If $c \leq k - 1$ and $n \geq 2k + 2$, then X is not weakly 2-Fano.
(iii) If $n = 2k$ then X is (weakly) 2-Fano if and only if $c = 1$ ($c = 1, 2$).
(iv) If $n = 2k + 1$, then X is not 2-Fano; X is weakly 2-Fano if and only if $c = 1$, and possibly when $X = H_1 \cap H_2 \subset G(2, 5)$.

Proof. By Proposition 6.1, if $c \geq n - 2k + 2$, and if we are not in the case when $n = 2k$ and $c = 2$, then X is not weakly 2-Fano. This gives (i). For (ii), note that if $c \leq k - 1$ then, by Example 5.1, the natural map $u_2 : \overline{NE}_2(X) \to \overline{NE}_2(G(k, n))$ is surjective. If $n - 2k - 2 + c > 0$ and $n > 2k$, then part (ii) follows, since

$$\mathrm{ch}_2(X) \cdot \sigma_{1,1}^* = -\frac{n - 2k - 2 + c}{2} < 0.$$

Part (iii) follows immediately, since if $n = 2k$ then

$$\mathrm{ch}_2(X) = \frac{2 - c}{2}(\sigma_2 + \sigma_{1,1}) = \frac{2 - c}{2}\sigma_1^2.$$

We now prove (iv). Assume that $n = 2k + 1$. We have:

$$\mathrm{ch}_2(X) = \frac{3 - c}{2}\sigma_2 + \frac{1 - c}{2}\sigma_{1,1}.$$

By (i), if $c \geq 3$ then X is not weakly 2-Fano. Assume that $c \leq 2$. If $k \geq 3$, then $c \leq 2 \leq k - 1$. By Example 5.1, the natural map $u_2 : \overline{NE}_2(X) \to \overline{NE}_2(G(k, n))$ is surjective. It follows that, in this case, X is weakly 2-Fano if and only if the coefficients of σ_2 and $\sigma_{1,1}$ in the formula for $\mathrm{ch}_2(X)$ are non-negative, i.e., $c = 1$. Note that X is not 2-Fano in this case, as $\mathrm{ch}_2(X) \cdot \sigma_{1,1}^* = 0$.

We are left to analyze what happens in the case when $k < 3$, i.e., the case of $G(2, 5)$. If $c = 1$ then $\mathrm{ch}_2(X) = \sigma_2 \geq 0$, and X is weakly 2-Fano. By Example 5.1, the natural map $u_2 : \overline{NE}_2(X) \to \overline{NE}_2(G(2, 5))$ is surjective, and X is not 2-Fano since $\mathrm{ch}_2(X) \cdot \sigma_{1,1}^* = 0$. Now assume $c = 2$. Then we have:

$$\mathrm{ch}_2(X) = \frac{1}{2}\sigma_2 - \frac{1}{2}\sigma_{1,1}.$$

By Example 5.1, X contains a surface S with class $\sigma_2^* + \sigma_{1,1}^*$. Clearly, X is not 2-Fano, since $\mathrm{ch}_2(X) \cdot S = 0$. $\square$

6.2. Orthogonal Grassmannians.

We fix Q a nondegenerate symmetric bilinear form on the n-dimensional vector space V. Let $OG(k, n)$ be the subvariety of the Grassmannian $G(k, n)$ parametrizing linear subspaces that are isotropic with respect to Q.

If $n \neq 2k$ then $OG(k, n)$ is a Fano manifold of dimension $\frac{k(2n - 3k - 1)}{2}$ and $\rho = 1$. On the other hand, $OG(k, 2k)$ has two connected components [GH, p. 737]: If $\Sigma \subset V$ is a fixed isotropic subspace of dimension k in V, then one component $OG_+(k, 2k)$, corresponds to $[W] \in OG(k, 2k)$ such that $\dim(W \cap \Sigma) \equiv k \pmod 2$, while the other component $OG_-(k, 2k)$

corresponds to those $[W] \in OG(k, 2k)$ such that $\dim(W \cap \Sigma) \not\equiv k \pmod 2$. The two components are disjoint and isomorphic. Note also that

$$OG(k-1, 2k-1) \cong OG_+(k, 2k).$$

The orthogonal Grassmannian $OG(k, n)$ is the zero locus in $G(k, n)$ of a global section of the vector bundle $\mathrm{Sym}^2(\mathcal{S}^*)$. Using this description and the formula for $\mathrm{ch}(G(k, n))$ described in 3.4, standard Chern class computations show that for any component X of $OG(k, n)$ we have:

$$\mathrm{ch}(X) = \frac{k(2n - 3k - 1)}{2} + (n - k - 1)\sigma_1 +$$

$$+ \left(\frac{n - 3k - 1}{2}\sigma_2 - \frac{n - 3k - 3}{2}\sigma_{1,1} \right) +$$

$$+ \left(\frac{n - 3k - 7}{6}\sigma_3 - \frac{n - 3k - 4}{6}\sigma_{2,1} + \frac{n - 3k - 1}{6}\sigma_{1,1,1} \right) + \dots.$$

6.2.1. **Complete intersections in $OG_+(k, 2k)$.** Our main reference in what follows is [Cos09]. We consider now one component $OG_+(k, 2k)$ of the orthogonal Grassmannian $OG(k, 2k)$. For the reader's convenience, we recall the description of Schubert varieties in $OG_+(k, 2k)$. Let

$$F_1 \subset F_2 \subset \dots \subset F_k$$

be an isotropic flag in V, with $[F_k] \in OG_+(k, 2k)$. This induces a second flag

$$F_{k-1} \subset F_{k-1}^\perp \subset F_{k-2}^\perp \subset \dots \subset F_1^\perp \subset V.$$

Here, by abuse of notation, we denote by $F_{k-1}^\perp$ an isotropic subspace of dimension k parametrized by $OG_-(k, 2k)$ and such that $F_{k-1} \subset F_{k-1}^\perp$.

For each decreasing sequence

$$\lambda : k - 1 \geq \lambda_1 > \lambda_2 > \dots > \lambda_s \geq 0 \quad (s \leq k),$$

(where we assume $k - s$ is even) we denote by

$$\mu : k - 1 \geq \mu_{s+1} > \mu_{s+2} > \dots > \mu_k \geq 0$$

the sequence obtained by removing $k - 1 - \lambda_i$ from $k - 1, \dots, 0$. For each sequence λ as above, we have a Schubert variety of codimension $\sum \lambda_i$:

$$\Omega_\lambda^0 = \{ [W] \in OG(k, 2k) \mid \dim(W \cap F_{k - \lambda_i}) = i, \ \dim(W \cap F_{\mu_j}^\perp) = j \}.$$

Let Ω_λ be the closure of Ω_λ^0 and denote by τ_λ its cohomology class. The cohomology of $OG_+(k, 2k)$ is generated by the classes τ_λ. In particular, $b_4(OG_+(k, 2k)) = 1$.

Claim 6.3. On $OG_+(k, 2k)$ we have $\sigma_2 = \sigma_{1,1} = \frac{1}{2}\sigma_1^2$.

Proof. Since $b_4 = 1$, it is enough to find a surface S in $OG_+(k, 2k)$ such that $\sigma_2 \cdot S = \sigma_{1,1} \cdot S$. Let $S = \Omega_{k-1, k-2, \dots, 3, 1}$ (the unique Schubert variety of dimension 2). One can show that $\sigma_2 \cdot S = \sigma_{1,1} \cdot S = 2$. We leave this fun computation to the reader. $\square$

Proposition 6.4. $OG_+(k, 2k)$ *is a 2-Fano manifold.*

Consider a smooth complete intersection

$$X = (d_1 H) \cap \ldots \cap (d_c H) \subseteq OG_+(k, 2k) \quad (k \geq 3),$$

where $H = \frac{1}{2}\sigma_1$ denotes a hyperplane section of the half-spinor embedding of $OG_+(k, 2k)$. The Chern character of X is given by:

$$ch(X) = \frac{k(k-1)}{2} + \left(2k - 2 - \sum d_i\right)H + \left(\frac{4 - \sum d_i^2}{2}\right)H^2 + \ldots.$$

Then X is Fano if and only if $\sum d_i < 2k - 2$. Moreover, X is 2-Fano if and only if all $d_i = 1$ and $c \leq 3$. The only other cases when X is weakly 2-Fano are when $c = 4$, $d_1 = \ldots = d_4 = 1$ and $c = 2$, $d_1 = d_2 = 2$.

Proof. Since $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$, by Claim 6.3, we obtain

$$ch_2(OG_+(k, 2k)) = \frac{1}{2}\sigma_1^2.$$

In particular, $OG_+(k, 2k)$ is 2-Fano. Recall that a hyperplane section of $OG_+(k, 2k)$ via the Plücker embedding is linearly equivalent to $2H$, where H is a hyperplane section of the spinor embedding [Muk95, Proposition 1.7]. It follows that $2H = \sigma_1$. The result now follows from the formula for the Chern character of $OG(k, n)$. $\qquad\square$

6.3. Symplectic Grassmannians.

We fix ω a non-degenerate antisymmetric bilinear form on the n-dimensional vector space V, n even. Let $SG(k, n)$ be the subvariety of the Grassmannian $G(k, n)$ parametrizing linear subspaces that are isotropic with respect to ω. Then $SG(k, n)$ is a Fano manifold of dimension $\frac{k(2n - 3k + 1)}{2}$ and $\rho(X) = 1$. Notice that X is the zero locus in $G(k, n)$ of a global section of the vector bundle $\wedge^2(\mathcal{S}^*)$. Using this description and the formula for $ch\big(G(k, n)\big)$ described in 3.4, standard Chern class computations show that

$$ch(SG(k, n)) = \frac{k(2n - 3k + 1)}{2} + (n - k + 1)\sigma_1 +$$

$$+ \left(\frac{n - 3k + 3}{2}\sigma_2 - \frac{n - 3k + 1}{2}\sigma_{1,1}\right) +$$

$$+ \left(\frac{n - 3k + 1}{6}\sigma_3 - \frac{n - 3k + 4}{6}\sigma_{2,1} + \frac{n - 3k + 7}{6}\sigma_{1,1,1}\right) + \ldots.$$

6.3.1. Complete intersections in $SG(k, 2k)$.

The symplectic Grassmannian $SG(k, 2k)$ is a Fano manifold with $b_4 = 1$. For example, note that $b_4\big(SG(k, 2k)\big) = b_4\big(OG(k, 2k + 1)\big)$ (see for instance [BS02, Section 3.1]), and $b_4\big(OG(k, 2k + 1)\big) = 1$ (see Section 6.2.1).

Claim 6.5. On $SG(k, 2k)$ we have $\sigma_2 = \sigma_{1,1} = \frac{1}{2}\sigma_1^2$.

Proof. Since $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$, it is enough to prove that on $SG(k, 2k)$ we have $\sigma_2 = \sigma_{1,1}$. Since $b_4(SG(k, 2k)) = 1$, we are done if we find a surface S in $SG(k, 2k)$ such that $S \cdot \sigma_2 = S \cdot \sigma_{1,1}$. Let x be a general point on $SG(k, 2k)$ and let H_x denote the space of lines on $SG(k, 2k)$ that pass through x. Recall from [AC12, 5.5]) that

$$H_x \cong \mathbf{P}^{k-1} \subset \mathbf{P}^{k-1} \times \mathbf{P}^{k-1}$$

is the diagnoal embedding. Let S be the surface in $SG(k, 2k)$ corresponding to a line in $H_x \cong \mathbf{P}^{k-1}$ via the map $T_1 : \overline{\mathrm{NE}}_1(H_x) \to \overline{\mathrm{NE}}_2(SG(k, 2k))$. It follows that the class of S is $\sigma_2^* + \sigma_{1,1}^*$. Clearly, $S \cdot \sigma_2 = S \cdot \sigma_{1,1} = 1$.   □

It follows from 6.3 that

$$\mathrm{ch}(SG(k, 2k)) = \left(\frac{k(k+1)}{2}\right) + (k+1)\sigma_1 + \frac{1}{2}\sigma_1^2 + \dots$$

In particular, $SG(k, 2k)$ is 2-Fano (as proved also in [AC12, 5.5]) and we have the following consequence:

Proposition 6.6. *Consider a smooth complete intersection*

$$X = (d_1 H) \cap \dots \cap (d_c H) \subseteq SG(k, 2k) \quad (k \geq 2, c \geq 1),$$

where $H = \sigma_1$ is a hyperplane section under the Plucker embedding.
The Chern character of X is given by:

$$ch(X) = \left(\frac{k(k+1)}{2} - r\right) + (k + 1 - \sum d_i)\sigma_1 + \frac{1 - \sum d_i^2}{2}\sigma_1^2 + \dots$$

Then X is Fano if and only if $\sum d_i < k$. Moreover, X is weakly 2-Fano if and only if $c = d_1 = 1$. In this case X is not 2-Fano.

6.4. **Complete intersections in homogeneous spaces** G_2/P_2**.** If G is a group of type G_2, there exist two maximal parabolic subgroups P_1 and P_2 in G. The quotient variety G/P_1 is isomorphic to a 5-dimensional quadric $Q \subset \mathbf{P}^6$, and G/P_2 is a Mukai variety of genus $g = 10$ (see Theorem 2.3):

$$G_2/P_2 \subset \mathbf{P}^{13}.$$

One has $b_4(G_2/P_2) = 1$ (see for instance [And11, Proposition 4.5 and Appendix A.3]), and G_2/P_2 is 2-Fano by [AC12, 5.7.].

Recall from [Hwa01, 1.4.5] the polarized minimal family of rational curves through a general point $y \in G_2/P_2$ is

$$(H_y, L_y) \cong (\mathbf{P}^1, \mathcal{O}(3)).$$

Let H denote the hyperplane class (the generator of the Picard group). We claim that

$$\mathrm{ch}_2(G_2/P_2) = \frac{1}{2}H^2.$$

This will follow from the following more general remark, applied to $Y = G_2/P_2$.

Remark 6.7 (Complete intersections in varieties with $\rho(Y) = b_4(Y) = 1$)**.** Let Y be a Fano manifold with $\rho(X) = 1$, and H an ample generator of $\text{Pic}(Y)$. Let $y \in Y$ be a general point, and (H_y, L_y) a polarized minimal family of rational curves through y, as defined in Section 5. Suppose that $\dim(H_y) \geq 1$. If $b_4(Y) = 1$, then the map

$$T_1 : \overline{\text{NE}}_1(H_y) \to \overline{\text{NE}}_2(Y)$$

is clearly surjective. Let $C \subset H_y$ be a complete curve, and $S = T_1([C])$ the corresponding surface class on Y. By (5.1), the second Chern character of Y is given by:

$$\text{ch}_2(Y) = aH^2, \quad a \in \frac{1}{2}\mathbf{Z}, \quad a(H^2 \cdot S) = -(K_{H_y} - \frac{d}{2}L_y) \cdot C.$$

In particular, $a \leq -(K_{H_y} - \frac{d}{2}L_y) \cdot C$.

Now consider a complete intersection:

$$X = (d_1 H) \cap \ldots \cap (d_c H) \subset Y.$$

The natural map $u_2 : \overline{\text{NE}}_2(X) \to \overline{\text{NE}}_2(Y)$ is surjective. Thus, by (3.1), X is 2-Fano (respectively weakly 2-Fano) if and only if it is Fano and $\sum d_i^2 < 2a$ (respectively $\leq 2a$).

We have the following consequence:

Proposition 6.8. *A linear section $H_1 \subset G_2/P_2$ is weakly 2-Fano, but not 2-Fano. A linear section $H_1 \cap H_2 \subset G_2/P_2$ is not 2-Fano.*

7. FANO MANIFOLDS WITH HIGH INDEX AND $\rho = 1$

In this section we address n-dimensional Fano manifolds X with index $i_X \geq n - 2$ and $\rho(X) = 1$. We also treat those with bigger Picard number for $n > 4$.

Recall from Section 3 that $\mathbf{P}^n$ and $Q^n \subset \mathbf{P}^{n+1}$ are 2-Fano for $n \geq 3$.

7.1. **Del Pezzo manifolds.** We go through the classification in Theorem 2.2. We first consider manifolds with $\rho = 1$.

7.1.1. **Degree $d = 5$.** We saw in Section 3.4 that the Grassmannian $G(2,5)$ is 2-Fano. Consider now a linear section

$$X = H_1 \cap \ldots \cap H_c \subset G(2,5) \quad (c \geq 1).$$

By Proposition 6.2(iv), if $c = 3$, the threefold X is not weakly 2-Fano. If $c = 1$, then X is weakly 2-Fano, but not 2-Fano. If $c = 2$, then X is not 2-Fano. We could not decide if in this case X is weakly 2-Fano. We raise the following:

Question 7.1. *Is a linear section $\mathbf{P}^7 \cap G(2,5) \subset \mathbf{P}^9$ weakly 2-Fano?*

7.1.2. **Degree $d = 4$.** By 3.3.1, a Del Pezzo variety of type Y_4 is 2-Fano if and only if $n \geq 6$ and weakly 2-Fano if and only if $n \geq 5$.

7.1.3. Degree $d = 3$. By 3.3.1, a Del Pezzo variety of type Y_3 is 2-Fano if and only if $n \geq 8$ and weakly 2-Fano if and only if $n \geq 7$.

7.1.4. Degree $d = 2$. By Corollary 4.2(ii) or 3.3.2, Del Pezzo varieties of type Y_2 are 2-Fano (respectively weakly 2-Fano) if and only if $n > 11$ (respectively $n \geq 11$).

7.1.5. Degree $d = 1$. By Corollary 3.3.2, Del Pezzo varieties of type Y_1 are (weakly) 2-Fano if and only if $n > 23$ ($n \geq 23$).

7.1.6. Del Pezzo manifolds with $\rho > 1$. All the Del Pezzo manifolds with $\rho > 1$ are weakly 2-Fano but not 2-Fano. For $\mathbf{P}^2 \times \mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ this follows from Lemma 4.7, for $\mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1))$ from Corollary 4.5, and for $\mathbf{P}(T_{\mathbf{P}^2})$ from Example 4.6.

Remark 7.2. We get the following classification of weakly 2-Fano Del Pezzo manifolds:

- All the Del Pezzo manifolds with $\rho > 1$ are weakly 2-Fano.
- The only Del Pezzo manifolds with $\rho = 1$ that are weakly 2-Fano are:
  - ($d = 5$) $G(2,5)$ and its linear sections of codimension 1 (and possibly codimension 2, see Question 7.1);
  - ($d = 4$) Complete intersections of quadrics $Q \cap Q' \subset \mathbf{P}^{n+2}$ if $n \geq 5$;
  - ($d = 3$) Cubic hypersurfaces $Y_3 \subset \mathbf{P}^{n+1}$ if $n \geq 7$;
  - ($d = 2$) Degree 4 hypersurfaces in $\mathbf{P}(2, 1, \ldots, 1)$ if $n \geq 11$;
  - ($d = 1$) Degree 6 hypersurfaces in $\mathbf{P}(3, 2, 1, \ldots, 1)$ if $n \geq 23$.

7.2. Mukai manifolds.

7.2.1. Mukai manifolds of dimension > 4 and $\rho > 1$. Recall that the only Mukai manifolds of dimension > 4 and $\rho > 1$ are $\mathbf{P}^3 \times \mathbf{P}^3$, $\mathbf{P}^2 \times Q^3$, $\mathbf{P}(T_{\mathbf{P}^3})$ and $\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1) \oplus \mathcal{O})$. The manifolds $\mathbf{P}^3 \times \mathbf{P}^3$ and $\mathbf{P}^2 \times Q^3$ are weakly 2-Fano but not 2-Fano by Lemma 4.7, while $\mathbf{P}(T_{\mathbf{P}^3})$ and $\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1) \oplus \mathcal{O})$ are not weakly Fano by Example 4.6 and Corollary 4.5, respectively.

Next we go through the classification of Mukai manifolds with $\rho = 1$ in Theorem 2.3.

7.2.2. Genus $g \leq 5$. Consider the case of complete intersections. By 3.3.1:

- $X_4 \subset \mathbf{P}^{n+1}$ is 2-Fano (respectively weakly 2-Fano) if and only if $n \geq 15$ (respectively $n \geq 14$).
- $X_{2 \cdot 3} \subset \mathbf{P}^{n+2}$ is 2-Fano (respectively weakly 2-Fano) if and only if $n \geq 11$ (respectively $n \geq 10$)
- $X_{2 \cdot 2 \cdot 2} \subset \mathbf{P}^{n+3}$ is 2-Fano (respectively weakly 2-Fano) if and only if $n \geq 9$ (respectively $n \geq 8$).

Consider the case of double covers. Using Corollary 4.2, we have:

(i)   A double cover $X \to \mathbf{P}^n$ branched a long a sextic is 2-Fano (respectively weakly 2-Fano) if and only if $n \geq 27$ (respectively $n \geq 26$).

(ii) A double cover $X \to Q \subset \mathbf{P}^{n+1}$ branched a long the intersection of the quadric Q with a quartic hypersurafce, is 2-Fano (respectively weakly 2-Fano) if and only if $n \geq 15$ (respectively $n \geq 14$).

7.2.3. Genus 6. A linear section X of Σ_{10}^6 is isomorphic to one of the following ([IP99, Proposition 5.2.7]):

(i)   A complete intersection in $G(2,5)$ of a linear subspace and a quadric.

(ii) A double cover of a smooth linear section Y of $G(2,5)$, branched along a quadric section B of Y.

In Case (i), it follows from Proposition 6.1 that X is not weakly 2-Fano. Consider now Case (ii). Set $c := \mathrm{codim}(Y)$. Then X is not weakly 2-Fano by Corollary 4.2, since we have:

$$\mathrm{ch}_2(Y) - \frac{3}{8}B^2 = \frac{3-c}{2}\sigma_2 + \frac{1-c}{2}\sigma_{1,1} - \frac{3}{8}(2\sigma_1)^2 = -\frac{c}{2}\sigma_2 - \frac{c+2}{2}\sigma_{1,1},$$

$$\left(\mathrm{ch}_2(Y) - \frac{3}{8}B^2\right) \cdot \sigma_{1|Y}^{\dim(Y)-2} = \left(-\frac{c}{2}\sigma_2 - \frac{c+2}{2}\sigma_{1,1}\right) \cdot \sigma_1^4 < 0.$$

7.2.4. Genus 7. By 6.2.1, the manifold $OG_+(5,10)$ is 2-Fano and a linear section of codimension c is 2-Fano (respectively weakly 2-Fano) if and only if $c < 4$ (respectively $c \leq 4$).

7.2.5. Genus 8. We saw in Section 3.4 that the Grassmannian $G(2,6)$ is weakly 2-Fano. Let $X \subset G(2,6)$ be a linear section of codimension c. By Proposition 6.2, if $c \geq 4$ or $c = 1$, then X is not weakly 2-Fano. Assume that $c = 3$. Then

$$\mathrm{ch}_2(X) = \frac{1}{2}(\sigma_2 - 3\sigma_{1,1}).$$

By a straightforward calculation, $\sigma_1^6 = 9\sigma_2^* + 5\sigma_{1,1}^*$. It follows that

$$\mathrm{ch}_2(X) \cdot \sigma_{1|X}^3 = \mathrm{ch}_2(X) \cdot \sigma_1^6 = -3 < 0.$$

In particular, X is not weakly 2-Fano.

Assume now that $c = 2$. Then $\mathrm{ch}_2(X) = \sigma_2 - \sigma_{1,1}$. By Example 5.1, the variety $H_x \subset \mathbf{P}^1 \times \mathbf{P}^3$ defined in Section 5 is isomorphic to a smooth quadric surface in $\mathbf{P}^3$ (via the projection π_2). The map $T_1 : \overline{\mathrm{NE}}_1(H_x) \to \overline{\mathrm{NE}}_2(G(k,n))$ sends the classes of the lines in the two rulings of the quadric surface H_x to the classes σ_2^* and $\sigma_2^* + \sigma_{1,1}^*$. In particular, X is not 2-Fano, as $\mathrm{ch}_2(X) \cdot (\sigma_2^* + \sigma_{1,1}^*) = 0$. We could not decide if in this case X is weakly 2-Fano. We raise the following:

Question 7.3. *Is a linear section* $\mathbf{P}^{12} \cap G(2,6) \subset \mathbf{P}^{14}$ *weakly 2-Fano?*

7.2.6. Genus 9. We saw in Section 6.3 that the symplectic Grassmannian $SG(k,2k)$ is 2-Fano. By 6.3.1, a codimension $c \geq 1$ linear section X of $SG(k,2k)$ is not 2-Fano. The only other case when X is weakly 2-Fano is for $c = 1$.

7.2.7. Genus 10. We saw in Section 6.4 that the variety G_2/P_2 is 2-Fano. By Proposition 6.8, a codimension $c \geq 1$ linear section in G_2/P_2 is not 2-Fano and it is weakly 2-Fano if and only if $c = 1$.

7.2.8. Genus $g = 12$. By Remark 3.2,

$$c_1(\wedge^2(\mathcal{S}^*)) = 2\sigma_1, \quad \mathrm{ch}_2(\wedge^2(\mathcal{S}^*)) = \sigma_2.$$

It follows from Lemma 3.1 and the computation of $\mathrm{ch}(G(3,7))$ made in Section 3.4 that the Chern characters of the threefold $X = X_{22}$ are given by:

$$c_1(X) = \left(c_1\big(G(3,7)\big) - 3c_1\big(\wedge^2(\mathcal{S}^*)\big)\right)_{|X} = \sigma_{1|X}$$

$$\mathrm{ch}_2(X) = \left(\mathrm{ch}_2\big(G(3,7)\big) - 3\mathrm{ch}_2\big(\wedge^2(\mathcal{S}^*)\big)\right)_{|X} = -\frac{3}{2}\sigma_{2|X} + \frac{1}{2}\sigma_{1,1|X}.$$

Since $\rho = 1$ and $\dim(X) = 3$, $b_4(X) = 1$. In particular, the restrictions $\sigma_{2|X}$ and $\sigma_{1,1|X}$ are multiples of the positive codimension 2-cycle

$$A := (\sigma_1^2)_{|X}.$$

We claim that

$$\sigma_{1,1|X} = \frac{5}{11}A, \quad \sigma_{2|X} = \frac{6}{11}A.$$

To see this, it is enough to prove that

$$(\sigma_2 \cdot \sigma_1)_X = 12, \quad (\sigma_{1,1} \cdot \sigma_1)_X = 10,$$

where $(,)_X$ denotes the intersection on X. Since X is the zero locus of a global section of the rank 9 vector bundle $\mathcal{E} = (\wedge^2(\mathcal{S}^*))^{\oplus 3}$, it follows that

$$(\sigma_2 \cdot \sigma_1)_X = \sigma_2 \cdot \sigma_1 \cdot c_9(\mathcal{E}).$$

By a standard computation with Chern classes,

$$c_9(\mathcal{E}) = c_3(\wedge^2(\mathcal{S}^*))^3 = \big(c_1(\mathcal{S}^*)c_2(\mathcal{S}^*) - c_3(\mathcal{S}^*)\big)^3 = \sigma_{2,1}^3.$$

It is a straightforward exercise in Schubert calculus to check that

$$\sigma_{2,1}^3 = 4\sigma_{4,4,1} + 8\sigma_{4,3,2} + 2\sigma_{3,3,3}.$$

It follows that

$$\sigma_1 \cdot \sigma_{2,1}^3 = 12\sigma_2^* + 10\sigma_{1,1}^*.$$

Then $\mathrm{ch}_2(X) \cdot A = -13 < 0$. Hence, X_{22} is not weakly 2-Fano.

Remark 7.4. We get the following classification of weakly 2-Fano Mukai manifolds with $\rho = 1$:

    (1) Complete interesection in projective spaces:
    ($g = 3$) Degree 4 hypersurafces in $\mathbf{P}^{n+1}$ if $n \geq 15$;
    ($g = 4$) Complete intersections $X_{2\cdot3} \subset \mathbf{P}^{n+2}$ if $n \geq 11$;
    ($g = 5$) Complete intersections $X_{2\cdot2\cdot2} \subset \mathbf{P}^{n+3}$ if $n \geq 9$.

    (2) Complete interesection in weighted projective spaces:
    ($g = 2$) Degree 6 hypersurafces in $\mathbf{P}(3, 1, \ldots, 1)$ if $n \geq 26$;

($g = 3$) Complete intersections of two quadrics in $\mathbf{P}(2, 1, \ldots, 1)$, $n \geq 14$.

(3) With genus $g \geq 6$:
($g = 7$) $OG_+(5, 10)$ and linear sections of codimension $c \leq 4$;
($g = 8$) $G(2, 6)$ and possibly a linear section of codimension 2 in $G(2, 6)$ (see Question 7.3);
($g = 9$) $SG(3, 6)$ and linear sections of codimension 1;
($g = 10$) G_2/P_2 and linear sections of codimension 1.

8. FANO THREEFOLDS WITH PICARD NUMBER $\rho \geq 2$

By the results of Mori-Mukai [MM81] (see also [MM03]) there are 88 types of Fano threefolds with Picard number $\rho(X) \geq 2$, up to deformation. We will go through the list in [MM81] and check that none of them is 2-Fano. We point out those that are weakly 2-Fano. We recall the terminology and notation from [MM81]:

(i) V_d ($1 \leq d \leq 5$) denotes a Fano 3-fold of index 2, with $\rho(X) = 1$ and degree d (See Thm. 2.2).

(ii) W is a smooth divisor of $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1, 1)$. It is isomorphic to the $\mathbf{P}^1$-bundle $\mathbf{P}(T_{\mathbf{P}^2})$ over $\mathbf{P}^2$, and appears as (32) in the following list.

(iii) The blow-up of $\mathbf{P}^3$ at a point is denoted by V_7. It appears as (35) in the following list. The smooth quadric in $\mathbf{P}^4$ is denoted by Q.

(iv) S_d ($1 \leq d \leq 7$) is a Del Pezzo surface of degree d. $\mathbf{F}_1$ is the blow-up of $\mathbf{P}^2$ at a point.

(v) All curves are understood to be smooth and irreducible, and all intersections are understood to be scheme theoretic.

(vi) A divisor D (respectively a curve C) on the product variety
$$M = \mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_m}$$
is of multi-degree $(a_1, \ldots, a_m)$ if $\mathcal{O}_M(D) \cong \otimes_{i=1}^m \pi_i^* \mathcal{O}_{\mathbf{P}^{n_i}}(a_i)$ (respectively if
$$C \cdot \pi_i^* \mathcal{O}_{\mathbf{P}^{n_i}}(a_i) = a_i$$
for all $i = 1, \ldots, m$), where π_i is the projection of M onto the i-th factor.

8.1. **Fano 3-folds with $\rho = 2$.** We go through the list in [MM81, Table 2] and check that each Fano 3-fold in the list is not 2-Fano. We point out the cases in which the 3-fold is weakly 2-Fano.

(**1**) The blow-up of V_1 with center an elliptic curve which is an intersection of two members of $\left| -\frac{1}{2} K_{V_1} \right|$. This is not weakly 2-Fano by Corollary 4.16.

(**2**) A double cover of $\mathbf{P}^1 \times \mathbf{P}^2$ whose branch locus is a divisor of bidegree $(2,4)$. Since $\mathbf{P}^1 \times \mathbf{P}^2$ is not 2-Fano, this is not weakly 2-Fano by Corollary 4.2(ii).

(**3**) The blow-up of V_2 with center an elliptic curve which is an intersection of two members of $\left| -\frac{1}{2} K_{V_2} \right|$. This is not weakly 2-Fano by Corollary 4.16.

(**4**) The blow-up of $\mathbf{P}^3$ with center an intersection of two cubics. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 4.16.

(**5**) The blow-up of $V_3 \subset \mathbf{P}^4$ with center a plane cubic in it. This is not weakly 2-Fano by Corollary 4.16.

(**6a**) A divisor on $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(2,2)$ is not weakly 2-Fano by Example 4.8.

(**6b**) A double cover of W whose branch locus is a member of $|-K_W|$. Since $W = \mathbf{P}(T_{\mathbf{P}^2})$ is not 2-Fano by Lemma 4.4, its double cover is not weakly 2-Fano by Corollary 4.2(ii).

(**7**) The blow-up of $Q \subset \mathbf{P}^4$ with center an intersection of two members of $\left| \mathcal{O}_Q(2) \right|$. Since Q is a Fano threefold with index 3, this is not weakly 2-Fano by Corollary 4.16.

(**8**) A double cover of V_7 whose branch locus is a member B of $|-K_{V_7}|$ such that either (a) $B \cap D$ is smooth, or (b) $B \cap D$ is reduced but not smooth, where D is the exceptional divisor of the blow-up $V_7 \to \mathbf{P}^3$. Since $V_7 = \mathbf{P}_{\mathbf{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$ is not 2-Fano by Lemma 4.4, this 3-fold is not weakly 2-Fano by Corollary 4.2(ii).

(**9**) The blow-up of $\mathbf{P}^3$ with center a curve of degree 7 and genus 5 which is an intersection of cubics. This is not weakly 2-Fano by Corollary 4.16.

(**10**) The blow-up of $V_4 \subset \mathbf{P}^5$ with center an elliptic curve which is an intersection of two hyperplane sections. This is not weakly 2-Fano by Corollary 4.16.

(**11**) The blow-up of $V_3 \subset \mathbf{P}^4$ with center a line on it. Since V_3 has Picard number 1 and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 4.18.

(**12**) The blow-up of $\mathbf{P}^3$ with center a curve of degree 6 and genus 3 which is an intersection of cubics. This is not weakly 2-Fano by Corollary 4.16.

(**13**) The blow-up of $Q \subset \mathbf{P}^4$ with center a curve of degree 6 and genus 2. This is not weakly 2-Fano by Corollary 4.16.

(**14**) The blow-up of $V_5 \subset \mathbf{P}^6$ with center an elliptic curve which is an intersection of two hyperplane sections. This is not weakly 2-Fano by Corollary 4.16.

(**15**) The blow-up of $\mathbf{P}^3$ with center an intersection of a quadric A and a cubic B such that either (a) A is smooth, or (b) A is reduced, but not smooth. This is not weakly 2-Fano by Corollary 4.16.

(**16**) The blow-up of $V_4 \subset \mathbf{P}^5$ with center a conic on it. Since V_4 has $\rho = 1$ and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 4.18.

(**17**) The blow-up of $Q \subset \mathbf{P}^4$ with center an elliptic curve of degree 5 on it. This is not weakly 2-Fano by Corollary 4.16.

(**18**) A double cover of $\mathbf{P}^1 \times \mathbf{P}^2$ whose branch locus is a divisor of bidegree $(2, 2)$. Since $\mathbf{P}^1 \times \mathbf{P}^2$ is not 2-Fano by Lemma 4.7, this is not weakly 2-Fano by Corollary 4.2(ii).

(**19**) The blow-up of $V_4 \subset \mathbf{P}^5$ with center a line on it. Since V_4 has $\rho = 1$ and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 4.18.

(**20**) The blow-up of $V_5 \subset \mathbf{P}^6$ with center a twisted cubic on it. Since V_5 has $\rho = 1$ and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 4.18.

(**21**) The blow-up of $Q \subset \mathbf{P}^4$ with center a twisted quartic (i.e., a smooth rational curve of degree 4 which spans $\mathbf{P}^4$) on it. Since Q is a Fano threefold with index 3, this is not weakly 2-Fano by Corollary 4.16.

(**22**) The blow-up of $V_5 \subset \mathbf{P}^6$ with center a twisted conic on it. Since V_5 has $\rho = 1$ and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 4.18.

(**23**) The blow-up of $Q \subset \mathbf{P}^4$ with center an intersection of $A \in |\mathcal{O}_Q(1)|$ and $B \in |\mathcal{O}_Q(2)|$ such that either (a) A is smooth or, (2) A is not smooth. Since Q has index 3, this is not weakly 2-Fano by Corollary 4.16.

(**24**) A divisor on $\mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1, 2)$ is not weakly 2-Fano by Example 4.8.

(**25**) The blow-up of $\mathbf{P}^3$ with center an elliptic curve which is an intersection of two quadrics. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 4.16.

(**26**) The blow-up of $V_5 \subset \mathbf{P}^6$ with center a line on it. Since V_5 has $\rho = 1$ and is not weakly 2-Fano (see Section 7.1), this is not weakly 2-Fano by Corollary 4.18.

(**27**) The blow-up of $\mathbf{P}^3$ with center a twisted cubic. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 4.16.

(**28**) The blow-up of $\mathbf{P}^3$ with center a plane cubic. This is not weakly 2-Fano by Corollary 4.16.

(**29**) The blow-up of $Q \subset \mathbf{P}^4$ with center a conic on it. Since Q has index 3, this is not weakly 2-Fano by Corollary 4.16.

(**30**) The blow-up of $\mathbf{P}^3$ with center a conic. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 4.16.

(**31**) The blow-up of $Q \subset \mathbf{P}^4$ with center a line on it. Since Q has index 3, this is not weakly 2-Fano by Corollary 4.16.

(**32**) $W \cong \mathbf{P}(T_{\mathbf{P}^2})$. This is not 2-Fano, but weakly 2-Fano by Example 4.6.

(**33**) The blow-up of $\mathbf{P}^3$ with center a line. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 4.16.

(**34**) The product $\mathbf{P}^1 \times \mathbf{P}^2$ is not 2-Fano, but weakly 2-Fano by Lemma 4.7.

(**35**) $V_7 \cong \mathbf{P}_{\mathbf{P}^2}(\mathcal{O} \oplus \mathcal{O}(1))$. This is not 2-Fano, but weakly 2-Fano by Corollary 4.5.

(**36**) The blow-up of the Veronese cone $W_4 \subseteq \mathbf{P}^6$ with center the vertex, that is the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2))$ over $\mathbf{P}^2$. This is not 2-Fano, but weakly 2-Fano by Corollary 4.5.

8.2. **Fano 3-folds with $\rho = 3$.** We go through the list in [MM81, Table 3] and check that each Fano 3-fold in the list is not 2-Fano. We point out the cases in which the 3-fold is weakly 2-Fano.

(**1**) A double cover of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ whose branch locus is a divisor of tridegree $(2, 2, 2)$. Since $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ is not 2-Fano by Lemma 4.7, this is not weakly 2-Fano by Corollary 4.2(ii).

(**2**) A member X of the linear system $|\mathcal{O}_\pi(1)^{\otimes 2} \otimes \mathcal{O}(2, 3)|$ on the $\mathbf{P}^2$-bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1, -1)^{\oplus 2})$ over $\mathbf{P}^1 \times \mathbf{P}^1$ such that $X \cap Y$ is irreducible. Here Y is a member of $|\mathcal{O}_\pi(1)|$.

We prove that X is not weakly 2-Fano by a direct computation. Set $\mathcal{E} := \mathcal{O} \oplus \mathcal{O}(-1, -1)^{\oplus 2}$ and let $\pi : \mathbf{P}(\mathcal{E}) \to \mathbf{P}^1 \times \mathbf{P}^1$ be the corresponding projection map. If π_1, π_2 are the two projections from $\mathbf{P}^1 \times \mathbf{P}^1$, we let $H_i = \pi_i^* \mathcal{O}(1)$ $(i = 1, 2)$. Let $\xi = c_1(\mathcal{O}_\pi(1))$. By Lemma 4.3 and the formula (3.1) we have

$$\mathrm{ch}_2(\mathbf{P}(\mathcal{E})) = 6\pi^*(H_1 \times H_2) + 2\pi^*(H_1 + H_2) \cdot \xi + \frac{3}{2}\xi^2,$$

$$\mathrm{ch}_2(X) = \left(2\pi^*(-H_1 - 2H_2) \cdot \xi - \frac{1}{2}\xi^2\right)_{|X}.$$

We claim that $\mathrm{ch}_2(X) \cdot (\pi^* H_1)_{|X} < 0$. This is a direct computation:

$$\mathrm{ch}_2(X) \cdot (\pi^* H_1)_X = \mathrm{ch}_2(X) \cdot \pi^* H_1 \cdot \left(\pi^*(2H_1 + 3H_2) + 2\xi\right) = -\frac{15}{2}.$$

(**3**) A divisor on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ of tridegree $(1, 1, 2)$ is not weakly 2-Fano by Example 4.9.

(**4**) The blow-up of Y (No. (18) in the list for $\rho = 2$) with center a smooth fiber of $Y \to \mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^2$. Recall that $Y \to \mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^2$ is a double cover branched along a divisor of bidegree $(2, 2)$. Apply Corollary 4.19 to deduce that this is not weakly 2-Fano.

(**5**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ with center a curve C of bidegree $(5, 2)$ such that the composition $C \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^2$ is an embedding. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^2} \cdot C = 16 > 2$, this is not weakly 2-Fano by Corollary 4.16.

(**6**) The blow-up of $\mathbf{P}^3$ with center a disjoint union of a line and an elliptic curve of degree 4. This is not weakly 2-Fano by Corollary 4.16.

(**7**) The blow-up of W with center an elliptic curve of degree 4. This is not weakly 2-Fano by Corollary 4.16.

(**8**) A member X of the linear system $|\pi_1^* g^* \mathcal{O}(1) \otimes \pi_2^* \mathcal{O}(2)|$ on $\mathbf{F}_1 \times \mathbf{P}^2$, where π_i $(i = 1, 2)$ is the projection to the i-th factor and $g : \mathbf{F}_1 \to \mathbf{P}^2$ is the blow-up map. We prove that X is not weakly 2-Fano by a direct computation. Set $h_1 := c_1(\pi_1^* g^* \mathcal{O}(1))$ and $h_2 := c_1(\pi_2^* \mathcal{O}(1))$. By (4.3.1), $\mathrm{ch}_2(\mathbf{F}_1) = 0$. By (3.1), we have:

$$\mathrm{ch}_2(X) = \big(\mathrm{ch}_2(\mathbf{F}_1 \times \mathbf{P}^2) - \frac{1}{2}X^2\big)_{|X} = \big(\frac{3}{2}h_2^2 - \frac{1}{2}(h_1 + 2h_2)^2\big)_{|X} =$$

$$= \big(-\frac{1}{2}h_1^2 - \frac{1}{2}h_2^2 - 2h_1 h_2\big)_{|X}.$$

We claim that $\mathrm{ch}_2(X) \cdot h_{2|X} < 0$. This is a direct computation:

$$\mathrm{ch}_2(X) \cdot h_{2|X} = \mathrm{ch}_2(X) \cdot h_2 \cdot X = \mathrm{ch}_2(X) \cdot h_2 \cdot (h_1 + 2h_2) = -3.$$

(**9**) The blow-up of the cone $W_4 \subset \mathbf{P}^6$ over the Veronese surface $R_4 \subset \mathbf{P}^5$ with center a disjoint union of the vertex and a quartic curve in $R_4 \cong \mathbf{P}^2$. Since the center of the blow-up is a curve of genus 3, this is not weakly 2-Fano by Corollary 4.16.

(**10**) The blow-up of $Q \subset \mathbf{P}^4$ with center a disjoint union of two conics on it. Since Q has index 3, this is not weakly 2-Fano by Corollary 4.16.

(**11**) The blow-up of V_7 with center an elliptic curve which is an intersection of two members of of $|-\frac{1}{2}K_{V_7}|$. This is not weakly 2-Fano by Corollary 4.16.

(**12**) The blow-up of $\mathbf{P}^3$ with center a disjoint union of a line and a twisted cubic. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 4.16.

(**13**) The blow-up of $W \subset \mathbf{P}^2 \times \mathbf{P}^2$ with center a curve C of bidegree $(2,2)$ on it such that the composition of $C \hookrightarrow W \hookrightarrow \mathbf{P}^2 \times \mathbf{P}^2$ with the projection $\pi_i : \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^2$ is an embedding for both $i = 1, 2$. Since $-K_W \cdot C = 8$, this is not weakly 2-Fano by Corollary 4.16.

(**14**) The blow-up of $\mathbf{P}^3$ with center a union of a cubic in a plane S and a point not in S. This is not weakly 2-Fano by Corollary 4.16.

(**15**) The blow-up of $Q \subset \mathbf{P}^4$ with center a disjoint union of a line and a conic on it. Since Q has index 3, this is not weakly 2-Fano by Corollary 4.16.

(**16**) The blow-up of V_7 with center the strict transform of a twisted cubic passing through the center of the blow-up $V_7 \to \mathbf{P}^3$. Since $-K_{V_7} \cdot C = 10$, this is not weakly 2-Fano by Corollary 4.16.

(**17**) A smooth divisor on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ of tridegree $(1,1,1)$ is not weakly 2-Fano by Example 4.9.

(**18**) The blow-up of $\mathbf{P}^3$ with center a disjoint union of a line and a conic. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 4.16.

(**19**) The blow-up X of $Q \subset \mathbf{P}^4$ with center two points p and q on it which are not collinear. By 3.3.1,

$$\mathrm{ch}_2(Q) = \frac{1}{2}h_{|Q}^2.$$

It $\tilde{T}$ is the proper transform of a general hyperplane section T of Q that passes through p, by Lemma 4.15, we have

$$\mathrm{ch}_2(X) \cdot \tilde{T} = \mathrm{ch}_2(Q) \cdot T - 2 = \frac{1}{2}h^3 - 2 = -1.$$

In particular, X is not weakly 2-Fano.

Remark 8.1. Moreover, note that $\tilde{T}$ is a base-point free divisor on X. It follows from Corollary 4.17 that no blow-up of X along points and disjoint smooth curves is weakly 2-Fano.

(**20**) The blow-up of $Q \subset \mathbf{P}^4$ with center two disjoint lines on it. Since Q has index 3, this is not weakly 2-Fano by Corollary 4.16.

(**21**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ with center a curve C of bidegree $(2,1)$. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^2} \cdot C = 7$, this is not weakly 2-Fano by Corollary 4.16.

(**22**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ with center a conic C in $\{t\} \times \mathbf{P}^2$ ($t \in \mathbf{P}^1$). Since $-K_{\mathbf{P}^1 \times \mathbf{P}^2} \cdot C = 6$, this is not weakly 2-Fano by Corollary 4.16.

(**23**) The blow-up of V_7 with center a conic C passing through the center of the blow-up $V_7 \to \mathbf{P}^3$. Recall that V_7 is the blow-up of $\mathbf{P}^3$ at a point. Since $-K_{V_7} \cdot C = 6$, this is not weakly 2-Fano by Corollary 4.16.

(**24**) The fiber product $X = W \times_{\mathbf{P}^2} \mathbf{F}_1$ where $W \to \mathbf{P}^2$ is the $\mathbf{P}^1$-bundle $\mathbf{P}(T_{\mathbf{P}^2})$ and $\pi : \mathbf{F}_1 \to \mathbf{P}^2$ is the blow-up map. This is not weakly 2-Fano: Since $X = \mathbf{P}_{\mathbf{F}_1}(\pi^* T_{\mathbf{P}^2})$, by Lemma 4.4, $\mathrm{ch}_2(X) \geq 0$ if and only if

$$\mathrm{ch}_2(\mathbf{F}_1) + \frac{1}{2}\pi^*\big(c_1(\mathbf{P}^2)^2 - 4c_2(\mathbf{P}^2)\big) \geq 0.$$

By Lemma 4.12, $\mathrm{ch}_2(\mathbf{F}_1) = \pi^*\mathrm{ch}_2(\mathbf{P}^2) + \frac{3}{2}E^2$, where E is the exceptional divisor of $\mathbf{F}_1$. Hence, X is not weakly 2-Fano, since

$$\pi^*\big(c_1(\mathbf{P}^2)^2 - 3c_2(\mathbf{P}^2)\big) + \frac{3}{2}E^2 = -\frac{3}{2} < 0.$$

(**25**) The blow-up of $\mathbf{P}^3$ with center two disjoint lines, that is, $\mathbf{P}(\mathcal{O}(1,0) \oplus \mathcal{O}(0,1))$ over $\mathbf{P}^1 \times \mathbf{P}^1$. This is not weakly 2-Fano by Corollary 4.5.

(**26**) The blow-up of $\mathbf{P}^3$ with center a disjoint union of a point and a line. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 4.16.

(**27**) $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ is weakly 2-Fano and not 2-Fano by Lemma 4.7.

(**28**) $\mathbf{P}^1 \times \mathbf{F}_1$ is weakly 2-Fano by Lemma 4.7.

(**29**) The blow-up X of V_7 with center a line L on the exceptional divisor $E \cong \mathbf{P}^2$ of the blow-up $\pi : V_7 \to \mathbf{P}^3$ at a point p. The line L corresponds to a plane $\Lambda \subset \mathbf{P}^3$ passing through p. By Lemma 4.12, we have:

$$\mathrm{ch}_2(V_7) = 2(\pi^* h)^2 + 2E^2.$$

Let T be a plane through the point p, not containing the plane Λ. The proper transform $\tilde{T}$ of T in X intersects L in a point. By Lemma 4.15,

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(V_7) \cdot T - \frac{3}{2} = (2(\pi^* h)^2 + 2E^2) \cdot (\pi^* h - E) - \frac{3}{2} = -\frac{3}{2}.$$

In particular, $\tilde{X}$ is not weakly 2-Fano.

(**30**) The blow-up X of V_7 along the proper transform of a line passing through the center of the blow-up $V_7 \to \mathbf{P}^3$. By Lemma 4.13, we have

$$\mathrm{ch}_2(X) \cdot E = 0,$$

where E is the exceptional divisor corresponding to the line. In particular, X is not 2-Fano. This is the only case of a Fano threefold where we could not decide if X is weakly 2-Fano.

(**31**) The blow-up of the cone over a smooth quadric surface in $\mathbf{P}^3$ with center the vertex, that is, the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1,1))$ over $\mathbf{P}^1 \times \mathbf{P}^1$. This is weakly 2-Fano and not 2-Fano by Corollary 4.5.

8.3. **Fano 3-folds with $\rho = 4$.** We go through the list in [MM81, Table 4], [MM03] and check that each Fano 3-fold in the list is not 2-Fano. We point out the cases in which the 3-fold is weakly 2-Fano.

(**1**) A smooth divisor on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ of tridegree $(1, 1, 1, 1)$ is not weakly 2-Fano by Example 4.10.

(**2**) The blow-up of the cone over a smooth quadric surface $S \subset \mathbf{P}^3$ with center a disjoint union of the vertex and an elliptic curve on S. This is not weakly 2-Fano by Corollary 4.16.

(**3**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with center a curve of tridegree $(1, 1, 2)$. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1} \cdot C = 8$, this is not weakly 2-Fano by Corollary 4.16.

(**4**) The blow-up of Y (No. (19) in the list for $\rho = 3$) with center the strict transform of a conic on Q passing through p and q. This is not 2-Fano by Remark 8.1.

(**5**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^2$ with center two disjoint curves C_1 and C_2 of bidegree $(2, 1)$ and $(1, 0)$ respectively. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^2} \cdot C_1 = 7$, this is not weakly 2-Fano by Corollary 4.16.

(**6**) The blow-up of $\mathbf{P}^3$ with center three disjoint lines, that is, the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with center the tridiagonal curve. Since $\mathbf{P}^3$ has index 4, this is not weakly 2-Fano by Corollary 4.16.

(**7**) The blow-up of $W \subset \mathbf{P}^2 \times \mathbf{P}^2$ with center two disjoint curves C_1 and C_2 of bidegree $(0, 1)$ and $(1, 0)$. Since $-K_W \cdot C_i = 3$, this is not weakly 2-Fano by Corollary 4.16.

(**8**) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with center a curve C of tridegree $(0, 1, 1)$. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1} \cdot C = 4$, this is not weakly 2-Fano by Corollary 4.16.

(**9**) The blow-up X of Y (No. (25) in the list for $\rho = 3$) with center an exceptional line of the blowing up $Y \to \mathbf{P}^3$. Recall that Y is the blow-up of $\mathbf{P}^3$ along two disjoint lines. If T is the proper transform in Y of a plane in $\mathbf{P}^3$ intersecting the two lines at general points, then by Corollary 4.15 we have:

$$\mathrm{ch}_2(Y) \cdot T = \mathrm{ch}_2(\mathbf{P}^3) \cdot H - 3 = -1.$$

Since T is disjoint from the exceptional line blown-up,

$$\mathrm{ch}_2(X) \cdot \tilde{T} = \mathrm{ch}_2(Y) \cdot T = -1.$$

In particular, X is not weakly 2-Fano.

(**10**) $\mathbf{P}^1 \times S_7$ is not weakly 2-Fano by 4.3.1 and Lemma 4.7.

(**11**) The blow-up X of $\mathbf{P}^1 \times \mathbf{F}_1$ with center $\{t\} \times C$, where $t \in T$ and C is the exceptional curve of the first kind on $\mathbf{F}_1$. If $\mathbf{F}_1 \to \mathbf{P}^2$ is the blow-up of a point $p \in \mathbf{P}^2$, let T be the surface $\mathbf{P}^1 \times L$, where L is the proper transform of a general line trough the point p. Since T intersects $\{t\} \times C$ in one point, it follows from Corollary 4.15, that

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(\mathbf{P}^1 \times \mathbf{F}_1) \cdot T - \frac{3}{2} = -\frac{3}{2}.$$

In particular, X is not weakly 2-Fano.

(**12**) The blow-up X of Y (No. (33) in the list for $\rho = 2$) with center two exceptional lines of the blowing-up $Y \to \mathbf{P}^3$ along a line L. Let T be

the proper transform on Y of a plane in $\mathbf{P}^3$ that contains L. It follows from Corollary 4.15 that

$$\mathrm{ch}_2(Y) \cdot T = \mathrm{ch}_2(\mathbf{P}^3) \cdot H + \frac{3}{2}\left(L^2\right)_T - \deg(N_{L|\mathbf{P}^3}) = \frac{3}{2}.$$

Let $\tilde{T}$ be the proper transform of T in X. Since $\tilde{T}$ intersects the blown-up curves in two points, it follows by Corollary 4.15 that

$$\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(Y) \cdot T - 3 = -\frac{3}{2}.$$

In particular, X is not weakly 2-Fano.

(**13**) (See [MM03].) The blow-up of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with center a curve of tridegree $(1, 1, 3)$. Since $-K_{\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1} \cdot C = 10$, this is not weakly 2-Fano by Corollary 4.16.

8.4. **Fano 3-folds with $\rho \geq 5$.** We go through the list in [MM81, Table 3] and check that each Fano 3-fold in the list is not 2-Fano. We point out the cases in which the 3-fold is weakly 2-Fano.

(**1**) The blow-up X of Y (No. (29) in the list for $\rho = 2$) with center three exceptional lines of the blowing-up $Y \to Q$ along a conic C.

Let T be the proper transform on Y of a general hyperplane section of Q. Note that T will intersect C in two points. It follows from Corollary 4.15 that

$$\mathrm{ch}_2(Y) \cdot T = \mathrm{ch}_2(Q) \cdot H - 3 = -2.$$

Since T is disjoint from the three exceptional lines of the blow-up, it follows that $\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(Y) \cdot T = -2$. In particular, X is not weakly 2-Fano.

(**2**) The blow-up X of Y (No. (25) in the list for $\rho = 3$) with center two exceptional lines l, l' of the blowing-up

$$\phi : Y \to \mathbf{P}^3$$

such that l, l' lie on the same irreducible component of the exceptional set of ϕ. Recall that ϕ is the blow-up of two disjoint lines L_1 and L_2 in $\mathbf{P}^3$.

Let T be the proper transform on Y of a general plane in $\mathbf{P}^3$ that intersects both L_1 and L_2. It follows from Corollary 4.15 that

$$\mathrm{ch}_2(Y) \cdot T = \mathrm{ch}_2(\mathbf{P}^3) \cdot H - 3 = -1.$$

Since T is disjoint from l and l', $\mathrm{ch}_2(\tilde{X}) \cdot \tilde{T} = \mathrm{ch}_2(Y) \cdot T = -1$. In particular, X is not weakly 2-Fano.

(**3**) $\mathbf{P}^1 \times S_6$ is not weakly 2-Fano by 4.3.1 and Lemma 4.7.

(**4**) $\mathbf{P}^1 \times S_{11-\rho}$ is not weakly 2-Fano by 4.3.1 and Lemma 4.7.

9. Fano fourfolds with index $i \geq 2$ and Picard number $\rho \geq 2$

The classification of Fano fourfolds of index 2 and $\rho > 1$ can be found in [IP99, Table 12.7]. We go through this list, check that none of them is 2-Fano, and point out the cases in which the 4-fold is weakly 2-Fano. We use the same notation as in the previous section.

(**1**) $\mathbf{P}^1 \times V_1$. This is not weakly 2-Fano by Lemma 4.7.

(**2**) $\mathbf{P}^1 \times V_2$. This is not weakly 2-Fano by Lemma 4.7.

(**3**) $\mathbf{P}^1 \times V_3$. This is not weakly 2-Fano by Lemma 4.7.

(**4**) A double cover of $\mathbf{P}^2 \times \mathbf{P}^2$ whose branch locus is a divisor of bidegree $(2, 2)$. Since $\mathbf{P}^1 \times \mathbf{P}^2$ is not 2-Fano by Lemma 4.7, this is not weakly 2-Fano by Corollary 4.2(ii).

(**5**) A divisor of $\mathbf{P}^2 \times \mathbf{P}^3$ of bidegree $(1, 2)$. This is not weakly 2-Fano by Example 4.8.

(**6**) $\mathbf{P}^1 \times V_4$. This is not weakly 2-Fano by Lemma 4.7.

(**7**) An intersection of two divisors of bidegree $(1, 1)$ on $\mathbf{P}^3 \times \mathbf{P}^3$. This is not weakly 2-Fano by Example 4.11.

(**8**) A divisor of $\mathbf{P}^2 \times Q^3$ of bidegree $(1, 1)$. By making a computation similar to those in 4.2.1, one can check that this is not weakly 2-Fano.

(**9**) $\mathbf{P}^1 \times V_5$. This is not weakly 2-Fano by Lemma 4.7.

(**10**) The blow-up of Q^4 along a conic C which is not contained in a plane lying on Q^4. We claim that this is not weakly 2-Fano.

The normal bundle of C in Q^4 is $N \cong \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)$. Let $\pi : X \to Q^4$ denote the blow-up, and $E \cong \mathbf{P}(N^*)$ the exceptional divisor. Consider the surface S in E, ruled over C, corresponding to a surjection

$$N^* \cong \mathcal{O}(-2) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2) \twoheadrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-2).$$

Using the formula for ch_2 from Lemma 4.12, one gets that $\mathrm{ch}_2(X) \cdot S = -2$.

(**11**) $\mathbf{P}_{\mathbf{P}^3}(\mathcal{E})$, where $\mathcal{E}$ is the null-correlation bundle on $\mathbf{P}^3$. Recall that $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = h^2$. Therefore this is weakly 2-Fano but not 2-Fano by Lemma 4.4.

(**12**) The blow-up $Q^4 \subset \mathbf{P}^5$ along a line ℓ. We claim that this is not weakly 2-Fano.

The normal bundle of ℓ in Q^4 is $N \cong \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}$. Let $\pi : X \to Q^4$ denote the blow-up, and $E \cong \mathbf{P}(N^*)$ the exceptional divisor. Consider the surface S in E, ruled over ℓ, corresponding to the surjection

$$N^* \cong \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \twoheadrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

Using the formula for ch_2 from Lemma 4.12, one gets that $\mathrm{ch}_2(X) \cdot S = -2$.

(**13**) $\mathbf{P}_{Q^3}(\mathcal{O}(-1) \oplus \mathcal{O})$, where $Q^3 \subset \mathbf{P}^4$ is a smooth quadric. This is weakly 2-Fano but not 2-Fano by Lemma 4.4.

(**14**) $\mathbf{P}^1 \times \mathbf{P}^3$. This is weakly 2-Fano but not 2-Fano by Lemma 4.7.

(**15**) $\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(-1) \oplus \mathcal{O}(1))$. This weakly 2-Fano but not 2-Fano by Lemma 4.4.

(**16**) $\mathbf{P}^1 \times W$. This is weakly 2-Fano but not 2-Fano by Lemma 4.7.

(**17**) $\mathbf{P}^1 \times V_7$. This is weakly 2-Fano but not 2-Fano by Lemma 4.7.

(**18**) $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. This is weakly 2-Fano but not 2-Fano by 4.7.

References

[And11]   D. Anderson, *Chern class formulas for G_2 Schubert loci*, Trans. Amer. Math. Soc. Vol. **363** (2011), no. 12, 6615–6646.

[AC12]   C. Araujo and A.-M. Castravet, *Minimal rational curves on higher Fano manifolds*, Amer. J. of Math. Vol. **134** (2012), no. 1, 87–107.

[BS02]   N. Bergeron and F. Sottile, *A Pieri-type formula for isotropic flag manifolds*, Trans. Amer. Math. Soc. Vol. **354** (2002), no. 7, 2659–2705.

[Cam92]   F. Campana, *Connexité rationnelle des variétés de Fano*, Ann. Sci. École Norm. Sup. (4) **35** (1992), no. 5, 539–545.

[Cos09]   I. Coskun, *Restriction varieties and geometric branching rules*. Adv. Math. Vol. **228** (2011), no. 4, 2441–2502.

[deJS06]   A. J. de Jong and J. Starr, *A note on Fano manifolds whose second Chern character is positive*, preprint arXiv:math/0602644v1 [math.AG], (2006).

[deJS07]   A. J. de Jong and J. Starr, *Higher Fano manifolds and rational surfaces*, Duke Math. J. Vol. **139** (2007), no. 1, 173–183.

[deJHS08]   A. J. de Jong, X. He, and J. Starr, *Families of rationally simply connected varieties over surfaces and torsors for semisimple groups*, preprint arXiv:0809.5224v1 [math.AG], (2008).

[Fuj82a]   T. Fujita, *Classification of projective varieties of Δ-genus one*, Proc. Japan Acad. Ser. A Math. Sci. Vol. **58** (1982), no. 3, 113–116.

[Fuj82b]   T. Fujita, *On polarized varieties of small Δ-genera*, Tohoku Math. J. (2) Vol. **34** (1982), no. 3, 319–341.

[Ful98]   W. Fulton, *Intersection theory*. Second Edition. Ergebnisse der Mathemayik und ihrer Grenzgebeite. 3. Folge., **2**Springer-Verlag, Berlin, 1998.

[GH]   P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. Wiley Classics Library. John Wiley Sons, Inc., New York, 1994.

[GHMS02]   T. Graber, J. Harris, B. Mazur, and J. Starr, *Rational connectivity and sections of families over curves*, preprint arXiv:0809.5224v1 [math.AG], (2008).

[GHS03]   T. Graber, J. Harris, and J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. Vol. **16** (2003), no. 1, 57–67.

[Hwa01]   J.-M. Hwang, *Geometry of minimal rational curves on Fano manifolds*, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), ICTP Lect. NotesVol. **6** (2001), 335–393.

[Isk77]   V.A. Iskovskikh, *Fano 3-folds I*, English translation, Math. USSR Izvestija Vol. **11** (1977), no. 3, 485–527.

[Isk78]   V.A. Iskovskikh, *Fano 3-folds II*, English translation, Math. USSR Izvestija Vol. **12** (1987), no. 3, 469–506.

[IP99]   V.A. Iskovskikh and Yu. G. Prokhorov, *Algebraic Geometry V, Fano varieties*, Encyclopaedia Math. Sci. Vol. **47**, Springer, Berlin (1999).

[KMM92a]   J. Kollár, Y. Miyaoka, and S. Mori, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. Vol. **36** (1992), no. 3, 765–779.

[KMM92b]   J. Kollár, Y. Miyaoka, and S. Mori, *Rationally connected varieties*, J. Algebraic Geom. Vol. **1** (1992), no. 3, 429–448.

[KO73]   S. Kobayashi, and T. Ochiai, *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ. Vol. **13** (1973), 31–47.

[MM81]   S. Mori and S. Mukai, *Classification of Fano 3-folds with $B_2 \geq 2$*, Manuscripta Math. Vol. **36** (1981), 147–162.

[MM03]   S. Mori and S. Mukai, *Erratum: Classification of Fano 3-folds with $B_2 \geq 2$*, Manuscripta Math. Vol. **110** (2003), 407.

[Muk89]   S. Mukai, *Birational classification of Fano 3-folds and Fano manifolds of coindex 3*, Proc. Nat. Acad. Sci. U.S.A. Vol. **86** (1992), no. 9, 3000–3002.

[Muk95]   S. Mukai, *Curves and symmetric spaces, I*, Amer. J. of Math. Vol. **117** (1995), no. 6, 1627–1644.

[Nob11]   E. E. Nobili, *Classification of Toric 2-Fano 4-folds*, Bulletin of the Brazilian Mathematical Society Vol. **42**, no. 3, (2011), 399–414.

[Nob12]   E. E. Nobili, *Birational geometry of toric varieties*, IMPA Ph.D. Thesis, arXiv:1204.3883 [math.AG](2012).

[Sat11]   H. Sato, *The numerical class of a surface on a toric manifold*, preprint arXiv:1106.5949v1 [math.AG], (2011).

[Wiś91]   J. Wiśniewski, *On Fano Manifolds of Large Index*, Manuscripta Math. Vol. **70** (1991), 145–152.

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