Cycles and sorting index for matchings and restricted permutations

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Abstract

We prove that the Mahonian-Stirling pairs of permutation statistics (sor, cyc) and (inv, rlmin) are equidistributed on the set of permutations that correspond to arrangements of n non-atacking rooks on a Ferrers board with n rows and n columns. The proofs are combinatorial and use bijections between matchings and Dyck paths and a new statistic, sorting index for matchings, that we define. We also prove a refinement of this equidistribution result which describes the minimal elements in the permutation cycles and the right-to-left minimum letters. Moreover, we define a sorting index for bicolored matchings and use it to show analogous equidistribution results for restricted permutations of type B_n and D_n.

1 Introduction

An inversion in a permutation σ is a pair $\sigma(i) > \sigma(j)$ such that $i < j$. The number of inversions in σ is denoted by $\mathrm{inv}(\sigma)$. The distribution of inv over the symmetric group S_n was first found by Rodriguez [9] in 1837 and is well known to be

$$\sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Much later, MacMahon [7] defined the major index maj and proved that it has the same distribution as inv. In his honor, all permutation statistics that are equally distributed with inv are called Mahonian. MacMahon's remarkable result initiated a systematic research of permutation statistics and in particular many more Mahonian statistics have been described in the literature since then.

Another classical permutation statistic is the number of cycles, cyc. Its distribution is given by

$$\sum_{\sigma\in S_n} t^{\mathrm{cyc}(\sigma)} = t(t+1)(t+2)\cdots(t+n-1)$$

and the coefficients of this polynomial are known as the unsigned Stirling numbers of the first kind.

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Given these two distributions, it is natural then to ask which "Mahonian-Stirling" pairs of statistics $(\mathrm{stat}_1, \mathrm{stat}_2)$ have the distribution

$$\sum_{\sigma\in S_n} q^{\mathrm{stat}_1(\sigma)} t^{\mathrm{stat}_2(\sigma)} = t(t+q)(t+q+q^2)\cdots(t+q+\cdots+q^{n-1}). \tag{1.1}$$

As proved by Björner and Wachs [2], $(\mathrm{inv}, \mathrm{rlmin})$ and $(\mathrm{maj}, \mathrm{rlmin})$ are two such pairs, where rlmin is the number of right-to-left minimum letters. A right-to-left minimum letter of a permutation σ is a letter $\sigma(i)$ such that $\sigma(i) < \sigma(j)$ for all $j > i$. The set of all right-to-left minimum letters in σ will be denoted by $\mathrm{Rlminl}(\sigma)$. In fact, Björner and Wachs proved the following stronger result

$$\sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)} \prod_{i\in \mathrm{Rlminl}(\sigma)} t_i = \sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma)} \prod_{i\in \mathrm{Rlminl}(\sigma)} t_i = t_1(t_2+q)(t_3+q+q^2)\cdots(t_n+q+\cdots+q^{n-1}). \tag{1.2}$$

A natural Mahonian partner for cyc was found by Petersen [8]. For a given permutation $\sigma \in S_n$ there is a unique expression

$$\sigma = (i_1 j_1)(i_2 j_2)\cdots(i_k j_k)$$

as a product of transpositions such that $i_s < j_s$ for $1 \le s \le k$ and $j_1 < \cdots < j_k$. The sorting index of σ is defined to be

$$\mathrm{sor}(\sigma) = \sum_{s=1}^{k} (j_s - i_s).$$

The sorting index can also be described as the total distance the elements in σ travel when σ is sorted using the Straight Selection Sort algorithm [6] in which, using a transposition, we move the largest number to its proper place, then the second largest to its proper place, etc. For example, the steps for sorting $\sigma = 6571342$ are

$$65\mathbf{7}1342 \xrightarrow{(37)} \mathbf{6}521347 \xrightarrow{(16)} 4\mathbf{5}21367 \xrightarrow{(25)} \mathbf{4}321567 \xrightarrow{(14)} 1\mathbf{3}24567 \xrightarrow{(23)} 1234567$$

and therefore $\sigma = (2\ 3)(1\ 4)(2\ 5)(1\ 6)(3\ 7)$ and $\mathrm{sor}(\sigma) = (3-2)+(4-1)+(5-2)+(6-1)+(7-3) = 16$. The relationship to other Mahonian statistics and the Eulerian partner for sor were studied by Wilson [10] who called the sorting index DIS.

Petersen showed that

$$\sum_{\sigma\in S_n} q^{\mathrm{sor}(\sigma)} t^{\mathrm{cyc}(\sigma)} = t(t+q)(t+q+q^2)\cdots(t+q+\cdots+q^{n-1}),$$

which implies equidistribution of the pairs $(\mathrm{inv}, \mathrm{rlmin})$ and $(\mathrm{sor}, \mathrm{cyc})$.

In this article we show that the pairs $(\mathrm{inv}, \mathrm{rlmin})$ and $(\mathrm{sor}, \mathrm{cyc})$ have the same distribution on the set of restricted permutations

$$S_{\mathbf{r}} = \{\sigma \in S_n : \sigma(k) \le r_k, 1 \le k \le n\}$$

for a nondecreasing sequence of integers $1 \le r_1 \le r_2 \le \cdots \le r_n \le n$. These can be described as permutations that correspond to arrangements of n non-atacking rooks on a Ferrers board with rows of length $r_1, \ldots, r_n$. To obtain the results, in Section 2 we define a sorting index and cycles for perfect matchings and study the distributions of these statistics over matchings of fixed type.

We use bijections between matchings and weighted Dyck paths which enable us to keep track of set-valued statistics and obtain more refined results similar to (1.2) for restricted permutations.

Analogously to sor, Petersen defined sorting index for signed permutations of type B_n and D_n. Using algebraic methods he proved that

$$\sum_{\sigma\in B_n} q^{\mathrm{sor}_B(\sigma)} t^{\ell'_B(\sigma)} = \sum_{\sigma\in B_n} q^{\mathrm{inv}_B(\sigma)} t^{\mathrm{nmin}_B(\sigma)} = \prod_{i=1}^{n}(1+t[2i]_q - t), \tag{1.3}$$

where $\ell'_B(\sigma)$ is the reflection length of σ, i.e., the minimal number of transpositions in

$$\{(ij) : 1 \le i < j \le n\} \cup \{(\bar{i}j) : 1 \le i < j \le n\}$$

needed to represent σ; $\mathrm{inv}_B(\sigma)$ is the number of type B_n inversions, which is known to be equal to the length of σ and is given by

$$\mathrm{inv}_B(\sigma) = |\{1 \le i < j \le n : \sigma(i) > \sigma(j)\}| + |\{1 \le i < j \le n : -\sigma(i) > \sigma(j)\}| + N(\sigma), \tag{1.4}$$

where

$$N(\sigma) = \text{ number of negative signs in } \sigma.$$

Finally,

$$\mathrm{nmin}_B(\sigma) = |\{i : \sigma(i) > |\sigma(j)| \text{ for some } j > i\}| + N(\sigma). \tag{1.5}$$

Petersen also defined sor_D, a sorting index for type D_n permutations and showed that it is equidistributed with the number of type D_n inversions:

$$\sum_{\sigma\in D_n} q^{\mathrm{sor}_D(\sigma)} = \sum_{\sigma\in D_n} q^{\mathrm{inv}_D(\sigma)} = [n]_q \cdot \prod_{i=1}^{n-1}[2i]_q. \tag{1.6}$$

In Section 3 we define a sorting index and cycles for bicolored matchings and give a combinatorial proof that the pairs $(\mathrm{sor}_B, \ell'_B)$ and $(\mathrm{inv}_B, \mathrm{nmin}_B)$ are equidistributed on the set of restricted signed permutations

$$B_{\mathbf{r}} = \{\sigma \in B_n : |\sigma(k)| \le r_k, 1 \le k \le n\}$$

for a nondecreasing sequence of integers $1 \le r_1 \le r_2 \le \cdots \le r_n \le n$. Using bijections between bicolored matchings and weighted Dyck paths with bicolored rises, we in fact prove equidistribution of set-valued statistics and their generating functions. Moreover, we find natural Stirling partners for sor_D and inv_D and prove equidistribution of the two Mahonian-Stirling pairs on sets of restricted permutations of type D_n:

$$D_{\mathbf{r}} = \{\sigma \in D_n : |\sigma(k)| \le r_k, 1 \le k \le n\}.$$

2 Statistics on perfect matchings

A matching is a partition of a set in blocks of size at most two and if it has no single-element blocks the matching is said to be perfect. The set of all perfect matchings with n blocks is denoted by $\mathcal{M}_n$. All matchings in this work will be perfect and henceforth we will omit this adjective.

2.1 Statistics based on crossings and nestings

A matching in $\mathcal{M}_n$ can be represented by a graph with $2n$ labeled vertices and n edges in which each vertex has a degree 1. The vertices $1, 2, \ldots, 2n$ are drawn on a horizontal line in natural order and two vertices that are in a same block are connected by a semicircular arc in the upper half-plane. We will use $i \cdot j$ to denote an arc with vertices $i < j$. The vertex i is said to be the opener while j is said to be the closer of the arc. For a vertex i, we will denote by $M(i)$ the other vertex which is in the same block in the matching M as i. Two arcs $i \cdot j$ and $k \cdot l$ with $i < k$ can be in three different relative positions. We say that they form a crossing if $i < k < j < l$, they form a nesting if $i < k < l < j$, and they form an alignment if $i < j < k < l$. The arc with the smaller opener will be called the left arc of the crossing, nesting, or the alignment, respectively, while the arc with the larger opener will be called the right arc. The numbers of crossings, nestings, and alignements in a matching M are denoted by $\mathrm{cr}(M)$, $\mathrm{ne}(M)$, and $\mathrm{al}(M)$, respectively.

If $o_1 < \cdots < o_n$ and $c_1 < \cdots < c_n$ are the openers and the closers in M, respectively, let

$$\mathrm{Long}(M) = \{k : o_k \cdot M(o_k) \text{ is not a right arc in a nesting}\}$$

and

$$\mathrm{Short}(M) = \{k : M(c_k) \cdot c_k \text{ is not a left arc in a nesting}\}.$$

Similarly, let

$$\mathrm{Left}(M) = \{k : o_k \cdot M(o_k) \text{ is not a right arc in a crossing}\}.$$

We will use lower-case letters to denote the cardinalities of the sets. For example, $\mathrm{long}(M) = |\mathrm{Long}(M)|$.

Example 2.1. *For the matching M in Figure 1 we have* $\mathrm{ne}(M) = \mathrm{cr}(M) = \mathrm{al}(M) = 5$, $\mathrm{Long}(M) = \{1, 2\}$, $\mathrm{Short}(M) = \{1, 2, 3, 5\}$, *and* $\mathrm{Left}(M) = \{1, 5\}$.

The pair of sets $(\{o_1, \ldots, o_n\}, \{c_1, \ldots, c_n\})$ of openers and closers of a matching M is called the type of M. There is a natural one-to-one correspondence between types of matchings in $\mathcal{M}_n$ and Dyck paths of semilength n, i.e., lattice paths that start at $(0, 0)$, end at $(2n, 0)$, use steps $(1, 1)$ (rises) and $(1, -1)$ (falls), and never go below the x-axis. The set of all such Dyck paths will be denoted by $\mathcal{D}_n$. Namely, the openers in the type correspond to the rises in the Dyck path while the closers correspond to the falls. Therefore, for convenience, we will say that a matching in $\mathcal{M}_n$ is of type D, for some Dyck path $D \in \mathcal{D}_n$, and we will denote the set of all matchings of type D by $\mathcal{M}_n(D)$.

The height of a rise of a Dyck path is the y-coordinate of the right endpoint of the corresponding $(1, 1)$ segment. The sequence $(h_1, \ldots, h_n)$ of the heights of the rises of a $D \in \mathcal{D}_n$ when read from left to right will be called shortly the height sequence of D. For example, the height sequence of the Dyck path in Figure 1 is $(1, 2, 3, 3, 3, 4)$. A weighted Dyck path is a pair $(D, (w_1, \ldots, w_n))$ where $D \in \mathcal{D}_n$ with height sequence $(h_1, \ldots, h_n)$ and $w_i \in \mathbb{Z}$ with $1 \leq w_i \leq h_i$. There is a well-known bijection φ_1 from the set $\mathcal{WD}_n$ of weighted Dyck paths of semilength n to $\mathcal{M}_n$ [3]. Namely, the openers $o_1 < o_2 < \cdots < o_n$ of the matching that corresponds to a given $(D, (w_1, \ldots, w_n)) \in \mathcal{WD}_n$ are determined according to the type D. To construct the corresponding matching M, we connect the openers from right to left, starting from o_n. After $o_n, o_{n-1}, \ldots, o_{k+1}$ are connected to a closer, there are exactly h_k unconnected closers that are larger than o_k. We connect o_k to the w_k-th of the available closers, when they are listed in decreasing order (see Figure 1).

Via the bijection φ_1 we immediately get the following generating function.

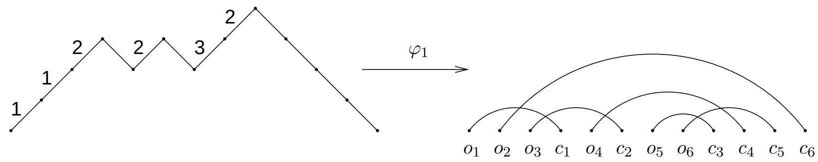

Figure 1: The bijection φ_1 between weighted Dyck paths and matchings.

Theorem 2.2. *If $D \in \mathcal{D}_n$ has a height sequence $(h_1, \ldots, h_n)$, then*

$$\sum_{M\in\mathcal{M}_n(D)} p^{\mathrm{cr}(M)} q^{\mathrm{ne}(M)} \prod_{i\in\mathrm{Left}(M)} s_i \prod_{i\in\mathrm{Long}(M)} t_i = \prod_{k=1}^{n} (t_k p^{h_k-1} + p^{h_k-2} q + \cdots + p q^{h_k-2} + s_k q^{h_k-1}). \quad (2.1)$$

Proof. The edge $o_k \cdot M(o_k)$ will be a right arc in exactly $w_k - 1$ nestings and exactly $h_k - w_k$ crossings in $M = \varphi_1(D, (w_1, \ldots, w_n))$. So, $k \in \mathrm{Long}(M)$ if and only if $w_k = 1$ while the closer that is connected to o_k is in $\mathrm{Left}(M)$ if and only if $w_k = h_k$. $\square$

The map φ_1 also has the following property. The definition of Rlminl was given for permutations but it extends to words in a straightforward way.

Proposition 2.3. *Let $(D, (w_1, \ldots, w_n)) \in \mathcal{WD}_n$ and $M = \varphi_1(D, (w_1, \ldots, w_n))$. Then*

$$\mathrm{Short}(M) = \mathrm{Rlminl}(2 - w_1, 3 - w_2, \ldots, n + 1 - w_n). \quad (2.2)$$

Proof. The proof is by induction on n, the number of arcs in the matching. If $n = 1$, the only matching with one arc is $M = \{o_1 \cdot c_1\}$, and $\mathrm{Short}(M) = \{1\}$. The corresponding weighted Dyck path has only one rise with weight $w_1 = 1$. So, $\mathrm{Rlminl}(2 - w_1) = \mathrm{Rlminl}(1) = \{1\}$.

Suppose (2.2) holds for all matchings with $n - 1$ arcs. If M is a matching with n arcs, openers $o_1 < \cdots < o_n$ and closers $c_1 < \cdots < c_n$, let M' be the matching obtained from M by deleting the arc $o_n \cdot M(o_n)$. The weight sequence associated to M' via the map φ_1^{-1} is $(w_1, \ldots, w_{n-1})$, since $w_k - 1$ is the number of nestings in which the arc of the k-th opener is a right arc, and this number is the same in both M and M'. Not also that $M(o_n) = c_{n+1-w_n}$.

Let the closers in M' be $c'_1 < \cdots < c'_{n-1}$. Then for $i < n$ and $1 \le k < n + 1 - w_n$, $o_i \cdot c_k$ is an arc in M if and only if $o_i \cdot c'_k$ is an arc in M'. On the other hand, for $n + 1 - w_n < k \le n$, $o_i \cdot c_k$ is an arc in M if and only if $o_i \cdot c'_{k-1}$ is an arc in M'.

For a number $k \in [n]$ there are two possibilities:

1. $k \in \mathrm{Rlminl}(2 - w_1, 3 - w_2, \ldots, n + 1 - w_n)$

   If $k = n + 1 - w_n$ then $k \in \mathrm{Short}(M)$ because $o_n \cdot c_{n+1-w_n}$ is an arc in M and there are no arcs nested below it.

   If $k \neq n+1-w_n$ then necessarily $k < n+1-w_n$. Also $k \in \mathrm{Rlminl}(2-w_1, 3-w_2, \ldots, n-w_{n-1}) = \mathrm{Short}(M')$, which implies that the arc $M'(c'_k) \cdot c'_k$ in M' has no arcs nested below it. But then $M(c_k) \cdot c_k$ is an arc in M and the additional arc $o_n \cdot c_{n+1-w_n}$ in M is not nested below it. So, $k \in \mathrm{Short}(M)$.

2. $k \notin \mathrm{Rlminl}(2-w_1, 3-w_2, \ldots, n+1-w_n)$

   Necessarily, $k \neq n+1-w_n$.

   If $k < n+1-w_n$, then $k \notin \mathrm{Rlminl}(2-w_1, 3-w_2, \ldots, n-w_{n-1})$ and, by the induction hypothesis, there is an arc $o_r \cdot c'_s$ nested below $M'(c'_k) \cdot c'_k$ in M'. But then the arc $o_r \cdot c_s$ is nested below $M(c_k) \cdot c_k$ in M, and consequently, $k \notin \mathrm{Short}(M)$.

   If $k > n+1-w_n$, then since o_n is the largest opener in M and $c_k > c_{n+1-w_n}$, the arc $o_n \cdot c_{n+1-w_n}$ is nested below $M(c_k) \cdot c_k$, and so $k \notin \mathrm{Short}(M)$.

□

2.2 Cycles and sorting index for matchings

Let M_0 be a matching in $\mathcal{M}_n(D)$. For $M \in \mathcal{M}_n(D)$ define $\mathrm{cyc}(M, M_0)$ as the number of cycles in the graph $G = (M, M_0)$ on $2n$ vertices in which the arcs from M are drawn in the upper half-plane as usual and the arcs of M_0 are drawn in the lower half-plane, reflected about the number axis. If the openers of M are $o_1 < \cdots < o_n$, we define

$$\mathrm{Cyc}(M, M_0) = \{k : o_k \text{ is a minimal vertex in a cycle in the graph } (M, M_0)\}.$$

Figure 2 shows the calculation of cyc and Cyc for all matchings of type with respect to the nonnesting matching of that type.

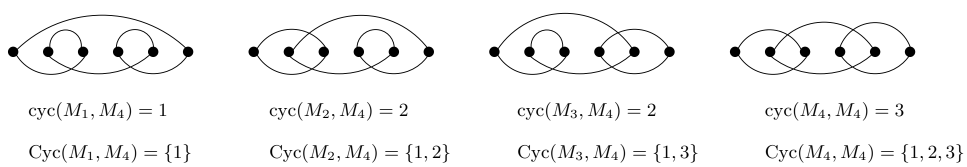

Figure 2: Counting cycles in matchings.

For $M, M_0 \in \mathcal{M}_n(D)$, we define the sorting index of M with respect to M_0, denoted by $\mathrm{sor}(M, M_0)$, in the following way. Let $o_1 < o_2 < \cdots < o_n$ be the openers in M and M_0. We construct a sequence of matchings $M_n, M_{n-1}, \ldots, M_2, M_1$ as follows. First, set $M_n = M$. Then, if $M_k(o_k) = M_0(o_k)$, set $M_{k-1} = M_k$. Otherwise, set M_{k-1} to be the matching obtained by replacing the edges $o_k \cdot M_k(o_k)$ and $M_k(M_0(o_k)) \cdot M_0(o_k)$ in the matching M_k by the edges $o_k \cdot M_0(o_k)$ and $M_k(M_0(o_k)) \cdot M_k(o_k)$. It follows from the definition that $M_1 = M_0$. In other words, we gradually sort the matching M by reconnecting the openers to the closers as "prescribed" by M_0. Note that when swapping of edges takes place, it is always true that $M_k(M_0(o_k)) < o_k$ and therefore all the intermediary matchings we get in the process are of type D. Define

$$\mathrm{sor}_k(M, M_0) = \begin{cases} |\{c : c > o_k, c \in [M_k(o_k), M_0(o_k)] \text{ and } M_0(c) < o_k\}|, & \text{if } M_k(o_k) \leq M_0(o_k) \\ |\{c : c > o_k, c \notin (M_0(o_k), M_k(o_k)) \text{ and } M_0(c) < o_k\}|, & \text{if } M_0(o_k) < M_k(o_k) \end{cases}$$

and

$$\mathrm{sor}(M, M_0) = \sum_{k=1}^{n} \mathrm{sor}_k(M, M_0).$$

Example 2.4. *Figure 3 shows the intermediate matchings that are obtained when $M = M_6$ is sorted to $M_0 = M_1$. So,*

$$\begin{array}{lll} \mathrm{sor}_6(M, M_0) = |\{c_3, c_5, c_6\}| = 3, & \mathrm{sor}_5(M, M_0) = |\{c_3, c_5\}| = 2, & \mathrm{sor}_4(M, M_0) = |\{c_2, c_5\}| = 2, \\ \mathrm{sor}_3(M, M_0) = |\emptyset| = 0, & \mathrm{sor}_2(M, M_0) = |\{c_5\}| = 1, & \mathrm{sor}_1(M, M_0) = |\emptyset| = 0, \end{array}$$

and $\mathrm{sor}(M, M_0) = 0 + 1 + 0 + 2 + 2 + 3 = 8$.

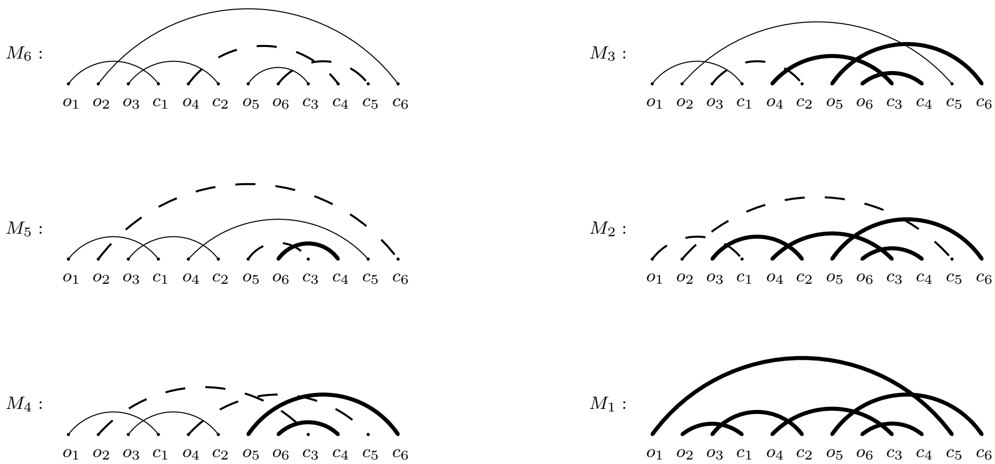

Figure 3: Sorting of the matching $M = M_6$ to the matching $M_0 = M_1$. The dashed lines indicate arcs that are about to be swapped while the bold lines represent arcs that have been placed in correct position.

Theorem 2.5. *Let D be a Dyck path with height sequence $(h_1, \ldots, h_n)$. For each $M_0 \in \mathcal{M}_n(D)$, there is a bijection*

$$\phi_1 : \{(w_1, w_2, \ldots, w_n) : 1 \leq w_i \leq h_i\} \to \mathcal{M}_n(D)$$

which depends on M_0 such that

(a) $\mathrm{sor}(\phi_1(w_1, \ldots, w_n), M_0) = \sum_{i=1}^{n}(w_i - 1)$,

(b) $\mathrm{Cyc}(\phi_1(w_1, \ldots, w_n), M_0) = \{k : w_k = 1\}$.

Additionally, if M_0 is the unique nonnesting matching of type D, then

(c) $\mathrm{Short}(\phi_1(w_1, \ldots, w_n)) = \mathrm{Rlminl}(2 - w_1, 3 - w_2, \ldots, n + 1 - w_n)$.

Proof. Fix $M_0 \in \mathcal{M}_n(D)$. We construct the bijection ϕ_1 in the following way. Draw the matching M_0 with arcs in the lower half-plane. Suppose $o_1 < \cdots < o_n$ are the openers of M_0. To construct $M = \phi_1(w_1, \ldots, w_n)$, we draw arcs in the upper half plane by connecting the openers from right to left to closers as follows.

Suppose that the openers $o_n, o_{n-1}, \ldots, o_{k+1}$ are already connected to a closer and denote the partial matching in the upper half-plane by N_k. To connect o_k, we consider all the closers c with the property $c > o_k$ and $M_0(c) \leq o_k$. There are exactly h_k such closers, call them candidates for o_k.

Let c_{k_0} be the closer which is w_k-th on the list when all those h_k candidates are listed starting from $M_0(o_k)$ and then going cyclically to left. If c_{k_0} is not connected to an opener by an arc in the upper half-plane, draw the arc $o_k \cdot c_{k_0}$. Otherwise, there is a maximal path in the graph of the type: $c_{k_0}, N_k(c_{k_0}), M_0(N_k(c_{k_0})), N_k(M_0(N_k(c_{k_0}))), \ldots, c^*$ which starts with c_{k_0}, follows arcs in N_k and M_0 alternately and ends with a closer c^* which has not been connected to an opener yet (see Figure 4). Due to the order in which we have been drawing the arcs in the upper half-plane, all vertices in the aforementioned path are to the right of o_k. In particular, c^* is to the right of o_k and is not one of the candidates for o_k. Draw an arc in the upper half-plane connecting o_k to c^*. After all openers are connected in this manner, the resulting matching in the upper half-plane is $M = \phi_1(w_1, \ldots, w_n)$.

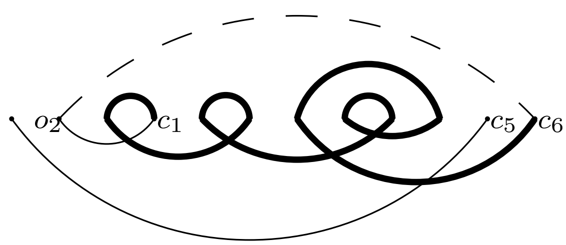

Figure 4: The solid arcs in the top half-plane represent the partial matching N_2. The candidates for o_2 are c_1 and c_5. If $w_2 = 1$, o_2 will try to connect to c_1, but since it is already connected to an opener, we follow the bold path that starts with c_1 to reach $c^* = c_6$ and connect it to o_2.

Let $M_n = M, M_{n-1}, \ldots, M_2, M_1 = M_0$ be the intermediary sequence of matchings constructed when sorting M to M_0. Then $M_k(o_k)$ is exactly the closer c_{k_0} defined above. This means that $\mathrm{sor}_k(M, M_0) = w_k - 1$ and therefore $\mathrm{sor}(M, M_0) = \sum_{k=1}^{n}(w_k - 1)$. This property also gives us a way of finding the sequence $w_1, \ldots, w_n)$ which corresponds to a given $M \in \mathcal{M}_n(D)$. Namely, $w_k = \mathrm{sor}_k(M, M_0) + 1$.

To prove the second property of ϕ_1, we analyze when connecting o_k by an arc will close a cycle. There are two cases.

1. The closer c_{k_0} which was w_k-th on the list of candidates for o_k was not incident to an arc in the partial matching N_k and we drew the arc $o_k \cdot c_{k_0}$. If $w_k = 1$, then $c_{k_0} = M_0(o_k)$ and the arcs connecting o_k and c_{k_0} in the upper and lower half-planes close a cycle. Otherwise, $M_0(c_{k_0}) < o_k$ and therefore $M_0(c_{k_0})$ is not incident to an arc in N_k and the arc $o_k \cdot c_{k_0}$ will not close a cycle.

2. The closer c_{k_0} which was w_k-th on the list of candidates for o_k was incident to an arc in the partial matching N_k and we drew the arc $o_k \cdot c^*$. If $w_k = 1$, the path traced from c_{k_0} to c^*, the arc $o_k \cdot c_{k_0}$ in M_0, and the newly added arc $o_k \cdot c^*$ form a cycle. Otherwise, connecting o_k to c^* does not close a cycle since the opener $M_0(c_{k_0})$ is in the same connected component of the graph (M, M_0) as o_k, but is not connected to a closer yet, since $M_0(c_{k_0}) < o_k$.

We conclude that a cycle is closed exactly when $w_k = 1$ and therefore

$$\mathrm{Cyc}(\phi_1(w_1, \ldots, w_n), M_0) = \{k : w_k = 1\}.$$

Finally, we prove the third property of ϕ_1. If M_0 is a nonnesting matching, its edges are $o_k \cdot c_k$ where the openers and closers are indexed in ascending order. Let $M = \phi_1(w_1, \ldots, w_n)$. The following observations are helpful. When connecting o_k in the construction of M, the first choice for o_k, i.e., the w_k-th candidate for o_k is exactly c_{k+1-w_k}. Also, $M(o_k) \geq c_{k+1-w_k}$. Furthermore, if c_k was not a candidate for $M(c_k)$, i.e. if the edge c_k was chosen as a partner for $M(c_k)$ by following a path in the graph as described above, then $k \notin \mathrm{Short}(M)$. Namely the edge $M(c_{k_0}) \cdot c_{k_0}$, where c_{k_0} was the first choice when the opener $M(c_k)$ was connected in the construction of M, is nested below it.

For a number $k \in [n]$ there are three possibilities:

1. $k \notin \{2 - w_1, 3 - w_2, \ldots, n + 1 - w_n\}$

   In this case, c_k was not a first choice for any of the openers and therefore must have been connected to an opener by following a path in the graph (M, M_0). It follows from the observation above that $k \notin \mathrm{Short}(M)$.

2. $k \in \{2 - w_1, 3 - w_2, \ldots, n + 1 - w_n\}$ and $k \in \mathrm{Rlminl}(2 - w_1, 3 - w_2, \ldots, n + 1 - w_n)$

   Then c_k was a first choice for at least one opener. Let o be the largest one. Then all openers to the right of o got connected to a closer which is greater than c_k, so no edge is nested below $o \cdot c_k \in M$. Consequently, $k \in \mathrm{Short}(M)$.

3. $k \in \{2 - w_1, 3 - w_2, \ldots, n + 1 - w_n\}$ but $k \notin \mathrm{Rlminl}(2 - w_1, 3 - w_2, \ldots, n + 1 - w_n)$

   In this case, let $m+1-w_m$ be the rightmost number in the sequence $(2-w_1, \ldots, n+1-w_n)$ which is smaller than k. It is necessarily to the right of k in this sequence and belongs to $\mathrm{Rlminl}(2 - w_1, \ldots, n + 1 - w_n)$. This implies that the edge $o_m \cdot c_{m+1-w_m}$ is in M, while $M(o_l) > c_k$ for all $l > m$. So, $M(c_k) < o_m$ and therefore the edge $o_m \cdot c_{m+1-w_m}$ is nested below $M(c_k) \cdot c_k$, which means that $k \notin \mathrm{Rlminl}(2 - w_1, 3 - w_2, \ldots, n + 1 - w_n)$.

□

Corollary 2.6. *Let $M_0 \in \mathcal{M}_n(D)$ and let $(h_1, \ldots, h_n)$ be the height sequence of D. Then*

$$\sum_{M \in \mathcal{M}_n(D)} q^{\mathrm{sor}(M,M_0)} \prod_{i \in \mathrm{Cyc}(M,M_0)} t_i = \prod_{k=1}^{n} (t_k + q + \cdots + q^{h_k - 1}). \tag{2.3}$$

Combining Theorem 2.2 and Corollary 2.6 we get the following corollary.

Corollary 2.7. *Let $M_0 \in \mathcal{M}_n(D)$ and let $(h_1, \ldots, h_n)$ be the height sequence of D. Then*

$$\sum_{M \in \mathcal{M}_n(D)} q^{\mathrm{sor}(M,M_0)} \prod_{i \in \mathrm{Cyc}(M,M_0)} t_i = \sum_{M \in \mathcal{M}_n(D)} q^{\mathrm{ne}(M)} \prod_{i \in \mathrm{Long}(M)} t_i.$$

Corollary 2.8. *If M_0 is the unique nonnesting matching of type D then the multisets*

$$\{(\mathrm{sor}(M, M_0), \mathrm{Cyc}(M, M_0), \mathrm{Short}(M)) : M \in \mathcal{M}_n(D)\}$$

and

$$\{(\mathrm{ne}(M), \mathrm{Long}(M), \mathrm{Short}(M)) : M \in \mathcal{M}_n(D)\}$$

are equal.

Proof. Follows from Proposition 2.3 and Theorem 2.5. □

2.3 Connections with restricted permutations

For a fixed n, let $\mathbf{r}$ denote the non-decreasing sequence of integers $1 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq n$. Let

$$S_{\mathbf{r}} = \{\sigma \in S_n : \sigma(k) \leq r_k, 1 \leq k \leq n\}.$$

Note that $S_{\mathbf{r}} \neq \emptyset$ precisely when $r_k \geq k$, for all k, so we will consider only the sequences that satisfy this condition without explicitly mentioning it. Let $D(\mathbf{r})$ be the unique Dyck path whose k-th fall is preceded by exactly r_k rises. Consider the following bijection $f_{\mathbf{r}} : S_{\mathbf{r}} \to \mathcal{M}_n(D(\mathbf{r}))$. If $\sigma \in S_{\mathbf{r}}$, then $f_{\mathbf{r}}(\sigma)$ is the matching in $\mathcal{M}_n(D(\mathbf{r}))$ with edges $o_{\sigma(k)} \cdot c_k$, where $o_1 < \cdots < o_n$ are the openers and $c_1 < \cdots < c_n$ are the closers. It is not difficult to see that $f_{\mathbf{r}}$ is well defined and that it is a bijection.

Two arcs $o_{\sigma(j)} \cdot c_j$ and $o_{\sigma(k)} \cdot c_k$ in $f_{\mathbf{r}}(\sigma)$ with $j < k$ form a nesting if and only if $\sigma(j) > \sigma(k)$. So, $\mathrm{ne}(f_{\mathbf{r}}(\sigma)) = \mathrm{inv}(\sigma)$. Moreover, $\sigma(j) \in \mathrm{Rlminl}(\sigma)$ if and only if $\sigma(j)$ does not form an inversion with a $\sigma(k)$ for any $k > j$, which means if and only if $o_{\sigma(j)} \cdot c_j$ is not nested within anything in $f_{\mathbf{r}}(\sigma)$, i.e., $\sigma(j) \in \mathrm{Long}(f_{\mathbf{r}}(\sigma))$. From Theorem 2.2 we get the following corollary.

Corollary 2.9. *Let $\mathbf{r}$ be a non-decreasing sequence of integers $1 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq n$ with $r_k \geq k$, for all k. Then*

$$\sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{inv}(\sigma)} \prod_{i \in \mathrm{Rlminl}(\sigma)} t_i = \prod_{k=1}^{n} (t_k + q + q^2 + \cdots + q^{h_k - 1})$$

where $(h_1, \ldots, h_n)$ is the height sequence of $D(\mathbf{r})$. In particular,

$$\sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{rlminl}(\sigma)} = \prod_{k=1}^{n} (t + q + q^2 + \cdots + q^{r_k - k}).$$

Proof. The first result follows directly from the discussion above and Theorem 2.2. For the second equality, note that the height sequence $(h_1, \ldots, h_n)$ of the Dyck path $D(\mathbf{r})$ is a permutation of the sequence of the heights of the falls in $D(\mathbf{r})$, where the height of a fall is the y-coordinate of the higher end of the corresponding $(1, -1)$ step. The height of the k-th fall is easily seen to be $r_k - k + 1$. $\square$

In particular, when $r_1 = r_2 = \cdots = r_n = n$, we have $S_{\mathbf{r}} = S_n$. The height sequence of $D(\mathbf{r})$ is $(1, 2, \ldots, n)$ and we recover the result of Björner and Wachs about the distribution of $(\mathrm{inv}, \mathrm{Rlmin})$ given in (1.2).

If $M_0 \in \mathcal{M}(D(\mathbf{r}))$ the sorting index $\mathrm{sor}(\,\cdot\,, M_0)$ induces a permutation statistic on $S_{\mathbf{r}}$. Namely, if $\sigma, \sigma_0 \in S_{\mathbf{r}}$, define

$$\mathrm{sor}_{\mathbf{r}}(\sigma, \sigma_0) = \mathrm{sor}(f_{\mathbf{r}}^{-1}(\sigma), f_{\mathbf{r}}^{-1}(\sigma_0)).$$

Equivalently, the statistic $\mathrm{sor}_{\mathbf{r}}(\sigma, \sigma_0)$ on $S_{\mathbf{r}}$ can be defined directly via a sorting algorithm similar to Straight Selection Sort. Namely, permute the elements in $\sigma \in S_{\mathbf{r}}$ by applying transpositions which place the largest element n in position $\sigma_0^{-1}(n)$, then the element $n-1$ in position $\sigma_0^{-1}(n-1)$, etc. Let $\sigma_n = \sigma, \sigma_{n-1}, \ldots, \sigma_1 = \sigma_0$, be the sequence of permutations obtained in this way. Specifically, $\sigma_k^{-1}(i) = \sigma_0^{-1}(i)$ for $i > k$, and σ_{k-1} is obtained by swapping k and $\sigma_k(\sigma_0^{-1}(k))$ in σ_k.

Let $l = \sigma_k^{-1}(k)$ and $m = \sigma_0^{-1}(k)$. Define

$$a_k = \begin{cases} |\{i : l \leq i \leq m, \sigma_0(i) < k\}|, & l < m \\ 0, & l = m \\ |\{i : r_i \geq k, i \notin (m,l), \sigma_0(i) < k\}|, & l > m. \end{cases} \tag{2.4}$$

Then

$$\mathrm{sor}_{\mathbf{r}}(\sigma, \sigma_0) = \sum_{k=1}^{n} a_k.$$

Note that, $\mathrm{sor}_{\mathbf{r}}(\sigma, \sigma_0)$ in general depends on $\mathbf{r}$. However, the case when σ_0 is the identity permutation is an exception.

Lemma 2.10. *Let $\mathbf{r}$ be a non-decreasing sequence of integers $1 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq n$ with $r_k \geq k$, for all k. Let $\sigma \in S_{\mathbf{r}}$. Then*

$$\mathrm{sor}_{\mathbf{r}}(\sigma, \mathbf{id}) = \mathrm{sor}(\sigma).$$

Proof. First note that the case $l > m$ in (2.4) cannot occur. Namely, in the case when $\sigma_0 = \mathbf{id}$, we have $m = k$ and if $l > k$, $\sigma_k^{-1}(l) = \sigma_0^{-1}(l) = l$. This contradicts $l = \sigma_k^{-1}(k)$. Therefore, the definition of a_k simplifies to

$$a_k = |\{i : l \leq i < k\}|.$$

This is precisely the "distance" that k travels when being placed in its correct position with the Straight Selection Sort algorithm. $\square$

Corollary 2.11. *Let $\mathbf{r}$ be a non-decreasing sequence of integers $1 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq n$ with $r_k \geq k$, for all k. Let $\sigma_0 \in S_{\mathbf{r}}$. Then*

$$\sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{sor}_{\mathbf{r}}(\sigma,\sigma_0)} \prod_{i \in \mathrm{Cyc}(\sigma\sigma_0^{-1})} t_i = \prod_{i=1}^{n} (t_i + q + \cdots + q^{h_i - 1}), \tag{2.5}$$

where $(h_1, \ldots, h_n)$ is the height sequence of $D(\mathbf{r})$ and $\mathrm{Cyc}(\sigma)$ is the set of the minimal elements in the cycles of σ. In particular,

$$\sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{sor}(\sigma)} \prod_{i \in \mathrm{Cyc}(\sigma)} t_i = \prod_{i=1}^{n} (t_k + q + \cdots + q^{h_k - 1}) \tag{2.6}$$

and

$$\sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{sor}(\sigma)} t^{\mathrm{cyc}(\sigma)} = \sum_{\sigma \in S_{\mathbf{r}}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{rlminl}(\sigma)} \tag{2.7}$$

Proof. Let $f_{\mathbf{r}}(\sigma_0) = M_0$ and $f_{\mathbf{r}}(\sigma) = M$. The cycle $k \to \sigma_0\sigma^{-1}(k) \to \cdots \to (\sigma_0\sigma^{-1})^s(k) = k$ of the permutation $\sigma_0\sigma^{-1}$ corresponds to the cycle $o_k \frown M(o_k) \smile M_0(M(o_k)) \frown \cdots \smile o_k$ in the graph (M, M_0). So, $k \in \mathrm{Cyc}(\sigma_0\sigma^{-1})$ if and only if $k \in \mathrm{Cyc}(M, M_0)$. Now, (2.5) follows from (2.3) and the fact that the cycles of $\sigma\sigma_0^{-1}$ are equal to the cycles of $\sigma_0\sigma^{-1}$ reversed. Since $\mathbf{id} \in S_{\mathbf{r}}$ for every sequence $\mathbf{r}$, we get (2.6) as a corollary of Lemma 2.10. $\square$

Let $\mathrm{Lrmaxp}(\sigma)$ denote the set of left-to-right maximum places in the permutation σ, i.e,

$$\mathrm{Lrmaxp}(\sigma) = \{k : \sigma(k) > \sigma(j) \text{ for all } j < k\}.$$

From Corollary 2.8 we get the following result for restricted permutations.

Corollary 2.12. *The triples* $(\mathrm{inv}, \mathrm{Rlminl}, \mathrm{Lrmaxp})$ *and* $(\mathrm{sor}, \mathrm{Cyc}, \mathrm{Lrmaxp})$ *are equidistributed on* $S_{\mathbf{r}}$*. That is, the multisets*

$$\{(\mathrm{inv}(\sigma), \mathrm{Rlminl}(\sigma), \mathrm{Lrmaxp}(\sigma)) : \sigma \in S_{\mathbf{r}}\}$$

and

$$\{(\mathrm{sor}(\sigma), \mathrm{Cyc}(\sigma), \mathrm{Lrmaxp}(\sigma)) : \sigma \in S_{\mathbf{r}}\}$$

are equal.

The equidistribution of the pairs $(\mathrm{Rlminl}, \mathrm{Lrmaxp})$ and $(\mathrm{Cyc}, \mathrm{Lrmaxp})$ on $S_{\mathbf{r}}$ for the special case when the corresponding Dyck path $D(\mathbf{r})$ is of the form $u^{k_1}d^{k_1}u^{k_2}d^{k_2}\cdots u^{k_s}d^{k_s}$ was shown by Foata and Han [4] .

Corollary 2.13. *Let* $\sigma_0 \in S_{\mathbf{r}}$*. Then*

$$\sum_{\sigma \in S_{\mathbf{r}}} t^{\mathrm{cyc}(\sigma\sigma_0^{-1})} = \prod_{k=1}^{n} (t + r_k - k). \tag{2.8}$$

In particular, the left-hand side of (2.8) *does not depend on* σ_0*.*

We remark that the sets $\{\sigma\sigma_0^{-1} : \sigma \in S_{\mathbf{r}}\}$ and $S_{\mathbf{r}}$ are in general not equal. For example, let $\sigma_0 = 143265 \in S_{[4,4,4,6,6,6]}$. Then $\sigma = 231546 \in S_{[4,4,4,6,6,6]}$ but $\sigma\sigma_0^{-1} = 251364 \notin S_{[4,4,4,6,6,6]}$.

The polynomial $\prod_{k=1}^{n}(t + r_k - k)$ is well-known in rook theory. It is equal [5] to the polynomial

$$\sum_{k=0}^{n} r_{n-k}(t-1)(t-2)\cdots(t-k)$$

where r_k is the number of placements of k non-atacking rooks on a Ferrers board with rows of length $r_1, r_2, \ldots, r_n$.

3 Bicolored matchings

In this section we consider statistics on the set $\mathcal{M}_n^{(2)}$ of bicolored matchings on $[2n]$, whose n edges are colored with one of two colors: red or blue.

3.1 Bicolored crossings and nestings

Bicolored matchings have four types of crossings, depending on the color of the right and the left edge that form the crossing, as well as four types of nestings and four types of alignments. Let $\mathrm{cr}_{*r}(M)$ be the number of crossings in M in which the right edge is red, regardless of the color of the left edge, and analogously define the numbers $\mathrm{cr}_{*b}(M)$, $\mathrm{ne}_{*r}(M)$, $\mathrm{ne}_{*b}(M)$, $\mathrm{al}_{*r}(M)$, $\mathrm{al}_{*b}(M)$.

Additionally, let $\mathrm{b}(M)$ denote the total number of blue edges in M, and let $\mathrm{longr}(M)$ denote the number of long red edges – red edges in M that are not nested within any other edge, while

$$\mathrm{Longr}(M) = \{k : o_k \cdot M(o_k) \text{ is a large red edge}\}.$$

The generating function of these refined statistics for bicolored matchings of type D

$$P_D(\mathbf{q}, p, \mathbf{t}) = \sum_{M \in \mathcal{M}_n^{(2)}(D)} q_1^{\mathrm{ne}_{*r}(M)} q_2^{\mathrm{ne}_{*b}(M)} q_3^{\mathrm{cr}_{*r}(M)} q_4^{\mathrm{cr}_{*b}(M)} q_5^{\mathrm{al}_{*r}(M)} q_6^{\mathrm{al}_{*b}(M)} p^{\mathrm{b}(M)} \prod_{i \in \mathrm{Longr}(M)} t_i$$

is given by the following theorem. In the proof we will use the set $\mathcal{WD}_n^{(2)}$ of all (partially) bicolored weighted Dyck paths whose rises are colored red or blue. The elements in $\mathcal{WD}_n^{(2)}$ can be written as triples $(D, (w_1, \ldots, w_n), (\epsilon_1, \ldots, \epsilon_n))$ where D is a Dyck path of semilength n, $w_i \in \mathbb{Z}$ with $1 \leq w_i \leq h_i$, where $(h_1, \ldots, h_n)$ is the height sequence of D, and $\epsilon_i \in \{0, 1\}$, for $1 \leq i \leq n$. Here we are using $\epsilon_i = 0$ to represent a red rise and $\epsilon_i = 1$ to represent a blue rise.

Theorem 3.1. *Let D be a Dyck path with height sequence $(h_1, \ldots, h_n)$. Then*

$$P_D(\mathbf{q}, p, \mathbf{t}) = \prod_{i=1}^{n} \sum_{k=1}^{h_i} (q_1^{k-1} q_3^{h_i-k} q_5^{i-h_i} t_i^{\delta_{k,1}} + q_2^{h_i-k} q_4^{k-1} q_6^{i-h_i} p),$$

where $\delta_{i,j}$ is the Kronecker delta function.

Proof. To find $P_D(q_1, q_2, q_3, q_4, q_5, q_6, p, t)$ we will describe an appropriate bijection φ_2 from $\mathcal{WD}_n^2$ to $\mathcal{M}_n^{(2)}$. Let $(D, (w_1, \ldots, w_n), (\epsilon_1, \ldots, \epsilon_n)) \in \mathcal{WD}_n^{(2)}$. The corresponding matching M has type D and is constructed in the following way. The openers $o_1 < \cdots < o_n$ of M are connected to closers from right to left starting with o_n. If $\epsilon_k = 0$, connect the opener o_k to the w_i-th available closer from right to left and color the edge red. If $\epsilon_k = 1$, connect the opener to the $(h_i - w_i + 1)$-st available closer to the right of o_k counted from right to left and color the edge blue. Then if $\epsilon_k = 0$ the corresponding red edge $o_k \cdot M(o_k)$ will be a right edge in $w_i - 1$ nestings, $h_i - w_i$ crossings, and $i - h_i$ alignments. It will be a long edge if and only if $w_i = 1$. Similarly, if $\epsilon_k = 1$, then the blue edge $o_k \cdot M(o_k)$ will be a right edge in $h_i - w_i$ nestings, $w_i - 1$ crossings, and $i - h_i$ alignments. It will not be a long edge but it will contribute to $\mathrm{b}(M)$. The theorem follows by summing over all possible colorings and weightings of the path D. □

This theorem gives the generating function for statistics that can be defined in terms of nestings, crossing, and alignments with right edges of specified color. In particular, define

$$\mathrm{mix}(M) = \mathrm{ne}(M) + 2\mathrm{cr}_{*b}(M) + 2\mathrm{al}_{*b}(M) + \mathrm{b}(M).$$

Corollary 3.2. *Let D be a Dyck path with height sequence $(h_1, \ldots, h_n)$.*

$$\sum_{M \in \mathcal{M}_n^{(2)}(D)} q^{\mathrm{mix}(M)} \prod_{i \in \mathrm{Longr}(M)} t_i = \prod_{k=1}^{n} \left(t_k + q[h_k - 1]_q + q^{2k-h_k}[h_k]_q \right).$$

Proof. It follows from Theorem 3.1 by setting $q_1 = q_2 = q$, $q_4 = q_6 = q^2$, $q_3 = q_5 = 1$, and $p = q$. □

3.2 Sorting index and cycles for bicolored matchings

For a vertex v in a bicolored matching M, denote

$$\operatorname{col}(v,M)=\begin{cases}0, & \text{if the edge in } M \text{ incident with } v \text{ is red}\\ 1, & \text{if the edge in } M \text{ incident with } v \text{ is blue.}\end{cases}$$

Let M_0 be a matching with only red edges of type D. For $M \in \mathcal{M}_n^{(2)}(D)$ we define $\operatorname{sor}(M, M_0)$ similarly as for monochromatic matchings. Namely, let $o_1 < o_2 < \cdots < o_n$ be the openers in M (and consequently in M_0 as well). Define the sequence of bicolored matchings

$$M = M_n, M_{n-1}, \ldots, M_2, M_1$$

as follows. Supose $M_n, \ldots, M_k$ are defined for some $k \le n$. Then, if $M_k(o_k) = M_0(o_k)$ the matching M_{k-1} has the same edges as M_k with colors

$$\operatorname{col}(o_i, M_{k-1})=\begin{cases}\operatorname{col}(o_i, M_k), & \text{if } i \neq k\\ 0, & \text{if } i = k.\end{cases}$$

Otherwise, M_{k-1} is the bicolored matching obtained by replacing the two edges $o_k \cdot M_k(o_k)$ and $M_k(M_0(o_k)) \cdot M_0(o_k)$ in the matching M_k by the edges $o_k \cdot M_0(o_k)$ and $M_k(M_0(o_k)) \cdot M_k(o_k)$ and setting their colors to be

$$\operatorname{col}(o_i, M_{k-1})=\begin{cases}\operatorname{col}(o_i, M_k), & \text{if } o_i \neq o_k \text{ and } o_i \neq M_k(M_0(o_k))\\ \operatorname{col}(o_i, M_k)+\operatorname{col}(o_k, M_k) (\operatorname{mod} 2), & \text{otherwise.}\end{cases}$$

In other words, the sequence of matchings $M = M_n, M_{n-1}, \ldots, M_2, M_1$ is the one we get when we gradually sort the matching M from right to left by connecting the openers to the closers as prescribed by M_0 and recoloring edges depending on the color of the edge which is currently being "processed". Note that $\operatorname{col}(o_k, M_i) = 0$ for $i < k$ and so the final matching that we get after n steps is indeed the desired M_0.

The sorting index $\operatorname{sor}(M, M_0)$ is now defined in the following way.
If $\operatorname{col}(o_k, M_k) = 0$ define

$$\operatorname{sor}_k(M, M_0)=\begin{cases}|\{c : c > o_k, c \in [M_k(o_k), M_0(o_k)] \text{ and } M_0(c) < o_k\}|, & \text{if } M_k(o_k) \le M_0(o_k)\\ |\{c : c > o_k, c \notin (M_0(o_k), M_k(o_k)) \text{ and } M_0(c) < o_k\}|, & \text{if } M_0(o_k) < M_k(o_k)\end{cases}$$

while if $\operatorname{col}(o_k, M_k) = 1$ define

$$\operatorname{sor}_k(M, M_0)=\begin{cases}2k-1-|\{c : c > o_k, c \in [M_k(o_k), M_0(o_k)], M_0(c) < o_k\}|, & \text{if } M_k(o_k) \le M_0(o_k)\\ 2k-1-|\{c : c > o_k, c \notin (M_0(o_k), M_k(o_k)), M_0(c) < o_k\}|, & \text{if } M_0(o_k) < M_k(o_k).\end{cases}$$

Then $\operatorname{sor}(M, M_0)$ is defined as

$$\operatorname{sor}(M, M_0) = \sum_{k=1}^{n} \operatorname{sor}_k(M, M_0).$$

In particular, if M has only red edges, $\mathrm{sor}(M, M_0)$ is equal to $\mathrm{sor}(M, M_0)$ defined for monochromatic matchings in Section 2.

Similarly to the monochromatic case, we define $\mathrm{cyc}(M, M_0)$ to be the number of cycles in the graph (M, M_0). However, in this case we can distinguish between two types of cycles: $\mathrm{cyc}_0(M, M_0)$ will denote the number of cycles with even number of blue edges, while $\mathrm{cyc}_1(M, M_0)$ will denote the number of cycles with odd number of blue edges. We will denote by $\mathrm{Cyc}_0(M, M_0)$ and $\mathrm{Cyc}_1(M, M_0)$ the indices of the minimal openers in the respective sets.

Theorem 3.3. *Let $M_0 \in \mathcal{M}_n^{(2)}(D)$ be a matching with all red edges of type D and let $(h_1, \dots, h_n)$ be the height sequence of D. There is a bijection which depends on M_0*

$$\phi_2 : \{((w_1, \dots, w_n), (\epsilon_1, \dots, \epsilon_n)) : w_i \in \mathbb{Z}, 1 \le w_i \le h_i, \epsilon_i \in \{0, 1\}\} \to M_n^{(2)}(D)$$

such that the matching $M = \phi_2((w_1, \dots, w_n), (\epsilon_1, \dots, \epsilon_n))$ has the following properties:

$$\mathrm{sor}(M, M_0) = \sum_{k=1}^{n} (w_k + \epsilon_k(2k - h_k) - 1), \tag{3.1}$$

and

$$\mathrm{cyc}(M, M_0) = |\{k : w_k = 1 + \epsilon_k(h_k - 1)\}|. \tag{3.2}$$

Moreover, $\mathrm{Cyc}_0(M, M_0) = \{k : (w_k, \epsilon_k) = (1, 0)\}$ *and* $\mathrm{Cyc}_1(M, M_0) = \{k : (w_k, \epsilon_k) = (h_k, 1)\}$.

Proof. We construct ϕ_2 in the following way. Draw the matching M_0 with red arcs in the lower half-plane. Let $o_1 < \cdots < o_n$ be the vertices that are to be openers in $M = \varphi((w_1, \dots, w_n), (\epsilon_1, \dots, \epsilon_n))$ as determined by the type D. We draw arcs in the upper half plane by connecting the openers from right to left to closers as follows.

Suppose that the openers $o_n, o_{n-1}, \dots, o_{k+1}$ are already connected to a closer and denote the partial matching in the upper half-plane by N_k. In particular, N_n is the empty matching. To connect o_k, we consider all the closers c with the property $c > o_k$ and $M_0(c) \le o_k$. Note that there are exactly h_k such closers, call them candidates for o_k.

If $\epsilon_k = 0$, let c_{k_0} be the closer which is w_k-th on the list when all those h_k candidates are listed starting from $M_0(o_k)$ and then going cyclically to the left. Otherwise, let c_{k_0} be the closer which is $(h_k - w_k + 1)$-st on that list. If c_{k_0} is not connected to an opener by an arc in the upper half-plane, draw the arc $o_k \cdot c_{k_0}$ with color $\mathrm{col}(o_k, N_{k-1}) = \epsilon_k$. Otherwise, there is a maximal path in the graph of the type

$$c_{k_0}, N_k(c_{k_0}), M_0(N_k(c_{k_0})), N_k(M_0(N_k(c_{k_0}))), \dots, c^* \tag{3.3}$$

which starts with c_{k_0}, follows arcs in N_k and M_0 alternately and ends with a closer c^* which has not been connected to an opener yet. Note that, due to the order in which the arcs in the upper half-plane are drawn, all vertices in the aforementioned path are to the right of o_k. In particular, c^* is to the right of o_k and is not one of the candidates for o_k. Draw an arc in the upper half-plane connecting o_k to c^* and set its color to be

$$\mathrm{col}(o_k, N_{k-1}) = \epsilon_k + \text{the sum of the colors of the edges in that path (mod 2)}. \tag{3.4}$$

When all the openers are connected in this manner, the resulting matching in the upper half-plane is $M = \phi_2((w_1, \dots, w_n), (\epsilon_1, \dots \epsilon_n))$.

Let $M_n = M, M_{n-1}, \ldots, M_2, M_1$ be the sequence of intermediary matchings constructed when M is sorted to M_0. Then $M_k(o_k)$ is exactly the closer c_{k_0} defined above and we claim that $\mathrm{col}(o_k, M_k) = \epsilon_k$. To prove this claim, consider the following graph G which represents the effects of color changing of arcs. The vertices of G are $\{1, 2, \ldots, n\}$. Two vertices $l < k$ are connected by an edge if the $M_k(M_0(o_k))) = o_l$. This means that

$$\mathrm{col}(o_l, M_{k-1}) = \mathrm{col}(o_l, M_k) + \mathrm{col}(o_k, M_k).$$

Let G_k be the induced subgraph of G on the vertices $\{k, k+1, \ldots, n\}$. One can prove by induction that when connecting the opener o_k in the construction of ϕ_2, the arcs that are traced in the path (3.3) are exactly the ones that have openers that are in the connected component of k in G_k and $\mathrm{col}(o_k, M_k) = \epsilon_k$.

Now that we know how the sequence of matchings $M_n = M, M_{n-1}, \ldots, M_2, M_1$ relates to $(w_1, \ldots, w_n), (\epsilon_1, \ldots \epsilon_n)$, it is not difficult to find $\mathrm{sor}_k(M, M_0)$. Suppose first that $M_k(o_k) \leq M_0(o_k)$ and consider the closers in the set

$$\{c : c > o_k, c \in [M_k(o_k), M_0(o_k)] \text{ and } M_0(c) \leq o_k\}.$$

If $\epsilon_k = 0$, the elements in this set are exactly the first w_k candidates for o_k in the construction of M. If $\epsilon_k = 1$, this set contains the first $h_k - w_k + 1$ candidates for o_k. So, in this case

$$\mathrm{sor}_k(M, M_0) = \begin{cases} w_k - 1, & \text{if } \epsilon_k = 0 \\ 2k - 1 - h_k + w_k, & \text{if } \epsilon_k = 1. \end{cases} \tag{3.5}$$

The case $M_0(o_k) < M_k(o_k)$ is similar and we get again (3.5). This proves (3.1).

We note that the inverse map ϕ_2^{-1} is not difficult to construct. To recover w_k and ϵ_k that correspond to a given matching M, sort M to M_0. If $M_n = M, M_{n-1}, \ldots, M_1$ is the sequence of intermediary matchings obtained in the process, set $\epsilon_k = \mathrm{col}(o_k, M_k)$ and

$$w_k = \mathrm{sor}_k(M, M_0) + 1 - \mathrm{col}(o_k, M_k)(2k - h_k).$$

Similarly as in the case of monochromatic matchings, a cycle in the graph (M, M_0) is closed exactly when $c_{k_0} = M_0(o_k)$, which means when $w_k = 1$ and $\epsilon_k = 0$ or when $w_k = h_k$ and $\epsilon_k = 1$. This proves the second property of ϕ_2. Moreover, it follows from (3.4) that if the edge $o_k \cdot M(o_k)$ is the edge with the smallest opener in its own cycle (i.e. the edge that closes a cycle when $\phi_2((w_1, \ldots, w_n), (\epsilon_1, \ldots, \epsilon_n))$ is constructed), then

$$\epsilon_k = \text{the sum of the colors of the edges in that cycle(mod2)}.$$

Hence the minimal openers of the cycles with even number of blue edges correspond to the pairs $(1, 0)$ among $(w_1, \epsilon_1), (w_2, \epsilon_2), \ldots, (w_n, \epsilon_n)$ while the minimal openers of the cycles with odd number of blue edges correspond to the pairs $(w_k, \epsilon_k) = (h_k, 1)$ in that list. $\square$

From the properties of the map ϕ_2, we get the following generating function.

Corollary 3.4. *Let $(h_1, \ldots, h_n)$ be the height sequence of the Dyck path D and let $M_0 \in \mathcal{M}_n^{(2)}(D)$ be a matching with all red edges. Then*

$$\sum_{M\in\mathcal{M}_n^{(2)}(D)} q^{\mathrm{sor}(M,M_0)} \prod_{i\in\mathrm{Cyc}_0(M,M_0)} t_i \prod_{i\in\mathrm{Cyc}_1(M,M_0)} s_i = \prod_{k=1}^{n}\left(t_k + (q+q^{2k-h_k})[h_k-1]_q + s_k q^{2k-1}\right) \tag{3.6}$$

Corollary 3.5. *Let $(h_1,\ldots,h_n)$ be the height sequence of the Dyck path D and let $M_0 \in \mathcal{M}_n^{(2)}(D)$ be a matching with all red edges. Then*

$$\sum_{M\in\mathcal{M}_n^{(2)}(D)} q^{\mathrm{sor}(M,M_0)} \prod_{i\in\mathrm{Cyc}_0(M,M_0)} t_i = \sum_{M\in\mathcal{M}_n^{(2)}(D)} q^{\mathrm{mix}(M)} \prod_{i\in\mathrm{Longr}(M)} t_i.$$

3.3 Connections with restricted signed permutations

Petersen [8] defined a sorting index for signed permutations. Every signed permutation $\sigma \in B_n$ can be uniquely written as a product

$$\sigma = (i_1 j_1)(i_2 j_2)\cdots(i_k j_k)$$

of transpositions such that $i_s < j_s$ for $1 \le s \le k$ and $0 < j_1 < \cdots < j_k$. Here the transposition (ij) means to swap both i with j and $\bar{i}$ with $\bar{j}$ (provided $i \neq \bar{j}$). The type B_n sorting index is defined to be

$$\mathrm{sor}(\sigma) = \sum_{s=1}^{k}(j_s - i_s - \chi(i_s < 0)).$$

As before, the sorting index keeps track of the total distance the elements in σ move when σ is sorted using a "type B" Straight Selection Sort algorithm in which, using a transposition, the largest number is moved to its proper place, then the second largest, and so on. For example, the steps for sorting $\sigma = \bar{5}13\bar{4}\bar{2}$ are

$$24\bar{3}\bar{1}\mathbf{5}\ \bar{5}13\bar{4}\bar{2} \xrightarrow{(\bar{1}5)} \bar{5}\mathbf{4}\bar{3}\bar{1}\bar{2}\ 213\bar{4}5 \xrightarrow{(\bar{4}4)} \bar{5}\bar{4}\bar{3}\bar{1}\bar{2}\ \mathbf{2}1345 \xrightarrow{(12)} \bar{5}\bar{4}\bar{3}\bar{2}\bar{1}\ 12345$$

and therefore $\sigma = (12)(\bar{4}4)(\bar{1}5)$ and $\mathrm{sor}(\sigma) = (2-1) + (4-(-4)-1) + (5-(-1)-1) = 13$.

Let $\mathbf{r}$ be a nondecreasing sequence of positive integers $r_1 \le r_2 \le \cdots \le r_n \le n$ and let

$$B_{\mathbf{r}} = \{\sigma \in B_n : \sigma(i) \le r_i\}.$$

As before, only sequences $\mathbf{r}$ for which $r_i \ge i$ are of interest as otherwise the set $B_{\mathbf{r}}$ is empty. There is a canonical bijection

$$g_{\mathbf{r}} : B_{\mathbf{r}} \to \mathcal{M}_n^{(2)}(D(\mathbf{r}))$$

that takes a permutation $\sigma \in B_{\mathbf{r}}$ to the matching with edges $o_{|\sigma(k)|}\cdot c_k$ colored red if $\sigma(k) > 0$ and colored blue if $\sigma(k) < 0$, for $k \in \{1,2,\ldots,n\}$.

As in the monochromatic case, for each $\sigma_0 \in B_{\mathbf{r}}$ with $\sigma(k) > 0$ for $k > 0$, this map induces a statistic on $B_{\mathbf{r}}$, which we will denote $\mathrm{sor}_{\mathbf{r}}(\,\cdot\,,\sigma_0)$:

$$\mathrm{sor}_{\mathbf{r}}(\sigma,\sigma_0) = \mathrm{sor}(g_{\mathbf{r}}^{-1}(\sigma), g_{\mathbf{r}}^{-1}(\sigma_0)).$$

The direct definition of $\mathrm{sor}_{\mathbf{r}}(\sigma, \sigma_0)$ on $B_{\mathbf{r}}$ via a sorting algorithm similar to Straight Selection Sort is as follows. Permute the elements in $\sigma \in B_{\mathbf{r}}$ by applying transpositions which place the largest element n in position $\sigma_0^{-1}(n)$, then the element $n-1$ in position $\sigma_0^{-1}(n-1)$, etc. Let $\sigma = \sigma_n, \sigma_{n-1}, \ldots, \sigma_1$, be the sequence of intermediary permutations obtained in this way. In particular, $\sigma_k^{-1}(i) = \sigma_0^{-1}(i)$ for $|i| > k$, and σ_{k-1} is obtained by swapping k and $\sigma_k(\sigma_0^{-1}(k))$ in σ_k.

Let $l = \sigma_k^{-1}(k)$ and $m = \sigma_0^{-1}(k) > 0$. Define

$$b_k = \begin{cases} 0, & l = m \\ |\{i : l \leq i \leq m, \sigma_0(i) < k\}|, & 0 < l < m \\ |\{i : r_i \geq k, i \notin (m,l), \sigma_0(i) < k\}|, & l > m \\ 2k-1, & l = -m \\ 2k-1-|\{i : l \leq i \leq m, \sigma_0(i) < k\}|, & 0 > l > -m \\ 2k-1-|\{i : r_i \geq k, i \notin (m,l), \sigma_0(i) < k\}|, & l < -m. \end{cases} \tag{3.7}$$

Then

$$\mathrm{sor}_{\mathbf{r}}(\sigma, \sigma_0) = \sum_{k=1}^{n} b_k.$$

As before, $\mathrm{sor}_{\mathbf{r}}(\sigma, \sigma_0)$ in general depends not only on σ_0, but on $\mathbf{r}$ as well. However, the case when σ_0 is the identity permutation is an exception.

Lemma 3.6. *Let $\mathbf{r}$ be a non-decreasing sequence of integers $1 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq n$ with $r_k \geq k$, for all k. Let $\sigma \in B_{\mathbf{r}}$. Then*

$$\mathrm{sor}_{\mathbf{r}}(\sigma, \mathbf{id}) = \mathrm{sor}(\sigma).$$

Proof. The case $|l| > m$ in (3.7) cannot occur. Namely, in the case when $\sigma_0 = \mathbf{id}$, we have $m = k$ and if $|l| > k$, $\sigma_k^{-1}(l) = \sigma_0^{-1}(l) = l$. This contradicts $l = \sigma_k^{-1}(k)$. Therefore, the definition of b_k simplifies to

$$b_k = \begin{cases} |\{i : l \leq i < k\}|, & 0 < l \leq k \\ 2k-1-|\{i : l \leq i < k\}|, & 0 > l > -k. \end{cases}$$

This is precisely the "distance" that k travels when being placed in its correct position with the sorting algorithm. $\square$

Signed permutations can be decomposed into two types of cycles. The cycles can be of the form $(a_1, \ldots, a_k)$ (this cycle also takes $\bar{a_1}$ to $\bar{a_2}$, etc.) or of the form $(a_1, \ldots, a_k, \bar{a_1}, \ldots, \bar{a_k})$, for $k \geq 1$ and all $a_1, \ldots, a_k$ different. The former cycles are called balanced and the letter ones unbalanced. Let

$$\mathrm{Cyc}_0(\sigma) = \{|k| : k \text{ is the minimal number in absolute value in a balanced cycle of } \sigma\},$$

$$\mathrm{Cyc}_1(\sigma) = \{|k| : k \text{ is the minimal number in absolute value in a unbalanced cycle of } \sigma\},$$

and let $\mathrm{cyc}_0(\sigma) = |\mathrm{Cyc}_0(\sigma)|$ and $\mathrm{cyc}_1(\sigma) = |\mathrm{Cyc}_1(\sigma)|$. For example, the permutation $\sigma = \bar{3}\bar{9}\bar{5}\bar{7}1\bar{6}\bar{4}\bar{8}2$ can be decomposed into $\sigma = (1\bar{3}5)(2\bar{9}\bar{2}9)(4\bar{7})(6\bar{6})(8)$, so $\mathrm{Cyc}_0(\sigma) = \{1, 4, 8\}$ and $\mathrm{Cyc}_1(\sigma) = \{2, 6\}$.

Corollary 3.7. *Let $\sigma_0 \in B_{\mathbf{r}}$ with $\sigma(k) > 0$ for $k > 0$. Then*

$$\sum_{\sigma \in B_{\mathbf{r}}} q^{\mathrm{sor}_{\mathbf{r}}(\sigma,\sigma_0)} \prod_{i \in \mathrm{Cyc}_0(\sigma\sigma_0^{-1})} t_i \prod_{i \in \mathrm{Cyc}_1(\sigma\sigma_0^{-1})} s_i = \prod_{k=1}^{n} \left(t_i + (q + q^{2k-h_k})[h_k - 1]_q + s_i q^{2k-1} \right). \tag{3.8}$$

Proof. As in the monochromatic case, there is a natural correspondence between the cycles of the permutation $\sigma\sigma_0^{-1}$ and the cycles in the bicolored graph $(g_{\mathbf{r}}(\sigma), g_{\mathbf{r}}(\sigma_0))$. □

Corollary 3.8. *Let $\sigma_0 \in B_{\mathbf{r}}$ with $\sigma(k) > 0$ for $k > 0$. Then*

$$\sum_{\sigma \in B_{\mathbf{r}}} q^{\mathrm{sor}_{\mathbf{r}}(\sigma,\sigma_0)} t^{\ell'_B(\sigma\sigma_0^{-1})} = \prod_{k=1}^{n} \left(1 + q[h_k - 1]_q t + q^{2k-h_k}[h_k]_q t \right). \tag{3.9}$$

In particular,

$$\sum_{\sigma \in B_{\mathbf{r}}} q^{\mathrm{sor}_{\mathbf{r}}(\sigma)} t^{\ell'_B(\sigma)} = \prod_{k=1}^{n} \left(1 + q[h_k - 1]_q t + q^{2k-h_k}[h_k]_q t \right). \tag{3.10}$$

Proof. It is not difficult to see that the reflection length of a balanced cycle $(a_1, \ldots, a_k)$ is $k-1$, while the reflection length of an unbalanced cycle is k. Therefore, $\ell'_B(\sigma) = n - \mathrm{cyc}_0(\sigma)$. The result follows from (3.8). □

The minimal number of terms in $\{(1\bar{1})\} \cup \{(i\ i+1) : 1 \le i \le n\}$ needed to express $\sigma \in B_n$ is called the length of σ. It is known to be equal to the type B_n inversion number inv_B given in (1.4).

Lemma 3.9. *Let $\sigma \in B_{\mathbf{r}}$ and $M = g_{\mathbf{r}}(\sigma)$. Then*

$$\mathrm{inv}_B(\sigma) = \mathrm{mix}(M).$$

Proof. Clearly, $N(\sigma)$ is equal to the number of blue edges in M. Moreover, two arcs $M(c_i) \cdot c_i$ and $M(c_j) \cdot c_j$ with $i < j$ can be in three different relative positions.

(i) They form a nesting. If the right arc is red, we have $\sigma(i) > \sigma(j)$ but not $-\sigma(i) > \sigma(j)$. If the right arc is blue, we have $-\sigma(i) > \sigma(j)$, but not $\sigma(i) > \sigma(j)$.

(ii) They form a crossing. If the right arc is blue, we have both $\sigma(i) > \sigma(j)$ and $-\sigma(i) > \sigma(j)$. If the right arc is red, neither $\sigma(i) > \sigma(j)$ nor $\sigma(i) > \sigma(j)$ is true.

(iii) They form an alignment. This case is the same as (ii).

□

For a signed permutation σ we define the set of positive right-to-left minimum letters to be

$$\mathrm{Prlminl}(\sigma) = \{k : 0 < \sigma(k) < |\sigma(l)| \text{ for all } l > k\}.$$

It is not difficult to see that $o_k \cdot g_{\mathbf{r}}(\sigma)(o_k)$ is a large red edge if and only if $k \in \mathrm{Prlminl}(\sigma)$. Therefore we get the following corollary.

Corollary 3.10. *Let $\sigma_0 \in B_{\mathbf{r}}$ with $\sigma(k) > 0$ for $k > 0$. Then*

$$\sum_{\sigma\in B_{\mathbf{r}}} q^{\mathrm{sor}_{\mathbf{r}}(\sigma,\sigma_0)} \prod_{i\in \mathrm{Cyc}_0(\sigma\sigma_0^{-1})} t_i = \sum_{\sigma\in B_{\mathbf{r}}} q^{\mathrm{inv}_B(\sigma)} \prod_{i\in \mathrm{Prlminl}(\sigma)} t_i = \prod_{k=1}^{n} \left(t_i + q[h_k-1]_q + q^{2k-h_k}[h_k]_q\right).$$

Proof. The positive right-to-left minimum letters in the signed permutation σ correspond to the openers of the long red edges in $g_{\mathbf{r}}(\sigma)$. So, the result follows from Corollary 3.7, Lemma 3.9, and Corollary 3.2. □

Recall that

$$\mathrm{nmin}_B(\sigma) = |\{i : \sigma(i) > |\sigma(j) \text{ for some } j > i\}| + N(\sigma).$$

It is readily seen that

$$\mathrm{nmin}_B(\sigma) = n - |\mathrm{Prlminl}(\sigma)|.$$

Corollary 3.11. *Let $\sigma_0 \in B_{\mathbf{r}}$ with $\sigma(k) > 0$ for $k > 0$. Then*

$$\sum_{\sigma\in B_{\mathbf{r}}} q^{\mathrm{inv}_B(\sigma)} t^{\mathrm{nmin}_B(\sigma)} = \sum_{\sigma\in B_{\mathbf{r}}} q^{\mathrm{sor}_{\mathbf{r}}(\sigma,\sigma_0)} t^{\ell'_B(\sigma\sigma_0^{-1})}. \tag{3.11}$$

Since $\mathbf{id} \in B_{\mathbf{r}}$ for every sequence $\mathbf{r}$, this generalizes the equidistribution result (1.3) for signed permutations [8].

3.4 Type D_n permutations

The type D_n permutations can be defined as signed permutations with an even number of minus signs. The type D_n inversion number is defined as

$$\mathrm{inv}_D(\sigma) = |\{1 \le i < j \le n : \sigma(i) > \sigma(j)\}| + |\{1 \le i < j \le n : -\sigma(i) > \sigma(j)|.$$

It is well known [1] that $\mathrm{inv}_D(\sigma)$ is equal to the length of σ, i.e., the minimal number of transpositions in $\{(\bar{1}2), (12), (23), \ldots, (n-1\ n)\}$ needed to express σ and that

$$\sum_{\sigma\in D_n} q^{\mathrm{inv}_D(\sigma)} = \prod_{i=1}^{n-1}(1+q^i)[i+1]_q = [n]_q \cdot \prod_{i=1}^{n-1}[2i]_q.$$

For type D_n permutations, we define

$$\mathrm{Prlminl}'(\sigma) = \{\sigma(k) : 1 < \sigma(k) < |\sigma(l)| \text{ for all } l > k\}.$$

If $\sigma = (i_1 j_1)\cdots(i_k j_k)$ is the unique factorization with $1 < j_1 < \cdots < j_k$, the type D_n sorting index is defined to be

$$\mathrm{sor}_D(\sigma) = \sum_{s=1}^{k} \left(j_s - i_s - 2\cdot\chi(i_s < 0)\right).$$

Petersen [8] proved that

$$\sum_{\sigma\in D_n} q^{\mathrm{sor}_D(\sigma)} = \sum_{\sigma\in D_n} q^{\mathrm{inv}_D(\sigma)}. \tag{3.12}$$

The type D_n permutations correspond to bicolored matchings with an even number of blue edges via the map $g_{\mathbf{r}}$. Therefore, results for type D_n permutations analogous to those given for signed permutations can be derived by considering appropriate restrictions of the bijections φ_2 and ϕ_2. Namely, the color of the first rise of a bicolored weighted Dyck path that corresponds to a bicolored matching with an even number of blue edges via any of these two bijections is determined by the colors of the other rises. The type D_n inversions correspond to a modified mix statistic of bicolored matchings. Let

$$\mathrm{mix}'(M) = \mathrm{ne}(M) + 2\mathrm{cr}_{*b}(M) + 2\mathrm{al}_{*b}(M) \qquad \text{and} \qquad \mathrm{Longr}'(M) = \mathrm{Longr}(M)\backslash\{1\}.$$

Corollary 3.12. *Let D be a Dyck path with height sequence $(h_1, \ldots, h_n)$. Then*

$$\sum_M q^{\mathrm{mix}'(M)} \prod_{i \in \mathrm{Longr}'(M)} t_i = \prod_{i=2}^{n} \left(t_i + q[h_i - 1]_q + q^{2i-h_i-1}[h_i]_q \right), \tag{3.13}$$

where the sum is over all bicolored matchings of type D with an even number of blue edges.

Proof. We use the bijection φ_2 restricted to the bicolored weighted Dyck paths with even number of blue edges. Similarly as in the proof of Theorem 3.1, we get that

$$\sum_M q_1^{\mathrm{ne}_{*r}(M)} q_2^{\mathrm{ne}_{*b}(M)} q_3^{\mathrm{cr}_{*r}(M)} q_4^{\mathrm{cr}_{*b}(M)} q_5^{\mathrm{al}_{*r}(M)} q_6^{\mathrm{al}_{*b}(M)} \prod_{i \in \mathrm{Longr}'(M)} t_i,$$

where the sum is over all bicolored matchings of type D with an even number of blue edges, is equal to

$$\prod_{i=2}^{n} \sum_{k=1}^{h_i} (q_1^{k-1} q_3^{h_i-k} q_5^{i-h_i} t_i^{\delta_{k,1}} + q_2^{h_i-k} q_4^{k-1} q_6^{i-h_i}).$$

Setting $q_1 = q_2 = q$, $q_4 = q_6 = q^2$, and $q_3 = q_5 = 1$ yields (3.13). $\square$

The type D_n sorting index of a permutation corresponds to the following modified sorting index of bicolored matchings. The matching M is sorted to M_0 as before and

$$\mathrm{sor}'(M) = \sum_{k=2}^{n} \mathrm{sor}'_k(M, M_0)$$

where, for $2 \le k \le n$,

$$\mathrm{sor}'_k(M, M_0) = \begin{cases} \mathrm{sor}_k(M, M_0), & \text{if } \mathrm{col}(o_k, M_k) = 0 \\ \mathrm{sor}_k(M, M_0) - 1, & \text{if } \mathrm{col}(o_k, M_k) = 1. \end{cases} \tag{3.14}$$

Also, let $\mathrm{Cyc}'_0(M, M_0) = \mathrm{Cyc}_0(M, M_0)\backslash\{1\}$ and $\mathrm{Cyc}'_1(M, M_0) = \mathrm{Cyc}_1(M, M_0)\backslash\{1\}$.

Lemma 3.13. *Let M_0 be a matching with all red edges of type D and let $(h_1, \ldots, h_n)$ be the height sequence of D. Suppose $M = \phi_2((w_1, \ldots, w_n), (\epsilon_1, \ldots, \epsilon_n))$ is a bicolored matching of type D with an even number of blue edges, where ϕ_2 depends on M_0. Then*

$$\mathrm{sor}'(M, M_0) = \sum_{k=2}^{n} (w_k + \epsilon_k(2k - 1 - h_k) - 1).$$

Proof. It follows from the property (3.5) of the bijection ϕ_2 and (3.14). $\square$

Therefore, via the bijection ϕ_2 we get the following multivariate generating function.

Corollary 3.14. *Let D be a Dyck path with height sequence $(h_1, \ldots, h_n)$ and M_0 a matching of type D with only red edges.*

$$\sum_{M} q^{\mathrm{sor}'(M,M_0)} \prod_{i \in \mathrm{Cyc}'_0(M,M_0)} t_i \prod_{i \in \mathrm{Cyc}'_1(M,M_0)} s_i = \prod_{i=2}^{n} \Big(t_i + (q + q^{2i-h_i-1})[h_i - 1]_q + q^{2i-2} s_i) \Big),$$

where the sum is over all bicolored matchings of type D with an even number of blue edges.

The results for the bicolored matchings with even number of blue edges yield results for restricted permutations of type D_n. Let

$$D_n(\mathbf{r}) = \{\sigma \in D_n : |\sigma(k)| \le r_k, 1 \le k \le n\}.$$

For $\sigma \in D_n$, define $\mathrm{Cyc}'_0(\sigma) = \mathrm{Cyc}_0(\sigma) \backslash \{1\}$ and $\mathrm{Cyc}'_1(\sigma) = \mathrm{Cyc}_1(\sigma) \backslash \{1\}$.

Corollary 3.15. *Let $\mathbf{r} : 1 \le r_1 \le \cdots r_n \le n$ be a nondecreasing integer sequence such that $r_k \ge k$. Let $(h_1, \ldots, h_n)$ be the height sequence of the corresponding Dyck path $D(\mathbf{r})$.*

$$\sum_{\sigma \in D_n(\mathbf{r})} q^{\mathrm{inv}_D(\sigma)} \prod_{i \in \mathrm{Prlminl}'(\sigma)} t_i = \prod_{i=2}^{n} \Big(t_i + q[h_i - 1]_q + q^{2i-h_i-1} [h_i]_q \Big) \tag{3.15}$$

Moreover, suppose that $\sigma_0 \in D_n(\mathbf{r})$ with $\sigma(k) > 0$ for all $k \ge 1$. Then

$$\sum_{\sigma \in D_n(\mathbf{r})} q^{\mathrm{sor}_D(\sigma\sigma_0^{-1})} \prod_{i \in \mathrm{Cyc}'_0(\sigma\sigma_0^{-1})} t_i \prod_{i \in \mathrm{Cyc}'_1(\sigma\sigma_0^{-1})} s_i = \prod_{i=2}^{n} \Big(t_i + (q + q^{2i-h_i-1})[h_i - 1]_q + q^{2i-2} s_i \Big). \tag{3.16}$$

Proof. Note that $\mathrm{inv}_D(\sigma) = \mathrm{inv}_B(\sigma) - N(\sigma) = \mathrm{mix}(M) - \mathrm{b}(M) = \mathrm{mix}'(M)$, where $M = g_{\mathbf{r}}(\sigma)$. $\square$

Corollary 3.16. *Let $\mathbf{r} : 1 \le r_1 \le \cdots r_n \le n$ be a nondecreasing integer sequence such that $r_k \ge k$. Let $(h_1, \ldots, h_n)$ be the height sequence of the corresponding Dyck path $D(\mathbf{r})$. Also let $\sigma_0 \in D_n(\mathbf{r})$ with $\sigma(k) > 0$ for all $k \ge 1$. Then*

$$\sum_{\sigma \in D_n(\mathbf{r})} q^{\mathrm{sor}_D(\sigma\sigma_0^{-1})} \prod_{i \in \mathrm{Cyc}'_0(\sigma\sigma_0^{-1})} t_i = \sum_{\sigma \in D_n(\mathbf{r})} q^{\mathrm{inv}_D(\sigma)} \prod_{i \in \mathrm{Prlminl}'(\sigma)} t_i. \tag{3.17}$$

In particular, when $r_1 = \cdots = r_n = n$ we have $D_n(\mathbf{r}) = D_n$, $h_k = k$, and

$$\sum_{\sigma \in D_n} q^{\mathrm{inv}_D(\sigma)} \prod_{i \in \mathrm{Prlminl}'(\sigma)} t_i = \sum_{\sigma \in D_n} q^{\mathrm{sor}_D(\sigma)} \prod_{i \in \mathrm{Cyc}'_0(\sigma)} t_i = \prod_{i=2}^{n} \big(t_i + q[i-1]_q + q^{i-1}[i]_q \big). \tag{3.18}$$

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