arXiv:1206.1306v1 [math.DG] 6 Jun 2012

ON THE CLASSIFICATION OF HOMOGENEOUS EINSTEIN METRICS ON GENERALIZED FLAG MANIFOLDS WITH $b_2(M) = 1$

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Abstract. Let G be a simple compact connected Lie group. We study homogeneous Einstein metrics for a class of compact homogeneous spaces, namely generalized flag manifolds G/H with second Betti number $b_2(G/H) = 1$. There are 33 such manifolds which have some common geometric features; for example they admit a unique invariant complex structure which gives rise to unique invariant Kähler–Einstein metric. The most typical examples are the compact isotropy irreducible Hermitian symmetric spaces, for which the Killing form is the unique homogeneous Einstein metric (which is Kähler). For the remaining 26 cases, the first results were obtained by I. Ohmura (cf. ([Sak]) and M. Kimura ([Kim]) (these results have been recently verified by a joint work of first author with S. Anastassiou ([AnC]), where homogeneous Einstein metrics are studied from the viewpoint of the normalized Ricci flow). Nowadays the classification of homogeneous Einstein metrics has completed for the 24 spaces by A. Arvanitoyeorgos and first author ([AC2], [AC3]). In this paper we construct the Einstein equation for the two unexamined spaces (both corresponding to the Lie group E_8), namely the cosets $\mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SU}(5)$ and $\mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$. We determine Ricci tensors of E_8-invariant metrics explicitly by computing the non-zero structure constants. We use a method based on comparison of left-invariant metrics of E_8 which arise from different reductive decompositions. For the first space we classify all homogeneous Einstein metrics. For the second one, we see that the Einstein equation reduces to an algebraic system of five non-linear equations, but we fail to solve the algebraic system of equations. Since, for the rest members of the examined class, we know that there always exists a finite number of non-Kähler Einstein metrics, we conjecture that the space $\mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$ admits a finite number of homogeneous non-Kähler Einstein metrics.

2000 *Mathematics Subject Classification.* Primary 53C25; Secondary 53C30.

Keywords: Homogeneous Einstein metric, flag manifold, second Betti number, finiteness conjecture, twistor fibration.

Introduction

Given a Riemannian manifold M, the question whether M carries an Einstein metric, that is a Riemannian metric g of constant Ricci curvature, is a fundamental one in Riemannian geometry. The Einstein equation $\mathrm{Ric}_g = \lambda \cdot g$ ($\lambda \in \mathbb{R}$) reduces to a system of a non-linear second order PDEs, and a good understanding of its solutions in the general case seems far from being attained. If M is compact, then Einstein metrics (of volume 1) become in a natural way privileged metrics, since they are characterized variational as the critical points of the total scalar curvature functional $T : \mathcal{M} \to \mathbb{R}$, given by $T(g) = \int_M S_g dV_g$, restricted to the set $\mathcal{M}_1$ of Riemannian metrics of volume 1. However, even in this case general existence results are difficult to obtained. If we consider a homogeneous G-space $M = G/H$, then it is natural to work with G-invariant Riemannian metrics. For such a metric the Einstein equation reduces to an algebraic system which is more manageable, and in some cases can been solved explicity. Most known examples of Einstein manifolds are homogeneous.

A generalized flag manifold is an adjoint orbit $M = \mathrm{Ad}(G)w$ ($w \in \mathfrak{g}$) of a compact connected semi-simple Lie group G and can be represented as a compact homogeneous space of the form $M = G/H = G/C(S)$, where $C(S)$ is the centralizer of a torus S in G (and thus $\mathrm{rank}\, G = \mathrm{rank}\, H$). Generalized flag manifolds have been classified in terms of painted Dynkin diagrams and these have Kähler metrics, that is, the homogeneous manifolds $M = G/H$ can be expressed as $G^{\mathbb{C}}/U$, where $G^{\mathbb{C}}$ is the complexification of G and U a parabolic subgroup of $G^{\mathbb{C}}$. Thus on M we can define a finite number of invariant complex structures, and for any such structure there is a compatible G-invariant Kähler–Einstein metric. In this paper we investigate invariant Einstein metrics on generalized flag manifolds $M = G/H$ of a compact connected simple Lie group G with second Betti number $b_2(M) = 1$. Such a space is determined by painting black in the Dynkin diagram of G only one simple root. By [BHi] it is known that $M = G/H$ admits a unique invariant complex structure, and thus a unique Kähler-Einstein metric. Compact isotropy irreducible Hermitian symmetric spaces are the most typical examples of this category, and these are the only flag manifolds for which the Kähler-Einstein metric is given by the Killing form. Generalized flag manifolds $M = G/H$ with $b_2(M) = 1$ can be divided

The first author was full-supported by Masaryk University under the Grant Agency of Czech Republic, project no. P 201/ 12/ G028.

into following six classes, with respect to the height of the painted black simple root (see §2), or equivalently, with respect to the decomposition of the associated isotropy representation (see Table 1):

(A) The compact isotropy irreducible Hermitian symmetrics spaces $M = G/H$, which admit (up to scalling) a unique invariant Einstein metric. In this case the height of the simple root is equal to 1.

(B) The flag manifolds $M = G/H$ for which the isotropy representation decomposes into two inequivalent irreducible $\mathrm{Ad}(H)$-submodules, i.e., $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. These spaces are determined by painting black a simple root with height 2 and their classification was obtained in [AC1] (see also [Sak]).

(C) Seven flag manifolds $M = G/H$ with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$. These spaces were determined by painting black a simple root with height 3 [Kim].

(D) Four flag manifolds $M = G/H$ with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$. These spaces are determined by a simple root with height 4 [AC3].

(E) The flag manifold $M = G/H = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SU}(5)$. It is determined by painting black the simple root α_4 and is such that with $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_5$.

(F) The flag manifold $M = G/H = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$. It is determined by painting black the simple root α_5 and is such that $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_6$.

As we mention in Table, homogeneous Einstein metrics of the first four classes (B)-(D) have been completely classified in [Sak], [Kim], [AC2], and [AC3] (see also the recent work [AnC], where invariant Einstein metrics were studied under the more general context of Ricci flow). In particular, only the spaces corresponding to the cases (E) and (F), have not been examined yet. In this article we focus on these two flag manifolds and we construct the homogeneous Einstein equation. Next we treat the associated algebraic systems with the goal to prove the existence of positive real solutions and if possible to classify them. For case (E) we obtain the full classification of homogeneous Einstein metrics.

Theorem A. *The generalized flag manifold $M = G/H = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SU}(5)$ admits (up to a scale) precisely five non-Kähler E_8-invariant Einstein metrics.*

TABLE 1. The number of invariant Einstein metrics on generalized flag manifolds with $b_2(M) = 1$.

$M = G/H$ with $b_2(M) = 1$	$\mathcal{E}(M)$	$M = G/H$ with $b_2(M) = 1$	$\mathcal{E}(M)$
(A) Hermitian Symmetric Spaces ([Wo1])		**(C)** $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ ([Kim])	
$\mathrm{SU}(\ell)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(\ell - p))$	$= 1$	$\mathrm{F}_4 / \mathrm{U}(3) \times \mathrm{SU}(2)$	$= 3$
$\mathrm{SO}(2\ell + 1)/\mathrm{SO}(2) \times \mathrm{SO}(2\ell - 1)$	$= 1$	$\mathrm{E}_6 / \mathrm{U}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$	$= 3$
$\mathrm{Sp}(\ell)/\mathrm{U}(\ell)$	$= 1$	$\mathrm{E}_7 / \mathrm{U}(3) \times \mathrm{SU}(5)$	$= 3$
$\mathrm{SO}(2\ell)/\mathrm{SO}(2) \times \mathrm{SO}(2\ell - 2)$	$= 1$	$\mathrm{E}_7 / \mathrm{SU}(2) \times \mathrm{SU}(6) \times \mathrm{U}(1)$	$= 3$
$\mathrm{SO}(2\ell)/\mathrm{U}(\ell)$	$= 1$	$\mathrm{E}_8 / \mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$= 3$
$\mathrm{E}_6 / \mathrm{U}(1) \times \mathrm{SO}(10)$	$= 1$	$\mathrm{E}_8 / \mathrm{U}(8)$	$= 3$
$\mathrm{E}_7 / \mathrm{U}(1) \times \mathrm{E}_6$	$= 1$	$\mathrm{G}_2 / \mathrm{U}(2)$ (U(2) represented by the long root)	$= 3$
(B) $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ ([DiK], [AC2])		**(D)** $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$ ([AC3])	
$\mathrm{SO}(2\ell + 1)/\mathrm{U}(\ell - m) \times \mathrm{SO}(2m + 1)$ $(\ell - m \neq 1)$	$= 2$	$\mathrm{F}_4 / \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$= 3$
$\mathrm{Sp}(\ell)/\mathrm{U}(\ell - m) \times \mathrm{Sp}(m)$ $(m \neq 0)$	$= 2$	$\mathrm{E}_7 / \mathrm{SU}(4) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$= 3$
$\mathrm{SO}(2\ell)/\mathrm{U}(\ell - m) \times \mathrm{SO}(2m)$ $(\ell - m \neq 1,\ m \neq 0)$	$= 2$	$\mathrm{E}_8 / \mathrm{SU}(7) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$= 3$
$\mathrm{G}_2 / \mathrm{U}(2)$ (U(2) represented by the short root)	$= 2$	$\mathrm{E}_8 / \mathrm{SO}(10) \times \mathrm{SU}(3) \times \mathrm{U}(1)$	$= 5$
$\mathrm{F}_4 / \mathrm{SO}(7) \times \mathrm{U}(1)$	$= 2$	**(E)** $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5$	(new)
$\mathrm{F}_4 / \mathrm{Sp}(3) \times \mathrm{U}(1)$	$= 2$	$\mathrm{E}_8 / \mathrm{SU}(5) \times \mathrm{SU}(4) \times \mathrm{U}(1)$	$= 6$
$\mathrm{E}_6 / \mathrm{SU}(6) \times \mathrm{U}(1)$	$= 2$	**(F)** $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$	
$\mathrm{E}_6 / \mathrm{SU}(2) \times \mathrm{SU}(5) \times \mathrm{U}(1)$	$= 2$	$\mathrm{E}_8 / \mathrm{SU}(5) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	?
$\mathrm{E}_7 / \mathrm{SU}(7) \times \mathrm{U}(1)$	$= 2$		
$\mathrm{E}_7 / \mathrm{SU}(2) \times \mathrm{SO}(10) \times \mathrm{U}(1)$	$= 2$		
$\mathrm{E}_7 / \mathrm{SO}(12) \times \mathrm{U}(1)$	$= 2$		
$\mathrm{E}_8 / \mathrm{E}_7 \times \mathrm{U}(1)$	$= 2$		
$\mathrm{E}_8 / \mathrm{SO}(14) \times \mathrm{U}(1)$	$= 2$		

For the second space the construction of the Einstein equation of an E_8-invariant Riemannian metric is more complicated than case (E), since we find 9 non-zero structure constants with respect to the decomposition $\mathfrak{g} = \mathfrak{e}_8 = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$, a result that was presented also in [Chr] in terms of symmetric-$\mathfrak{t}$-triples. To obtain the values of non-zero structure constants, we define a fibration of $G/H = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$ over the space $G/K = \mathrm{E}_8 / \mathrm{SU}(6) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ with three isotropy summands. This gives rise to a new decomposition $\mathfrak{g} = \mathfrak{e}_8 = \mathfrak{k} \oplus \mathfrak{n} = \mathfrak{k} \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$, such that $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_6$. By comparing the Ricci components of E_8 for the left-invariant metrics arising from these different decompositions, we obtain certain conditions between the structure constants (the same approach was used also in case (E)). Next, we use the twistor fibration of G/H over the symmetric space $\mathrm{E}_8 / \mathrm{E}_7 \times \mathrm{SU}(2)$ to obtain some further relations (the contribution of the twistor fibration in the construction of the homogeneous Einstein equation was first time highlighted in [AC2]). In this way we can write down explicitly the Ricci tensor for an E_8-invariant metric on G/H (as well as, the Ricci tensor for E_8).

For $\mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$, the system of algebraic equations which gives the homogeneous Einstein equation consists of five non-linear polynomial equations. We have tried to compute a Gröbner basis, but we fail to obtain it. Since for the spaces arising from cases (B)-(E) we have proved that they always admit a finite number of (non-Kahler) invariant Einstein metrics (see Table 1, for the number $\mathcal{E}(M)$), we conjecture the following:

Conjecture B. *Let G a compact connected simple Lie group and let $M = G/H$ be a generalized flag manifold with first Betti number equal to 1, which is not an irreducible Hermitian symmetric space of compact type. Then M admits a finite number of non-isometric G-invariant Einstein metrics, which are not Kähler.*

The paper is organized as follows: In §1 we describe the Ricci tensor on a reductive homogeneous space, and in §2 we discuss the algebraic setting of flag manifolds. In §3 we treat the space $M = G/H = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SU}(5)$, we write down explicitly the homogeneous Einstein equation and we prove Theorem A. The last section §4 is about the space $M = G/H = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$.

1. The Ricci tensor of a G-invariant metric

Let G be a compact connected semi-simple Lie group with Lie algebra $\mathfrak{g}$, and let H be a closed subgroup of G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. We denote by B the negative of the Killing form of $\mathfrak{g}$. Then B is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. Let $\mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant orthogonal complement of $\mathfrak{h}$ with respect to B, that means $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$. As usual we identify $\mathfrak{m} = T_oG/H$, where $o = eH \in G/H$. Let now $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p$ be a decomposition of $\mathfrak{h}$ into its ideals, i.e. $[\mathfrak{h}, \mathfrak{h}_i] \subset \mathfrak{h}_i$, for any $i = 0, 1, \ldots, p$, where $\mathfrak{h}_0$ is the center of $\mathfrak{h}$ and $\mathfrak{h}_i$ $(i = 1, \cdots, p)$ are simple ideals of $\mathfrak{h}$. We assume that $\mathfrak{h}_i$ $(i = 1, \cdots, p)$ are mutually non-isomorphic and that $\dim \mathfrak{h}_0 \leq 1$. We also assume that $\mathfrak{m} = T_oG/H$ admits a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ into q irreducible $\mathrm{Ad}(H)$-modules $\mathfrak{m}_j$ $(j = 1, \cdots, q)$, which are mutually non-equivalent.

Let us consider now the following left-invariant Riemannian metric on G:

$$< \, , \, >= u_0 \cdot B|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + \cdots + u_p \cdot B|_{\mathfrak{h}_p} + x_1 \cdot B|_{\mathfrak{m}_1} + \cdots + x_q \cdot B|_{\mathfrak{m}_q}, \quad u_0, u_1, \cdots, u_p, x_1, \cdots, x_q \in \mathbb{R}_+. \tag{1}$$

This metric is also $\mathrm{Ad}(H)$-invariant. We also consider the G-invariant Riemannian metric on G/H, given by

$$(\, , \,) = x_1 \cdot B|_{\mathfrak{m}_1} + \cdots + x_q \cdot B|_{\mathfrak{m}_q}, \quad x_1, \cdots, x_q \in \mathbb{R}_+. \tag{2}$$

Because $\mathfrak{m}_i \not\cong \mathfrak{m}_j$ for any $i \neq j$, any G-invariant metric on G/H is given by (2). For practical reasons, next we will write the decomposition $\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_p \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ (resp. $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$) as $\mathfrak{g} = \mathfrak{w}_0 \oplus \mathfrak{w}_1 \oplus \cdots \oplus \mathfrak{w}_p \oplus \mathfrak{w}_{p+1} \oplus \cdots \oplus \mathfrak{w}_{p+q}$ (resp. $\mathfrak{m} = \mathfrak{w}_{p+1} \oplus \cdots \oplus \mathfrak{w}_{p+q}$). Then, the space of left invariant symmetric covariant 2-tensors on G which are $\mathrm{Ad}(H)$-invariant is given by

$$\{v_0 \cdot B|_{\mathfrak{w}_0} + v_1 \cdot B|_{\mathfrak{w}_1} + \cdots + v_{p+q} \cdot B|_{\mathfrak{w}_{p+q}} \mid v_0, v_1, \cdots, v_{p+q} \in \mathbb{R}\} \tag{3}$$

and the space of G-invariant symmetric covariant 2-tensors on G/H is given by

$$\{z_{p+1} \cdot B|_{\mathfrak{w}_{p+1}} + \cdots + z_{p+q} \cdot B|_{\mathfrak{w}_{p+q}} \mid \; z_{p+1}, \cdots, z_{p+q} \in \mathbb{R}\}. \tag{4}$$

In particular, the Ricci tensor r of a left invariant Riemannian metric $< \, , \, >$ on G is a left invariant symmetric covariant 2-tensor on G which is $\mathrm{Ad}(H)$-invariant and thus r is of the form (3), and the Ricci tensor $\bar{r}$ of a G-invariant Riemannian metric on G/H is a G-invariant symmetric covariant 2-tensor on G/H, and thus $\bar{r}$ is of the form (4). Let $\{e_\alpha\}$ be a B-orthonormal basis adapted to the decomposition of $\mathfrak{g}$, i.e., $e_\alpha \in \mathfrak{w}_i$ for some i, and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{w}_i$ and $e_\beta \in \mathfrak{w}_j$). We set $A^\gamma_{\alpha\beta} = B([e_\alpha, e_\beta], e_\gamma)$ so that $[e_\alpha, e_\beta] = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$, and set $c^k_{ij} = \begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{w}_i, \; e_\beta \in \mathfrak{w}_j, \; e_\gamma \in \mathfrak{w}_k$. Then c^k_{ij} is independent of the B-orthonormal bases chosen for $\mathfrak{w}_i, \mathfrak{w}_j, \mathfrak{w}_k$, and symmetric in all three indices, i.e. $c^k_{ij} = c^k_{ji} = c^j_{ki}$ (see [WZ2]). Now, we write a left-invariant metric on G of the form (1), by

$$g = y_0 \cdot B|_{\mathfrak{w}_0} + y_1 \cdot B|_{\mathfrak{w}_1} + \cdots + y_p \cdot B|_{\mathfrak{w}_p} + y_{p+1} \cdot B|_{\mathfrak{w}_{p+1}} + \cdots + y_{p+q} \cdot B|_{\mathfrak{w}_{p+q}}, \quad y_0, y_1, \cdots, y_{p+q} \in \mathbb{R}_+, \tag{5}$$

and a G-invariant Riemannian metric on G/H of the form (2), by

$$h = w_{p+1} \cdot B|_{\mathfrak{w}_{p+1}} + \cdots + w_{p+q} \cdot B|_{\mathfrak{w}_{p+q}}, \quad w_{p+1}, \cdots, w_{p+q} \in \mathbb{R}_+. \tag{6}$$

Theorem 1. ([AMS]) *Let $d_k = \dim \mathfrak{w}_k$. (1) The components $r_0, r_1, \cdots, r_{p+q}$ of the Ricci tensor r of the metric g of the form (5) on G, are given by*

$$r_k = \frac{1}{2y_k} + \frac{1}{4d_k}\sum_{j,i}\frac{y_k}{y_j y_i}\begin{bmatrix}k\\ ji\end{bmatrix} - \frac{1}{2d_k}\sum_{j,i}\frac{y_j}{y_k y_i}\begin{bmatrix}j\\ ki\end{bmatrix} \quad (k = 0, 1, \cdots, p+q), \tag{7}$$

where the sum is taken over all $i, j = 0, 1, \cdots, p+q$. Moreover, for each k we have $\sum_{i,j}\begin{bmatrix}j\\ ki\end{bmatrix} = d_k$.

(2) *The components $\bar{r}_{p+1}, \cdots, \bar{r}_{p+q}$ of the Ricci tensor $\bar{r}$ of the metric h given by (6) on G/H, are given by*

$$\bar{r}_k = \frac{1}{2w_k} + \frac{1}{4d_k}\sum_{j,i}\frac{w_k}{w_j w_i}\begin{bmatrix}k\\ ji\end{bmatrix} - \frac{1}{2d_k}\sum_{j,i}\frac{w_j}{w_k w_i}\begin{bmatrix}j\\ ki\end{bmatrix} \quad (k = p+1, \cdots, p+q), \tag{8}$$

where the sum is taken over all $i, j = p+1, \cdots, p+q$.

2. Decomposition associated to generalized flag manifolds

In this section we review briefly the Lie theoretic description of a flag manifold in terms of painted Dynkin diagrams, and next we recall some notions from the geometry and the topology of such a space.

Let G be a compact semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{t}$ a maximal abelian subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{t}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Then $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We assume that $\dim_{\mathbb{C}} \mathfrak{t}^{\mathbb{C}} = \ell = \operatorname{rank} \mathfrak{g}^{\mathbb{C}}$. We identify an element of the root system Δ of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{t}^{\mathbb{C}}$ with an element of $\sqrt{-1}\mathfrak{t}$, by the duality defined by the Killing form of $\mathfrak{g}^{\mathbb{C}}$. Consider the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{t}^{\mathbb{C}}$, i.e., $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^{\mathbb{C}}_{\alpha}$. Let $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ be a fundamental system of Δ and $\{\Lambda_1, \cdots, \Lambda_\ell\}$ the fundamental weights of $\mathfrak{g}^{\mathbb{C}}$ corresponding to Π, that is $2(\Lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$, for any $1 \le i, j \le \ell$. We choose a subset $\Pi_0 \subset \Pi$ and we set $\Pi_M = \Pi\backslash\Pi_0 = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$ $(1 \le \alpha_{i_1} < \cdots < \alpha_{i_r} \le \ell)$. We put $[\Pi_0] = \Delta \cap \operatorname{span}_{\mathbb{Z}}\{\Pi_0\}$ and $[\Pi_0]^+ = \Delta^+ \cap \operatorname{span}_{\mathbb{Z}}\{\Pi_0\}$, where $\operatorname{span}_{\mathbb{Z}}\{\Pi_0\}$ denotes the subspace of $\sqrt{-1}\mathfrak{t}$ generated by Π_0 with integer coefficients, and Δ^+ is the set of all positive roots relative to Π. Take a Weyl basis $\{E_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha : \alpha \in \Delta\}$ (see [AC2]), and set $A_\alpha = E_\alpha + E_{-\alpha}$ and $B_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$. Then the Lie algebra $\mathfrak{g}$, is a real form of $\mathfrak{g}^{\mathbb{C}}$ which can be identified with the fixed-point set $\mathfrak{g}^\tau$ of the complex conjugation τ in $\mathfrak{g}^{\mathbb{C}}$, that means $\mathfrak{g}^\tau = \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta^+}\{\mathbb{R}A_\alpha + \mathbb{R}B_\alpha\}$ (see [Hel]). Now, the subalgebra $\mathfrak{u} \subset \mathfrak{g}^{\mathbb{C}}$ defined by (9), is a *parabolic subalgebra* $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$, since it contains the *Borel subalgebra* $\mathfrak{b} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^{\mathbb{C}}_\alpha \subset \mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{u} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in[\Pi_0]\cup\Delta^+} \mathfrak{g}^{\mathbb{C}}_\alpha. \tag{9}$$

Let $G^{\mathbb{C}}$ be a simply connected complex semi-simple Lie group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$ and U the parabolic subgroup of $G^{\mathbb{C}}$ generated by $\mathfrak{u}$. Since U is connected, the complex homogeneous manifold $G^{\mathbb{C}}/U$ is simply connected (and compact). Moreover, G acts transitively on $G^{\mathbb{C}}/U$ with isotropy group the connected closed subgroup $H = G \cap U \subset G$, thus $G^{\mathbb{C}}/U = G/H$ as C^∞-manifolds. This identification implies that $G^{\mathbb{C}}/U$ carries a G-invariant Kähler metric. Moreover $H = G \cap U$ is the centralizer of a torus $S \subset T$ in G, where T is the maximal torus generated from the ad-diagonal subalgebra $\mathfrak{t}$. Thus $\operatorname{rank} G = \operatorname{rank} H$. The homogeneous space $M = G^{\mathbb{C}}/U = G/H$ is called *generalized flag manifold*, and any generalized flag manifold is constructed in this way. Let $\mathfrak{h}$ be the Lie algebra of H and let $\mathfrak{h}^{\mathbb{C}}$ be its complexification. In view of (9) and due to the inclusion $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}} \subset \mathfrak{u}$, we obtain a direct sum decomposition $\mathfrak{u} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{n}$, such that $\mathfrak{g} \cap \mathfrak{u} = \mathfrak{h}$, where the the nilradical $\mathfrak{n}$ of $\mathfrak{u}$ and the subalgebra $\mathfrak{h}^{\mathbb{C}}$ are given by $\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+-[\Pi_0]^+} \mathfrak{g}^{\mathbb{C}}_\alpha$ and $\mathfrak{h}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in[\Pi_0]} \mathfrak{g}^{\mathbb{C}}_\alpha$, respectively. The real subalgebra $\mathfrak{h}$ is given by $\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha\in[\Pi_0]^+}\{\mathbb{R}A_\alpha + \mathbb{R}B_\alpha\}$.

Let $\widetilde{\alpha} = \sum_{k=1}^{\ell} c_k\alpha_k$ be the highest (or maximal) root of $\mathfrak{g}^{\mathbb{C}}$, that is $c_k \ge m_k$ for any positive root $\alpha = \sum_{k=1}^{\ell} m_k\alpha_k \in R^+$. The postive integer c_i is called the *height* of the simple root $\alpha_i \in \Pi$, for any $i = 1, \ldots, \ell$. Next we will use the map $\operatorname{ht} : \Pi \to \mathbb{Z}_+$, $\alpha_i \mapsto \operatorname{ht}(\alpha_i) := c_i$.

Proposition 1. ([BuR, Proposition 4.3]) *Let $\mathfrak{z}$ be the center of the nilpotent Lie algebra $\mathfrak{n}$. Then we have $\operatorname{ad}(\mathfrak{h}^{\mathbb{C}})(\mathfrak{z}) \subset \mathfrak{z}$ and the action of $\mathfrak{h}^{\mathbb{C}}$ on $\mathfrak{z}$ is irreducible. Moreover, the $\operatorname{ad}(\mathfrak{h}^{\mathbb{C}})$-module $\mathfrak{z}$ is generated by the highest root space $\mathfrak{g}^{\mathbb{C}}_{\widetilde{\alpha}}$.*

We denote by $\mathfrak{h}_0$ the center of $\mathfrak{h}$, and $\mathfrak{h}^{\mathbb{C}}_0$ its complexification. Since $\mathfrak{h}^{\mathbb{C}}$ is a reductive subalgebra of $\mathfrak{g}^{\mathbb{C}}$, it admits the decomposition $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}}_0 \oplus \mathfrak{h}^{\mathbb{C}}_{ss}$, where $\mathfrak{h}^{\mathbb{C}}_{ss}$ is the semisimple part of $\mathfrak{h}^{\mathbb{C}}$, given by $\mathfrak{h}^{\mathbb{C}}_{ss} = [\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}] = \bigoplus_{\alpha\in\Pi_0}\mathbb{C}\alpha \bigoplus_{\alpha\in[\Pi_0]}\mathfrak{g}^{\mathbb{C}}_\alpha$. The set $[\Pi_0]$ is the root system of $\mathfrak{h}^{\mathbb{C}}_{ss}$ and Π_0 is a basis of simple roots for it. For convenience, we will denote the set $[\Pi_0]$ by Δ_H. We set $\Delta_M = \Delta\backslash\Delta_H$. Roots belong to Δ_M are called *complementary roots* and they have a significant role in the geometry of $M = G/H$. For example, let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to B. Then we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $[\,\mathfrak{h},\, \mathfrak{m}\,] \subset \mathfrak{m}$, and we identify $\mathfrak{m}$ with the tangent space T_oG/H in $o = eH \in G/H$. Set $\Delta^+_M = \Delta^+ \backslash \Delta^+_H$, where Δ^+_H is the system of positive roots of $\mathfrak{h}^{\mathbb{C}}$, i.e., $\Delta^+_H = [\Pi_0]^+$. Then

$$\mathfrak{m} = \bigoplus_{\alpha \in \Delta^+_M} \{\mathbb{R}A_\alpha + \mathbb{R}B_\alpha\}. \tag{10}$$

The complexified tangent space $\mathfrak{m}^{\mathbb{C}}$ is given by $\mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in \Delta_M} \mathfrak{g}^{\mathbb{C}}_\alpha$, and the set $\{E_\alpha : \alpha \in \Delta_M\}$ is a basis of $\mathfrak{m}^{\mathbb{C}}$. Note that although the set Π_M consists of all these complerentary roots which are simple, is not in general a basis of Δ_M, that is Δ_M is not in general a root system.

Generalized flag manifolds $M = G/H$ of a compact connected simple Lie group G are classified by using the Dynkin diagram of G, as follows: Let $\Gamma = \Gamma(\Pi)$ be the Dynkin diagram corresponding to the base of simple roots Π of the root system Δ of $\mathfrak{g}^{\mathbb{C}}$ relative to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$.

Definition 1. *The painted Dynkin diagram of $M = G/H$ is obtained from the Dynkin diagram $\Gamma = \Gamma(\Pi)$ by painting black the nodes which correspond to the simple roots of Π_M. The subdiagram of white nodes with the connecting lines between them determines the semisimple part $\mathfrak{h}_{ss}$ of the Lie algebra $\mathfrak{h}$ of H, and each black node gives rise to one $\mathfrak{u}(1)$-summand (their totality forms the center $\mathfrak{h}_0$ of $\mathfrak{h}$).*

Thus, the painted Dynkin diagram determines the isotropy group H and the space $M = G/H$ completely. It should be noted that the resulting painted Dynkin diagram does not depend on the choise of a maximal abelian subalgebra $\mathfrak{t}$ and hence of Δ. On the other hand the necessity of making a choise of a base Π for Δ (or equivalently of an ordering Δ^+ in Δ) reduces the number of painted Dynkin diagrams. By using certain rules to determine whether different painted Dynkin diagrams define isomorphic flag manifolds, one can obtain all flag manifolds G/H of a compact connected simple Lie group G (cf. [AA]).

Remark 1. The (real) dimension of the center $\mathfrak{h}_0$ of the subalgebra $\mathfrak{h}$ is equal to the number of black nodes in the painted Dynkin diagram of $M = G/H$, or equivalent equal to the number of $\mathfrak{u}(1)$ summands in the decomposition of $\mathfrak{h}$. By assuming that $\Pi_M = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$, it follows that the fundamental weights $\Lambda_{i_1}, \ldots, \Lambda_{i_r}$ form a basis of the dual space $\mathfrak{h}_0^*$ of $\mathfrak{h}_0$. Since $\mathfrak{h}_0^* \cong \mathfrak{h}_0$, via the Killing form of $\mathfrak{g}$, we obtain $\dim \mathfrak{h}_0 = r = |\Pi_M|$, where $|\Pi_M|$ is the cardinality of Π_M (cf. [APe]). From [BHi, p. 507] we have that $H^2(M;\mathbb{R}) = H^1(H;\mathbb{R}) = \mathfrak{h}_0$. Thus, the second Betti number $b_2(M)$ of the flag manifold $M = G/H$ is equal to $\dim \mathfrak{h}_0$, and it is obtained directly from the painted Dynkin diagram. Moreover, any flag manifold $M = G/H$ of a simple Lie group G with $b_2(M) = r$, is determined by a subset $\Pi_M \subset \Pi$ with $|\Pi_M| = r$ and it is constructed in the above way.

From now on we assume that G is simple. Moreover, we choose a subset $\Pi_0 \subset \Pi$ such that $\Pi_M = \Pi - \Pi_0 = \{\alpha_i\}$, for some fixed i with $1 \le i \le \ell$. Then the corresponding flag manifold $M = G/H$ is such that $\dim \mathfrak{h}_0 = 1$ and $b_2(M) = 1$. We also assume that $\mathrm{ht}(\alpha_i) = N \in \mathbb{Z}^+$. To an integer k with $1 \le k \le N$ we associate the set $\Delta^+(\alpha_i, k) = \left\{\alpha \in \Delta^+ \,\middle|\, \alpha = \sum_{j=1}^{\ell} m_j\alpha_j,\ m_i = k\right\}$. Then it is obvious that $\Delta^+_M = \Delta^+ \backslash \Delta^+_H = \bigcup_{1 \le k \le N} \Delta^+(\alpha_i, k)$. We define a subspace $\mathfrak{n}_k$ of the nilradical $\mathfrak{n}$ by $\mathfrak{n}_k = \bigoplus_{\alpha \in \Delta^+(\alpha_i,k)} \mathbb{C}E_\alpha$. Then $\mathfrak{n}_k$ $(k = 1, \cdots, N)$ are $\mathrm{ad}(\mathfrak{h}^{\mathbb{C}})$-invariant subspaces, and $\mathfrak{n} = \bigoplus_{j=1}^N \mathfrak{n}_j$ is an irreducible decomposition of $\mathfrak{n}$ (see [Wo2]). In view of Proposition 1 we have that $\mathfrak{z} = \mathfrak{n}_N$. We also define subspaces $\mathfrak{m}_k$ of $\mathfrak{m}$, given by

$$\mathfrak{m}_k = \bigoplus_{\alpha \in \Delta^+(\alpha_i,k)} \{\mathbb{R}(E_\alpha + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_\alpha - E_{-\alpha})\}. \tag{11}$$

Note that $\mathfrak{m}_k$ are $\mathrm{Ad}(H)$-invariant submodules of $\mathfrak{m}$ which are matually inequivalent each other, for any $k = 1, \cdots, N$ ([Kim]). We also recall the following useful inclusions (see for example [AC2]):

$$[\mathfrak{h}, \mathfrak{m}_i] \subset \mathfrak{m}_i, \quad [\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h} + \mathfrak{m}_{2i}, \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{|i-j|} \ \ (i \neq j). \tag{12}$$

By using (10), we get a characterization of $\mathfrak{m}$ in terms of the submodules $\mathfrak{m}_k$:

Lemma 1. *Let $M = G^{\mathbb{C}}/U = C/H$ be a flag manifold of a compact connected simple Lie group G, defined by a subset $\Pi_M = \{\alpha_i : \mathrm{ht}(\alpha_i) = N\} \subset \Pi$. Then, $\mathfrak{m} = T_oM$ admits a decomposition $\mathfrak{m} = \bigoplus_{k=1}^N \mathfrak{m}_k$ into N irreducible, inequivalent $\mathrm{Ad}(H)$-submodules $\mathfrak{m}_k$ defined by (11). Moreover, it is $d_k = \dim_{\mathbb{R}} \mathfrak{m}_k = 2\cdot|\Delta^+(\alpha_i, k)|$, for any $1 \le k \le N$.*

Note that according to the notation of §1, for the space $M = G^{\mathbb{C}}/U = G/H$ in Lemma 1, it is $N = q$.

Remark 2. It is well known (cf. [Tak], [APe], [AC3]) that for a flag manifold G/H, there is a 1-1 correspondence between G-invariant complex structures J and compatible G-invariant Kähler-Einstein metrics h_J, given by $J \leftrightarrow h_J = \{h_\alpha = (\delta_{\mathfrak{m}}, \alpha) : \alpha \in \Delta_M^+\}$, where $h_\alpha = h_J(E_\alpha, E_{-\alpha})$ are the components of the metric h_J with respect to the base $\{E_\alpha : \alpha \in \Delta_M\}$ of $\mathfrak{m}^{\mathbb{C}}$. The weight $\delta_{\mathfrak{m}} = (1/2)\sum_{\beta\in\Delta_M^+}\beta \in \sqrt{-1}\mathfrak{t}$ is called *Koszul form*. If we assume that M is defined by a subset $\Pi_M = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$, then the following relation holds: $2\delta_{\mathfrak{m}} = u_{i_1}\cdot\Lambda_{\alpha_{i_1}} + \cdots + u_{i_r}\cdot\Lambda_{\alpha_{i_r}}$. The positive integers $u_{i_1} > 0, \ldots, u_{i_r} > 0$ are called *Koszul numbers*.

Proposition 2. ([BHi], [Tak]) *Let $M = G^{\mathbb{C}}/U = G/H$ be a flag manifold defined as in Lemma 1. Then M admits a unique G-invariant Kähler-Einstein metric given by*

$$h_J = B|_{\mathfrak{m}_1} + 2\cdot B|_{\mathfrak{m}_2} + \cdots + N\cdot B|_{\mathfrak{m}_N}. \tag{13}$$

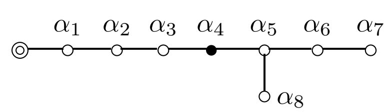
Proof. We give a short proof here since one is difficult to find it in the literature. By [BHi, Proposition 13.8], we know that M admits a unique G-invariant complex structure J, induced by the invariant ordering $\Delta_M^+ = \Delta^+/\Delta_H^+$ (we identify J with its conjugate $\bar{J}$ which is induced by the invariant ordering $\Delta_M^- = -\Delta_M^+$). The complex structure J is described by an $\mathrm{ad}(\mathfrak{h}^{\mathbb{C}})$-invariant endomorphism J_o on $\mathfrak{m}^{\mathbb{C}}$ with $J_o^2 = -\mathrm{Id}_{\mathfrak{m}^{\mathbb{C}}}$, explicitly determined by the formulae $J_oE_{\pm\alpha} = \pm\sqrt{-1}E_{\pm\alpha}$, for any $\alpha \in \Delta_M^+$. In view of Remark 2, M admits a unique G-inavriant Kähler-Einstein metric h_J compatible with J. Because $\Pi_M = \{\alpha_i : \mathrm{ht}(\alpha_i) = N\}$, (where i is fixed, $1 \le i \le \ell$), we have $\delta_{\mathfrak{m}} = u_i/2\cdot\Lambda_i$, where $u_i > 0$. From Lemma 1 it is $\mathfrak{m} = \bigoplus_{k=1}^N \mathfrak{m}_k$, thus the G-invariant metric h_J on M has the form (2), that is $h_J = \sum_{k=1}^N h_k\cdot B|_{\mathfrak{m}_k}$, with $(h_1, \ldots, h_N) \in \mathbb{R}_+^N$. Here we denote by h_k the components of the metric h_J on $\mathfrak{m}_k$, that means $h_k = h_J(E_\alpha, E_{-\alpha})$ where $\alpha \in \Delta^+(\alpha_i, k)$, for any $1 \le k \le N$, By applying Remark 2, we get $h_k = h_J(E_\alpha, E_{-\alpha}) = (\delta_{\mathfrak{m}}, \alpha)$, where $\alpha \in \Delta^+(\alpha_i, k)$. Because $(\Lambda_i, \alpha_i) = (\alpha_i, \alpha_i)/2$, it is

$$h_k = (\delta_{\mathfrak{m}}, \alpha) = (\frac{u_i}{2}\cdot\Lambda_i, m_1\alpha_1 + \cdots + k\alpha_i + \cdots + m_\ell\alpha_\ell) = (\frac{u_i}{2}\cdot\Lambda_i, k\alpha_i) = k\cdot u_i\cdot(\alpha_i, \alpha_i).$$

Since the simple root α_i is fixed, the number $u_i\cdot(\alpha_i, \alpha_i)$ is constant and indepedent of the integer k, for any $1 \le k \le N$. By normalizing the metric the proof is complete. $\square$

3. Homogeneous Einstein metrics on $E_8/U(1)\times SU(4)\times SU(5)$

3.1. **The flag manifold $\mathrm{E}_8\,/\,\mathrm{U}(1)\times\mathrm{SU}(4)\times\mathrm{SU}(5)$.** Let $G = \mathrm{E}_8$. A basis of simple roots for the root system of E_8 is given by $\Pi = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_7 = e_7 - e_8, \alpha_8 = e_6 + e_7 + e_8\}$, and $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$ (cf. [AA], [AC3]). We set $\Pi_M = \{\alpha_4\}$, thus $\Pi_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$. So we obtain the (extended) painted Dynkin diagram (the douple circle denotes the negative of $\widetilde{\alpha}$)



It defines the flag manifold $M = G/H = \mathrm{E}_8\,/\,\mathrm{U}(1)\times\mathrm{SU}(4)\times\mathrm{SU}(5)$. Let $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$ with respect to B. Because $\mathrm{ht}(\alpha_4) = 5$, from Lemma 1 it follows that $N = 5 = q$, that is $\mathfrak{m} = \mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3\oplus\mathfrak{m}_4\oplus\mathfrak{m}_5$. In this way we find a pair (Π, Π_0) for $\mathfrak{g} = \mathfrak{e}_8$, which has an irreducible decomposition

$$\mathfrak{g} = \mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\mathfrak{h}_2\oplus\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3\oplus\mathfrak{m}_4\oplus\mathfrak{m}_5 \tag{14}$$

as $\mathrm{Ad}(H)$-modules, where $\mathfrak{h}_0$ is the center of $\mathfrak{h}$ and $\mathfrak{h}_1 = \mathfrak{su}(4)$, $\mathfrak{h}_2 = \mathfrak{su}(5)$. Note that $d_0 = \dim\mathfrak{h}_0 = 1$, $d_1 = \dim\mathfrak{h}_1 = 15$ and $d_2 = \dim\mathfrak{h}_2 = 24$. Also, by applying the second part of Lemma 1 we obtain that $d_3 = \dim\mathfrak{m}_1 = 80$, $d_4 = \dim\mathfrak{m}_2 = 60$, $d_5 = \dim\mathfrak{m}_3 = 40$, $d_6 = \dim\mathfrak{m}_4 = 20$ and $d_7 = \dim\mathfrak{m}_5 = 8$.

Proposition 3. *In the decomposition* (14) we can take the ideal $\mathfrak{h}_2$ such that $[\mathfrak{h}_2, \mathfrak{m}_5] = \{0\}$.

Proof. We can assume that $\mathfrak{h}_2 \ne \{0\}$. Note that there is only a simple root $\alpha_{j_0} = \alpha_8$ with $(\alpha_{j_0}, \widetilde{\alpha}) \ne 0$ and thus we can take the ideal $\mathfrak{h}_2$ so that $\left[\mathfrak{h}_2^{\mathbb{C}}, E_{\widetilde{\alpha}}\right] = \{0\}$. Since $\mathfrak{n}_5 = [\mathfrak{h}^{\mathbb{C}}, E_{\widetilde{\alpha}}]$, we have that $\left[\mathfrak{h}_2^{\mathbb{C}}, \mathfrak{n}_5\right] = \left[\mathfrak{h}_2^{\mathbb{C}}, [\mathfrak{h}^{\mathbb{C}}, E_{\widetilde{\alpha}}]\right] \subset \left[\left[\mathfrak{h}_2^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}\right], E_{\widetilde{\alpha}}\right] + \left[\mathfrak{h}^{\mathbb{C}}, \left[\mathfrak{h}_2^{\mathbb{C}}, E_{\widetilde{\alpha}}\right]\right] = \{0\}$. By the definition of $\mathfrak{m}_5$, we get the result. $\square$

3.2. **Proof of Theorem 1.** Following the notation of §1, we consider left-invariant Riemannian metrics on the compact Lie group E_8, given by

$$\begin{aligned} <\,,\,> = {} & u_0\cdot B|_{\mathfrak{h}_0} + u_1\cdot B|_{\mathfrak{h}_1} + u_2\cdot B|_{\mathfrak{h}_2} \\ & + x_1\cdot B|_{\mathfrak{m}_1} + x_2\cdot B|_{\mathfrak{m}_2} + x_3\cdot B|_{\mathfrak{m}_3} + x_4\cdot B|_{\mathfrak{m}_4} + x_5\cdot B|_{\mathfrak{m}_5}, \end{aligned} \tag{15}$$

for positive real numbers $u_0, u_1, u_2, x_1, x_2, x_3, x_4, x_5$. Note that the left-invariant metric (15) is also $\mathrm{Ad}(H)$-invariant.

Lemma 2. *For a left invariant metric $<\ ,\ >$ on E_8 given by (15), the non-zero structure constants $\begin{bmatrix} k \\ ij \end{bmatrix}$ are the following (and their symmetries):*

$$\begin{bmatrix} 3 \\ 03 \end{bmatrix}, \begin{bmatrix} 4 \\ 04 \end{bmatrix}, \begin{bmatrix} 5 \\ 05 \end{bmatrix}, \begin{bmatrix} 6 \\ 06 \end{bmatrix}, \begin{bmatrix} 7 \\ 07 \end{bmatrix}, \begin{bmatrix} 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \end{bmatrix}, \begin{bmatrix} 4 \\ 14 \end{bmatrix}, \begin{bmatrix} 5 \\ 15 \end{bmatrix}, \begin{bmatrix} 6 \\ 16 \end{bmatrix}, \begin{bmatrix} 7 \\ 17 \end{bmatrix},$$

$$\begin{bmatrix} 2 \\ 22 \end{bmatrix}, \begin{bmatrix} 3 \\ 23 \end{bmatrix}, \begin{bmatrix} 4 \\ 24 \end{bmatrix}, \begin{bmatrix} 5 \\ 25 \end{bmatrix}, \begin{bmatrix} 6 \\ 26 \end{bmatrix}, \begin{bmatrix} 4 \\ 33 \end{bmatrix}, \begin{bmatrix} 5 \\ 34 \end{bmatrix}, \begin{bmatrix} 6 \\ 35 \end{bmatrix}, \begin{bmatrix} 7 \\ 36 \end{bmatrix}, \begin{bmatrix} 6 \\ 44 \end{bmatrix}, \begin{bmatrix} 7 \\ 45 \end{bmatrix}.$$

Proof. This fact follows from relations $[\mathfrak{h}_i, \mathfrak{h}_i] \subset \mathfrak{h}_i$, $[\mathfrak{h}_i, \mathfrak{h}_0] = \{0\}$ and the inclusions given by (12). We also mention that an important consequence of Proposition 3 is the following one: $\begin{bmatrix} 7 \\ 27 \end{bmatrix} = \begin{bmatrix} 7 \\ 72 \end{bmatrix} = \begin{bmatrix} 2 \\ 77 \end{bmatrix} = 0.$ □

Proposition 4. *The components of the Ricci tensor r for a left-invariant Riemannian metric $<\ ,\ >$ on E_8 defined by* (15)*, are given as follows:*

$$\left\{\begin{aligned}
r_0 &= \frac{u_0}{4\,{x_1}^2}\begin{bmatrix} 0 \\ 33 \end{bmatrix} + \frac{u_0}{4\,{x_2}^2}\begin{bmatrix} 0 \\ 44 \end{bmatrix} + \frac{u_0}{4\,{x_3}^2}\begin{bmatrix} 0 \\ 55 \end{bmatrix} + \frac{u_0}{4\,{x_4}^2}\begin{bmatrix} 0 \\ 66 \end{bmatrix} + \frac{u_0}{4\,{x_5}^2}\begin{bmatrix} 0 \\ 77 \end{bmatrix}\\
r_1 &= \frac{1}{4\,d_1\,u_1}\begin{bmatrix} 1 \\ 11 \end{bmatrix} + \frac{u_1}{4\,d_1\,{x_1}^2}\begin{bmatrix} 1 \\ 33 \end{bmatrix} + \frac{u_1}{4\,d_1\,{x_2}^2}\begin{bmatrix} 1 \\ 44 \end{bmatrix} + \frac{u_1}{4\,d_1\,{x_3}^2}\begin{bmatrix} 1 \\ 55 \end{bmatrix}\\
&\quad + \frac{u_1}{4\,d_1\,{x_4}^2}\begin{bmatrix} 1 \\ 66 \end{bmatrix} + \frac{u_1}{4\,d_1\,{x_5}^2}\begin{bmatrix} 1 \\ 77 \end{bmatrix}\\
r_2 &= \frac{1}{4\,d_2\,u_2}\begin{bmatrix} 2 \\ 22 \end{bmatrix} + \frac{u_2}{4\,d_2\,{x_1}^2}\begin{bmatrix} 2 \\ 33 \end{bmatrix} + \frac{u_2}{4\,d_1\,{x_2}^2}\begin{bmatrix} 2 \\ 44 \end{bmatrix} + \frac{u_2}{4\,d_1\,{x_3}^2}\begin{bmatrix} 2 \\ 55 \end{bmatrix} + \frac{u_2}{4\,d_1\,{x_4}^2}\begin{bmatrix} 2 \\ 66 \end{bmatrix}\\
r_3 &= \frac{1}{2x_1} - \frac{1}{2\,d_3}\begin{bmatrix} 4 \\ 33 \end{bmatrix}\frac{x_2}{{x_1}^2} + \frac{1}{2\,d_3}\begin{bmatrix} 3 \\ 45 \end{bmatrix}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right)\\
&\quad + \frac{1}{2\,d_3}\begin{bmatrix} 3 \\ 56 \end{bmatrix}\left(\frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3}\right) + \frac{1}{2\,d_3}\begin{bmatrix} 3 \\ 67 \end{bmatrix}\left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4}\right)\\
&\quad - \frac{1}{2\,d_3\,{x_1}^2}\left(u_0\begin{bmatrix} 0 \\ 33 \end{bmatrix} + u_1\begin{bmatrix} 1 \\ 33 \end{bmatrix} + u_2\begin{bmatrix} 2 \\ 33 \end{bmatrix}\right)\\
r_4 &= \frac{1}{2x_2} + \frac{1}{4\,d_4}\begin{bmatrix} 4 \\ 33 \end{bmatrix}\left(\frac{x_2}{{x_1}^2} - \frac{2}{x_2}\right) - \frac{1}{2\,d_4}\begin{bmatrix} 6 \\ 44 \end{bmatrix}\frac{x_4}{{x_2}^2}\\
&\quad + \frac{1}{2\,d_4}\begin{bmatrix} 4 \\ 35 \end{bmatrix}\left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_2x_1}\right) + \frac{1}{2\,d_4}\begin{bmatrix} 4 \\ 57 \end{bmatrix}\left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3}\right)\\
&\quad - \frac{1}{2\,d_4\,{x_2}^2}\left(u_0\begin{bmatrix} 0 \\ 44 \end{bmatrix} + u_1\begin{bmatrix} 1 \\ 44 \end{bmatrix} + u_2\begin{bmatrix} 2 \\ 44 \end{bmatrix}\right)\\
r_5 &= \frac{1}{2x_3} + \frac{1}{2\,d_5}\begin{bmatrix} 5 \\ 34 \end{bmatrix}\left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_3x_1} - \frac{x_1}{x_3x_2}\right) + \frac{1}{2\,d_5}\begin{bmatrix} 5 \\ 36 \end{bmatrix}\left(\frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3}\right)\\
&\quad + \frac{1}{2\,d_5}\begin{bmatrix} 5 \\ 47 \end{bmatrix}\left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_3x_2}\right) - \frac{1}{2\,d_5\,{x_3}^2}\left(u_0\begin{bmatrix} 0 \\ 55 \end{bmatrix} + u_1\begin{bmatrix} 1 \\ 55 \end{bmatrix} + u_2\begin{bmatrix} 2 \\ 55 \end{bmatrix}\right)\\
r_6 &= \frac{1}{2x_4} + \frac{1}{4\,d_6}\begin{bmatrix} 6 \\ 44 \end{bmatrix}\left(\frac{x_4}{{x_2}^2} - \frac{2}{x_4}\right) + \frac{1}{2\,d_6}\begin{bmatrix} 6 \\ 35 \end{bmatrix}\left(\frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4} - \frac{x_3}{x_4x_1}\right)\\
&\quad + \frac{1}{2\,d_6}\begin{bmatrix} 6 \\ 37 \end{bmatrix}\left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4}\right) - \frac{1}{2\,d_6\,{x_4}^2}\left(u_0\begin{bmatrix} 0 \\ 66 \end{bmatrix} + u_1\begin{bmatrix} 1 \\ 66 \end{bmatrix} + u_2\begin{bmatrix} 2 \\ 66 \end{bmatrix}\right)\\
r_7 &= \frac{1}{2x_5} + \frac{1}{2\,d_7}\begin{bmatrix} 7 \\ 45 \end{bmatrix}\left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5}\right)\\
&\quad + \frac{1}{2\,d_7}\begin{bmatrix} 7 \\ 36 \end{bmatrix}\left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5}\right) - \frac{1}{2\,d_7\,{x_5}^2}\left(u_0\begin{bmatrix} 0 \\ 77 \end{bmatrix} + u_1\begin{bmatrix} 1 \\ 77 \end{bmatrix}\right).
\end{aligned}\right. \tag{16}$$

Proof. This is an immediate application of Theorem 1, (1). Ofcourse we use the results of Lemma 2 and the relation $\sum_{i,j}\begin{bmatrix} j \\ ki \end{bmatrix} = d_k$. □

Now, an E_8-invariant Riemannian metric on $G/H = \mathrm{E}_8 / \mathrm{U}(1)\times\mathrm{SU}(4)\times\mathrm{SU}(5)$ is determined completely by an $\mathrm{Ad}(H)$-invariant inner product on the tangent space $\mathfrak{m} = T_oG/H$, which we will denote by $(\ ,\ )$. Because of the decomposition $\mathfrak{m} = \mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3\oplus\mathfrak{m}_4\oplus\mathfrak{m}_5$, it depends on five real positive parameters. In particular, and since the irrducible submodules $\mathfrak{m}_i$ are pairwise inequivalent for any $i = 1,\dots,5$, any G-invariant Riemannian metric on G/H will be expressed from relation (6), that means

$$(\ ,\ ) = x_1\cdot B|_{\mathfrak{m}_1} + x_2\cdot B|_{\mathfrak{m}_2} + x_3\cdot B|_{\mathfrak{m}_3} + x_4\cdot B|_{\mathfrak{m}_4} + x_5\cdot B|_{\mathfrak{m}_5},\quad (x_1,x_2,x_3,x_4,x_5)\in\mathbb{R}^5_+. \tag{17}$$

By applying a similar procedure like as Proposition 4 and by using Lemma 2, we also obtain that:

Proposition 5. *The components $\bar{r}_i$ of the Ricci tensor $\bar{r}$ for the G-invariant metric $(\ ,\ )$ on G/H defined by* (17)*, are given as follows*

$$\left\{\begin{aligned}
\bar{r}_1 &= \frac{1}{2x_1} - \frac{1}{2\,d_3}\begin{bmatrix}4\\33\end{bmatrix}\frac{x_2}{{x_1}^2} + \frac{1}{2\,d_3}\begin{bmatrix}3\\45\end{bmatrix}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right)\\
&\quad + \frac{1}{2\,d_3}\begin{bmatrix}3\\56\end{bmatrix}\left(\frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3}\right) + \frac{1}{2\,d_3}\begin{bmatrix}3\\67\end{bmatrix}\left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4}\right)\\
\bar{r}_2 &= \frac{1}{2x_2} + \frac{1}{4\,d_4}\begin{bmatrix}4\\33\end{bmatrix}\left(\frac{x_2}{{x_1}^2} - \frac{2}{x_2}\right) - \frac{1}{2\,d_4}\begin{bmatrix}6\\44\end{bmatrix}\frac{x_4}{{x_2}^2}\\
&\quad + \frac{1}{2\,d_4}\begin{bmatrix}4\\35\end{bmatrix}\left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_2x_1}\right) + \frac{1}{2\,d_4}\begin{bmatrix}4\\57\end{bmatrix}\left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3}\right)\\
\bar{r}_3 &= \frac{1}{2x_3} + \frac{1}{2\,d_5}\begin{bmatrix}5\\34\end{bmatrix}\left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_3x_1} - \frac{x_1}{x_3x_2}\right) + \frac{1}{2\,d_5}\begin{bmatrix}5\\36\end{bmatrix}\left(\frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3}\right)\\
&\quad + \frac{1}{2\,d_5}\begin{bmatrix}5\\47\end{bmatrix}\left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_3x_2}\right)\\
\bar{r}_4 &= \frac{1}{2x_4} + \frac{1}{4\,d_6}\begin{bmatrix}6\\44\end{bmatrix}\left(\frac{x_4}{{x_2}^2} - \frac{2}{x_4}\right) + \frac{1}{2\,d_6}\begin{bmatrix}6\\35\end{bmatrix}\left(\frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4} - \frac{x_3}{x_4x_1}\right)\\
&\quad + \frac{1}{2\,d_6}\begin{bmatrix}6\\37\end{bmatrix}\left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4}\right)\\
\bar{r}_5 &= \frac{1}{2x_5} + \frac{1}{2\,d_7}\begin{bmatrix}7\\45\end{bmatrix}\left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5}\right)\\
&\quad + \frac{1}{2\,d_7}\begin{bmatrix}7\\36\end{bmatrix}\left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5}\right).
\end{aligned}\right. \tag{18}$$

From Proposition 2, we known that the metric $B|_{\mathfrak{m}_1} + 2\cdot B|_{\mathfrak{m}_2} + 3\cdot B|_{\mathfrak{m}_3} + 4\cdot B|_{\mathfrak{m}_4} + 5\cdot B|_{\mathfrak{m}_5}$ is the unique Kähler-Einstein on G/H. By substituting these values in the system $\{\bar{r}_1 = \bar{r}_2 = \bar{r}_3 = \bar{r}_4 = \bar{r}_5\}$, we obtain

$$\begin{aligned}
\frac{1}{2} - \frac{1}{d_3}\left(\begin{bmatrix}4\\33\end{bmatrix} + \begin{bmatrix}5\\34\end{bmatrix} + \begin{bmatrix}6\\35\end{bmatrix} + \begin{bmatrix}7\\36\end{bmatrix}\right) &= \frac{1}{4} + \frac{1}{d_4}\left(\frac{1}{4}\begin{bmatrix}4\\33\end{bmatrix} - \frac{1}{2}\begin{bmatrix}5\\34\end{bmatrix} - \frac{1}{2}\begin{bmatrix}6\\44\end{bmatrix} - \frac{1}{2}\begin{bmatrix}7\\45\end{bmatrix}\right) =\\
\frac{1}{6} + \frac{1}{d_5}\left(\frac{1}{3}\begin{bmatrix}5\\34\end{bmatrix} - \frac{1}{3}\begin{bmatrix}6\\35\end{bmatrix} - \frac{1}{3}\begin{bmatrix}7\\45\end{bmatrix}\right) &= \frac{1}{8} + \frac{1}{d_6}\left(\frac{1}{4}\begin{bmatrix}6\\35\end{bmatrix} - \frac{1}{4}\begin{bmatrix}7\\36\end{bmatrix} + \frac{1}{8}\begin{bmatrix}6\\44\end{bmatrix}\right) = \frac{1}{10} + \frac{1}{d_7}\left(\frac{1}{5}\begin{bmatrix}7\\36\end{bmatrix} + \frac{1}{5}\begin{bmatrix}7\\45\end{bmatrix}\right).
\end{aligned} \tag{19}$$

3.2.1. **Comparison of left-invariant metrics.** From (19) we obtain a system with four equations and six unknowns, namely the triples $\begin{bmatrix}4\\33\end{bmatrix}$, $\begin{bmatrix}5\\34\end{bmatrix}$, $\begin{bmatrix}6\\35\end{bmatrix}$, $\begin{bmatrix}7\\36\end{bmatrix}$, $\begin{bmatrix}6\\44\end{bmatrix}$, and $\begin{bmatrix}7\\45\end{bmatrix}$ (the fact that G/H has six non-zero structures constants with respect to the decomposition (14), has been recently proved by the first author in terms of symmetric $\mathfrak{t}$-triples, see [Chr] but be aware of a different enumeration). In order to compute them explicitly, we need two more equations. In this direction, we will construct a new left-invariant metric $<<\ ,\ >>$ on $G = \mathrm{E}_8$, corresponding to a different decomposition of our Lie algebra $\mathfrak{g} = \mathfrak{e}_8$. By comparing this metric with the previous one $<\ ,\ >$, one can obtain crucial relations between the non-zero structure constants.

We put $\mathfrak{k} = \mathfrak{h}\oplus\mathfrak{m}_5$, $\mathfrak{k}_1 = \mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\mathfrak{m}_5$, $\mathfrak{n}_1 = \mathfrak{m}_1\oplus\mathfrak{m}_4$ and $\mathfrak{n}_2 = \mathfrak{m}_2\oplus\mathfrak{m}_3$. Then $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, and from Propositon 3, we see that $\mathfrak{k}_1$ is also a subalgebra of $\mathfrak{g}$. In particular it is $\mathfrak{k} = \mathfrak{k}_1\oplus\mathfrak{h}_2$, where $\mathfrak{h}_2 = \mathfrak{su}(5)$, and for dimensional reasons we also obtain that $\mathfrak{k}_1 = \mathfrak{su}(5)$. By using (12) we get that

$$[\mathfrak{n}_1,\ \mathfrak{n}_1]\subset\mathfrak{n}_2\oplus\mathfrak{k},\quad [\mathfrak{n}_1,\ \mathfrak{n}_2]\subset\mathfrak{n}_1\oplus\mathfrak{n}_2,\quad [\mathfrak{n}_2,\ \mathfrak{n}_2]\subset\mathfrak{n}_1\oplus\mathfrak{k}. \tag{20}$$

Thus, we obtain an irreducible decomposition $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2$ as $\mathrm{Ad}(K)$-modules, which are mutually non-equivalent (cf. [WZ1, p. 575]). Since $\mathfrak{h} \subset \mathfrak{k}$ we determine a fibration $G/H \to G/K$, given by $\mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SU}(5) \to \mathrm{E}_8 / \mathrm{SU}(5) \times \mathrm{SU}(5)$. The base space $G/K = \mathrm{E}_8 / \mathrm{SU}(5) \times \mathrm{SU}(5)$ has two isotropy summands, namely $\mathfrak{n}_1$ and $\mathfrak{n}_2$. We consider the following left-invariant metrics on $G = \mathrm{E}_8$ which are also $\mathrm{Ad}(K)$-invariant:

$$<< \ , \ >>= y_1 \cdot B|_{\mathfrak{k}_1} + y_2 \cdot B|_{\mathfrak{h}_2} + y_3 \cdot B|_{\mathfrak{n}_1} + y_4 \cdot B|_{\mathfrak{n}_2}, \quad (y_1, y_2, y_3, y_4) \in \mathbb{R}^4_+. \tag{21}$$

Next, we will use the notation $f_1 = \dim \mathfrak{k}_1 = 24$, $f_2 = \dim \mathfrak{h}_2 = 24$, $f_3 = \dim \mathfrak{n}_1 = 100$ and $f_4 = \dim \mathfrak{n}_2 = 100$.

Lemma 3. *For a left-invariant metric $<< \ , \ >>$ on E_8 given by (21), the non-zero structure constants are the following (and their symmetries):* $\left[\!\left[{1 \atop 11} \right]\!\right], \left[\!\left[{3 \atop 13} \right]\!\right], \left[\!\left[{4 \atop 14} \right]\!\right], \left[\!\left[{2 \atop 22} \right]\!\right], \left[\!\left[{3 \atop 23} \right]\!\right], \left[\!\left[{4 \atop 24} \right]\!\right], \left[\!\left[{4 \atop 33} \right]\!\right], \left[\!\left[{4 \atop 34} \right]\!\right]$.

Proof. This result follows from the decomposition $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{h}_2$ and the relations given in (20). Note that since $[\mathfrak{h}_2, \mathfrak{n}_1] \subset \mathfrak{n}_1$, and $\mathfrak{h}_2 \perp \mathfrak{k}_1$, it is $\left[\!\left[{1 \atop 23} \right]\!\right] = \left[\!\left[{2 \atop 13} \right]\!\right] = \left[\!\left[{3 \atop 12} \right]\!\right] = 0$. □

Proposition 6. *The components $\widetilde{r}_i$ of the Ricci tensor $\widetilde{r}$ of the left-invariant metric $<< \ , \ >>$ on E_8 defined by* (21)*, are given as follows:*

$$\begin{cases} \widetilde{r}_1 = \dfrac{1}{4\, f_1\, y_1} \left[\!\left[{1 \atop 11} \right]\!\right] + \dfrac{y_1}{4\, f_1\, {y_3}^2} \left[\!\left[{1 \atop 33} \right]\!\right] + \dfrac{y_1}{4\, f_1\, {y_4}^2} \left[\!\left[{1 \atop 44} \right]\!\right] \\ \widetilde{r}_2 = \dfrac{1}{4\, f_2\, y_2} \left[\!\left[{2 \atop 22} \right]\!\right] + \dfrac{y_2}{4\, f_2\, {y_3}^2} \left[\!\left[{2 \atop 33} \right]\!\right] + \dfrac{y_2}{4\, f_2\, {y_4}^2} \left[\!\left[{2 \atop 44} \right]\!\right] \\ \widetilde{r}_3 = \dfrac{1}{2y_3} + \dfrac{y_3}{4\, f_3\, {y_4}^2} \left[\!\left[{3 \atop 44} \right]\!\right] - \dfrac{1}{2\, f_3} \left(\dfrac{y_1}{{y_3}^2} \left[\!\left[{1 \atop 33} \right]\!\right] + \dfrac{y_2}{{y_3}^2} \left[\!\left[{2 \atop 33} \right]\!\right] + \dfrac{y_4}{{y_3}^2} \left[\!\left[{4 \atop 33} \right]\!\right] + \dfrac{1}{y_3} \left[\!\left[{4 \atop 34} \right]\!\right] \right) \\ \widetilde{r}_4 = \dfrac{1}{2y_4} + \dfrac{y_4}{4\, f_4\, {y_3}^2} \left[\!\left[{4 \atop 33} \right]\!\right] - \dfrac{1}{2\, f_4} \left(\dfrac{y_1}{{y_4}^2} \left[\!\left[{1 \atop 44} \right]\!\right] + \dfrac{y_2}{{y_4}^2} \left[\!\left[{2 \atop 44} \right]\!\right] + \dfrac{y_3}{{y_4}^2} \left[\!\left[{3 \atop 44} \right]\!\right] + \dfrac{1}{y_4} \left[\!\left[{3 \atop 34} \right]\!\right] \right). \end{cases} \tag{22}$$

Proof. We use the relations $\left[\!\left[{1 \atop 11} \right]\!\right] + \left[\!\left[{3 \atop 13} \right]\!\right] + \left[\!\left[{4 \atop 14} \right]\!\right] = f_1$, $\left[\!\left[{2 \atop 22} \right]\!\right] + \left[\!\left[{3 \atop 23} \right]\!\right] + \left[\!\left[{4 \atop 24} \right]\!\right] = f_2$ and Lemma 3. Then, the result is a straightforward application of Theorem 1, (1). □

Observe that equations (22) are obtained from equations (16) by setting $y_1 = u_0 = u_1 = x_5$, $y_2 = u_2$, $y_3 = x_1 = x_4$ and $y_4 = x_2 = x_3$. In fact, for these values the metrics $< \ , \ >$ and $<< \ , \ >>$ on E_8 coincide, so the components of the corresponding Ricci tensors must be equal. Thus, from relation $y_3 = x_1 = x_4$ it follows that $\widetilde{r}_3 = r_3 = r_6$, and relation $y_4 = x_2 = x_3$ implies that $\widetilde{r}_4 = r_4 = r_5$. By using the first relation, we obtain the following equations:

$$\left\{ \frac{1}{2\, f_3} \left[\!\left[{4 \atop 34} \right]\!\right] = \frac{1}{d_3} \begin{bmatrix} 5 \\ 34 \end{bmatrix} = \frac{1}{2d_6} \begin{bmatrix} 6 \\ 44 \end{bmatrix}, \quad \frac{1}{2\, f_3} \left[\!\left[{4 \atop 33} \right]\!\right] = \frac{1}{2d_3} \begin{bmatrix} 4 \\ 33 \end{bmatrix} + \frac{1}{2d_3} \begin{bmatrix} 6 \\ 35 \end{bmatrix} = \frac{1}{2d_6} \begin{bmatrix} 6 \\ 35 \end{bmatrix} \right\}. \tag{23}$$

So, from equations (19) and (23), we get a system of equations:

$$\left. \begin{array}{l|l} 60 - 4\begin{bmatrix} 4 \\ 33 \end{bmatrix} - \begin{bmatrix} 5 \\ 34 \end{bmatrix} - 3\begin{bmatrix} 6 \\ 35 \end{bmatrix} - 3\begin{bmatrix} 7 \\ 36 \end{bmatrix} + 2\begin{bmatrix} 6 \\ 44 \end{bmatrix} + \begin{bmatrix} 7 \\ 45 \end{bmatrix} = 0 & 4 + 2\begin{bmatrix} 6 \\ 35 \end{bmatrix} - 6\begin{bmatrix} 7 \\ 36 \end{bmatrix} + \begin{bmatrix} 6 \\ 44 \end{bmatrix} - 4\begin{bmatrix} 7 \\ 45 \end{bmatrix} = 0 \\ 20 + \begin{bmatrix} 4 \\ 33 \end{bmatrix} - 4\begin{bmatrix} 5 \\ 34 \end{bmatrix} + 2\begin{bmatrix} 6 \\ 35 \end{bmatrix} - 2\begin{bmatrix} 6 \\ 44 \end{bmatrix} = 0 & \begin{bmatrix} 4 \\ 33 \end{bmatrix} - 3\begin{bmatrix} 6 \\ 35 \end{bmatrix} = 0 \\ 20 + 4\begin{bmatrix} 4 \\ 33 \end{bmatrix} - 10\begin{bmatrix} 6 \\ 35 \end{bmatrix} + 6\begin{bmatrix} 7 \\ 36 \end{bmatrix} - 3\begin{bmatrix} 6 \\ 44 \end{bmatrix} - 4\begin{bmatrix} 7 \\ 45 \end{bmatrix} = 0 & \begin{bmatrix} 5 \\ 34 \end{bmatrix} - 2\begin{bmatrix} 6 \\ 44 \end{bmatrix} = 0. \end{array} \right\} \tag{24}$$

By solving system (24), we can obtain the explicit values of all non-zero triples.

Proposition 7. *For the G-invariant metric $(\ , \ )$ on $M = G/H = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SU}(5)$, the non-zero structure constants* $\begin{bmatrix} k \\ ij \end{bmatrix}$ *are given by* $\begin{bmatrix} 4 \\ 33 \end{bmatrix} = 12$, $\begin{bmatrix} 5 \\ 34 \end{bmatrix} = 8$, $\begin{bmatrix} 6 \\ 35 \end{bmatrix} = 4$, $\begin{bmatrix} 7 \\ 36 \end{bmatrix} = 4/3$, $\begin{bmatrix} 6 \\ 44 \end{bmatrix} = 4$, *and* $\begin{bmatrix} 7 \\ 45 \end{bmatrix} = 2$.

3.2.2. **Solutions of the homogeneous Einstein equation.** It is obvious that due to Proposition 7, the components $\bar{r}_i$ $(1 \leq i \leq 5)$ of the Ricci tensor are completely determined by equation (18). Thus, a G-invariant metric on G/H given by (17), is an Einstein metric, if and only if it is a positive real solution of the system equations $\Big\{\bar{r}_1 - \bar{r}_2 = 0, \quad \bar{r}_2 - \bar{r}_3 = 0, \quad \bar{r}_3 - \bar{r}_4 = 0, \quad \bar{r}_4 - \bar{r}_5 = 0\Big\}$. We normalize our equations by setting $x_1 = 1$. Then, we obtain the following system of polynomial equations:

$$\left\{\begin{array}{l}
f_1 = -15{x_2}^3x_3x_4x_5 - 14{x_2}^3x_4x_5 - 2{x_2}^3x_4 - 3{x_2}^2{x_3}^2x_5 - {x_2}^2x_3{x_4}^2 + 60{x_2}^2x_3x_4x_5 \\
+{x_2}^2x_3 - 3{x_2}^2{x_4}^2x_5 + 3{x_2}^2x_5 + 2x_2{x_3}^2x_4x_5 + 2x_2{x_3}^2x_4 - x_2{x_5}^2(x_2x_3 - 2x_4) \\
-48x_2x_3x_4x_5 + 14x_2x_4x_5 + 4x_3{x_4}^2x_5 = 0, \\
f_2 = 6{x_2}^3x_3x_4x_5 + 20{x_2}^3x_4x_5 + 5{x_2}^3x_4 - 6{x_2}^2{x_3}^2x_5 + 6{x_2}^2{x_4}^2x_5 - 60{x_2}^2x_4x_5 + 6{x_2}^2x_5 \\
-20x_2{x_3}^2x_4x_5 - 5x_2{x_3}^2x_4 + 48x_2x_3x_4x_5 + x_2x_4{x_5}^2 + 4x_2x_4x_5 - 4x_3{x_4}^2x_5 = 0, \\
f_3 = -12{x_2}^3x_4x_5 - 3{x_2}^3x_4 + 18{x_2}^2{x_3}^2x_5 - 4{x_2}^2x_3{x_4}^2 - 48{x_2}^2x_3x_5 + 4{x_2}^2x_3 \\
-18{x_2}^2{x_4}^2x_5 + 60{x_2}^2x_4x_5 + 6{x_2}^2x_5 + 12x_2{x_3}^2x_4x_5 + 3x_2{x_3}^2x_4 + x_2{x_5}^2(4x_2x_3 - 3x_4) \\
-12x_2x_4x_5 - 6x_3{x_4}^2x_5 = 0, \\
f_4 = 15{x_2}^3x_4 - 12{x_2}^2{x_3}^2x_5 + 14{x_2}^2x_3{x_4}^2 - 60{x_2}^2x_3x_4 + 48{x_2}^2x_3x_5 + 6{x_2}^2x_3 + 12{x_2}^2{x_4}^2x_5 \\
-12{x_2}^2x_5 + 15x_2{x_3}^2x_4 - x_2{x_5}^2(14x_2x_3 + 15x_4) + 6x_3{x_4}^2x_5 = 0
\end{array}\right. \tag{25}$$

To find non zero solutions of equations (25), we consider a polynomial ring $R = \mathbb{Q}[y, x_2, x_3, x_4, x_5]$ and an ideal I generated by $\{f_1,\ f_2,\ f_3,\ f_4,\ y\, x_2x_3x_4x_5 - 1\}$. We take a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5$ for a monomial ordering on R. Then a Gröbner basis for the ideal I contains the following polynomials:

$$(x_5 - 5)\, h_1(x_5),$$

where

$$\begin{array}{l}
h_1(x_5) = 47250369629121459608121860582842245939840000000000000000000000000000{x_5}^{80} \\
-4450272132623670759500175246684941765409639091200000000000000000000000{x_5}^{79} \\
-127927899803066910819626144206249935264735100081391520000000000000000000000{x_5}^{78} \\
+30984441136208124756805023456236711943789624683679786321327360000000000000000{x_5}^{77} \\
-94076670928013079147642943409263410662568734535795309001147102105440000000000000{x_5}^{76} \\
+653826144084182079268079901461204888968952996494327125389790471739985280000000000{x_5}^{75} \\
-197449903712626658670096070942650709064323241115231399455675397800949280620000000000{x_5}^{74} \\
+2954179066092568535575816419664189912798961603433559819729506406567466803680360000000{x_5}^{73} \\
-17622853307760226092030136830695230253063778779286983185241153806856431974468906000000{x_5}^{72} \\
-104623618546717019981045445808477609543759689966066277913643857876648516458458853200000{x_5}^{71} \\
+2731279642902266521491411566709779382366433660701184897058613249161440553160967615734000{x_5}^{70} \\
-21119606293500087129977290373008177567342904134793738827022405993991898712436895910061750{x_5}^{69} \\
+5667556243748995459386240850806848873153508627491406324575978221778369034430564358535255{x_5}^{68} \\
+369167960121909858399957670689430160227686551890079210737186311645689848231806296551657740{x_5}^{67} \\
-4800434272271033861684102369191505326677319053017937756263343751001429851641030408944274180{x_5}^{66} \\
+26694668655707097575912032278619692882088183884533875988758709456783168658330557773310678370{x_5}^{65} \\
-91393702195154280625869259712557851082690270108399432962846794216067922561183749783087383850{x_5}^{64} \\
+183768037597866907590678194291093892354428780236016242623045280050331760150997717166031980530{x_5}^{63} \\
-44345987442360490373269217506930653415547181057735851355097192971947241792693324159013041150{x_5}^{62} \\
-1229424539998751316910827056265027987825432734223697172278904607353976671235868139038632561370{x_5}^{61} \\
+5570446137699193588818314758597758492974147657857976123421310720208157593452080966091598994630{x_5}^{60} \\
-15821123080151667815115530707059183306450944580162830318658019644375031832817165315949415513640{x_5}^{59} \\
+34644727366300405259708245695215424352583487605273975428875037540988297558312647440307641569920{x_5}^{58} \\
-62506702585078796618685138074684381309153253904491978278065904633023396188606314939835437732830{x_5}^{57} \\
+95429576614023796268856038752848533197745616076395000425026028547262529540330260344573228091510{x_5}^{56} \\
-123800049098239044235196095236955615862388289184128163420696462266075514128299302481344075938840{x_5}^{55} \\
+134010221743576088993284463244065539349673944480149667773579038651458459427043701046439312729260{x_5}^{54} \\
-114548591627230324368437240031107292473584150629631953333195578293737653599757930068255710809410{x_5}^{53} \\
+65315542211839644662746416120499183235835782104450144009103079638298498253751624105854895706210{x_5}^{52} \\
-31664144533161575832348073162811438049746131915615733642608903277748685821166255390297951839400{x_5}^{51} \\
-45113602239479431686755397437754351143622061518045759353565500451008635025487168203331347526940{x_5}^{50} \\
+60871156883295700723384351203738117165452360428785992287440177045040893690837193626199681835550{x_5}^{49} \\
-46758932502669186802024531791246105785910318603998977789256656001618142324381664832449854318700{x_5}^{48} \\
+21434294453410317863892655494343914222067273830495940328932716473723089865147929378704448155710{x_5}^{47} \\
-23125304503094779111264403751282842157224193469540154492693355969504874716445700428339820453300{x_5}^{46} \\
-53230182694915830072039263239444962764639840463394447677303907368769169567525896527624751389100{x_5}^{45} \\
+55208249415800490041034565736174687250402447334237248897195468725679491695230260224259731893700{x_5}^{44} \\
-35485830031557517663611611401771052903915662188774305043536660421820740829300009392981399074800{x_5}^{43}
\end{array}$$

$$
\begin{aligned}
&+1849784974820480105095148375922808958446648207368901466965133581618407627724999786430512279 00{x_5}^{42}\\
&-754085921495897323276568577004387458191676481301432659604032396423172622740053362241241856 41{x_5}^{41}\\
&+105310023850078697613894380602709264284686205451436363024325695990487693854466633989852540 01{x_5}^{40}\\
&+181870172255500376909273974286158589310208860949283867763807157174529618560638612863315431 50{x_5}^{39}\\
&-206644174700331897370846658463374277649520801731981127000889300646943656171835327862751427 68{x_5}^{38}\\
&+119160559708088549228996443073317772535523869861561005620128511923799386882396901089230608 87{x_5}^{37}\\
&-370855532120533134842942653673109892345421857451909425940635501801252910858379990569262617 5{x_5}^{36}\\
&-964808057390569762946012399928135992542695088463384189233661858886548098235502962583883 34{x_5}^{35}\\
&+779521396809818762783053712322840780796118975861466278102820782746103634907574259253279 890{x_5}^{34}\\
&-442928305317101231218698431501006044423392364446878989062913546917526691594291366407500 029{x_5}^{33}\\
&+132177537460041732222335574951671464735043143702524016111303113595306993873098919838904 985{x_5}^{32}\\
&-161141475772004603841996353675578012474441386489807951216327299688073993952802170140956 45{x_5}^{31}\\
&-384952687994868987213095535269692279622433159877945191386949638011640242365435548214168 5{x_5}^{30}\\
&+235161056483650091126749543625130100146170456203654734348190243446870149463602863080182 5{x_5}^{29}\\
&-712516460616616751913578501294263306534891480500617074968174235211114694431876638085575{x_5}^{28}\\
&+282452075662793320138187529511595841424627288321648629298768091107556621045836294535500{x_5}^{27}\\
&-138374750045771562844828515941070490939815457547440766945879714422270211889392119468000{x_5}^{26}\\
&+50124765380811508159567577887326533840520730559017278454760109996379339300985947734875{x_5}^{25}\\
&-10762356490866992628456505828708095865331486302712083316693729128231697768588472709375{x_5}^{24}\\
&+770995102742362625030991844772593954271351734760437853063704471596407378827318100000{x_5}^{23}\\
&+238672534050707994950682405938563685589475241026765736303369800772306441196663718750{x_5}^{22}\\
&-77015369860349488230676579817358388095252750079341229113657799438981492710341796875{x_5}^{21}\\
&+16203407364044735966231105435072556199261185091779727549069469576750633917154296875{x_5}^{20}\\
&-7776172850621403205385230811141442027882804598089143631163075780822473346652343750{x_5}^{19}\\
&+3517442589328862894719995442588428798753284776977135499703475875542210752011718750{x_5}^{18}\\
&-931918476193213037106761643540495870645939634042925070304993345925047085888671875{x_5}^{17}\\
&+130192292265798214494340656423328372967173237511484789254427003490699178466796875{x_5}^{16}\\
&-4556905640331850011402935119725159001214289709469798473424494855591307373046875{x_5}^{15}\\
&-9500407036402926682898734997636942064251029306658760434514752293076843261718 75{x_5}^{14}\\
&-347153014704255595832633502840904724714534626503734418852910994651184082031 25{x_5}^{13}\\
&+343778203598032765977079413997677088544371891853083166911763419395446777343 75{x_5}^{12}\\
&+294091362421033343366544009941357927003366923843725365386724944152832031250 0{x_5}^{11}\\
&-236310210771511826488728108033008080539349857151991195021726939392089843750 0{x_5}^{10}\\
&+2285416774831245042671111661633091154565410434706808975874598121643066406 25{x_5}^{9}\\
&+6898535613919469568260400817112052062091344278689689488240957260131835937 5{x_5}^{8}\\
&-2196990136301435439979845790333017648816522849148073463676929473876953125 0{x_5}^{7}\\
&+2145911716144516228364784597830820394965441727377596086263656616210937500{x_5}^{6}\\
&-1358716205625614110795214995721268845734304446270008087158203125000000{x_5}^{5}\\
&-113579338554211660050349064914773078472948595595288276672363281250000 00{x_5}^{4}\\
&+7380993932820199496090852212620704634399632835388183593750000000000000{x_5}^{3}\\
&-152886115741814936180194597692737011995334625244140625000000000000000 0{x_5}^{2}\\
&-59644112767739370853341375818406829833984375000000000000000000000000x_5\\
&+21965642106656068055461849306869506835937500000000000000000000000000,
\end{aligned}
$$

and polynomials of the form

$$b_2x_2 + v_2(x_5), \quad b_3x_3 + v_3(x_5), \quad b_4x_4 + v_4(x_5) \tag{26}$$

where b_2, b_3, b_4 are integers and $v_2(x_5), v_3(x_5), v_4(x_5)$ are polynomials of x_5 with degree 80 of integer coefficients.

For the case when $x_5 - 5 = 0$, we consider ideals I_1 of the polynomial ring $R = \mathbb{Q}[y, x_2, x_3, x_4, x_5]$ generated by

$$\{f_1,\ f_2,\ f_3,\ f_4,\ y,\ x_2x_3x_4x_5 - 1, x_5 - 5\}.$$

Then, by taking a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5$ for a monomial ordering on R, we obtain a Gröbner basis for the ideals I_1 that contains polynomials

$$\{x_2 - 2, x_3 - 3, x_4 - 4, x_5 - 5\}.$$

This solution corresponds to the Kähler Einstein metric.

For the case $h_1(x_5) = 0$, we see that there are 18 positive solutions for x_5. After substituting these values in the equations $b_2x_2 + v_2(x_5) = 0$, $b_3x_3 + v_3(x_5) = 0$,$b_4x_4 + v_4(x_5) = 0$, we see that there are 5 cases that all values for x_2, x_3 and x_4 are positive.

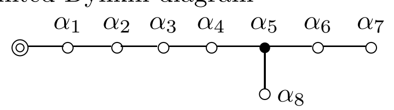
Thus we get :

Theorem A. *The generalized flag manifold* $M = G/H = \mathrm{E}_8 \,/\, \mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SU}(5)$ *admits (up to a scale) precisely five non-Kähler* E_8*-invariant Einstein metrics. These* E_8*-invariant Einstein metrics* $g = (x_1, x_2, x_3, x_4, x_5)$ *are given approximately by*

$$\begin{array}{lllll}
(1)\ x_1 = 1, & x_2 \approx 1.0213742, & x_3 \approx 0.54600746, & x_4 \approx 1.0535169, & x_5 \approx 1.1087938, \\
(2)\ x_1 = 1, & x_2 \approx 1.0373227, & x_3 \approx 1.0471761, & x_4 \approx 1.0308150, & x_5 \approx 0.29861996, \\
(3)\ x_1 = 1, & x_2 \approx 0.59978523, & x_3 \approx 1.0837088, & x_4 \approx 0.90182312, & x_5 \approx 1.2229122, \\
(4)\ x_1 = 1, & x_2 \approx 0.72071315, & x_3 \approx 1.0254588, & x_4 \approx 0.47523403, & x_5 \approx 1.0709463, \\
(5)\ x_1 = 1, & x_2 \approx 1.0829413, & x_3 \approx 1.0408835, & x_4 \approx 0.53261506, & x_5 \approx 1.1035115.
\end{array}$$

4. Homogeneous Einstein metrics on $\mathrm{E}_8 \,/\, \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$

4.1. **The flag manifold** $\mathbf{E_8 \,/\, U(1) \times SU(2) \times SU(3) \times SU(5)}$**.** We will exam now the case (F). We consider again the Lie group $G = \mathrm{E}_8$ and we set $\Pi_M = \{\alpha_5\}$, thus $\Pi_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8\}$. This choice gives rise to the following (extended) painted Dynkin diagram



It defines the flag manifold $M = G/H = \mathrm{E}_8 \,/\, \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$. From Lemma 1 and since we have $\mathrm{ht}(\alpha_5) = 6$, it follows that $N = 6 = q$, that is $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$. Thus we can find a pair (Π, Π_0) for $\mathfrak{g} = \mathfrak{e}_8$, which has an irreducible decomposition

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6 \tag{27}$$

as $\mathrm{Ad}(H)$-modules, where $\mathfrak{h}_0$ is the center of $\mathfrak{h}$ and $\mathfrak{h}_1 = \mathfrak{su}(2)$, $\mathfrak{h}_2 = \mathfrak{su}(3)$, $\mathfrak{h}_3 = \mathfrak{su}(5)$. Note that $d_0 = \dim \mathfrak{h}_0 = 1$, $d_1 = \dim \mathfrak{h}_1 = 3$, $d_2 = \dim \mathfrak{h}_2 = 8$ and $d_3 = \dim \mathfrak{h}_3 = 24$. Also from Lemma 1, we obtain thet $d_4 = \dim \mathfrak{m}_1 = 60$, $d_5 = \dim \mathfrak{m}_2 = 60$, $d_6 = \dim \mathfrak{m}_3 = 40$, $d_7 = \dim \mathfrak{m}_4 = 30$, $d_8 = \dim \mathfrak{m}_5 = 12$ and $d_9 = \dim \mathfrak{m}_6 = 10$.

Proposition 8. *In the decomposition* (2) we can take the ideals $\mathfrak{h}_1$, and $\mathfrak{h}_2$, such that $[\mathfrak{h}_1, \mathfrak{m}_6] = [\mathfrak{h}_2, \mathfrak{m}_6] = \{0\}$.

Proof. Since $\mathfrak{h}_1 = \mathfrak{su}(2)$, and $\mathfrak{h}_2 = \mathfrak{su}(3)$, we can assume that $\mathfrak{h}_1 \neq \{0\}$ and $\mathfrak{h}_2 \neq \{0\}$. Note that there is only a simple root $\alpha_{j_0} = \alpha_8$ with $(\alpha_{j_0}, \widetilde{\alpha}) \neq 0$ and thus we can take the ideals $\mathfrak{h}_1$ and $\mathfrak{h}_2$ such that $\left[\mathfrak{h}_1^{\mathbb{C}}, E_{\widetilde{\alpha}}\right] = \left[\mathfrak{h}_2^{\mathbb{C}}, E_{\widetilde{\alpha}}\right] = \{0\}$. Since $\mathfrak{n}_6 = [\mathfrak{h}^{\mathbb{C}}, E_{\widetilde{\alpha}}]$, we have that $\left[\mathfrak{h}_1^{\mathbb{C}}, \mathfrak{n}_6\right] = \left[\mathfrak{h}_1^{\mathbb{C}}, [\mathfrak{h}^{\mathbb{C}}, E_{\widetilde{\alpha}}]\right] \subset \left[\left[\mathfrak{h}_1^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}\right], E_{\widetilde{\alpha}}\right] + \left[\mathfrak{h}^{\mathbb{C}}, \left[\mathfrak{h}_1^{\mathbb{C}}, E_{\widetilde{\alpha}}\right]\right] = \{0\}$. By the definition of $\mathfrak{m}_6$, we get the result. Similar for $\mathfrak{h}_2$. □

4.2. **The construction of the homogeneous Einstein equation.** Following the notation of §1, next we consider left-invariant Riemannian metrics on the compact Lie group E_8, given by

$$\begin{aligned} <\ ,\ > = & \ u_0 \cdot B|_{\mathfrak{h}_0} + u_1 \cdot B|_{\mathfrak{h}_1} + u_2 \cdot B|_{\mathfrak{h}_2} + u_3 \cdot B|_{\mathfrak{h}_3} \\ & + x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2} + x_3 \cdot B|_{\mathfrak{m}_3} + x_4 \cdot B|_{\mathfrak{m}_4} + x_5 \cdot B|_{\mathfrak{m}_5} + x_6 \cdot B|_{\mathfrak{m}_6}, \end{aligned} \tag{28}$$

for some positive real numbers $u_0, u_1, \ldots, u_3, x_1, x_2, \ldots, x_6$. Note that a metric (28) is also $\mathrm{Ad}(H)$-invariant.

Lemma 4. *For a left invariant metric* $<\ ,\ >$ *on* E_8 *given by (28), the non-zero structure constants* $\begin{bmatrix} k \\ ij \end{bmatrix}$ *are the following (and their symmetries):*

$$\begin{bmatrix} 4 \\ 04 \end{bmatrix}, \begin{bmatrix} 5 \\ 05 \end{bmatrix}, \begin{bmatrix} 6 \\ 06 \end{bmatrix}, \begin{bmatrix} 7 \\ 07 \end{bmatrix}, \begin{bmatrix} 8 \\ 08 \end{bmatrix}, \begin{bmatrix} 9 \\ 09 \end{bmatrix}, \begin{bmatrix} 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 4 \\ 14 \end{bmatrix}, \begin{bmatrix} 5 \\ 15 \end{bmatrix}, \begin{bmatrix} 6 \\ 16 \end{bmatrix}, \begin{bmatrix} 7 \\ 17 \end{bmatrix}, \begin{bmatrix} 8 \\ 18 \end{bmatrix}, \begin{bmatrix} 2 \\ 22 \end{bmatrix}, \begin{bmatrix} 4 \\ 24 \end{bmatrix}, \begin{bmatrix} 5 \\ 25 \end{bmatrix}, \begin{bmatrix} 6 \\ 26 \end{bmatrix}, \begin{bmatrix} 7 \\ 27 \end{bmatrix},$$
$$\begin{bmatrix} 8 \\ 28 \end{bmatrix}, \begin{bmatrix} 3 \\ 33 \end{bmatrix}, \begin{bmatrix} 4 \\ 34 \end{bmatrix}, \begin{bmatrix} 5 \\ 35 \end{bmatrix}, \begin{bmatrix} 6 \\ 36 \end{bmatrix}, \begin{bmatrix} 7 \\ 37 \end{bmatrix}, \begin{bmatrix} 8 \\ 38 \end{bmatrix}, \begin{bmatrix} 9 \\ 39 \end{bmatrix}, \begin{bmatrix} 5 \\ 44 \end{bmatrix}, \begin{bmatrix} 6 \\ 45 \end{bmatrix}, \begin{bmatrix} 7 \\ 46 \end{bmatrix}, \begin{bmatrix} 8 \\ 47 \end{bmatrix}, \begin{bmatrix} 9 \\ 48 \end{bmatrix}, \begin{bmatrix} 7 \\ 55 \end{bmatrix}, \begin{bmatrix} 8 \\ 56 \end{bmatrix}, \begin{bmatrix} 9 \\ 57 \end{bmatrix}, \begin{bmatrix} 9 \\ 66 \end{bmatrix}.$$

Proof. We use the relations $[\mathfrak{h}_i, \mathfrak{h}_i] \subset \mathfrak{h}_i$, $[\mathfrak{h}_i, \mathfrak{h}_0] = \{0\}$ and the inclusions arising by applying (12). From Proposition 8 we also obtain that $\begin{bmatrix} 9 \\ 19 \end{bmatrix} = \begin{bmatrix} 9 \\ 91 \end{bmatrix} = \begin{bmatrix} 1 \\ 99 \end{bmatrix} = 0$ and $\begin{bmatrix} 9 \\ 29 \end{bmatrix} = \begin{bmatrix} 9 \\ 92 \end{bmatrix} = \begin{bmatrix} 2 \\ 99 \end{bmatrix} = 0$. □

Proposition 9. *The components of the Ricci tensor r for the left-invariant Riemannian metric $<\ ,\ >$ on* E_8 *defined by (28), are given as follows:*

$$
\left\{
\begin{aligned}
r_0 &= \frac{u_0}{4\,{x_1}^2}\begin{bmatrix}0\\44\end{bmatrix} + \frac{u_0}{4\,{x_2}^2}\begin{bmatrix}0\\55\end{bmatrix} + \frac{u_0}{4\,{x_3}^2}\begin{bmatrix}0\\66\end{bmatrix} + \frac{u_0}{4\,{x_4}^2}\begin{bmatrix}0\\77\end{bmatrix} + \frac{u_0}{4\,{x_5}^2}\begin{bmatrix}0\\88\end{bmatrix} + \frac{u_0}{4\,{x_6}^2}\begin{bmatrix}0\\99\end{bmatrix}\\
r_1 &= \frac{1}{4\,d_1\,u_1}\begin{bmatrix}1\\11\end{bmatrix} + \frac{u_1}{4\,d_1\,{x_1}^2}\begin{bmatrix}1\\44\end{bmatrix} + \frac{u_1}{4\,d_1\,{x_2}^2}\begin{bmatrix}1\\55\end{bmatrix} + \frac{u_1}{4\,d_1\,{x_3}^2}\begin{bmatrix}1\\66\end{bmatrix} + \frac{u_1}{4\,d_1\,{x_4}^2}\begin{bmatrix}1\\77\end{bmatrix} + \frac{u_1}{4\,d_1\,{x_5}^2}\begin{bmatrix}1\\88\end{bmatrix}\\
r_2 &= \frac{1}{4\,d_2\,u_2}\begin{bmatrix}2\\22\end{bmatrix} + \frac{u_2}{4\,d_2\,{x_1}^2}\begin{bmatrix}2\\44\end{bmatrix} + \frac{u_2}{4\,d_2\,{x_2}^2}\begin{bmatrix}2\\55\end{bmatrix} + \frac{u_2}{4\,d_2\,{x_3}^2}\begin{bmatrix}2\\66\end{bmatrix} + \frac{u_2}{4\,d_2\,{x_4}^2}\begin{bmatrix}2\\77\end{bmatrix} + \frac{u_2}{4\,d_2\,{x_5}^2}\begin{bmatrix}2\\88\end{bmatrix}\\
r_3 &= \frac{1}{4\,d_3\,u_3}\begin{bmatrix}3\\33\end{bmatrix} + \frac{u_3}{4\,d_3\,{x_1}^2}\begin{bmatrix}3\\44\end{bmatrix} + \frac{u_3}{4\,d_3\,{x_2}^2}\begin{bmatrix}3\\55\end{bmatrix} + \frac{u_3}{4\,d_3\,{x_3}^2}\begin{bmatrix}3\\66\end{bmatrix} + \frac{u_3}{4\,d_3\,{x_4}^2}\begin{bmatrix}3\\77\end{bmatrix} + \frac{u_3}{4\,d_3\,{x_5}^2}\begin{bmatrix}3\\88\end{bmatrix}\\
&\quad + \frac{u_3}{4\,d_3\,{x_6}^2}\begin{bmatrix}3\\99\end{bmatrix}\\
r_4 &= \frac{1}{2x_1} - \frac{1}{2\,d_4}\begin{bmatrix}5\\44\end{bmatrix}\frac{x_2}{{x_1}^2} + \frac{1}{2\,d_4}\begin{bmatrix}4\\56\end{bmatrix}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right) + \frac{1}{2\,d_4}\begin{bmatrix}4\\67\end{bmatrix}\left(\frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3}\right)\\
&\quad + \frac{1}{2\,d_4}\begin{bmatrix}4\\78\end{bmatrix}\left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4}\right) + \frac{1}{2\,d_4}\begin{bmatrix}4\\89\end{bmatrix}\left(\frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6} - \frac{x_6}{x_1x_5}\right)\\
&\quad - \frac{1}{2\,d_4\,{x_1}^2}\left(u_0\begin{bmatrix}0\\44\end{bmatrix} + u_1\begin{bmatrix}1\\44\end{bmatrix} + u_2\begin{bmatrix}2\\44\end{bmatrix} + u_3\begin{bmatrix}3\\44\end{bmatrix}\right)\\
r_5 &= \frac{1}{2x_2} + \frac{1}{4\,d_5}\begin{bmatrix}5\\44\end{bmatrix}\left(\frac{x_2}{{x_1}^2} - \frac{2}{x_2}\right) - \frac{1}{2\,d_5}\begin{bmatrix}7\\55\end{bmatrix}\frac{x_4}{{x_2}^2} + \frac{1}{2\,d_5}\begin{bmatrix}5\\46\end{bmatrix}\left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_2x_1}\right)\\
&\quad + \frac{1}{2\,d_5}\begin{bmatrix}5\\68\end{bmatrix}\left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3}\right) + \frac{1}{2\,d_5}\begin{bmatrix}5\\79\end{bmatrix}\left(\frac{x_2}{x_4x_6} - \frac{x_4}{x_2x_6} - \frac{x_6}{x_2x_4}\right)\\
&\quad - \frac{1}{2\,d_5\,{x_2}^2}\left(u_0\begin{bmatrix}0\\55\end{bmatrix} + u_1\begin{bmatrix}1\\55\end{bmatrix} + u_2\begin{bmatrix}2\\55\end{bmatrix} + u_3\begin{bmatrix}3\\55\end{bmatrix}\right)\\
r_6 &= \frac{1}{2x_3} - \frac{1}{2\,d_6}\begin{bmatrix}9\\66\end{bmatrix}\frac{x_6}{{x_3}^2} + \frac{1}{2\,d_6}\begin{bmatrix}6\\45\end{bmatrix}\left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_3x_1} - \frac{x_1}{x_3x_2}\right) + \frac{1}{2\,d_6}\begin{bmatrix}6\\47\end{bmatrix}\left(\frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3}\right)\\
&\quad + \frac{1}{2\,d_6}\begin{bmatrix}6\\58\end{bmatrix}\left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_3x_2}\right) - \frac{1}{2\,d_6\,{x_3}^2}\left(u_0\begin{bmatrix}0\\66\end{bmatrix} + u_1\begin{bmatrix}1\\66\end{bmatrix} + u_2\begin{bmatrix}2\\66\end{bmatrix} + u_3\begin{bmatrix}3\\66\end{bmatrix}\right)\\
r_7 &= \frac{1}{2x_4} + \frac{1}{4\,d_7}\begin{bmatrix}7\\55\end{bmatrix}\left(\frac{x_4}{{x_2}^2} - \frac{2}{x_4}\right) + \frac{1}{2\,d_7}\begin{bmatrix}7\\46\end{bmatrix}\left(\frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4} - \frac{x_3}{x_4x_1}\right)\\
&\quad + \frac{1}{2\,d_7}\begin{bmatrix}7\\48\end{bmatrix}\left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4}\right) + \frac{1}{2\,d_7}\begin{bmatrix}7\\59\end{bmatrix}\left(\frac{x_4}{x_2x_6} - \frac{x_2}{x_4x_6} - \frac{x_6}{x_2x_4}\right)\\
&\quad - \frac{1}{2\,d_7\,{x_4}^2}\left(u_0\begin{bmatrix}0\\77\end{bmatrix} + u_1\begin{bmatrix}1\\77\end{bmatrix} + u_2\begin{bmatrix}2\\77\end{bmatrix} + u_3\begin{bmatrix}3\\77\end{bmatrix}\right)\\
r_8 &= \frac{1}{2x_5} + \frac{1}{2\,d_8}\begin{bmatrix}8\\47\end{bmatrix}\left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5}\right) + \frac{1}{2\,d_8}\begin{bmatrix}8\\56\end{bmatrix}\left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5}\right)\\
&\quad + \frac{1}{2\,d_8}\begin{bmatrix}8\\49\end{bmatrix}\left(\frac{x_5}{x_1x_6} - \frac{x_1}{x_5x_6} - \frac{x_6}{x_1x_5}\right) - \frac{1}{2\,d_8\,{x_5}^2}\left(u_0\begin{bmatrix}0\\88\end{bmatrix} + u_1\begin{bmatrix}1\\88\end{bmatrix} + u_2\begin{bmatrix}2\\88\end{bmatrix} + u_3\begin{bmatrix}3\\88\end{bmatrix}\right)\\
r_9 &= \frac{1}{2x_6} + \frac{1}{4\,d_9}\begin{bmatrix}9\\66\end{bmatrix}\left(\frac{x_6}{{x_3}^2} - \frac{2}{x_6}\right) + \frac{1}{2\,d_9}\begin{bmatrix}9\\48\end{bmatrix}\left(\frac{x_6}{x_1x_5} - \frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6}\right)\\
&\quad + \frac{1}{2\,d_9}\begin{bmatrix}9\\57\end{bmatrix}\left(\frac{x_6}{x_2x_4} - \frac{x_2}{x_4x_6} - \frac{x_4}{x_2x_6}\right) - \frac{1}{2\,d_9\,{x_6}^2}\left(u_0\begin{bmatrix}0\\99\end{bmatrix} + u_3\begin{bmatrix}3\\99\end{bmatrix}\right).
\end{aligned}
\right.
\tag{29}
$$

Proof. We use Lemma 4 and relation $\sum_{i,j} \begin{bmatrix} j \\ ki \end{bmatrix} = d_k$. Next we apply Theorem 1, (1). □

Because the irreducible submodules $\mathfrak{m}_i$ $(i = 1, \ldots, 6)$ in the decomposition (27) are pairwise inequivalent, any E_8-invariant Riemannian metric has the form of (6), that means

$$(\ , \ ) = x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2} + x_3 \cdot B|_{\mathfrak{m}_3} + x_4 \cdot B|_{\mathfrak{m}_4} + x_5 \cdot B|_{\mathfrak{m}_5} + x_6 \cdot B|_{\mathfrak{m}_6}, \quad (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6_+. \quad (30)$$

Proposition 10. *The components $\bar{r}_i$ of the Ricci tensor $\bar{r}$ for the G-invariant metric $(\ , \ )$ on $G/H = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$ defined by (30), are given as follows:*

$$\left\{ \begin{aligned}
\bar{r}_1 &= \frac{1}{2x_1} - \frac{1}{2\,d_4}\begin{bmatrix} 5 \\ 44 \end{bmatrix}\frac{x_2}{{x_1}^2} + \frac{1}{2\,d_4}\begin{bmatrix} 4 \\ 56 \end{bmatrix}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right) + \frac{1}{2\,d_4}\begin{bmatrix} 4 \\ 67 \end{bmatrix}\left(\frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3}\right) \\
&\quad + \frac{1}{2\,d_4}\begin{bmatrix} 4 \\ 78 \end{bmatrix}\left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4}\right) + \frac{1}{2\,d_4}\begin{bmatrix} 4 \\ 89 \end{bmatrix}\left(\frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6} - \frac{x_6}{x_1x_5}\right) \\
\bar{r}_2 &= \frac{1}{2x_2} + \frac{1}{4\,d_5}\begin{bmatrix} 5 \\ 44 \end{bmatrix}\left(\frac{x_2}{{x_1}^2} - \frac{2}{x_2}\right) - \frac{1}{2\,d_5}\begin{bmatrix} 7 \\ 55 \end{bmatrix}\frac{x_4}{{x_2}^2} + \frac{1}{2\,d_5}\begin{bmatrix} 5 \\ 46 \end{bmatrix}\left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_2x_1}\right) \\
&\quad + \frac{1}{2\,d_5}\begin{bmatrix} 5 \\ 68 \end{bmatrix}\left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3}\right) + \frac{1}{2\,d_5}\begin{bmatrix} 5 \\ 79 \end{bmatrix}\left(\frac{x_2}{x_4x_6} - \frac{x_4}{x_2x_6} - \frac{x_6}{x_2x_4}\right) \\
\bar{r}_3 &= \frac{1}{2x_3} - \frac{1}{2\,d_6}\begin{bmatrix} 9 \\ 66 \end{bmatrix}\frac{x_6}{{x_3}^2} + \frac{1}{2\,d_6}\begin{bmatrix} 6 \\ 45 \end{bmatrix}\left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_3x_1} - \frac{x_1}{x_3x_2}\right) + \frac{1}{2\,d_6}\begin{bmatrix} 6 \\ 47 \end{bmatrix}\left(\frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3}\right) \\
&\quad + \frac{1}{2\,d_6}\begin{bmatrix} 6 \\ 58 \end{bmatrix}\left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_3x_2}\right) \\
\bar{r}_4 &= \frac{1}{2x_4} + \frac{1}{4\,d_7}\begin{bmatrix} 7 \\ 55 \end{bmatrix}\left(\frac{x_4}{{x_2}^2} - \frac{2}{x_4}\right) + \frac{1}{2\,d_7}\begin{bmatrix} 7 \\ 46 \end{bmatrix}\left(\frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4} - \frac{x_3}{x_4x_1}\right) \\
&\quad + \frac{1}{2\,d_7}\begin{bmatrix} 7 \\ 48 \end{bmatrix}\left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4}\right) + \frac{1}{2\,d_7}\begin{bmatrix} 7 \\ 59 \end{bmatrix}\left(\frac{x_4}{x_2x_6} - \frac{x_2}{x_4x_6} - \frac{x_6}{x_2x_4}\right) \\
\bar{r}_5 &= \frac{1}{2x_5} + \frac{1}{2\,d_8}\begin{bmatrix} 8 \\ 47 \end{bmatrix}\left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5}\right) + \frac{1}{2\,d_8}\begin{bmatrix} 8 \\ 56 \end{bmatrix}\left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5}\right) \\
&\quad + \frac{1}{2\,d_8}\begin{bmatrix} 8 \\ 49 \end{bmatrix}\left(\frac{x_5}{x_1x_6} - \frac{x_1}{x_5x_6} - \frac{x_6}{x_1x_5}\right) \\
\bar{r}_6 &= \frac{1}{2x_6} + \frac{1}{4\,d_9}\begin{bmatrix} 9 \\ 66 \end{bmatrix}\left(\frac{x_6}{{x_3}^2} - \frac{2}{x_6}\right) + \frac{1}{2\,d_9}\begin{bmatrix} 9 \\ 48 \end{bmatrix}\left(\frac{x_6}{x_1x_5} - \frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6}\right) \\
&\quad + \frac{1}{2\,d_9}\begin{bmatrix} 9 \\ 57 \end{bmatrix}\left(\frac{x_6}{x_2x_4} - \frac{x_2}{x_4x_6} - \frac{x_4}{x_2x_6}\right).
\end{aligned} \right. \quad (31)$$

From Proposition 2, we known that the unique E_8-invariant Kähler-Einstein metric on G/H is given by $B|_{\mathfrak{m}_1} + 2 \cdot B|_{\mathfrak{m}_2} + 3 \cdot B|_{\mathfrak{m}_3} + 4 \cdot B|_{\mathfrak{m}_4} + 5 \cdot B|_{\mathfrak{m}_5} + 6 \cdot B|_{\mathfrak{m}_6}$. We use these parameters to obtain the following equations:

$$\begin{aligned}
\frac{1}{2} - \frac{1}{d_4}\left(\begin{bmatrix} 5 \\ 44 \end{bmatrix} + \begin{bmatrix} 6 \\ 45 \end{bmatrix} + \begin{bmatrix} 7 \\ 46 \end{bmatrix} + \begin{bmatrix} 8 \\ 47 \end{bmatrix} + \begin{bmatrix} 9 \\ 48 \end{bmatrix}\right) &= \frac{1}{4} + \frac{1}{d_5}\left(\frac{1}{4}\begin{bmatrix} 5 \\ 44 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 6 \\ 45 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 7 \\ 55 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 8 \\ 56 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 9 \\ 57 \end{bmatrix}\right) = \\
\frac{1}{6} + \frac{1}{d_6}\left(\frac{1}{3}\begin{bmatrix} 6 \\ 45 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 7 \\ 46 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 8 \\ 56 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 9 \\ 66 \end{bmatrix}\right) &= \frac{1}{8} + \frac{1}{d_7}\left(\frac{1}{4}\begin{bmatrix} 7 \\ 46 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 8 \\ 47 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 7 \\ 55 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 9 \\ 57 \end{bmatrix}\right) = \quad (32) \\
\frac{1}{10} + \frac{1}{d_8}\left(\frac{1}{5}\begin{bmatrix} 8 \\ 47 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 9 \\ 48 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 8 \\ 56 \end{bmatrix}\right) &= \frac{1}{12} + \frac{1}{d_9}\left(\frac{1}{6}\begin{bmatrix} 9 \\ 48 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 9 \\ 57 \end{bmatrix} + \frac{1}{12}\begin{bmatrix} 9 \\ 66 \end{bmatrix}\right).
\end{aligned}$$

4.2.1. **Comparison of left-invariant metrics on $\mathbf{E_8}$.** From equations (32) we obtain a system with five equations and nine unknowns, namely the triples $\begin{bmatrix} 5 \\ 44 \end{bmatrix}$, $\begin{bmatrix} 6 \\ 45 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 46 \end{bmatrix}$, $\begin{bmatrix} 8 \\ 47 \end{bmatrix}$, $\begin{bmatrix} 9 \\ 48 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 55 \end{bmatrix}$, $\begin{bmatrix} 8 \\ 56 \end{bmatrix}$, $\begin{bmatrix} 9 \\ 57 \end{bmatrix}$, and $\begin{bmatrix} 9 \\ 66 \end{bmatrix}$. These are the only non-zero triples of G/H with respect to the decomposition (27) (see also [Chr]).

With the aim to obtain more conditions abouts these triples we follow the new method was applied also in case (E). Again our goal is to determine a new left-invariant metric on $<< \ , \ >>$ on $G = \mathrm{E}_8$. We put $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_6$, $\mathfrak{k}_1 = \mathfrak{h}_0 \oplus \mathfrak{h}_3 \oplus \mathfrak{m}_6$, $\mathfrak{n}_1 = \mathfrak{m}_1 \oplus \mathfrak{m}_5$, $\mathfrak{n}_2 = \mathfrak{m}_2 \oplus \mathfrak{m}_4$ and $\mathfrak{n}_3 = \mathfrak{m}_3$. Then $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, and from Propositon 8 we conclude that $\mathfrak{k}_1$ is also a subalgebra of $\mathfrak{g}$. In particular, we have $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_1 = \mathfrak{su}(2)$, and $\mathfrak{h}_2 = \mathfrak{su}(3)$. Also, for dimensional reasons it is $\mathfrak{k}_1 = \mathfrak{su}(6)$. Now, by using (12) we obtain the following inclusions:

$$\begin{array}{lll} [\mathfrak{n}_1, \ \mathfrak{n}_1] \subset \mathfrak{n}_2 \oplus \mathfrak{k}, & [\mathfrak{n}_1, \ \mathfrak{n}_3] \subset \mathfrak{n}_2, & [\mathfrak{n}_2, \mathfrak{n}_2] \subset \mathfrak{n}_2 \oplus \mathfrak{k}, \\ [\mathfrak{n}_1, \ \mathfrak{n}_2] \subset \mathfrak{n}_1 \oplus \mathfrak{n}_3, & [\mathfrak{n}_2, \mathfrak{n}_3] \subset \mathfrak{n}_1, & [\mathfrak{n}_3, \mathfrak{n}_3] \subset \mathfrak{k}. \end{array} \tag{33}$$

Thus we determine an irreducible decomposition $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ as $\mathrm{Ad}(K)$-modules, which are mutually non-equivalent. Since $\mathfrak{h} \subset \mathfrak{k}$ we can determine the fibration $G/H \to G/K$, explicity given by

$$\mathrm{E}_8 \,/\, \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5) \to \mathrm{E}_8 \,/\, \mathrm{SU}(6) \times \mathrm{SU}(2) \times \mathrm{SU}(3).$$

Note that the base $G/K = \mathrm{E}_8 \,/\, \mathrm{SU}(6) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ is a compact homogeneous manifold with three isotropy summands, namely $\mathfrak{n}_1, \mathfrak{n}_2$ and $\mathfrak{n}_3$. Let us consider now the following left-invariant metrics on E_8 which are also $\mathrm{Ad}(K)$-invariant:

$$<< \ , \ >>= y_1 \cdot B|_{\mathfrak{k}_1} + y_2 \cdot B|_{\mathfrak{h}_1} + y_3 \cdot B|_{\mathfrak{h}_2} + y_4 \cdot B|_{\mathfrak{n}_1} + y_5 \cdot B|_{\mathfrak{n}_2} + y_6 \cdot B|_{\mathfrak{n}_6}, \quad (y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6_+. \tag{34}$$

Lemma 5. *For a left invariant metric $<< \ , \ >>$ on E_8 given by (34), the non-zero structure constants are the following (and their symmetries):*

$$\left[\!\!\left[{1 \atop 11}\right]\!\!\right], \left[\!\!\left[{4 \atop 14}\right]\!\!\right], \left[\!\!\left[{5 \atop 15}\right]\!\!\right], \left[\!\!\left[{6 \atop 16}\right]\!\!\right], \left[\!\!\left[{2 \atop 22}\right]\!\!\right], \left[\!\!\left[{4 \atop 24}\right]\!\!\right], \left[\!\!\left[{5 \atop 25}\right]\!\!\right], \left[\!\!\left[{6 \atop 26}\right]\!\!\right], \left[\!\!\left[{3 \atop 33}\right]\!\!\right], \left[\!\!\left[{4 \atop 34}\right]\!\!\right], \left[\!\!\left[{5 \atop 35}\right]\!\!\right], \left[\!\!\left[{6 \atop 36}\right]\!\!\right], \left[\!\!\left[{5 \atop 44}\right]\!\!\right], \left[\!\!\left[{6 \atop 45}\right]\!\!\right].$$

Proof. This is an immediate consequence of the decomposition $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and relation (33). □

We set $f_1 = \dim \mathfrak{k}_1 = 35$, $f_2 = \dim \mathfrak{h}_1 = 3$, $f_3 = \dim \mathfrak{h}_2 = 8$, $f_4 = \dim \mathfrak{n}_1 = 72$, $f_5 = \dim \mathfrak{n}_2 = 90$ and $f_6 = \dim \mathfrak{n}_3 = 40$.

Proposition 11. *The components of the Ricci tensor $\widetilde{r}$ of the left-invariant metric $<< \ , \ >>$ on E_8 defined by (34), are given as follows:*

$$\left\{ \begin{array}{rl}
\widetilde{r}_1 = & \dfrac{1}{4\,f_1\,y_1} \left[\!\!\left[{1 \atop 11}\right]\!\!\right] + \dfrac{y_1}{4\,f_1\,{y_4}^2} \left[\!\!\left[{1 \atop 44}\right]\!\!\right] + \dfrac{y_1}{4\,f_1\,{y_5}^2} \left[\!\!\left[{1 \atop 55}\right]\!\!\right] + \dfrac{y_1}{4\,f_1\,{y_6}^2} \left[\!\!\left[{1 \atop 66}\right]\!\!\right] \\[3ex]
\widetilde{r}_2 = & \dfrac{1}{4\,f_2\,y_2} \left[\!\!\left[{2 \atop 22}\right]\!\!\right] + \dfrac{y_2}{4\,f_2\,{y_4}^2} \left[\!\!\left[{2 \atop 44}\right]\!\!\right] + \dfrac{y_2}{4\,f_2\,{y_5}^2} \left[\!\!\left[{2 \atop 55}\right]\!\!\right] + \dfrac{y_2}{4\,f_2\,{y_6}^2} \left[\!\!\left[{2 \atop 66}\right]\!\!\right] \\[3ex]
\widetilde{r}_3 = & \dfrac{1}{4\,f_3\,y_3} \left[\!\!\left[{3 \atop 33}\right]\!\!\right] + \dfrac{y_3}{4\,f_3\,{y_4}^2} \left[\!\!\left[{3 \atop 44}\right]\!\!\right] + \dfrac{y_3}{4\,f_3\,{y_5}^2} \left[\!\!\left[{3 \atop 55}\right]\!\!\right] + \dfrac{y_3}{4\,f_3\,{y_6}^2} \left[\!\!\left[{2 \atop 66}\right]\!\!\right] \\[3ex]
\widetilde{r}_4 = & \dfrac{1}{2y_4} + \dfrac{y_5}{4\,f_4\,{y_4}^2} \left[\!\!\left[{5 \atop 44}\right]\!\!\right] + \dfrac{1}{2\,f_4} \left[\!\!\left[{4 \atop 56}\right]\!\!\right] \left(\dfrac{y_4}{y_5\,y_6} - \dfrac{y_5}{y_4\,y_6} - \dfrac{y_6}{y_4\,y_5} \right) \\
& -\dfrac{1}{2\,f_4} \left(\dfrac{y_1}{{y_4}^2} \left[\!\!\left[{1 \atop 44}\right]\!\!\right] + \dfrac{y_2}{{y_4}^2} \left[\!\!\left[{2 \atop 44}\right]\!\!\right] + \dfrac{y_3}{{y_4}^2} \left[\!\!\left[{3 \atop 44}\right]\!\!\right] \right) \\[3ex]
\widetilde{r}_5 = & \dfrac{1}{2y_5} + \dfrac{1}{4\,f_5} \left[\!\!\left[{5 \atop 44}\right]\!\!\right] \left(\dfrac{y_5}{{y_4}^2} - \dfrac{2}{y_5} \right) + \dfrac{1}{2\,f_5} \left[\!\!\left[{5 \atop 46}\right]\!\!\right] \left(\dfrac{y_5}{y_4\,y_6} - \dfrac{y_4}{y_5\,y_6} - \dfrac{y_6}{y_4\,y_5} \right) \\
& -\dfrac{1}{2\,f_5} \left(\dfrac{y_1}{{y_5}^2} \left[\!\!\left[{1 \atop 55}\right]\!\!\right] + \dfrac{y_2}{{y_5}^2} \left[\!\!\left[{2 \atop 55}\right]\!\!\right] + \dfrac{y_3}{{y_5}^2} \left[\!\!\left[{3 \atop 55}\right]\!\!\right] \right) \\[3ex]
\widetilde{r}_6 = & \dfrac{1}{2y_6} + \dfrac{1}{2\,f_6} \left[\!\!\left[{6 \atop 45}\right]\!\!\right] \left(\dfrac{y_6}{y_4\,y_5} - \dfrac{y_4}{y_5\,y_6} - \dfrac{y_5}{y_4\,y_6} \right) - \dfrac{1}{2\,f_6} \left(\dfrac{y_1}{{y_6}^2} \left[\!\!\left[{1 \atop 66}\right]\!\!\right] + \dfrac{y_2}{{y_6}^2} \left[\!\!\left[{2 \atop 66}\right]\!\!\right] + \dfrac{y_3}{{y_6}^2} \left[\!\!\left[{3 \atop 66}\right]\!\!\right] \right)
\end{array} \right. \tag{35}$$

Proof. We use Lemma 5 and we apply again Theorem 1, (1). □

Observe that equations (35) are obtained from (29) by setting $y_1 = u_0 = u_3 = x_6$, $y_2 = u_1$, $y_3 = u_2$, $y_4 = x_1 = x_5$, $y_5 = x_2 = x_4$ and $y_6 = x_3$. For these values the metrics $< \ , \ >$ and $<< \ , \ >>$ on G coincide, so the components of the corresponding Ricci tensor must be equal. Therefore, from relation $y_4 = x_1 = x_5$ we conclude that $\widetilde{r}_4 = r_4 = r_8$, and from $y_5 = x_2 = x_4$ it must be $\widetilde{r}_5 = r_5 = r_7$. Thus we obtain the following equations.

$$\left.\begin{aligned} &\frac{1}{2\,f_4}\left[\!\!\left[\begin{matrix}6\\45\end{matrix}\right]\!\!\right] = \frac{1}{2d_4}\begin{bmatrix}6\\45\end{bmatrix} + \frac{1}{2d_4}\begin{bmatrix}7\\46\end{bmatrix} = \frac{1}{2d_8}\begin{bmatrix}8\\56\end{bmatrix} \\ &\frac{1}{2\,f_5}\left[\!\!\left[\begin{matrix}6\\45\end{matrix}\right]\!\!\right] = \frac{1}{2d_5}\begin{bmatrix}6\\45\end{bmatrix} + \frac{1}{2d_5}\begin{bmatrix}8\\56\end{bmatrix} = \frac{1}{2d_7}\begin{bmatrix}7\\46\end{bmatrix} \\ &\frac{1}{2\,f_5}\left[\!\!\left[\begin{matrix}5\\44\end{matrix}\right]\!\!\right] = \frac{1}{4d_5}\begin{bmatrix}5\\44\end{bmatrix} = \frac{1}{2d_7}\begin{bmatrix}8\\47\end{bmatrix} \end{aligned}\right\}. \tag{36}$$

4.2.2. **The contribution of the twistor fibration.** For the computation of the triples $\begin{bmatrix}7\\55\end{bmatrix}$ and $\begin{bmatrix}9\\57\end{bmatrix}$ we use the twistor fibration which admits any flag manifold $M = G/H$ of a compact (semi)-simple Lie group G, over an irreducible symmetric space G/L of compact type ([BuR, pp. 43-44]). This method was initially apllied in [AC2].

We set $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_6$ and $\mathfrak{p} = \mathfrak{m}_1 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_5$. Then, in view of the inclusions given by (12) we conclude that $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}$, $[\mathfrak{l}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l}$. Let L be the connected Lie subgroup of G with Lie algebra $\mathfrak{l}$. Then $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ is a reductive decomposition of G/L, and from the latter relations it follows that G/L is a locally symmetric space. In particular, since $G = \mathrm{E}_8$ is a simply connected Lie group, G/L is also simply connected and thus it is a symmetric space. Because G is simple (and compact), G/L is an irreducible symmetric space (of compact type). In our case we have that $\dim \mathfrak{l} = 136$, thus it must be $G/L = \mathrm{E}_8 / \mathrm{E}_7 \times \mathrm{SU}(2)$, since $\dim G/L = \dim G - \dim L = 278 - 136 = 112 = \dim \mathfrak{p}$. Due to the inlusion $\mathfrak{h} \subset \mathfrak{l}$ it follows that $H \subset L$, and thus we can determine the fibration $L/H \to G/H \stackrel{\pi}{\to} G/L$, explicitly given as follows:

$$\mathrm{E}_7 \times \mathrm{SU}(2)/\,\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5) \longrightarrow \mathrm{E}_8 \,/\, \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5) \stackrel{\pi}{\to} \mathrm{E}_8 \,/\, \mathrm{E}_7 \times \mathrm{SU}(2).$$

We observe that on the fiber L/H, the Lie group L does not act (almost) effectively, that is H contains some non-trivial normal subgroups of L. Let L' the normal subgroup of L which acts effectively on L/H with isotropy subgroup H'. Then $L/H = L'/H'$, that is

$$L/H = \mathrm{E}_7 \times \mathrm{SU}(2)/\,\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5) = \mathrm{E}_7 \,/\, \mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(5) = L'/H'.$$

The fiber L'/H' is a flag manifold with three isotropy summands ([Kim]): Let $\mathfrak{l}' = \mathfrak{h}' \oplus \mathfrak{f}$ be a reductive decomposition of $\mathfrak{l}'$ with repsect to B_{E_7}, the negative of the Killing form of E_7. Then $T_{o'}(L'/H') = \mathfrak{f} = \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3$, where $\mathfrak{f}_1 = \mathfrak{m}_2$, $\mathfrak{f}_2 = \mathfrak{m}_4$, and $\mathfrak{f}_3 = \mathfrak{m}_6$. We set $D_1 = \dim \mathfrak{f}_1 = 60$, $D_2 = \dim \mathfrak{f}_2 = 30$ and $D_3 = \dim \mathfrak{f}_3 = 10$ and we consider E_7-invariant metrics on $\mathrm{E}_7 \,/\, \mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$, of the form

$$g_{\mathfrak{f}} = w_1 \cdot B_{\mathrm{E}_7}\Big|_{\mathfrak{f}_1} + w_2 \cdot B_{\mathrm{E}_7}\Big|_{\mathfrak{f}_2} + w_3 \cdot B_{\mathrm{E}_7}\Big|_{\mathfrak{f}_3}, \quad (w_1, w_2, w_3) \in \mathbb{R}^3_+. \tag{37}$$

Lemma 6. *For a L'-invariant metric $g_{\mathfrak{f}}$ on the fiber L'/H' given by (37), the non-zero structure constants $\begin{bmatrix}k\\ij\end{bmatrix}_{\mathfrak{f}}$ are $\begin{bmatrix}2\\11\end{bmatrix}_{\mathfrak{f}}$ and $\begin{bmatrix}3\\12\end{bmatrix}_{\mathfrak{f}}$ (and their symmetries).*

Proof. This result follows from the inclusions $[\mathfrak{f}_1, \mathfrak{f}_1] \subset \mathfrak{h}' \oplus \mathfrak{f}_2$, $[\mathfrak{f}_1, \mathfrak{f}_2] \subset \mathfrak{f}_1 \oplus \mathfrak{f}_3$, $[\mathfrak{f}_1, \mathfrak{f}_3] \subset \mathfrak{f}_2$, $[\mathfrak{f}_2, \mathfrak{f}_2] \subset \mathfrak{h}'$, $[\mathfrak{f}_2, \mathfrak{f}_3] \subset \mathfrak{f}_1$, and $[\mathfrak{f}_3, \mathfrak{f}_3] \subset \mathfrak{h}'$, which are easily obtained from relations given in (12). ☐

Let R_i be the components of the Ricci tensor $\mathrm{Ric}_{g_{\mathfrak{f}}}$ for the E_7-invariant metric $g_{\mathfrak{f}}$ on the fiber $L'/H' = \mathrm{E}_7 \,/\, \mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$, defined by (37). Then, in view of Lemma 6 and by applying Theoren 1, (2), we obtain the following forms for the components R_i.

Proposition 12. *The components R_i of the Ricci tensor for an E_7-invariant metric $g_{\mathfrak{f}}$ on the fiber $L'/H' = \mathrm{E}_7 \,/\, \mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$ defined by (37), are given as follows:*

$$\left\{\begin{aligned} R_1 = &\ \frac{1}{2\,w_1} - \frac{1}{2\,D_1}\begin{bmatrix}2\\11\end{bmatrix}\frac{w_2}{{w_1}^2} + \frac{1}{2\,D_1}\begin{bmatrix}1\\23\end{bmatrix}\left(\frac{w_1}{w_2\,w_3} - \frac{w_2}{w_1\,w_3} - \frac{w_3}{w_1\,w_2}\right) \\ R_2 = &\ \frac{1}{2\,w_2} + \frac{1}{4\,D_2}\begin{bmatrix}2\\11\end{bmatrix}\left(\frac{w_2}{{w_1}^2} - \frac{2}{w_2}\right) + \frac{1}{2\,D_2}\begin{bmatrix}2\\13\end{bmatrix}\left(\frac{w_2}{w_1\,w_3} - \frac{w_1}{w_2\,w_3} - \frac{w_3}{w_1\,w_2}\right) \\ R_3 = &\ \frac{1}{2\,w_3} + \frac{1}{2\,D_3}\begin{bmatrix}3\\12\end{bmatrix}\left(\frac{w_3}{w_1\,w_2} - \frac{w_1}{w_2\,w_3} - \frac{w_2}{w_1\,w_3}\right) \end{aligned}\right. \tag{38}$$

From Proposition 2 we know that $\mathrm{E}_7 / \mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$ admits a unique Kähler-Einstein metric, explicitly given by $1 \cdot B_{\mathrm{E}_7}\Big|_{\mathfrak{f}_1} + 2 \cdot B_{\mathrm{E}_7}\Big|_{\mathfrak{f}_2} + 3 \cdot B_{\mathrm{E}_7}\Big|_{\mathfrak{f}_3}$. Thus, by solving the system $\Big\{ R_1 - R_2 = 0,\ R_2 - R_3 = 0 \Big\}$, we obtain the values $\begin{bmatrix} 2 \\ 11 \end{bmatrix}_{\mathfrak{f}} = 10$ and $\begin{bmatrix} 3 \\ 12 \end{bmatrix}_{\mathfrak{f}} = 10/3$.

Since $L' = \mathrm{E}_7$ is a simple Lie subgoup of E_8 there is a positive number c, such that $B_{\mathrm{E}_7} = c \cdot B_{\mathrm{E}_8}$, where $B_{\mathrm{E}_8} = B$ is the Killing form of E_8. In particular it is $c = B_{\mathrm{E}_7}/B_{\mathrm{E}_8} = 3/5$ (cf. [Brb]). Then, by applying an easy computation based on the definition of the structure constants $\begin{bmatrix} k \\ ij \end{bmatrix}$ we obtain that $\begin{bmatrix} 7 \\ 55 \end{bmatrix}$ and $\begin{bmatrix} 9 \\ 57 \end{bmatrix}$ are given as follows (see for example [AC3, Lemma 1]):

$$\begin{bmatrix} 7 \\ 55 \end{bmatrix} = c \cdot \begin{bmatrix} 2 \\ 11 \end{bmatrix}_{\mathfrak{f}} = 3/5 \cdot 10 = 6, \quad \begin{bmatrix} 9 \\ 57 \end{bmatrix} = c \cdot \begin{bmatrix} 3 \\ 12 \end{bmatrix}_{\mathfrak{f}} = 3/5 \cdot 10/3 = 2.$$

Now, from equations (32) and (36) we get the following system

$$\begin{cases} 60 - 5\begin{bmatrix} 5 \\ 44 \end{bmatrix} - 2\begin{bmatrix} 6 \\ 45 \end{bmatrix} - 4\begin{bmatrix} 7 \\ 46 \end{bmatrix} - 4\begin{bmatrix} 8 \\ 47 \end{bmatrix} - 4\begin{bmatrix} 9 \\ 48 \end{bmatrix} + 2\begin{bmatrix} 7 \\ 55 \end{bmatrix} + 2\begin{bmatrix} 8 \\ 56 \end{bmatrix} + 2\begin{bmatrix} 9 \\ 57 \end{bmatrix} = 0 \\ 20 + \begin{bmatrix} 5 \\ 44 \end{bmatrix} - 4\begin{bmatrix} 6 \\ 45 \end{bmatrix} + 2\begin{bmatrix} 7 \\ 46 \end{bmatrix} - 2\begin{bmatrix} 7 \\ 55 \end{bmatrix} - 2\begin{bmatrix} 9 \\ 57 \end{bmatrix} + 2\begin{bmatrix} 9 \\ 66 \end{bmatrix} = 0 \\ 10 + 2\begin{bmatrix} 6 \\ 45 \end{bmatrix} - 4\begin{bmatrix} 7 \\ 46 \end{bmatrix} + 2\begin{bmatrix} 8 \\ 47 \end{bmatrix} - \begin{bmatrix} 7 \\ 55 \end{bmatrix} - 2\begin{bmatrix} 8 \\ 56 \end{bmatrix} + 2\begin{bmatrix} 9 \\ 57 \end{bmatrix} - 2\begin{bmatrix} 9 \\ 66 \end{bmatrix} = 0 \\ 6 + 2\begin{bmatrix} 7 \\ 46 \end{bmatrix} - 6\begin{bmatrix} 8 \\ 47 \end{bmatrix} + 4\begin{bmatrix} 9 \\ 48 \end{bmatrix} + \begin{bmatrix} 7 \\ 55 \end{bmatrix} - 4\begin{bmatrix} 8 \\ 56 \end{bmatrix} - 2\begin{bmatrix} 9 \\ 57 \end{bmatrix} = 0 \\ 2 + 2\begin{bmatrix} 8 \\ 47 \end{bmatrix} - 4\begin{bmatrix} 9 \\ 48 \end{bmatrix} + 2\begin{bmatrix} 8 \\ 56 \end{bmatrix} - 2\begin{bmatrix} 9 \\ 57 \end{bmatrix} - \begin{bmatrix} 9 \\ 66 \end{bmatrix} = 0 \\ \begin{bmatrix} 6 \\ 45 \end{bmatrix} + \begin{bmatrix} 7 \\ 46 \end{bmatrix} - 5\begin{bmatrix} 8 \\ 56 \end{bmatrix} = 0 \\ \begin{bmatrix} 6 \\ 45 \end{bmatrix} - 2\begin{bmatrix} 7 \\ 46 \end{bmatrix} + \begin{bmatrix} 8 \\ 56 \end{bmatrix} = 0 \\ \begin{bmatrix} 5 \\ 44 \end{bmatrix} - 4\begin{bmatrix} 8 \\ 47 \end{bmatrix} = 0. \end{cases} \tag{39}$$

Thus, by substituting the values $\begin{bmatrix} 7 \\ 55 \end{bmatrix} = 6$, $\begin{bmatrix} 9 \\ 57 \end{bmatrix} = 2$, and solving equations (39) we get the explicit values of all non-zero triples of $\mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$ with respect to the reductive decomposition (27).

Proposition 13. *For the* E_8*-invariant metric* $(\ ,\ )$ *on* $M = G/H = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$, *the non-zero structure constants* $\begin{bmatrix} k \\ ij \end{bmatrix}$ *are given as follows:*

$$\begin{bmatrix} 5 \\ 44 \end{bmatrix} = 8, \ \begin{bmatrix} 6 \\ 45 \end{bmatrix} = 6, \ \begin{bmatrix} 7 \\ 46 \end{bmatrix} = 4, \ \begin{bmatrix} 8 \\ 47 \end{bmatrix} = 2, \ \begin{bmatrix} 9 \\ 48 \end{bmatrix} = 1, \ \begin{bmatrix} 7 \\ 55 \end{bmatrix} = 6, \ \begin{bmatrix} 8 \\ 56 \end{bmatrix} = 2, \ \begin{bmatrix} 9 \\ 57 \end{bmatrix} = 2, \ \begin{bmatrix} 9 \\ 66 \end{bmatrix} = 2.$$

4.2.3. **Solutions of the homogeneous Einstein equation.** By using Proposition 13 and the dimensions $d_i = \dim_{\mathbb{R}} \mathfrak{m}_i$ presented in §4.1, the components $\bar{r}_i$ $(1 \leq i \leq 6)$ of the Ricci tensor are completely determined by equation (31). In particular, a G-invariant metric $(\ ,\ ) = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6_+$ on $G/H = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$, is an Einstein metric, if and only if it is a positive real solution of the following system

$$\Big\{ \bar{r}_1 - \bar{r}_2 = 0, \ \ \bar{r}_2 - \bar{r}_3 = 0, \ \ \bar{r}_3 - \bar{r}_4 = 0, \ \ \bar{r}_4 - \bar{r}_5 = 0, \ \ \bar{r}_5 - \bar{r}_6 = 0 \Big\}, \tag{40}$$

where the components $\bar{r}_i$ are given as follows:

$$
\left\{
\begin{aligned}
&\bar{r}_1 = \frac{1}{2x_1} - \frac{x_2}{15{x_1}^2} + \frac{1}{20}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right) + \frac{1}{30}\left(\frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3}\right)\\
&+\frac{1}{60}\left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4}\right) + \frac{1}{120}\left(\frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6} - \frac{x_6}{x_1x_5}\right)\\
&\bar{r}_2 = \frac{1}{2x_2} + \frac{1}{30}\left(\frac{x_2}{{x_1}^2} - \frac{2}{x_2}\right) - \frac{1}{20}\frac{x_4}{{x_2}^2} + \frac{1}{20}\left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2}\right) + \frac{1}{60}\left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3}\right)\\
&+\frac{1}{60}\left(\frac{x_2}{x_4x_6} - \frac{x_4}{x_2x_6} - \frac{x_6}{x_2x_4}\right)\\
&\bar{r}_3 = \frac{1}{2x_3} - \frac{1}{40}\frac{x_6}{{x_3}^2} + \frac{3}{40}\left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_1x_3} - \frac{x_1}{x_3x_2}\right) + \frac{1}{20}\left(\frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3}\right)\\
&+\frac{1}{40}\left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_3x_2}\right)\\
&\bar{r}_4 = \frac{1}{2x_4} + \frac{1}{20}\left(\frac{x_4}{{x_2}^2} - \frac{2}{x_4}\right) + \frac{1}{15}\left(\frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4}\right) + \frac{1}{30}\left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4}\right)\\
&+\frac{1}{30}\left(\frac{x_4}{x_2x_6} - \frac{x_2}{x_4x_6} - \frac{x_6}{x_2x_4}\right)\\
&\bar{r}_5 = \frac{1}{2x_5} + \frac{1}{12}\left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5}\right) + \frac{1}{12}\left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5}\right) + \frac{1}{24}\left(\frac{x_5}{x_1x_6} - \frac{x_1}{x_5x_6} - \frac{x_6}{x_1x_5}\right)\\
&\bar{r}_6 = \frac{1}{2x_6} + \frac{1}{20}\left(\frac{x_6}{{x_3}^2} - \frac{2}{x_6}\right) + \frac{1}{20}\left(\frac{x_6}{x_1x_5} - \frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6}\right) + \frac{1}{10}\left(\frac{x_6}{x_2x_4} - \frac{x_2}{x_4x_6} - \frac{x_4}{x_2x_6}\right).
\end{aligned}
\right. \tag{41}
$$

We normalize our equations by setting $x_1 = 1$. We see that the system of equations (40) reduces to the following system of polynomial equations:

$$
\left\{
\begin{aligned}
&f_1 = -6x_3x_4^2x_5x_6 + 2x_2^3\big(x_4(1+6x_5)x_6 + x_3(x_5 + 6x_4x_5x_6)\big) - 2x_2\big(x_3^2x_4x_6 + x_4x_5(6+x_5)x_6\\
&+x_3x_5(x_4^2 - 26x_4x_6 + x_6^2)\big) + x_2^2\Big(4x_3^2x_5x_6 + 4(-1+x_4^2)x_5x_6 + x_3\big(2x_4^2x_6 + 2(-1+x_5^2)x_6\\
&+x_4(-1 + x_5^2 - 60x_5x_6 + x_6^2)\big)\Big) = 0\\
&f_2 = -6x_3^2x_4^2x_5x_6 + 3x_2^2x_5x_6\big(-2x_3^3 + 2x_3(1 - 10x_4 + x_4^2) + x_4x_6\big) + x_2^3x_3\big(5x_4(1+3x_5)x_6\\
&+2x_3(x_5 + 2x_4x_5x_6)\big) + x_2x_3\big(x_4x_5(3+x_5)x_6 - 5x_3^2x_4(1+3x_5)x_6 - 2x_3x_5(x_4^2 - 26x_4x_6 + x_6^2)\big) = 0\\
&f_3 = -6x_3^2x_4^2x_5x_6 + x_2^2x_6\big(14x_3^3x_5 + 2x_3(1 + 30x_4 - 7x_4^2)x_5 - 4x_3^2(-1 + x_4^2 + 12x_5 - x_5^2) - 3x_4x_5x_6\big)\\
&+x_2^3x_3\big(4x_3x_5 - 3x_4(1+3x_5)x_6\big) + x_2x_3\big(-3x_4x_5(3+x_5)x_6 + 3x_3^2x_4(1+3x_5)x_6 + 4x_3x_5(-x_4^2 + x_6^2)\big) = 0\\
&f_4 = 6x_3x_4^2x_5x_6 + x_2^3(-4x_3x_5 + 10x_4x_6) + 2x_2\big(5x_3^2x_4x_6 - 5x_4x_5^2x_6 + 2x_3x_5(x_4^2 - x_6^2)\big) + x_2^2\Big(-8x_3^2x_5x_6\\
&+8(-1+x_4^2)x_5x_6 + x_3\big(14x_4^2x_6 + 2(3 + 24x_5 - 7x_5^2)x_6 - 5x_4(-1 + x_5^2 + 12x_6 - x_6^2)\big)\Big) = 0\\
&f_5 = 2x_2^2x_3(6x_3x_5 - 5x_4x_6) - 2x_3\big(5x_3^2x_4x_6 - 5x_4x_5^2x_6 + 6x_3x_5(-x_4^2 + x_6^2)\big)\\
&+x_2\Big(-6x_4x_5x_6^2 + x_3^2\big(-10x_4^2x_6 + 10(-1+x_5^2)x_6 + x_4(1 - 48x_5 + 11x_5^2 + 60x_6 - 11x_6^2)\big)\Big) = 0.
\end{aligned}
\right. \tag{42}
$$

To find non zero solutions of equations (42), we consider a polynomial ring $R = \mathbb{Q}[y, x_2, x_3, x_4, x_5, x_6]$ and an ideal I generated by

$$\{f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ y\, x_2x_3x_4x_5x_6 - 1\}.$$

But we fail to compute a Gröbner basis for the ideal I. However, we conjecture that $\mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$ would admit a finite number non-Kähler invariant Einstein metrics, since the rest members of the examined class admit a finite number of non-Kähler Einstein metrics. Note that the Böhm-Wang-Ziller's conjecture ([BWZ]), the so called *finiteness conjecture*, states that there exist a finite number of invariant Einstein metrics on compact homogeneous space G/H with $\operatorname{rank} G = \operatorname{rank} H$.

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