HELIX SURFACES IN THE BERGER SPHERE

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Abstract. We characterize helix surfaces of the Berger sphere. In particular, we prove that, locally, a helix surface is invariant by the action of a 1-parameter group of isometries of the ambient space.

1. Introduction

We consider surfaces in the 3-dimensional Berger sphere whose unit normal vector forms a constant angle with the Hopf vector field. These surfaces are called *helix surfaces* or *constant angle surfaces* and they have been studied in most of the 3-dimensional geometries. In [2], Cermelli and Di Scala analyze the case of constant angle surfaces in $\mathbb{R}^3$ obtaining some remarkable relation with a Hamilton-Jacobi type equation and showing their application to equilibrium configurations of liquid crystals. Later, several authors have studied constant angle surfaces in most of the 3-dimensional homogeneous spaces, in particular: Dillen–Fastenakels–Van der Veken–Vrancken in $\mathbb{S}^2\times\mathbb{R}$ ([4]); Dillen–Munteanu in $\mathbb{H}^2\times\mathbb{R}$ ([3]); Fastenakels–Munteanu–Van Der Veken in the Heisenberg group ([7]); López–Munteanu in Sol_3 ([8]). Moreover, helix submanifolds have been studied in higher dimensional euclidean spaces and product spaces in [5, 6, 11].

We shall use the Hopf fibration to describe a model of the Berger sphere. Indeed, let $\mathbb{S}^2(1/2)=\{(z,t)\in\mathbb{C}\times\mathbb{R}\colon |z|^2+t^2=1/4\}$ be the usual 2-sphere and let $\mathbb{S}^3=\{(z,w)\in\mathbb{C}^2\colon |z|^2+|w|^2=1\}$ be the usual 3-sphere. Then the Hopf map $\psi:\mathbb{S}^3\to\mathbb{S}^2(1/2)$, given by

$$\psi(z,w)=\frac{1}{2}\,(2z\bar{w},|z|^2-|w|^2)\,,$$

is a Riemannian submersion and the vector fields

$$X_1(z,w)=(iz,iw),\quad X_2(z,w)=(-i\bar{w},i\bar{z}),\quad X_3(z,w)=(-\bar{w},\bar{z})$$

parallelize $\mathbb{S}^3$ with X_1 vertical and X_2, X_3 horizontal. The vector X_1 is called the *Hopf vector field*. The Berger sphere $\mathbb{S}^3_\varepsilon$, $\varepsilon>0$, is the sphere $\mathbb{S}^3$ endowed with the metric

$$g_\varepsilon(X,Y)=\langle X,Y\rangle+(\varepsilon^2-1)\langle X,X_1\rangle\,\langle Y,X_1\rangle\,,$$

where $\langle,\rangle$ represents the canonical metric of $\mathbb{S}^3$. Thus a helix surface in $\mathbb{S}^3_\varepsilon$ is such that its unit normal N satisfies

$$|g_\varepsilon(X_1,N)|=\varepsilon\cos\theta$$

for fixed $\theta\in[0,\pi/2]$.

From a classical result of Reeb [10], a compact surface in the Berger sphere cannot be transverse to the Hopf vector field everywhere. This means that the notion of helix surfaces in $\mathbb{S}^3_\varepsilon$, with $\theta\neq\pi/2$, is meaningful only in the non compact case. For this

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reason our study it will be local and will aim to the following characterization of helix surfaces which represents the main result of the paper.

Theorem 3.1. *Let M^2 be a helix surface in the Berger sphere $\mathbb{S}^3_\varepsilon$ with constant angle $\theta \neq \pi/2$. Then, locally, the surface is invariant by the action of a 1-parameter group of isometries of $\mathbb{S}^3_\varepsilon$. Moreover, there exists local coordinates on M^2 such that the position vector of M^2 in $\mathbb{R}^4$ is*

$$F(u,v) = A(v)\,\beta(u)\,,$$

where

$$\beta(u) = (\sqrt{c_1}\,\cos(\alpha_1 u), \sqrt{c_1}\,\sin(\alpha_1 u), \sqrt{c_2}\,\cos(\alpha_2 u), \sqrt{c_2}\,\sin(\alpha_2 u))$$

is a geodesic of the torus $\mathbb{S}^1(\sqrt{c_1}) \times \mathbb{S}^1(\sqrt{c_2}) \subset \mathbb{S}^3$ with

$$c_{1,2} = \frac{1}{2} \mp \frac{\varepsilon\cos\theta}{2\sqrt{B}}\,, \quad \alpha_1 = \frac{2B}{\varepsilon} c_2\,, \quad \alpha_2 = \frac{2B}{\varepsilon} c_1\,, \quad B = 1 + (\varepsilon^2 - 1)\cos^2\theta\,,$$

while $A(v)$ is a 1-parameter family of $4{\times}4$ orthogonal matrices such that $\hat{J}A(v) = A(v)\hat{J}$, where $\hat{J}$ is the canonic complex structure of $\mathbb{R}^4$.

2. Helix surfaces

With respect to the orthonormal basis on $\mathbb{S}^3_\varepsilon$ defined by

$$E_1 = \varepsilon^{-1} X_1, \quad E_2 = X_2, \quad E_3 = X_3, \tag{1}$$

the Levi-Civita connection ∇^ε of $(\mathbb{S}^3_\varepsilon, g_\varepsilon)$ is given by:

$$\begin{array}{lll} {\nabla^\varepsilon}_{E_1} E_1 = 0, & {\nabla^\varepsilon}_{E_2} E_2 = 0, & {\nabla^\varepsilon}_{E_3} E_3 = 0, \\ {\nabla^\varepsilon}_{E_1} E_2 = \varepsilon^{-1}(2-\varepsilon^2) E_3, & & {\nabla^\varepsilon}_{E_1} E_3 = -\varepsilon^{-1}(2-\varepsilon^2) E_2, \\ {\nabla^\varepsilon}_{E_2} E_1 = -\varepsilon E_3, & {\nabla^\varepsilon}_{E_3} E_1 = \varepsilon E_2, & {\nabla^\varepsilon}_{E_3} E_2 = -\varepsilon E_1 = -{\nabla^\varepsilon}_{E_2} E_3. \end{array} \tag{2}$$

Let M^2 be an oriented helix surface in $\mathbb{S}^3_\varepsilon$ and let N be a unit normal vector field. Then, by definition,

$$|g_\varepsilon(E_1, N)| = \cos\theta$$

for fixed $\theta \in [0, \pi/2]$. Note that $\theta \neq 0$. In fact, if it were then the vector fields E_2 and E_3 would be tangent to the surface M^2, which is absurd since the horizontal distribution of the Hopf map is not integrable. If $\theta = \pi/2$, we have that E_1 is always tangent to M and, therefore, M is a Hopf cylinder. Therefore, from now on we assume that the constant angle $\theta \neq \pi/2, 0$.

The Gauss and Weingarten formulas, for all $X, Y \in C(TM)$, are

$$\begin{aligned} {\nabla^\varepsilon}_X Y &= \nabla_X Y + \alpha(X,Y), \\ {\nabla^\varepsilon}_X N &= -A(X), \end{aligned} \tag{3}$$

where with A we have indicated the shape operator of M in $\mathbb{S}^3_\varepsilon$, with ∇ the induced Levi-Civita connection on M and by α the second fundamental form on M in $\mathbb{S}^3_\varepsilon$. Decomposing E_1 into its tangent and normal components we have

$$E_1 = T + \cos\theta\, N\,,$$

where T is the tangent component which satisfies $g_\varepsilon(T,T) = \sin^2\theta$.

For all $X \in C(TM)$, we have that

$$\begin{aligned} \nabla^{\varepsilon}{}_X E_1 &= \nabla^{\varepsilon}{}_X T - \cos\theta\, A(X) \\ &= \nabla_X T + g_\varepsilon(A(X),T)\, N - \cos\theta\, A(X). \end{aligned} \tag{4}$$

On the other hand (we refer to [1]), if $X = \sum X_i E_i$,

$$\begin{aligned} \nabla^{\varepsilon}{}_X E_1 &= \varepsilon\,(X_3 E_2 - X_2 E_3) \\ &= \varepsilon\, g_\varepsilon(JX,T)\, N - \varepsilon\, \cos\theta JX, \end{aligned} \tag{5}$$

where JX denotes the rotation of angle $\pi/2$ on TM. Identifying the tangent and normal components of (4) and (5) respectively, we obtain

$$\nabla_X T = \cos\theta\,(A(X) - \varepsilon\, JX) \tag{6}$$

and

$$g_\varepsilon(A(X) - \varepsilon\, JX, T) = 0. \tag{7}$$

Lemma 2.1. *Let M^2 be an oriented helix surface with constant angle θ in $\mathbb{S}^3_\varepsilon$. Then, we have that:*

(i) *with respect to the basis $\{T, JT\}$, the matrices associated to the shape operator A takes the following form*

$$A = \begin{pmatrix} 0 & -\varepsilon \\ -\varepsilon & \lambda \end{pmatrix},$$

for some function λ on M;

(ii) *the Levi-Civita connection ∇ of M is given by*

$$\nabla_T T = -2\varepsilon\cos\theta\, JT, \qquad \nabla_{JT} T = \lambda\cos\theta\, JT\,,$$

$$\nabla_T JT = 2\varepsilon\cos\theta\, T, \qquad \nabla_{JT} JT = -\lambda\cos\theta\, T\,;$$

(iii) *the Gauss curvature of M is constant and satisfies*

$$K = 4(1-\varepsilon^2)\,\cos^2\theta\,:$$

(iv) *the function λ satisfies the following equation*

$$T\lambda + \lambda^2\,\cos\theta + 4(\varepsilon^2-1)\,\cos^3\theta + 4\cos\theta = 0\,. \tag{8}$$

Proof. (i) follows directly from (7). From (6) and using

$$g_\varepsilon(T,T) = g_\varepsilon(JT,JT) = \sin^2\theta, \quad g_\varepsilon(T,JT) = 0\,,$$

we obtain (ii). From the Gauss equation of M in $\mathbb{S}^3_\varepsilon$ (see [1]), and taking into account (i), we have that the Gauss curvature of M is given by

$$\begin{aligned} K &= \det A + \varepsilon^2 + 4(1-\varepsilon^2)\,\cos^2\theta \\ &= 4(1-\varepsilon^2)\,\cos^2\theta\,. \end{aligned}$$

Finally, (8) follows from the Codazzi equation (see [1]):

$$\nabla_X A(Y) - \nabla_Y A(X) - A[X,Y] = 4(1-\varepsilon^2)\,\cos\theta\,(g_\varepsilon(Y,T)X - g_\varepsilon(X,T)Y)\,,$$

putting $X = T$, $Y = JT$ and using (ii).

□

Remark 2.2. We point out that if a helix surface is minimal then $\theta=\pi/2$. In fact, from (i) of Lemma 2.1 $\lambda=0$ and using (8) it follows that $\cos\theta(1+(\varepsilon^2-1)\cos^2\theta)=0$, which implies, since $1+(\varepsilon^2-1)\cos^2\theta$ is differnt from zero for all $\theta\in[0,\pi/2]$, that $\theta=\pi/2$.

As $g_\varepsilon(E_1,N)=\cos\theta$, there exists a smooth function φ on M so that

$$N=\cos\theta E_1+\sin\theta\cos\varphi\, E_2+\sin\theta\sin\varphi\, E_3.$$

Therefore

$$T=E_1-\cos\theta\, N=\sin\theta\,[\sin\theta\, E_1-\cos\theta\cos\varphi\, E_2-\cos\theta\sin\varphi\, E_3] \tag{9}$$

and

$$JT=\sin\theta\,(\sin\varphi\, E_2-\cos\varphi\, E_3)\,.$$

Also

$$\begin{aligned} A(T)&=-\nabla^\varepsilon{}_T N=(T\varphi+\varepsilon^{-1}(2-\varepsilon^2)\,\sin^2\theta+\varepsilon\cos^2\theta)\,JT,\\ A(JT)&=-\nabla^\varepsilon{}_{JT}N=(JT\varphi)\,JT-\varepsilon\, T\,. \end{aligned} \tag{10}$$

Comparing (10) with (i) of Lemma 2.1, it results that

$$\begin{cases} JT\varphi=\lambda\,,\\ \ \ T\varphi=-2\varepsilon^{-1}\,B\,, \end{cases} \tag{11}$$

where

$$B=1+(\varepsilon^2-1)\cos^2\theta\,. \tag{12}$$

We observe that, as

$$[T,JT]=\cos\theta\,(2\varepsilon\, T-\lambda\, JT),$$

the compatibility condition of system (11):

$$(\nabla_T JY-\nabla_{JT}T)\varphi=[T,JT]\varphi=T(JT\varphi)-JT(T\varphi)$$

is equivalent to (8).
We now choose local coordinates (u,v) on M such that

$$\partial_u=T. \tag{13}$$

Also, as ∂_v is tangent to M, it can be writen in the form $\partial_v=a\,T+b\,JT$, for certain functions $a=a(u,v)$ and $b=b(u,v)$. Since

$$0=[\partial_u,\partial_v]=(a_u+2\varepsilon b\cos\theta)\,T+(b_u-b\lambda\cos\theta)\,JT\,,$$

we obtain

$$\begin{cases} a_u=-2\varepsilon b\cos\theta\,,\\ b_u=b\lambda\cos\theta\,. \end{cases} \tag{14}$$

Moreover, writing (8) as

$$\lambda_u+\cos\theta\,\lambda^2+4(\varepsilon^2-1)\,\cos^3\theta+4\cos\theta=0\,,$$

after integration, one gets

$$\lambda(u,v)=2\sqrt{B}\tan(\eta(v)-2\cos\theta\sqrt{B}\,u)\,, \tag{15}$$

for some smooth function η depending on v. Replacing (15) in (14) and solving the system, we obtain

$$\begin{cases} a(u,v) = \dfrac{\varepsilon}{\sqrt{B}} \sin(\eta(v) - 2\cos\theta\sqrt{B}\,u), \\ b(u,v) = \cos(\eta(v) - 2\cos\theta\sqrt{B}\,u)\,. \end{cases} \tag{16}$$

Therefore (11) becomes

$$\begin{cases} \varphi_u = -2\varepsilon^{-1}B\,, \\ \varphi_v = 0\,, \end{cases} \tag{17}$$

of which the general solution is given by

$$\varphi(u,v) = -2\varepsilon^{-1}B\,u + c\,, \tag{18}$$

where c is a real constant.

With respect to the local coordinates (u,v) described above we have the following characterization of the position vector of a helix surface.

Proposition 2.3. *Let M^2 be a helix surface in $\mathbb{S}^3_\varepsilon$ with constant angle θ. Then, with respect to the local coordinates (u,v) on M defined in* (13)*, the position vector F of M^2 in $\mathbb{R}^4$ satisfies the equation*

$$\frac{\partial^4 F}{\partial u^4} + (\tilde{b}^2 - 2\tilde{a})\,\frac{\partial^2 F}{\partial u^2} + \tilde{a}^2\,F = 0\,, \tag{19}$$

where

$$\tilde{a} = \varepsilon^{-2}\sin^2\theta\,B\,, \qquad \tilde{b} = -2\varepsilon^{-1}\,B \tag{20}$$

and $B = 1 + (\varepsilon^2 - 1)\cos^2\theta$.

Proof. Let M^2 be a helix surface and let F be the position vector of M^2 in $\mathbb{R}^4$. Then, with respect to the local coordinates (u,v) on M defined in (13), we can write $F(u,v) = (F_1(u,v),\ldots,F_4(u,v))$. By definition, taking into account (9), we have that

$$\begin{aligned} \partial_u F &= (\partial_u F_1, \partial_u F_2, \partial_u F_3, \partial_u F_4) = T \\ &= \sin\theta\,[\sin\theta\,E_{1|F(u,v)} - \cos\theta\cos\varphi\,E_{2|F(u,v)} - \cos\theta\sin\varphi\,E_{3|F(u,v)}]\,. \end{aligned}$$

Using the expression of E_1, E_2 and E_3 with respect to the coordinates vector fields of $\mathbb{R}^4$, the latter implies that

$$\begin{cases} \partial_u F_1 = \sin\theta\,(-\varepsilon^{-1}\sin\theta\,F_2 + \cos\theta\cos\varphi\,F_4 + \cos\theta\sin\varphi\,F_3)\,, \\ \partial_u F_2 = \sin\theta\,(\varepsilon^{-1}\sin\theta\,F_1 + \cos\theta\cos\varphi\,F_3 - \cos\theta\sin\varphi\,F_4)\,, \\ \partial_u F_3 = -\sin\theta\,(\varepsilon^{-1}\sin\theta\,F_4 + \cos\theta\cos\varphi\,F_2 + \cos\theta\sin\varphi\,F_1)\,, \\ \partial_u F_4 = \sin\theta\,(\varepsilon^{-1}\sin\theta\,F_3 - \cos\theta\cos\varphi\,F_1 + \cos\theta\sin\varphi\,F_2)\,. \end{cases} \tag{21}$$

Moreover, taking the derivative with respect to u of (21) we find two constants $\tilde{a}$ and $\tilde{b}$ such that

$$\begin{cases} (F_1)_{uu} = \tilde{a}\,F_1 + \tilde{b}\,(F_2)_u\,, \\ (F_2)_{uu} = \tilde{a}\,F_2 - \tilde{b}\,(F_1)_u\,, \\ (F_3)_{uu} = \tilde{a}\,F_3 + \tilde{b}\,(F_4)_u\,, \\ (F_4)_{uu} = \tilde{a}\,F_4 - \tilde{b}\,(F_3)_u\,, \end{cases} \tag{22}$$

where, using (17),

$$\tilde{a} = -\frac{\varepsilon^{-1}\sin^2\theta}{2}\varphi_u = \varepsilon^{-2}\sin^2\theta B, \qquad \tilde{b} = \varphi_u = -2\varepsilon^{-1}B\,.$$

Finally, taking twice the derivative of (22) with respect to u and using (21)–(22) in the derivative we obtain the desired equation (19). □

Integrating (19), we have the following

Corollary 2.4. *Let M^2 be a helix surface in $\mathbb{S}^3_\varepsilon$. Then, with respect to the local coordinates (u,v) on M defined in (13), the position vector F of M^2 in $\mathbb{R}^4$ is given by*

$$F(u,v) = \cos(\alpha_1 u)\,g^1(v) + \sin(\alpha_1 u)\,g^2(v) + \cos(\alpha_2 u)\,g^3(v) + \sin(\alpha_2 u)\,g^4(v),$$

where

$$\alpha_{1,2} = \frac{1}{\varepsilon}(B \pm \varepsilon\sqrt{B}\cos\theta)$$

are real constant, while the $g^i(v)$, $i \in \{1,\dots,4\}$, are mutually orthogonal vectors fields in $\mathbb{R}^4$, depending only on v, such that

$$g_{11} = \langle g^1(v), g^1(v)\rangle = g_{22} = \langle g^2(v), g^2(v)\rangle = \frac{\varepsilon}{2B}\alpha_2\,,$$
$$g_{33} = \langle g^3(v), g^3(v)\rangle = g_{44} = \langle g^4(v), g^4(v)\rangle = \frac{\varepsilon}{2B}\alpha_1\,.$$

Proof. First, a direct integration of (19), gives the solution

$$F(u,v) = \cos(\alpha_1 u)\,g^1(v) + \sin(\alpha_1 u)\,g^2(v) + \cos(\alpha_2 u)\,g^3(v) + \sin(\alpha_2 u)\,g^4(v)\,,$$

where

$$\alpha_{1,2} = \sqrt{\frac{\tilde{b}^2 - 2\tilde{a} \pm \sqrt{\tilde{b}^4 - 4\tilde{a}\tilde{b}^2}}{2}}$$

are two constants, while the $g^i(v)$, $i \in \{1,\dots,4\}$, are vector fields in $\mathbb{R}^4$ which depend only on v. Now, taking into account the values of $\tilde{a}$ and $\tilde{b}$ given in (20), ones obtains

$$\alpha_{1,2} = \frac{1}{\varepsilon}(B \pm \varepsilon\sqrt{B}\cos\theta)\,.$$

Next, since $|F|^2 = 1$ and using (19), (21) and (22) we find that the position vector $F(u,v)$ and its derivatives must satisfy the relations:

$$\begin{array}{lll} \langle F, F\rangle = 1\,, & \langle F_u, F_u\rangle = \varepsilon^{-2}B\sin^2\theta\,, & \langle F, F_u\rangle = 0\,, \\ \langle F_u, F_{uu}\rangle = 0\,, & \langle F_{uu}, F_{uu}\rangle = D\,, & \langle F, F_{uu}\rangle = -\varepsilon^{-2}B\sin^2\theta, \\ \langle F_u, F_{uuu}\rangle = -D\,, & \langle F_{uu}, F_{uuu}\rangle = 0\,, & \langle F, F_{uuu}\rangle = 0\,, \\ \langle F_{uuu}, F_{uuu}\rangle = E, & & \end{array} \tag{23}$$

where

$$D = \varepsilon^{-2}B\,\tilde{b}^2\sin^2\theta - 3\tilde{a}^2\,, \qquad E = (\tilde{b}^2 - 2\tilde{a})\,D - \varepsilon^{-2}B\,\tilde{a}^2\sin^2\theta\,.$$

Putting $g_{ij}(v) = \langle g^i(v), g^j(v)\rangle$, and evaluating the relations (23) in $(0,v)$, we obtain:

$$g_{11} + g_{33} + 2g_{13} = 1\,, \tag{24}$$

$$\alpha_1^2\,g_{22} + \alpha_2^2\,g_{44} + 2\alpha_1\alpha_2\,g_{24} = \varepsilon^{-2}B\sin^2\theta\,, \tag{25}$$

$$\alpha_1\,g_{12} + \alpha_2\,g_{14} + \alpha_1\,g_{23} + \alpha_2\,g_{34} = 0\,, \tag{26}$$

$$\alpha_1^3\, g_{12} + \alpha_1\alpha_2^2\, g_{23} + \alpha_1^2\alpha_2\, g_{14} + \alpha_2^3 g_{34} = 0\,, \tag{27}$$

$$\alpha_1^4\, g_{11} + \alpha_2^4\, g_{33} + 2\alpha_1^2\alpha_2^2\, g_{13} = D\,, \tag{28}$$

$$\alpha_1^2\, g_{11} + \alpha_2^2\, g_{33} + (\alpha_1^2+\alpha_2^2)\, g_{13} = \varepsilon^{-2} B \sin^2\theta\,, \tag{29}$$

$$\alpha_1^4\, g_{22} + \alpha_1^3\alpha_2\, g_{24} + \alpha_1\alpha_2^3\, g_{24} + \alpha_2^4\, g_{44} = D\,, \tag{30}$$

$$\alpha_1^5\, g_{12} + \alpha_1^3\alpha_2^2\, g_{23} + \alpha_1^2\alpha_2^3\, g_{14} + \alpha_2^5\, g_{34} = 0\,, \tag{31}$$

$$\alpha_1^3\, g_{12} + \alpha_1^3\, g_{23} + \alpha_2^3\, g_{14} + \alpha_2^3\, g_{34} = 0\,, \tag{32}$$

$$\alpha_1^6\, g_{22} + \alpha_2^6\, g_{44} + 2\alpha_1^3\alpha_2^3\, g_{24} = E\,. \tag{33}$$

From (26), (27), (31), (32), it follows that

$$g_{12} = g_{14} = g_{23} = g_{34} = 0\,.$$

Also, from (24), (28) and (29), we obtain

$$g_{11} = \frac{\varepsilon^2\,(D+\alpha_2^4) - 2B\sin^2\theta\,\alpha_2^2}{\varepsilon^2(\alpha_1^2-\alpha_2^2)^2}\,, \qquad g_{13} = 0\,, \qquad g_{33} = \frac{\varepsilon^2\,(D+\alpha_1^4) - 2B\sin^2\theta\,\alpha_1^2}{\varepsilon^2(\alpha_1^2-\alpha_2^2)^2}.$$

Moreover, using (25), (30) and (33), we get

$$g_{22} = \frac{\varepsilon^2\,(E-2D\alpha_2^2) + B\sin^2\theta\,\alpha_2^4}{\varepsilon^2\alpha_1^2\,(\alpha_1^2-\alpha_2^2)^2}\,, \quad g_{24} = 0\,, \qquad g_{44} = \frac{\varepsilon^2\,(E-2D\alpha_1^2) + B\sin^2\theta\,\alpha_1^4}{\varepsilon^2\alpha_2^2\,(\alpha_1^2-\alpha_2^2)^2}\,.$$

Finally, a long but straightforward computation gives

$$g_{11} = g_{22} = \frac{\varepsilon}{2B}\alpha_2\,, \qquad g_{33} = g_{44} = \frac{\varepsilon}{2B}\alpha_1\,.$$

□

3. The main result

We are now in the right position to state the main result of the paper. Before doing this we recall that, looking at $(\mathbb{S}^3_\varepsilon, g_\varepsilon)$ in $\mathbb{R}^4$, its isometry group can be identified with:

$$\{A \in \mathrm{O}(4)\colon\, A\hat{J} = \pm \hat{J} A\}\,,$$

where $\hat{J}$ is the canonical complex structure of $\mathbb{R}^4$ defined by

$$\hat{J} = \begin{pmatrix} J_1 & 0 \\ 0 & J_1 \end{pmatrix}\,, \quad J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,,$$

while $\mathrm{O}(4)$ is the orthogonal group (see, for example, [12]).

Theorem 3.1. *Let M^2 be a helix surface in the Berger sphere $\mathbb{S}^3_\varepsilon$ with constant angle $\theta \neq \pi/2$. Then, locally, the surface is invariant by the action of a 1-parameter group of isometries of $\mathbb{S}^3_\varepsilon$. Moreover, the position vector of M^2 in $\mathbb{R}^4$, with respect to the local coordinates (u,v) on M defined in* (13)*, is*

$$F(u,v) = A(v)\,\beta(u)\,,$$

where

$$\beta(u) = (\sqrt{g_{11}}\,\cos(\alpha_1 u), \sqrt{g_{11}}\,\sin(\alpha_1 u), \sqrt{g_{33}}\,\cos(\alpha_2 u), \sqrt{g_{33}}\,\sin(\alpha_2 u))$$

is a geodesic in the torus $\mathbb{S}^1(\sqrt{g_{11}})\times\mathbb{S}^1(\sqrt{g_{33}})\subset\mathbb{S}^3$*, where* g_{11}*,* g_{33}*,* α_1*,* α_2 *are the four constants given in Corollary 2.4, and* $A(v)$ *is a 1-parameter family of* 4×4 *orthogonal matrices such that* $\hat{J}A(v)=A(v)\hat{J}$*.*

Proof. With respect to the local coordinates (u,v) on M defined in (13), Corollary 2.4 implies that the position vector of the helix surface in $\mathbb{R}^4$ is given by

$$F(u,v)=\cos(\alpha_1 u)\,g^1(v)+\sin(\alpha_1 u)\,g^2(v)+\cos(\alpha_2 u)\,g^3(v)+\sin(\alpha_2 u)\,g^4(v)\,,$$

where the vector fields $\{g^i(v)\}$ are mutually orthogonal and

$$||g^1(v)||=||g^2(v)||=\sqrt{g_{11}}=\text{constant}\,,$$
$$||g^3(v)||=||g^4(v)||=\sqrt{g_{33}}=\text{constant}\,.$$

Thus, if we put $e_i(v)=g^i(v)/||g^i(v)||$, $i\in\{1,\dots,4\}$, we can write:

$$\begin{aligned}F(u,v)=\;&\sqrt{g_{11}}\,(\cos(\alpha_1\,u)\,e_1(v)+\sin(\alpha_1\,u)\,e_2(v))\\&+\sqrt{g_{33}}\,(\cos(\alpha_2\,u)\,e_3(v)+\sin(\alpha_2\,u)\,e_4(v))\,.\end{aligned}\tag{34}$$

Denote by $\bar{J}$ the 4×4 matrices with entries $\bar{J}_{i,j}=\langle\hat{J}e_i,e_j\rangle$, $i,j=1,\dots,4$. We shall prove that $\bar{J}=(\hat{J})^T$. For this, since

$$\hat{J}F(u,v)=X_{1|F(u,v)}=\varepsilon\,E_{1|F(u,v)}=\varepsilon\,(F_u+\cos\theta\,N),$$

and using (19)–(23), we obtain the following identities

$$\begin{aligned}&\langle\hat{J}F,F_u\rangle=\varepsilon^{-1}\sin^2\theta,\\&\langle\hat{J}F,F_{uu}\rangle=0\,,\\&\langle F_u,\hat{J}F_{uu}\rangle=\varepsilon^{-3}B\sin^2\theta\,(\sin^2\theta-2B):=I\,,\\&\langle\hat{J}F_u,F_{uuu}\rangle=0\,,\\&\langle\hat{J}F_u,F_{uu}\rangle+\langle\hat{J}F,F_{uuu}\rangle=0\,,\\&\langle\hat{J}F_{uu},F_{uuu}\rangle+\langle\hat{J}F_u,F_{uuuu}\rangle=0\,.\end{aligned}\tag{35}$$

Evaluating (35) in $(0,v)$, they become respectively:

$$\alpha_1 g_{11}\langle\hat{J}e_1,e_2\rangle+\alpha_2\,g_{33}\langle\hat{J}e_3,e_4\rangle+\sqrt{g_{11}g_{33}}\,(\alpha_1\langle\hat{J}e_3,e_2\rangle+\alpha_2\langle\hat{J}e_1,e_4\rangle)=\varepsilon^{-1}\sin^2\theta,\tag{36}$$

$$\langle\hat{J}e_1,e_3\rangle=0\,,\tag{37}$$

$$\alpha_1^3\,g_{11}\langle\hat{J}e_1,e_2\rangle+\alpha_2^3\,g_{33}\langle\hat{J}e_3,e_4\rangle+\sqrt{g_{11}g_{33}}\,(\alpha_1\alpha_2^2\langle\hat{J}e_3,e_2\rangle+\alpha_1^2\alpha_2\langle\hat{J}e_1,e_4\rangle)=-I,\tag{38}$$

$$\langle\hat{J}e_2,e_4\rangle=0\,,\tag{39}$$

$$\alpha_1\langle\hat{J}e_2,e_3\rangle+\alpha_2\langle\hat{J}e_1,e_4\rangle=0\,,\tag{40}$$

$$\alpha_2\langle\hat{J}e_2,e_3\rangle+\alpha_1\langle\hat{J}e_1,e_4\rangle=0\,.\tag{41}$$

We point out that to obtain the previous identities we have divided by $\alpha_1^2-\alpha_2^2=4\varepsilon^{-1}\sqrt{B^3\cos^2\theta}$ which is, by the assumption on θ, always different from zero. From (40) and (41), taking into account the $\alpha_1^2-\alpha_2^2\neq0$, it results that

$$\langle\hat{J}e_3,e_2\rangle=0\,,\qquad\langle\hat{J}e_1,e_4\rangle=0\,.\tag{42}$$

Therefore

$$|\langle \hat{J}e_1, e_2\rangle| = 1 = |\langle \hat{J}e_3, e_4\rangle|.$$

Substituting (42) in (36) and (38), we obtain the system

$$\begin{cases} \alpha_1 g_{11}\langle \hat{J}e_1, e_2\rangle + \alpha_2\, g_{33}\langle \hat{J}e_3, e_4\rangle = \varepsilon^{-1}\sin^2\theta \\ \alpha_1^3 g_{11}\langle \hat{J}e_1, e_2\rangle + \alpha_2^3\, g_{33}\langle \hat{J}e_3, e_4\rangle = -I\,, \end{cases} \tag{43}$$

a solution of which is

$$\langle \hat{J}e_1, e_2\rangle = \frac{\varepsilon I + \alpha_2^2\sin^2\theta}{\varepsilon g_{11}\,\alpha_1(\alpha_2^2-\alpha_1^2)}\,, \qquad \langle \hat{J}e_3, e_4\rangle = -\frac{\varepsilon I + \alpha_1^2\sin^2\theta}{\varepsilon g_{33}\,\alpha_2(\alpha_2^2-\alpha_1^2)}\,.$$

Now, as

$$g_{11}\,g_{33} = \frac{\sin^2\theta}{4B}\,, \qquad \alpha_1\,\alpha_2 = \frac{B}{\varepsilon^2}\sin^2\theta\,, \qquad (\alpha_1^2-\alpha_2^2)^2 = \frac{16B^3}{\varepsilon^2}\cos^2\theta\,,$$

it results that

$$\langle \hat{J}e_1, e_2\rangle\langle \hat{J}e_3, e_4\rangle = 1\,.$$

Moreover, a direct check shows that $\langle \hat{J}e_1, e_2\rangle > 0$. Consequently, $\langle \hat{J}e_1, e_2\rangle = \langle \hat{J}e_3, e_4\rangle = 1$. We have thus proved that $\bar{J} = (\hat{J})^T$.
Then, if we fix the canonical orthonormal basis of $\mathbb{R}^4$ given by

$$E_1 = (1,0,0,0))\,, \quad E_2 = (0,1,0,0)\,, \quad E_3 = (0,0,1,0)\,, \quad E_4 = (0,0,0,1)\,,$$

there must exists a 1-parameter family of 4×4 orthogonal matrices $A(v) \in \mathrm{O}(4)$, with $\hat{J}A(v) = A(v)\hat{J}$, such that $e_i(v) = A(v)E_i$. Thus $A(v)$ is a 1-parameter group of isometries. Replacing $e_i(v) = A(v)E_i$ in (34) we obtain

$$F(u,v) = A(v)\beta(u)\,,$$

where the curve

$$\beta(u) = (\sqrt{g_{11}}\,\cos(\alpha_1 u), \sqrt{g_{11}}\,\sin(\alpha_1 u), \sqrt{g_{33}}\,\cos(\alpha_2 u), \sqrt{g_{33}}\,\sin(\alpha_2 u))\,,$$

is a geodesic of the torus $\mathbb{S}^1(\sqrt{g_{11}}) \times \mathbb{S}^1(\sqrt{g_{33}}) \subset \mathbb{S}^3$.
In conclusion, the surface M is locally invariant by the action of the 1-parameter group of isometries of $\mathbb{S}^3_\varepsilon$ given by $\{A(v)\}_v$. □

Remark 3.2. The geodesic β of the torus $\mathbb{S}^1(\sqrt{g_{11}}) \times \mathbb{S}^1(\sqrt{g_{33}}) \subset \mathbb{S}^3$ in Theorem 3.1 has slope

$$m = \frac{\alpha_2}{\alpha_1} = \frac{\sqrt{B} - \varepsilon\cos\theta}{\sqrt{B} - \varepsilon\cos\theta}$$

that, for fixed $\varepsilon > 0$, varying $\theta \in (0, \pi/2)$ assumes all possible values in $(0,1)$.

Example 3.3. We shall now find an explicit expression of the 1-parameter family $A(v)$ in Theorem 3.1. Since $A(v)$ is an orthogonal matrices that commutes with $\hat{J}$, from standard arguments (see, for example, [9, Lemma 2.19]), we can write $A(v)$ as

$$A_1(v) = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}\,, \quad \text{or} \quad A_2(v) = \begin{pmatrix} X & Y \\ -Y^T & X^T \end{pmatrix}$$

where X and Y are 2×2 matrices satisfying

$$X^T Y = Y^T X\,, \quad X^T X + Y^T Y = \mathrm{Id}\,. \tag{44}$$

Next, since $\hat{J}A(v)E_1 = A(v)E_2$ and $\hat{J}A(v)E_3 = A(v)E_4$, we deduce that the matrices X and Y can be written as

$$X = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad Y = \begin{pmatrix} c & -d \\ d & c \end{pmatrix},$$

where a, b, c, d are functions on v. Let assume that the matrices X and Y are not zero for any value of v where $A(v)$ is defined. Now, taking into account (44) we see that the functions a, b, c, d satisfy the system

$$\begin{cases} a^2 + b^2 + c^2 + d^2 = 1 \\ ad - bc = 0\,. \end{cases}$$

From the second equation there must exists a never zero function $\lambda(v)$ such that $(a, b) = \lambda(c, d)$. Using the first equation we conclude that $a^2 + b^2 = 1/(\lambda^2 + 1)$. Finally, taking $\lambda = 1$ there must exist a function $\xi(v)$ such that $a(v) = (\cos\xi(v))/\sqrt{2}$ and $b(v) = (\sin\xi(v))/\sqrt{2}$. The matrices $A(v)$ becomes one of the following two types:

$$A_1(v) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\xi(v) & -\sin\xi(v) & \cos\xi(v) & -\sin\xi(v) \\ \sin\xi(v) & \cos\xi(v) & \sin\xi(v) & \cos\xi(v)) \\ -\cos\xi(v) & \sin\xi(v) & \cos\xi(v) & -\sin\xi(v) \\ -\sin\xi(v) & -\cos\xi(v) & \sin\xi(v) & \cos\xi(v) \end{pmatrix},$$

$$A_2(v) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\xi(v) & -\sin\xi(v) & \cos\xi(v) & -\sin\xi(v) \\ \sin\xi(v) & \cos\xi(v) & \sin\xi(v) & \cos\xi(v)) \\ -\cos\xi(v) & -\sin\xi(v) & \cos\xi(v) & \sin\xi(v) \\ \sin\xi(v) & -\cos\xi(v) & -\sin\xi(v) & \cos\xi(v) \end{pmatrix}.$$

Using the notation of Theorem 3.1 the map

$$F_i(u, v) = A_i(v)\beta(u)\,, \quad i = 1, 2\,,$$

gives an explicit immersion of a surface into the Berger sphere. A tedious but standard calculation shows that F_2 defines a helix surface of constant angle $\theta \neq \pi/2$ for any function ξ. By way of contrast, F_1 defines a surface which is tangent everywhere to the Hopf vector field. In fact, the Hopf vector field X_1 results tangent to the orbits of the action of the 1-parameter group $A_1(v)$ on $\mathbb{S}^3$.

Remark 3.4. The immersion $F_1(u, v) = A_1(v)\beta(u)$ in Example 3.3 shows that a viceversa of Theorem 3.1 does not hold. In fact, $A_1(v)$ is a 1-parameter family of 4 orthogonal matrices that commute with $\hat{J}$ but the surface described by $F_1(u, v)$ is not a helix surfaces with constant angle $\theta \neq \pi/2$.

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