A HEREDITARILY INDECOMPOSABLE BANACH SPACE WITH RICH SPREADING MODEL STRUCTURE

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ABSTRACT. We present a reflexive Banach space $\mathfrak{X}_{\text{usm}}$ which is Hereditarily Indecomposable and satisfies the following properties. In every subspace Y of $\mathfrak{X}_{\text{usm}}$ there exists a weakly null normalized sequence $\{y_n\}_n$, such that every subsymmetric sequence $\{z_n\}_n$ is isomorphically generated as a spreading model of a subsequence of $\{y_n\}_n$. Also, in every block subspace Y of $\mathfrak{X}_{\text{usm}}$ there exists a seminormalized block sequence $\{z_n\}$ and $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ an isomorphism such that for every $n \in \mathbb{N}$ $T(z_{2n-1}) = z_{2n}$. Thus the space is an example of an HI space which is not tight by range in a strong sense.

INTRODUCTION

The aim of the present paper is to exhibit a space with the properties described in the abstract. The norming set W of the space $\mathfrak{X}_{\text{usm}}$ is saturated with constraints and it is very similar to the corresponding one in [3]. As it is pointed out in [3] the method of saturation under constraints is suitable for defining spaces with hereditary heterogeneous structure ([15], [16]). The basic ingredients of the norming set W are the following. First the unconditional frame is the ball of the dual T^* of Tsirelson space [9, 19]; namely W is a subset of B_{T^*} which satisfies the following properties. As in [3] it is closed in the operations $(\frac{1}{2^n}, \mathcal{S}_n, \alpha), (\frac{1}{2^n}, \mathcal{S}_n, \beta)$ which create the type I_α, type I_β functionals respectively. Furthermore it includes two types of special functionals denoted as type II_+ and type II_- functionals. The type II_- functionals are designed to impose the rich spreading model structure in the space $\mathfrak{X}_{\text{usm}}$, while the type II_+ functionals serve a double purpose. First they are a tool for finding c_0 spreading models in every subspace of $\mathfrak{X}_{\text{usm}}$. The c_0 spreading models are the fundamental initial ingredient for the ultimate construction. The second role of the type II_+ functionals is to show that the space $\mathfrak{X}_{\text{usm}}$ is not tight by range. We recall that recently V. Ferenczi and Th. Schlumprecht have presented in [8] a variant of Gowers-Maurey HI space ([11]) which is HI and not tight by range.

Since the norming set W is similar to the one in [3] many of the critical norm evaluations in the present paper are identical with the corresponding ones in [3]. The main difference of the present construction from the one

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in [3] concerns the "combinatorial result" which is a Ramsey type result yielding c_0 spreading models. For the proof of this result type II_+ functionals are a key ingredient.

We pass to a more detailed description of the properties of the space $\mathfrak{X}_{\text{usm}}$.

Theorem. The space $\mathfrak{X}_{\text{usm}}$ is reflexive, HI and hereditarily unconditional spreading model universal.

The latter means that there exists a universal constant $C > 0$ such that the following holds. For every subspace Y of $\mathfrak{X}_{\text{usm}}$ there exists a seminormalized weakly null sequence $\{x_n\}_n$ admitting spreading models C-equivalent to all normalized subsymmetric sequences. The fundamental property of $\{x_n\}_n$ deriving its spreading model universality is that for every Schreier set $F \subset \mathbb{N}$ the finite sequence $\{x_n\}_{n\in F} \overset{C}{\sim} \{u_n\}_{n\in F}$, where $\{u_n\}_n$ denotes Pelczynski's universal unconditional basis [17, 13].

The second property of $\mathfrak{X}_{\text{usm}}$ is that it is sequentially minimal. We recall, from [7], that a Banach space X with a basis is sequentially minimal, if in every infinite dimensional block subspace Y of X there exists a block sequence $\{x_n^{(Y)}\}_n$ satisfying the following. In every subspace Z of X there exists a Schauder basic sequence $\{z_k\}_k$ equivalent to a subsequence $\{x_{n_k}^{(Y)}\}_k$. A dichotomy of V. Ferenczi - Ch. Rosendal classification program [7] yields that every Banach space X with a Schauder basis $\{e_n\}_n$ either contains a block subspace which is tight by range or a sequentially minimal subspace. As consequence of this dichotomy, $\mathfrak{X}_{\text{usm}}$ is not tight by range. Moreover, the following stronger fact holds.

Theorem. Every Y block subspace of $\mathfrak{X}_{\text{usm}}$ contains a seminormalized block sequence $\{x_n\}_n$ satisfying the following. There exists an isomorphism $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ (necessarily onto) such that $T(x_{2n-1}) = x_{2n}$ for $n \in \mathbb{N}$.

The above result is a direct consequence of the structure imposed to the norming set W and hence to the space $\mathfrak{X}_{\text{usm}}$, in order to achieve the rich spreading model structure. In particular the following is proved.

Proposition. Let Y be a block subspace of $\mathfrak{X}_{\text{usm}}$. Then there exist $\{x_n, y_n\}_n$, $\{f_n, g_n\}_n$ such that f_n, g_n belong to W, $\operatorname{ran} x_n = \operatorname{ran} f_n$, $\operatorname{ran} y_n = \operatorname{ran} g_n$, $x_n < y_n < x_{n+1}$, $\{x_n\}_n, \{y_n\}_n$ are seminormalized, $f_n(x_n) = 1, g_n(y_n) = 1$ and $\{f_n + g_n\}_n$ generates a c_0 spreading model while $\{x_n - y_n\}_n$ does not generates an ℓ_1 spreading model.

The above proposition yields that there exists a strictly singular operator $S : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ with $S(x_n) = x_n - y_n$ and $S(y_n) = x_n - y_n$ (see [2]). As is explained in [7], the sequences $\{x_n\}_n$, $\{y_n\}_n$ are equivalent. It is also easy to see that $I - S$ is an isomorphism, satisfying the conclusion of the above theorem.

Every operator in the space $\mathfrak{X}_{\text{usm}}$ is of the form $T = \lambda I + S$ with S strictly singular. We recall that one of the main properties of the space in [3], is that the composition of any three strictly singular operators is a compact one. It is show that the space $\mathfrak{X}_{\rm usm}$ fails such a property, by proving that in any block subspace there exists a strictly singular operator, which is not polynomially compact. The proof of this result is directly linked to the variety of spreading models appearing in every block subspace of $\mathfrak{X}_{\rm usm}$.

The paper is organized as follows. The first section is devoted to the definition of the norming set W of the space $\mathfrak{X}_{\rm usm}$, a brief discussion is also included concerning the role of its ingredients. The second section concerns some basic norm evaluations on special convex combinations, which are identical to the corresponding estimates from [3]. The third section introduces the definition of the α, β indices, which are defined in the same manner as in [3] and related results. In the fourth section, a combinatorial result is stated and proven and it is used in the fifth section to establish the existence of c_0 spreading models. In the sixth section the structure of the spreading models of the space $\mathfrak{X}_{\rm usm}$ is studied. In the seventh and final section it is proven that the space is sequentially minimal, it is not tight by range it admits strictly singular non polynomially compact operators.

1. The norming set of the space $\mathfrak{X}_{\rm USM}$.

In this section we define the norming set W of the space $\mathfrak{X}_{\rm usm}$. As in [3], this set is defined with the use of the sequence $\{\mathcal{S}_n\}_n$ which we remind below and also families of $\mathcal{S}_n$-admissible functionals and the set W will be a subset of the norming set W_T of Tsirelson space. The key difference between the construction in [3] and the present one, is the way functionals of type II are defined, which yields the properties of the space $\mathfrak{X}_{\rm usm}$.

The Schreier families. The Schreier families is an increasing sequence of families of finite subsets of the naturals, first appeared in [1], inductively defined in the following manner.

Set $\mathcal{S}_0 = \big\{\{n\} : n \in \mathbb{N}\big\}$ and $\mathcal{S}_1 = \{F \subset \mathbb{N} : \#F \leqslant \min F\}$.

Suppose that $\mathcal{S}_n$ has been defined and set $\mathcal{S}_{n+1} = \{F \subset \mathbb{N} : F = \cup_{j=1}^k F_j$, where $F_1 < \cdots < F_k \in \mathcal{S}_n$ and $k \leqslant \min F_1\}$

If for $n, m \in \mathbb{N}$ we set $\mathcal{S}_n * \mathcal{S}_m = \{F \subset \mathbb{N} : F = \cup_{j=1}^k F_j$, where $F_1 < \cdots < F_k \in \mathcal{S}_m$ and $\{\min F_j : j = 1, \dots, k\} \in \mathcal{S}_n\}$, then it is well known that $\mathcal{S}_n * \mathcal{S}_m = \mathcal{S}_{n+m}$.

The suppression unconditional universal basis of Pełczyński. Let $\{x_k\}_k$ be a norm dense sequence in the unit sphere of $C[0,1]$. Denote by $\{u_k\}_k$ the unit vector basis of c_{00} and define $\|\cdot\|_u$ on c_{00} as follows.

$$\|\sum_{k=1}^n \alpha_k u_k\|_u = \sup\big\{\|\sum_{k\in F} \alpha_k x_k\| : F \subset \{1, \dots, n\}\big\}$$

Let U be the completion of $(c_{00}, \|\cdot\|_u)$. Then $\{u_k\}_k$ is a suppression unconditional Schauder basis for U, such that for any $\{y_k\}_k$ suppression unconditional Schauder basic sequence and $\varepsilon > 0$, there exists a subsequence of $\{u_k\}_k$, which is $(1+\varepsilon)$-equivalent to $\{y_k\}_k$.

The sequence $\{u_k\}_k$ is called the unconditional basis of Pełczyński (see [17]).

Notation. A sequence of vectors $x_1 < \cdots < x_k$ in c_{00} is said to be $\mathcal{S}_n$-admissible if $\{\min\operatorname{supp} x_i : i = 1, \dots, k\} \in \mathcal{S}_n$.

Let $G \subset c_{00}$. A vector $f \in G$ is said to be an average of size $s(f) = n$, if there exist $f_1, \dots, f_d \in G, d \leqslant n$, such that $f = \frac{1}{n}(f_1 + \cdots + f_d)$.

A sequence $\{f_j\}_j$ of averages in G is said to be very fast growing, if $f_1 < f_2 < \dots$, $s(f_j) > 2^{\max\operatorname{supp} f_{j-1}}$ and $s(f_j) > s(f_{j-1})$ for $j > 1$.

The coding function. Choose $L_0 = \{\ell_k : k \in \mathbb{N}\}, \ell_1 > 9$ an infinite subset of the naturals such that:

(i) For any $k \in \mathbb{N}$ we have that $\ell_{k+1} > 2^{2\ell_k}$ and
(ii) $\sum_{k=1}^{\infty} \frac{1}{2^{\ell_k}} < \frac{1}{1000}$.

Decompose L_0 into further infinite subsets L_1, L_2, L_3. Set

$$\begin{aligned}\mathcal{Q} \;=\; & \big\{\big(f_1, \dots, f_m\big) : m \in \mathbb{N}, f_1 < \dots < f_m \in c_{00} \\ & \text{with } f_k(i) \in \mathbb{Q}, \text{ for } i \in \mathbb{N}, k = 1, \dots, m\big\}\end{aligned}$$

Choose a one to one function $\sigma : \mathcal{Q} \to L_2$, called the coding function, such that for any $\big(f_1, \dots, f_m\big) \in \mathcal{Q}$, we have that

$$\sigma\big(f_1, \dots, f_m\big) > 2^{\frac{1}{\|f_m\|_0}} \cdot \max\operatorname{supp} f_m$$

Remark 1.1. If we set $L = L_1 \cup L_2$, For any $n \in \mathbb{N}$ we have that $\#L \cap \{n, \dots, 2^{2n}\} \leqslant 1$, moreover for every $n \in L_3$, we have that $L \cap \{n, \dots, 2^{2n}\} = \varnothing$.

The norming set. The norming set W is defined to be the smallest subset of c_{00} satisfying the following properties:

1. The set $\{\pm e_n\}_{n\in\mathbb{N}}$ is a subset of W, for any $f \in W$ we have that $-f \in W$, for any $f \in W$ and any E interval of the naturals we have that $Ef \in W$ and W is closed under rational convex combinations. Any $f = \pm e_n$ will be called a functional of type 0.

2. The set W is closed in the $(\frac{1}{2^n}, \mathcal{S}_n, \alpha)$ operation, i.e. it contains any functional f which is of the form $f = \frac{1}{2^n}\sum_{q=1}^{d} \alpha_q$, where $\{\alpha_q\}_{q=1}^{d}$ is an $\mathcal{S}_n$-admissible and very fast growing sequence of α-averages in W. If E is an interval of the naturals, then $g = \pm Ef$ is called a functional of type I_α, of weight $w(g) = n$.

3. The set W is closed in the $(\frac{1}{2^n}, \mathcal{S}_n, \beta)$ operation, i.e. it contains any functional f which is of the form $f = \frac{1}{2^n}\sum_{q=1}^{d} \beta_q$, $\{\beta_q\}_{q=1}^{d}$ is an $\mathcal{S}_n$-admissible and very fast growing sequence of β-averages in W. If E is an interval of the naturals, then $g = \pm Ef$ is called a functional of type I_β, of weight $w(g) = n$.

4. For any special sequence $\{f_q, g_q\}_{q=1}^{d}$ in W and $F \subset \{1, \dots, d\}$ such that $2(\#F) \leqslant \min\operatorname{supp} f_{\min F}$, the set W contains any functional f which is of the form $f = \frac{1}{2}\sum_{q\in F}(f_q + g_q)$.

If E is an interval of the naturals, then $g = \pm Ef$ is called a functional of type II_+ with weights $\widehat{w}(g) = \{w(f_q) : q \in F, \operatorname{ran}(f_q + g_q) \cap E \neq \varnothing\}$.

5. For any special sequence $\{f_q, g_q\}_{q=1}^d$ in W and $F \subset \{1, \dots, d\}$ such that $2(\#F) \leqslant \min\operatorname{supp} f_{\min F}$ and $\{\lambda_q\}_{q\in F} \subset \mathbb{Q}$ with $\|\sum_{q\in F} \lambda_q u_q^*\|_u \leqslant 1$, where $\{u_k^*\}_k$ denotes the biorthogonals of the unconditional basis of Pełczyński, the set W contains any functional f which is of the form $f = \frac{1}{2}\sum_{q\in F} \lambda_q (f_q - g_q)$.

If E is an interval of the naturals, then $g = \pm Ef$ is called a functional of type II_- with weights $\widehat{w}(g) = \{w(f_q) : q \in F, \operatorname{ran}(f_q - g_q) \cap E \neq \varnothing\}$.

We call a functional $f \in W$ which is either of type II_+ or of type II_-, a functional of type II.

For $d \in \mathbb{N}$, a sequence of pairs of functionals of type I_α $\{f_q, g_q\}_{q=1}^d$, is called a special sequence if

$$f_1 < g_1 < f_2 < g_2 < \cdots < f_d < g_d \tag{1}$$

$$w(f_q) = w(g_q) \quad \text{for} \quad q = 1, \dots, d \tag{2}$$

(3)
$$w(f_1) \in L_1 \quad \text{and} \quad \sigma(f_1, g_1, f_2, g_2 \dots, f_{q-1}, g_{q-1}) = w(f_q) \text{ for } 1 < q \leqslant d$$

We call an α-average any average $\alpha \in W$ of the form $\alpha = \frac{1}{n}\sum_{j=1}^d f_j, d \leqslant n$, where $f_1 < \cdots < f_d \in W$.

We call a β-average any average $\beta \in W$ of the form $\beta = \frac{1}{n}\sum_{j=1}^d f_j, d \leqslant n$, where $f_1, \dots, f_d \in W$ are functionals of type II, with pairwise disjoint weights $\widehat{w}(f_j)$.

In general, we call a convex combination any $f \in W$ that is not of type 0, I_α, I_β or II.

A sequence of pairs of functionals of type I_α $b = \{f_q, g_q\}_{q=1}^\infty$ is called a special branch, if $\{f_q, g_q\}_{q=1}^d$ is a special sequence for all $d \in \mathbb{N}$. We denote the set of all special branches by $\mathcal{B}$.

If $b = \{f_q, g_q\}_{q=1}^\infty \in \mathcal{B}$, we denote by $b_+ = \{f_q + g_q : q \in \mathbb{N}\}$ and $b_- = \{f_q - g_q : q \in \mathbb{N}\}$.

For $x \in c_{00}$ define $\|x\| = \sup\{f(x) : f \in W\}$ and $\mathfrak{X}_{\mathrm{usm}} = \overline{(c_{00}(\mathbb{N}), \|\cdot\|)}$. Evidently $\mathfrak{X}_{\mathrm{usm}}$ has a bimonotone basis.

The features of the space $\mathfrak{X}_{\mathrm{usm}}$. Before proceeding to the study of the space $\mathfrak{X}_{\mathrm{usm}}$, it is probably useful to explain the role of the specific ingredients in the definition of the norming set W. First, as we have mentioned in the introduction, we will use saturation under constraints in a similar manner as in [3]. This yields the type I_α, I_β functionals and the indices $\alpha(\{x_k\}_k), \beta(\{x_k\}_k)$ for block sequences $\{x_k\}_k$ in $\mathfrak{X}_{\mathrm{usm}}$, which are defined as in [3]. As the familiar reader would observe, the special functionals in the present construction differ from the corresponding ones in [3]. This is due to the desirable main property of the space $\mathfrak{X}_{\mathrm{usm}}$, namely that every subspace contains a sequence admitting all unconditional spreading sequences as a spreading model. This is related to property 5 in the above definition of the norming set W.

What requires further discussion are the type II_+ functionals. The primitive role of them is to allow to locate in every block subspace a seminormalized block sequence generating a c_0 spreading model. This follows from the next proposition.

Proposition. Let $\{x_k\}_k$ be a seminormalized block sequence in $\mathfrak{X}_{\mathrm{usm}}$ such that the following hold.

(i) $\alpha\big(\{x_k\}_k\big) = 0$ and $\beta\big(\{x_k\}_k\big) = 0$
(ii) For every special branch $b = \{f_q, g_q\}_{q=1}^\infty$

$$\lim_k \sup\big\{|f_q(x_k)| \vee |g_q(x_k)| : q \in \mathbb{N}\big\} = 0$$

Then there exists a subsequence $\{x_{k_n}\}_n$ of $\{x_k\}_k$ generating a c_0 spreading model.

Note that in [3], property (i) is sufficient for a sequence to have a subsequence generating a c_0 spreading model. However, in the present paper this is not the case and the special functionals of type II_+ are crucial for establishing property (ii) in the above proposition.

As consequence, we obtain that in every block subspace there exists a block sequence generating a c_0 spreading model. As in [3], from the c_0 spreading model one can pass to exact nodes (see Def. 6.4) $\{x_k, y_k, f_k, g_k\}$, with $\{f_k, g_k\}_{k=1}^\infty$ defining a special branch. The desired sequence is the sequence $\{x_k - y_k\}_k$. A secondary role of the type II_+ special functionals is to determine intertwined equivalent sequences $\{v_k, w_k\}_k$. Those are subsequences of the above described sequence $\{x_k, y_k\}_k$.

As in [3], the norming set of the space $\mathfrak{X}_{\mathrm{usm}}$ is a subset of the unit ball of the dual T^* of Tsirelson space (see [9]). Moreover, most of the critical norm evaluations are identical with those in [3].

2. BASIC EVALUATIONS FOR SPECIAL CONVEX COMBINATIONS

In this section we present some results concerning estimations of the norm of special convex combinations. These estimations are crucial throughout the rest of the paper, as like in [3], special convex combinations are one of the main tools used to establish the properties of the space $\mathfrak{X}_{\mathrm{usm}}$.

Definition 2.1. Let $x = \sum_{k\in F} c_k e_k$ be a vector in c_{00}. Then x is said to be a (n,ε) basic special convex combination (or a (n,ε) basic s.c.c.) if:

(i) $F \in \mathcal{S}_n, c_k \geqslant 0$, for $k \in F$ and $\sum_{k\in F} c_k = 1$.
(ii) For any $G \subset F, G \in \mathcal{S}_{n-1}$, we have that $\sum_{k\in G} c_k < \varepsilon$.

The proof of the next proposition can be found in [4], Chapter 2, Proposition 2.3.

Proposition 2.2. For any M infinite subset of the naturals, any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $F \subset M, \{c_k\}_{k\in F}$, such that $x = \sum_{k\in F} c_k e_k$ is a (n, ε) basic s.c.c.

Definition 2.3. Let $x_1 < \cdots < x_m$ be vectors in c_{00} and $\psi(k) = \min \operatorname{supp} x_k$, for $k = 1, \ldots, m$. Then $x = \sum_{k=1}^m c_k x_k$ is said to be a (n, ε) special convex combination (or (n, ε) s.c.c.), if $\sum_{k=1}^m c_k e_{\psi(k)}$ is a (n, ε) basic s.c.c.

The proof of the following result can be found in [3], Proposition 2.5.

Proposition 2.4. Let $x = \sum_{k\in F} c_k e_k$ be a (n, ε) basic s.c.c. and $G \subset F$. Then the following holds.

$$\|\sum_{k\in G} c_k e_k\|_T \leqslant \frac{1}{2^n} \sum_{k\in G} c_k + \varepsilon$$

The next proposition is identical to Corollary 2.8 from [3].

Proposition 2.5. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ such that $\|x_k\| \leqslant 1, \{c_k\}_k \subset \mathbb{R}$ and $\phi(k) = \max \operatorname{supp} x_k$ for all k. Then:

$$\|\sum_{k} c_k x_k\| \leqslant 6\|\sum_{k} c_k e_{\phi(k)}\|_T \tag{4}$$

The following corollary is an easy consequence of Propsitions 2.4 and 2.5 and its proof can be found in [3], Corollary 2.9.

Corollary 2.6. Let $x = \sum_{k=1}^m c_k x_k$ be a (n, ε) s.c.c. in $\mathfrak{X}_{\text{usm}}$, such that $\|x_k\| \leqslant 1$, for $k = 1, \ldots, m$. If $F \subset \{1, \ldots, m\}$, then

$$\|\sum_{k\in F} c_k x_k\| \leqslant \frac{6}{2^n} \sum_{k\in F} c_k + 12\varepsilon.$$

In particular, we have that $\|x\| \leqslant \frac{6}{2^n} + 12\varepsilon$.

The proof of the next corollary is based on Corollary 2.6. It's proof is identical to the one of Corollary 2.10 from [3].

Corollary 2.7. The basis of $\mathfrak{X}_{\text{usm}}$ is shrinking.

The definition of the norming set yields the following result, the proof of which can be found in [3], Corollary 2.11.

Proposition 2.8. The basis of $\mathfrak{X}_{\text{usm}}$ is boundedly complete.

Combining the previous two results with R. C. James' well known result [12], we conclude the following.

Corollary 2.9. The space $\mathfrak{X}_{\text{usm}}$ is reflexive.

Rapidly increasing sequences are defined in the exact same manner, as in [3], Definition 3.10.

Definition 2.10. Let $C \geqslant 1, \{n_k\}_k$ be strictly increasing naturals. A block sequence $\{x_k\}_k$ is called a $(C, \{n_k\}_k)$ α-rapidly increasing sequence (or $(C, \{n_k\}_k)$ α-RIS) if the following hold.

(i) $\|x_k\| \leqslant C, \quad \frac{1}{2^{n_{k+1}}} \max\operatorname{supp} x_k < \frac{1}{2^{n_k}} \quad$ for all k.

(ii) For any functional f in W of type I_α of weight $w(f) = n$, for any k such that $n < n_k$, we have that $|f(x_k)| < \frac{C}{2^n}$.

Definition 2.11. Let $n \in \mathbb{N}, C \geqslant 1, \theta > 0$. A vector $x \in \mathfrak{X}_{\text{usm}}$ is called a (C, θ, n) vector if the following hold. There exist $0 < \varepsilon < \frac{1}{32C2^{3n}}$, and $\{x_k\}_{k=1}^m$ with $\|x_k\| \leqslant C$ for $k = 1, \ldots, m$ such that

(i) $\min\operatorname{supp} x_1 \geqslant 8C2^{2n}$

(ii) There exist $\{c_k\}_{k=1}^m \subset [0,1]$ such that $\sum_{k=1}^m c_k x_k$ is a (n, ε) s.c.c.

(iii) $x = 2^n \sum_{k=1}^m c_k x_k$ and $\|x\| \geqslant \theta$

If moreover there exist $\{n_k\}_{k=1}^m$ strictly increasing naturals with $n_1 > 2^{2n}$ such that $\{x_k\}_{k=1}^m$ is $(C, \{n_k\}_{k=1}^m)$ α-RIS, then x is called a (C, θ, n) exact vector.

Remark 2.12. Let x be a (C, θ, n) vector in $\mathfrak{X}_{\text{usm}}$. Then, using Corollary 2.6 we conclude that $\|x\| < 7C$.

3. The α, β indices

The α and β indices concerning block sequences in $\mathfrak{X}_{\text{usm}}$, are identically defined, as in [3], Definitions 3.1 and 3.2. Note that in [3], the α, β indices are sufficient to fully describe the spreading models admitted by block sequences. In the present paper, this is not the case. However, the α, β indices retain an important role in determining what spreading models a block sequence generates.

Definition 3.1. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ that satisfies the following. For any $n \in \mathbb{N}$, for any very fast growing sequence $\{\alpha_q\}_q$ of α-averages in W and for any $\{F_k\}_k$ increasing sequence of subsets of the naturals, such that $\{\alpha_q\}_{q\in F_k}$ is $\mathcal{S}_n$-admissible, the following holds. For any $\{x_{n_k}\}_k$ subsequence of $\{x_k\}_k$, we have that $\lim_k \sum_{q\in F_k} |\alpha_q(x_{n_k})| = 0$.

Then we say that the α-index of $\{x_k\}_k$ is zero and write $\alpha\big(\{x_k\}_k\big) = 0$. Otherwise we write $\alpha\big(\{x_k\}_k\big) > 0$.

Definition 3.2. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ that satisfies the following. For any $n \in \mathbb{N}$, for any very fast growing sequence $\{\beta_q\}_q$ of β-averages in W and for any $\{F_k\}_k$ increasing sequence of subsets of the naturals, such that $\{\beta_q\}_{q\in F_k}$ is $\mathcal{S}_n$-admissible, the following holds. For any $\{x_{n_k}\}_k$ subsequence of $\{x_k\}_k$, we have that $\lim_k \sum_{q\in F_k} |\beta_q(x_{n_k})| = 0$.

Then we say that the β-index of $\{x_k\}_k$ is zero and write $\beta\big(\{x_k\}_k\big) = 0$. Otherwise we write $\beta\big(\{x_k\}_k\big) > 0$.

Remark 3.3. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ and $\{E_k\}_k$ be an increasing sequence of intervals of the natural numbers with $E_k \subset \operatorname{ran} x_k$ for all $k \in \mathbb{N}$. Set $y_k = E_k x_k$.

(i) If $\alpha\big(\{x_k\}_k\big)=0$, then $\alpha\big(\{y_k\}_k\big)=0$.
(ii) If $\beta\big(\{x_k\}_k\big)=0$, then $\beta\big(\{y_k\}_k\big)=0$.

Remark 3.4. Let $\{x_k\}_k$, $\{y_k\}_k$ be block sequence such that if $z_k=x_k+y_k$, $\{z_k\}_k$ is also a block sequence.

(i) If $\alpha\big(\{x_k\}_k\big)=0$ and $\alpha\big(\{y_k\}_k\big)=0$, then $\alpha\big(\{z_k\}_k\big)=0$.
(ii) If $\beta\big(\{x_k\}_k\big)=0$ and $\beta\big(\{y_k\}_k\big)=0$, then $\beta\big(\{z_k\}_k\big)=0$.

Remark 3.5. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ and $\{F_k\}_k$ be an increasing sequence of subsets of the natural numbers and $\{c_i\}_{i\in F_k}\subset[0,1]$ with $\sum_{i\in F_k}c_i=1$ for all $k\in\mathbb{N}$. Set $y_k=\sum_{i\in F_k}c_ix_i$.

(i) If $\alpha\big(\{x_k\}_k\big)=0$, then $\alpha\big(\{y_k\}_k\big)=0$.
(ii) If $\beta\big(\{x_k\}_k\big)=0$, then $\beta\big(\{y_k\}_k\big)=0$.

The following two Propositions are proven in [3], Proposition 3.3.

Proposition 3.6. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$. Then the following assertions are equivalent.

(i) $\alpha\big(\{x_k\}_k\big)=0$
(ii) For any $\varepsilon>0$ there exists $j_0\in\mathbb{N}$ such that for any $j\geqslant j_0$ there exists $k_j\in\mathbb{N}$ such that for any $k\geqslant k_j$, and for any $\{\alpha_q\}_{q=1}^d$ $\mathcal{S}_j$-admissible and very fast growing sequence of α-averages such that $s(\alpha_q)>j_0$, for $q=1,\dots,d$, we have that $\sum_{q=1}^d|\alpha_q(x_k)|<\varepsilon$.

Proposition 3.7. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$. Then the following assertions are equivalent.

(i) $\beta\big(\{x_k\}_k\big)=0$
(ii) For any $\varepsilon>0$ there exists $j_0\in\mathbb{N}$ such that for any $j\geqslant j_0$ there exists $k_j\in\mathbb{N}$ such that for any $k\geqslant k_j$, and for any $\{\beta_q\}_{q=1}^d$ $\mathcal{S}_j$-admissible and very fast growing sequence of β-averages such that $s(\beta_q)>j_0$, for $q=1,\dots,d$, we have that $\sum_{q=1}^d|\beta_q(x_k)|<\varepsilon$.

The next Proposition is similar to Proposition 3.5 from [3].

Proposition 3.8. Let $\{x_k\}_k$ be a seminormalized block sequence in $\mathfrak{X}_{\text{usm}}$, such that either $\alpha\big(\{x_k\}_k\big)>0$, or $\beta\big(\{x_k\}_k\big)>0$. Then there exists a subsequence $\{x_{n_k}\}_k$ of $\{x_k\}_{k\in\mathbb{N}}$, that generates an ℓ_1^n spreading model, for every $n\in\mathbb{N}$.

In particular, there exists $\theta>0$ such that for any $k_0,n\in\mathbb{N}$, there exists x a (C,θ,n) vector supported by $\{x_k\}_k$ with $\min\operatorname{supp}x\geqslant k_0$.

If moreover $\{x_k\}_k$ is $(C,\{n_k\})$ α-RIS, then for every $n,k_0\in\mathbb{N}$ there exists x a (C,θ,n) exact vector supported by $\{x_k\}_k$ with $\min\operatorname{supp}x\geqslant k_0$.

The proof of the following lemma, is identical to Lemma 3.6 from [3].

Lemma 3.9. Let $x = 2^n \sum_{k=1}^m c_k x_k$ be a (C,θ,n) vector in $\mathfrak{X}_{\text{usm}}$. Let also α be an α-average and set $G_\alpha = \{k : \operatorname{ran}\alpha \cap \operatorname{ran} x_k \neq \varnothing\}$. Then the following holds.
(5)

$$|\alpha(x)| < \min\Big\{\frac{C2^n}{s(\alpha)} \sum_{k\in G_\alpha} c_k,\ \frac{6C}{s(\alpha)} \sum_{k\in G_\alpha} c_k + \frac{1}{3\cdot 2^{2n}}\Big\} + 2C2^n \max\{c_k : k\in G_\alpha\}$$

The next lemma is proven in [3], Lemma 3.7.

Lemma 3.10. Let x be a (C,θ,n) vector in $\mathfrak{X}_{\text{usm}}$. Let also $\{\alpha_q\}_{q=1}^d$ be a very fast growing and $\mathcal{S}_j$-admissible sequence of α-averages with $j < n$. Then the following holds.

$$\sum_{q=1}^d |\alpha_q(x)| < \frac{6C}{s(\alpha_1)} + \frac{1}{2^n} \tag{6}$$

The following corollary is an immediate consequence of Lemma 3.10 and it is similar with Proposition 3.9 from [3].

Corollary 3.11. Let x be a (C,θ,n) vector in $\mathfrak{X}_{\text{usm}}$. Let also f be a functional of type I_α in W with $w(f) = j < n$. Then the following holds

$$|f(x)| < \frac{6C + 1/2^n}{2^j} \tag{7}$$

Combining Lemma 3.10 with Corollary 3.11 we conclude the following.

Corollary 3.12. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$, such that x_k is a (C,θ,n_k) vector and $\{n_k\}_k$ is strictly increasing. Then $\alpha\big(\{x_k\}_k\big) = 0$. Moreover, passing if necessary to a subsequence, $\{x_k\}_k$ is $(7C,\{n_k\}_k)$ α-RIS.

Notation. Let $x = 2^n \sum_{k=1}^m c_k x_k$ be a (C,θ,n) exact vector in $\mathfrak{X}_{\text{usm}}$, where $\{x_k\}_{k=1}^m$ is $(C,\{n_k\}_{k=1}^m)$ α-RIS. Let also $f = \frac{1}{2}\sum_{q\in F}(f_q + g_q)$ be a type II_+ functional (or $f = \frac{1}{2}\sum_{q\in F}\lambda_q(f_q - g_q)$ be a type II_- functional). Set $i_q = w(f_q)$ for $q \in F$ and

$$\begin{aligned}
E_0 &= \{q : n \leqslant i_q < 2^{2n}\}\\
E_1 &= \{q : i_q < n\}\\
E_2 &= \{q : 2^{2n} < i_q < n_1\}\\
J_k &= \{q : n_k \leqslant i_q < n_{k+1}\}, \text{ for } k < m \text{ and } \quad J_m = \{q : n_m \leqslant i_q\}
\end{aligned}$$

Note that from Remark 1.1 either $E_0 = \varnothing$ or $\#E_0 = 1$. Under the above notation the following lemma holds, which is similar to Lemma 3.13 from [3] and their proofs are almost identical.

Lemma 3.13. Let $x = 2^n \sum_{k=1}^m c_k x_k$ be a (C, θ, n) exact vector in $\mathfrak{X}_{\text{usm}}$, where $\{x_k\}_{k=1}^m$ is $(C, \{n_k\}_{k=1}^m)$ α-RIS.

Then if $f = \frac{1}{2}\sum_{q\in F}(f_q + g_q)$ is a functional of type II_+, there exists $F_f \subset \{k : \operatorname{ran} f \cap \operatorname{ran} x_k \neq \varnothing\}$ with $\{\min\operatorname{supp} x_k : k \in F_f\} \in \mathcal{S}_2$ such that

$$\begin{aligned} f(x) &< \frac{1}{2}\sum_{q\in E_0}(f_q+g_q)(x) + \sum_{q\in E_1}\frac{7C}{2^{i_q}} + \sum_{k=2}^m\sum_{q\in J_k}\frac{2^{n_k}}{2^{i_q+n_{k-1}}} \\ &+ \sum_{k=1}^{m-1}\sum_{q\in J_k}\frac{C2^n}{2^{i_q}} + \sum_{q\in E_2}\frac{C2^n}{2^{i_q}} + C2^n\sum_{k\in F_f}c_k \end{aligned} \tag{8}$$

Similarly, if $f = \frac{1}{2}\sum_{q\in F}\lambda_q(f_q - g_q)$ is a functional of type II_-, there exists $F_f \subset \{k : \operatorname{ran} f \cap \operatorname{ran} x_k \neq \varnothing\}$ with $\{\min\operatorname{supp} x_k : k \in F_f\} \in \mathcal{S}_2$ such that

$$\begin{aligned} f(x) &< \frac{1}{2}\sum_{q\in E_0}\lambda_q(f_q-g_q)(x) + \sum_{q\in E_1}\frac{7C}{2^{i_q}} + \sum_{k=2}^m\sum_{q\in J_k}\frac{2^{n_k}}{2^{i_q+n_{k-1}}} \\ &+ \sum_{k=1}^{m-1}\sum_{q\in J_k}\frac{C2^n}{2^{i_q}} + \sum_{q\in E_2}\frac{C2^n}{2^{i_q}} + C2^n\sum_{k\in F_f}c_k \end{aligned} \tag{9}$$

The next corollary is similar to Corollary 3.14 from [3].

Corollary 3.14. Let x be a (C, θ, n) exact vector in $\mathfrak{X}_{\text{usm}}$ and $f = \frac{1}{2}\sum_{q\in F}(f_q + g_q)$ be a type II_+ functional (or $f = \frac{1}{2}\sum_{q\in F}\lambda_q(f_q - g_q)$ be a type II_- functional), such that $\{n, \dots, 2^{2n}\} \cap \hat{w}(f) = \varnothing$. Set $i_q = w(f_q)$ for $q \in F$ and $E_1 = \{q : i_q < n\}$. Then the following holds.

$$|f(x)| < \sum_{q\in E_1}\frac{7C}{2^{i_q}} + \frac{2C}{2^n} \tag{10}$$

The lemma which follows is similar to Lemma 3.15 from [3].

Lemma 3.15. Let $x = 2^n \sum_{k=1}^m c_k x_k$ be a (C, θ, n) exact vector in $\mathfrak{X}_{\text{usm}}$ and β be a β-average in W. Then there exists $F_\beta \subset \{k : \operatorname{ran}\beta \cap \operatorname{ran} x_k \neq \varnothing\}$ with $\{\min\operatorname{supp} x_k : k \in F_f\} \in \mathcal{S}_2$ such that

$$|\beta(x)| < \frac{8C}{s(\beta)} + C2^n\sum_{k\in F_\beta}c_k \tag{11}$$

The next lemma is similar to Lemma 3.16 from [3].

Lemma 3.16. Let x be a (C,θ,n) exact vector in $\mathfrak{X}_{\text{usm}}$ and $\{\beta\}_{q=1}^d$ be a very fast growing and $\mathcal{S}_j$-admissible sequence of β-averages with $j \leqslant n-3$. Then the following holds

$$\sum_{q=1}^d |\beta_q(x)| < \sum_{q=1}^d \frac{8C}{s(\beta_q)} + \frac{1}{2^n} \tag{12}$$

If moreover $s(\beta_1) \geqslant \min\operatorname{supp} x$, then $\sum_{q=1}^d |\beta_q(x)| < \frac{2}{2^n}$

The next result uses the previous lemma and it is similar to Proposition 3.17 from [3].

Corollary 3.17. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$, such that x_k is a (C,θ,n_k) exact vector and $\{n_k\}_k$ is strictly increasing. Then $\beta\big(\{x_k\}_k\big) = 0$.

4. A combinatorial result

In this section we introduce a new condition concerning the behaviour of branches of special functionals on a block sequence $\{x_k\}_k$ (see the definition below). When this condition is satisfied, we shall write $\mathcal{B}\otimes\{x_k\}_k = 0$. We prove that one can find in every block subspace a normalized block sequence $\{x_k\}_k$ satisfying $\mathcal{B}\otimes\{x_k\}_k = 0$, as well as $\alpha\big(\{x_k\}_k\big) = 0$ and $\beta\big(\{x_k\}_k\big) = 0$. We then proceed to prove a Ramsey type result concerning block sequences with $\mathcal{B}\otimes\{x_k\}_k = 0$ and $\beta\big(\{x_k\}_k\big) = 0$. The above are used in the next section to show that a block sequence $\{x_k\}_k$ with $\mathcal{B}\otimes\{x_k\}_k = 0$, $\alpha\big(\{x_k\}_k\big) = 0$ and $\beta\big(\{x_k\}_k\big) = 0$, has a subsequence generating a c_0 spreading model.

Definition 4.1. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ and $b = \{f_q, g_q\}_{q=1}^\infty \in \mathcal{B}$ (see the definition of the norming set) satisfying the following. For every $\varepsilon > 0$ there exist $k_0, q_0 \in \mathbb{N}$, such that for every $k \geqslant k_0, q \geqslant q_0$ we have that $|(f_q \pm g_q)(x_k)| < \varepsilon$. Then we write $b\otimes\{x_k\}_k = 0$. If $b\otimes\{x_k\}_k = 0$ for every $b\in\mathcal{B}$, then we write $\mathcal{B}\otimes\{x_k\}_k = 0$.

Remark 4.2. If $b\otimes\{x_k\}_k \neq 0$, using a pigeon hole argument, it is easy to see that there exists M an infinite subset of the natural numbers and $\varepsilon > 0$ such that one of the following holds.

(i) For every $k \in M$, there exists $q\in\mathbb{N}$ such that $|(f_q + g_q)(x_k)| \geqslant \varepsilon$. In this case we say that b_+ ε-norms $\{x_k\}_k$.

(ii) For every $k \in M$, there exists $q\in\mathbb{N}$ such that $|(f_q - g_q)(x_k)| \geqslant \varepsilon$. In this case we say that b_- ε-norms $\{x_k\}_k$.

In either case we say that b ε-norms $\{x_k\}_k$.

Proposition 4.3. Let $\{x_k\}_k$ be a bounded block sequence and $b\in\mathcal{B}$ such that b_+ ε-norms $\{x_k\}_k$. Then there exists a subsequence of $\{x_k\}_k$ that generates an ℓ_1 spreading model.

Proof. If $b = \{f_q, g_q\}_{q=1}^\infty$ passing, if necessary, to a subsequence, we may assume the following.

(i) For every $k \in \mathbb{N}$ there exists $q_k \in \mathbb{N}$ such that $(f_{q_k} + g_{q_k})(x_k) > \varepsilon$ and $\min\operatorname{supp} f_{q_k} \geqslant 2k$.
(ii) For $k \neq m \in \mathbb{N}$, $\operatorname{ran}(f_{q_k} + g_{q_k}) \cap \operatorname{ran} x_m = \varnothing$

Then for $n \leqslant k_1 < \cdots < k_n$ natural numbers and $\{c_i\}_{i=1}^n$ non negative reals, we have that $f = \frac{1}{2}\sum_{i=1}^n (f_{q_{k_i}} + g_{q_{k_i}}) \in W$ and $f(\sum_{i=1}^n c_i x_{k_i}) > \frac{\varepsilon}{2}\sum_{i=1}^n c_i$, therefore

$$\|\sum_{i=1}^n c_i x_{k_i}\| > \frac{\varepsilon}{2}\sum_{i=1}^n c_i \tag{13}$$

Since $\{x_k\}_k$ is weakly null, every spreading model admitted by it must be unconditional. Combining this fact with (13), we conclude that every spreading model admitted by $\{x_k\}_k$ is equivalent to the usual basis of ℓ_1.

□

Lemma 4.4. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ with $\beta\big(\{x_k\}_k\big) = 0$ and $\varepsilon > 0$. Then there exists M an infinite subset of the natural numbers, such that the set $B_\varepsilon = \{b \in \mathcal{B} : b\ \varepsilon\text{-norms } \{x_k\}_{k\in M}\}$ is finite.

Proof. Towards a contradiction, assume that for every M infinite subset of the natural numbers, the set $\{b \in \mathcal{B} : b\ \varepsilon\text{-separates } \{x_k\}_{k\in M}\}$ is infinite. By using induction, choose $M_1 \supset M_2 \supset \cdots \supset M_n \supset \cdots$ infinite subsets of the natural numbers and $\{b_n : n \in \mathbb{N}\} \subset \mathcal{B}$ with $b_n \neq b_m$ for $n \neq m$, satisfying the following. For every $n \in \mathbb{N}$ and and $k \in M_n$, if $b_n = \{f_q^n, g_q^n\}_{q=1}^\infty$ there exists $q \in \mathbb{N}$ such that either $|(f_q^n + g_q^n)(x_k)| > \varepsilon$ or $|(f_q^n - g_q^n)(x_k)| > \varepsilon$. To simplify notation, from now on we will assume that $|(f_q^n + g_q^n)(x_k)| > \varepsilon$.

We are going to prove the following. For every $k_0, m \in \mathbb{N}$, there exists $k \geqslant k_0$ and β a β-average in W of size $s(\beta) = m$, such that $\beta(x_k) > \frac{\varepsilon}{2}$. By Proposition 3.7, this means that $\beta\big(\{x_k\}_k\big) > 0$ which yields a contradiction.

Let $k_0, m \in \mathbb{N}$. Since $b_n \neq b_l$ for $n \neq l$, there exists $q_0 \in \mathbb{N}$, such that for every $1 \leqslant n < l \leqslant m$, for every $q_1, q_2 \geqslant q_0$, $w(f_{q_1}^n) \neq w(f_{q_2}^l)$.

Choose $k \in M_m$ with $k \geqslant k_0$ and $\min\operatorname{supp} x_k \geq \max\{\max\operatorname{supp} g_{q_0}^n : n = 1, \ldots, m\}$. Then for $n = 1, \ldots, m$ there exists $q_n > q_0$ such that $|(f_{q_n}^n + g_{q_n}^n)(x_k)| > \varepsilon$. Set $h_n = \operatorname{sgn}\big((f_{q_n}^n + g_{q_n}^n)(x_k)\big)\frac{1}{2}(f_{q_n}^n + g_{q_n}^n)$ for $n = 1, \ldots, m$.

Then h_n is a functional of type II in W with $\hat{w}(h_n) = \{w(f_{q_n}^n)\}$ for $n = 1, \ldots, m$ and $h_n(x_k) > \frac{\varepsilon}{2}$. Since $\hat{w}(h_n) \cap \hat{w}(h_l) = \varnothing$ for $1 \leqslant n < l \leqslant m$, we have that $\beta = \frac{1}{m}\sum_{n=1}^m h_n$ is a β-average of size $s(\beta) = m$ with $\beta(x_k) > \frac{\varepsilon}{2}$. This completes the proof.

□

Lemma 4.5. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ with $\beta\big(\{x_k\}_k\big) = 0$. Then there exists M an infinite subset of the natural numbers, such that the set $B = \{b \in \mathcal{B} :$ there exists $\varepsilon > 0$ such that $b\ \ \varepsilon\text{-norms } \{x_k\}_{k\in M}\}$ is countable.

Proof. Apply Lemma 4.4 and choose $M_1 \supset M_2 \supset \cdots \supset M_n \supset \cdots$ infinite subsets of the natural numbers such that the set $B_n = \{b \in \mathcal{B} : b \ \ \frac{1}{n}\text{-norms } \{x_k\}_{k\in M_n}\}$ is finite, for every $n \in \mathbb{N}$. Choose M a diagonalization of $\{M_n\}_n$.

We will show that $B = \{b \in \mathcal{B} :$ there exists $\varepsilon > 0$ such that $b \ \ \varepsilon$-norms $\{x_k\}_{k\in M}\} \subset \cup_n B_n$.

Let $b \in B$. Then, there exists $n \in \mathbb{N}$, such that $b \ \frac{1}{n}$-norms $\{x_k\}_{k\in M}$. It easily follows that $b \in B_n$.

$\square$

Lemma 4.6. Let $\{x_k\}_k$ be a bounded block sequence in $\mathfrak{X}_{\text{usm}}$ with $\beta\big(\{x_k\}_k\big) = 0$. Then there exists $\{F_k\}_k$ an increasing sequence of subsets of the natural numbers with $\#F_k \leqslant \min F_k$ for all $k \in \mathbb{N}$ with $\lim_k \#F_k = \infty$ such that if $y_k = \frac{1}{\#F_k}\sum_{i\in F_k} x_i$, then $\mathcal{B} \otimes \{y_k\}_k = 0$.

Proof. Using Lemma 4.5 and passing, if necessary, to a subsequence, we may assume that if $B' = \{b \in \mathcal{B} :$ there exists $\varepsilon > 0$ such that $b\ \varepsilon$-norms $\{x_k\}_k\}$, then $B' = \{b_n : n \in \mathbb{N}\}$.

Let $b_n = \{f_q^n, g_q^n\}_{q=1}^\infty$ for all $n \in \mathbb{N}$ and choose $M_1 \supset M_2 \supset \cdots \supset M_n \supset \cdots$ infinite subsets of the natural numbers such that for every $n, q \in \mathbb{N}$, there exists at most one $k \in M_n$, with $\operatorname{ran}(f_q^n + g_q^n) \cap \operatorname{ran} x_k \neq \varnothing$.

Choose M a diagonalization of $\{M_n\}_n$. Then for every $n \in \mathbb{N}$ there exists $q_n \in \mathbb{N}$ such that for every $q \geqslant q_n$ there exists at most one $k \in M$ with $\operatorname{ran}(f_q^n + h_q^n) \cap \operatorname{ran} x_k \neq \varnothing$.

Choose $\{F_k\}_k$ an increasing sequence of subsets of the natural numbers with $\#F_k \leqslant \min F_k$ for all $k \in \mathbb{N}$ with $\lim_k \#F_k = \infty$ and set $y_k = \frac{1}{\#F_k}\sum_{i\in F_k} x_i$ for all $k \in \mathbb{N}$.

Towards a contradiction, assume that there exist $\varepsilon > 0$ and $b = \{f_q, g_q\}_{q=1}^\infty \in \mathcal{B}$, such that $b\ \varepsilon$-norms $\{y_k\}_k$. For convenience, assume that $b_+ \ \ \varepsilon$-norms $\{y_k\}_k$ and choose N an infinite subset of the naturals, such that for every $k \in N$ there exists $q_k \in \mathbb{N}$ with $|(f_{q_k} + g_{q_k})(y_k)| > \varepsilon$.

It follows that for every $k \in N$, there exists $i_k \in F_k$ such that $|(f_{q_k} + g_{q_k})(x_{i_k})| > \varepsilon$. We conclude that $b\ \varepsilon$-norms $\{x_k\}_k$ and hence $b \in B'$, i.e. $b = b_n$, for some $n \in \mathbb{N}$.

Choose $k \in N$ with $k > \max\operatorname{supp} g_{q_n}^n$ and $\#F_k > \varepsilon^{-1}\sup\{\|x_k\| : k \in \mathbb{N}\}$. Then for every $q \in \mathbb{N}$, there exists at most one $i \in F_k$, such that $\operatorname{ran}(f_q^n + g_q^n) \cap \operatorname{ran} x_i \neq \varnothing$ and hence for every $q \in \mathbb{N}$, we have that $|(f_q^n + h_q^n)(y_k)| < \frac{\sup\{\|x_k\|:k\in\mathbb{N}\}}{\#F_k} < \varepsilon$. This contradiction completes the proof.

$\square$

Proposition 4.7. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ such that x_k is a (C, θ, n_k) exact vector with $n_k \in L_3$ (see the definition of the coding function) and $\{n_k\}_k$ is strictly increasing. Then $\mathcal{B} \otimes \{x_k\}_k = 0$.

Proof. Let $b \in \mathcal{B}$ Observe that for $q \in \mathbb{N}, h_q = \frac{1}{2}(f_q \pm g_q)$ is a functional of type II and by Corollary 3.14, if $i_q = w(f_q)$ for $k \in \mathbb{N}$ we have that $|h_q(x_k)| < \frac{7C}{2^{i_q}} + \frac{2C}{2^{n_k}}$. From this it easily follows that $b \otimes \{x_k\}_k = 0$. $\square$

Proposition 4.8. Let $\{x_k\}_k$ be a normalized block sequence in $\mathfrak{X}_{\text{usm}}$. Then there exists $\{y_k\}_k$ a further normalized block sequence of $\{x_k\}_k$ such that $\alpha\big(\{y_k\}_k\big) = 0, \beta\big(\{y_k\}_k\big) = 0$ and $\mathcal{B} \otimes \{y_k\}_k = 0$.

Proof. Since $\mathfrak{X}_{\text{usm}}$ does not contain a copy of c_0, we may choose $\{z_k\}_k$ a normalized block sequence of $\{x_k\}_k$, such that if $z_k = \sum_{i\in G_k} c_i x_i$, then $\lim_k \max\{|c_i| : i \in G_k\} = 0$.

If $\alpha\big(\{z_k\}_k\big) = 0, \beta\big(\{z_k\}_k\big) = 0$ and $\mathcal{B} \otimes \{z_k\}_k = 0$, then $\{z_k\}_k$ is the desired sequence. Otherwise, we distinguish three cases.

Case 1: $\alpha\big(\{z_k\}_k\big) = 0, \beta\big(\{z_k\}_k\big) = 0$ and there exist $b \in \mathcal{B}$, $\varepsilon > 0$ such that b_+ ε-norms $\{z_k\}_k$.

Using Proposition 4.3 and passing, if necessary, to a subsequence, we may assume that $\{z_k\}_k$ generates an ℓ_1 spreading model. Apply Lemma 4.6 to find $\{F_k\}_k$ an increasing sequence of subsets of the natural numbers with $\#F_k \leqslant \min F_k$ for all $k \in \mathbb{N}$ with $\lim_k \#F_k = \infty$ such that if $y_k = \frac{1}{\#F_k}\sum_{i\in F_k} z_i$, then $\mathcal{B} \otimes \{y_k\}_k = 0$.

Since $\{z_k\}_k$ generates an ℓ_1 spreading model, we have that $\{y_k\}_k$ is seminormalized. Moreover Remark 3.5 yields that $\alpha\big(\{y_k\}_k\big) = 0$ as well as $\beta\big(\{y_k\}_k\big) = 0$. We conclude that if $y'_k = \frac{1}{\|y_k\|}y_k$, then $\{y'_k\}_k$ is the desired sequence.

Case 2: $\alpha\big(\{z_k\}_k\big) = 0, \beta\big(\{z_k\}_k\big) = 0$ and there exist $b \in \mathcal{B}$, $\varepsilon > 0$ such that b_- ε-norms $\{z_k\}_k$.

If $b = \{f_q, g_q\}_{q=1}^\infty$ passing if necessary to a subsequence, we may assume that for every $k \in \mathbb{N}$ there exists $q_k \in \mathbb{N}$ such that $|(f_{q_k} - g_{q_k})(z_k)| > \varepsilon$ and $\max\{|c_i| : i \in F_k\} < \frac{\varepsilon}{2}$.

Fix $k \in \mathbb{N}$ and set $i_k = \max\{i \in G_k : \operatorname{ran} f_{q_k} \cap \operatorname{ran} x_i \neq \varnothing\}, G_k^1 = \{i \in G_k : i \leqslant i_k\}$ and $G_k^2 = \{i \in G_k : i > i_k\}$. Set

$$z'_k = \operatorname{sgn}\big(f_{q_k}(z_k)\big)\sum_{i\in G_k^1} c_i x_i + \operatorname{sgn}\big(g_{q_k}(z_k)\big)\sum_{i\in G_k^2} c_i x_i$$

Observe the following.

$$\begin{aligned} f_{q_k}(z'_k) &= |f_{q_k}(z_k)| \\ g_{q_k}(z'_k) &> |g_{q_k}(z_k)| - |c_{i_k}| > |g_{q_k}(z_k)| - \frac{\varepsilon}{2} \end{aligned}$$

$$\frac{1}{2} \leqslant \|z'_k\| \leqslant 2$$

Combining the above we conclude that by setting $w_k = \frac{1}{\|z'_k\|}z'_k$, we have that $(f_{q_k} + g_{q_k})(w_k) > \frac{\varepsilon}{4}$, i.e. b_+ $\frac{\varepsilon}{4}$-norms $\{w_k\}_k$. Moreover Remarks 3.3 and 3.4 yield that $\alpha\big(\{w_k\}_k\big) = 0$ as well as $\beta\big(\{w_k\}_k\big) = 0$, hence this case has been reduced to the previous one.

Case 3: $\alpha\big(\{z_k\}_k\big) > 0$ or $\beta\big(\{z_k\}_k\big) > 0$.

Apply proposition 3.8 to construct a sequence $\{y'_k\}_k$ of (C,θ,n_k) vectors with $\{n_k\}_k$ strictly increasing. Set $y_k = \frac{1}{\|y_k\|}y'_k$. Corollary 3.12 yields that $\alpha\big(\{y_k\}_k\big) = 0$ and passing, if necessary to a subsequence, $\{y_k\}_k$ is $(C,\{n_k\}_k)$ α-RIS.

Assume that $\beta\big(\{y_k\}_k\big) = 0$. Then this case is reduced either to case 1, or to case 2.

If on the other hand $\beta\big(\{y_k\}_k\big) > 0$, apply proposition 3.8 to construct a sequence $\{w'_k\}_k$ of (C,θ,n_k) exact vectors with $n_k \in L_3$ for all $k \in \mathbb{N}$ and $\{n_k\}_k$ strictly increasing. Set $w_k = \frac{1}{\|w'_k\|}w_k$. Corollaries 3.12, 3.17 and Proposition 4.7 yield that $\{w_k\}_k$ is the desired sequence.

$\square$

The following definition is a slight variation of Definition 4.1 from [3].

Definition 4.9. Let $x_1 < x_2 < x_3$ be vectors in $\mathfrak{X}_{\text{usm}}$, $f = \pm E\big(\frac{1}{2}\sum_{q\in F}(f_q + g_q)\big)$ be a functional of type II_+ (or $f = \pm E\big(\frac{1}{2}\sum_{q\in F}\lambda_q(f_q - g_q)\big)$ be a functional of type II_-), such that $\operatorname{supp} f \cap \operatorname{ran} x_i \neq \varnothing$, for $i = 1,2,3$. Set $q_0 = \min\{q \in F : \operatorname{ran}(f_q + g_q) \cap \operatorname{ran} x_2 \neq \varnothing\}$. If $\operatorname{ran}(f_{q_0} + g_{q_0}) \cap \operatorname{ran} x_3 = \varnothing$, then we say that f separates x_1, x_2, x_3.

Lemma 4.10. Let $\{n_k\}_k$ be a strictly increasing sequence of natural numbers satisfying the following. For every $m \in \mathbb{N}$, there exists a special sequence $\{f_q^m, g_q^m\}_{q=1}^{d_m}$ such that $\{n_k : k = 1,\dots,m\} \subset \{w(f_q^m) : q = 1,\dots,d_m\}$. Then there exists $b = \{f_q, g_q\}_{q=1}^\infty \in \mathcal{B}$, such that $\{n_k : k \in \mathbb{N}\} \subset \{w(f_q) : q \in \mathbb{N}\}$.

Proof. We construct b by induction. Let $m \in \mathbb{N}$ and suppose that we have chosen natural numbers $1 \leqslant p_1 < \cdots < p_m$ and a special sequence $\{f_q, g_q\}_{q=1}^{p_m}$ such that the following are satisfied. For $1 \leqslant l \leqslant m$

(i) $\{n_k : k = 1,\dots,l\} \subset \{w(f_q) : q = 1,\dots,p_l\}$

(ii) $\sigma(f_1, g_1, f_2, g_2 \dots, f_{p_l}, g_{p_l}) = n_{l+1}$

Since $\{n_k : k = 1,\dots,m+2\} \subset \{w(f_q^{m+2}) : q = 1,\dots,d_{m+2}\}$, there exist $1 < q_0 < q_1 \leqslant d_{m+2}$, such that $w(f_{q_0}^{m+2}) = n_{m+1}$ and $w(f_{q_1}^{m+2}) = n_{m+2}$

Then
$$\sigma(f_1^{m+2}, g_1^{m+2}, \dots, f_{q_1-1}^{m+2}, g_{q_1-1}^{m+2}) = n_{m+2}$$
Set $p_{m+1} = q_1 - 1$. It remains to be shown that $p_m < p_{m+1}$ and that $\{f_q, g_q\}_{q=1}^{p_m} = \{f_q^{m+2}, g_q^{m+2}\}_{q=1}^{p_m}$.

Since
$$\begin{aligned} n_{m+1} &= \sigma(f_1, g_1, \dots, f_{p_m}, g_{p_m}) \\ w(f_{q_0}^{m+2}) &= \sigma(f_1^{m+2}, g_1^{m+2}, \dots, f_{q_0-1}^{m+2}, g_{q_0-1}^{m+2}) \end{aligned}$$
and $w(f_{q_0}^{m+2}) = n_{m+1}$, by the fact that σ is one to one, we conclude that $\{f_q, g_q\}_{q=1}^{p_m} = \{f_q^{m+2}, g_q^{m+2}\}_{q=1}^{q_0-1}$. Thus, it follows that $p_m = q_0 - 1 < q_1 - 1 = p_{m+1}$ and $\{f_q, g_q\}_{q=1}^{p_m} = \{f_q^{m+2}, g_q^{m+2}\}_{q=1}^{p_m}$.

$\square$

Proposition 4.11. Let $\{x_k\}_k$ be a bounded block sequence in $\mathfrak{X}_{\text{usm}}$, such that $\beta\big(\{x_k\}_k\big) = 0$ and $\mathcal{B} \otimes \{x_k\}_k = 0$. Then for any $\varepsilon > 0$, there exists M an infinite subset of the naturals, such that for any $k_1 < k_2 < k_3 \in M$, for any functional $f \in W$ of type II that separates $x_{k_1}, x_{k_2}, x_{k_3}$, we have that $|f(x_{k_i})| < \varepsilon$, for some $i \in \{1, 2, 3\}$.

Proof. Towards a contradiction, assume that this is not the case. By using Ramsey theorem [18], we may assume that there exists $\varepsilon > 0$ such that for any $k < l < m \in \mathbb{N}$, there exists $f_{k,l,m}$ a functional of type II that separates x_k, x_l, x_m and $|f_{k,l,m}(x_k)| > \varepsilon, |f_{k,l,m}(x_l)| > \varepsilon, |f_{k,l,m}(x_m)| > \varepsilon$. We may also assume that $f_{k,l,m}$ is of type II$_+$, for every $k < l < m \in \mathbb{N}$, or that $f_{k,l,m}$ is of type II$_-$, for every $k < l < m \in \mathbb{N}$. From now on we shall assume the first.

For $1 < k < m \in \mathbb{N}$, there exists $b_{k,m} = \{f_q^{k,m}, g_q^{k,m}\}_{q=1}^{\infty} \in \mathcal{B}$, with $f_{1,k,m} = E_{k,m}\Bigg(\frac{1}{2}\sum_{q \in F_{k,m}}(f_q^{k,m} + g_q^{k,m})\Bigg)$. Set

$$\begin{aligned} p_{k,m} &= \min\{q \in F_{k,m} : \operatorname{ran}(f_q^{k,m} + g_q^{k,m}) \cap x_1 \neq \varnothing\} \\ q_{k,m} &= \min\{q \in F_{k,m} : \operatorname{ran}(f_q^{k,m} + g_q^{k,m}) \cap x_k \neq \varnothing\} \end{aligned}$$

Notice, that for $1 < k < m$, since $|f_{1,k,m}(x_1)| > \varepsilon$, it follows that, if $w(f_{p_{k,m}}^{k,m}) = j_{k,m}$

$$\frac{1}{2^{j_{k,m}}} > \frac{\varepsilon}{\|x_1\| \max\operatorname{supp} x_1}$$

By applying Ramsey theorem once more, we may assume that there exists $j_1 \in \mathbb{N}$, such that for any $1 < k < m$, we have that $w(f_{p_{k,m}}^{k,m}) = j_1$.

Arguing in the same way and diagonalizing, we may assume that for any $k > 1$, there exists $j_k \in \mathbb{N}$ such that for any $m > k$, we have that $w(f_{q_{k,m}}^{k,m}) = j_k$.

Moreover, for every $1 < k < m \in \mathbb{N}$, the following holds.

$$2(\#F_{k,m}) \leqslant \min\operatorname{supp} f_{p_{k,m}}^{k,m} \leqslant \max\operatorname{supp} x_1$$

Setting $\varepsilon' = \frac{4\varepsilon}{\max\operatorname{supp} x_1}$, there exists $r_{k,m} \in F_{k,m}$ such that

$$|E_{k,m}\big(\frac{1}{2}(f_{r_{k,m}}^{k,m} + g_{r_{k,m}}^{k,m})\big)(x_m)| > \varepsilon' \tag{14}$$

Since $f_{1,k,m}$ separates x_1, x_k, x_m, it follows that $r_{k,m} > q_{k,m}$.

Set $i_{k,m} = w(f_{r_{k,m}}^{k,m})$ for all $1 < k < m \in \mathbb{N}$ and

$$A = \big\{\{k, l, m\} \in [\mathbb{N} \setminus \{1\}]^3 : i_{k,m} = i_{l,m}\big\}$$

Applying Ramsey theorem once more, we may assume that either $[\mathbb{N} \setminus \{1\}]^3 \subset A$ or $[\mathbb{N} \setminus \{1\}]^3 \subset A^c$.

Assume that $[\mathbb{N} \setminus \{1\}]^3 \subset A^c$. Then, for $m > 2$, we have that

$$h_k = \operatorname{sgn}\Bigg(E_{k,m}\big(\frac{1}{2}(f_{r_{k,m}}^{k,m} + g_{r_{k,m}}^{k,m})\big)(x_m)\Bigg)E_{k,m}\big(\frac{1}{2}(f_{r_{k,m}}^{k,m} + g_{r_{k,m}}^{k,m})\big)$$

are functionals of type II with pairwise disjoint weights $\hat{w}(h_k)$ and $h_k(x_m) > \varepsilon'$ for $k = 2, \dots, m-1$. We conclude that $\beta = \frac{1}{m-2}\sum_{k=2}^{m-1} h_k$ is a β-average in W of size $s(\beta) = m-2$ and $\beta(x_m) > \varepsilon'$. Proposition 3.7 yields that $\beta\big(\{x_k\}_k\big) > 0$, which is absurd.

Hence, we may assume that $[\mathbb{N} \setminus \{1\}]^3 \subset A$, i.e. for every $m > 2$, there exists $i_m \in \mathbb{N}$, such that for every $1 < k < m$, $i_{k,m} = i_m$. By the fact that σ is one to one, we conclude that for every $m > 2$, by setting $\{f_q^m, g_q^m\}_{q=1}^{r_m} = \sigma^{-1}(\{i_m\})$ the following holds.

$$\{f_q^{k,m}, g_q^{k,m}\}_{q=1}^{r_{k,m}-1} = \{f_q^m, g_q^m\}_{q=1}^{r_m}, \quad \text{for } 1 < k < m \tag{15}$$

Set

$$C = \big\{\{k,l\} \in [\mathbb{N} \setminus \{1\}]^2 : j_k \neq j_l\big\}$$

Applying Ramsey theorem once more, we may assume that either $[\mathbb{N} \setminus \{1\}]^2 \subset C$ or $[\mathbb{N} \setminus \{1\}]^2 \subset C^c$.

Assume that $[\mathbb{N} \setminus \{1\}]^2 \subset C^c$. Then there exists $j_0 \in \mathbb{N}$, such that $j_k = j_0$ for all $k > 1$. For $1 < k < m$, by (15) $\{f_{q_{k,m}}^{k,m}, g_{q_{k,m}}^{k,m}\} \in \{f_q^m, g_q^m : q = 1, \dots, r_m\}$. Since for $2 < k < m$, $j_2 = j_k$, we conclude that $\{f_{q_{2,m}}^{2,m}, g_{q_{2,m}}^{2,m}\} = \{f_{q_{k,m}}^{k,m}, g_{q_{k,m}}^{k,m}\}$.

Set $h_m = \frac{1}{2}(f_{q_{2,m}}^{2,m} + g_{q_{2,m}}^{2,m})$. By the fact that $f_{2,m}, f_{m-1,m}$ separate x_1, x_2, x_m and x_1, x_{m-1}, x_m respectively, we conclude that $\operatorname{ran} x_k \subset \operatorname{ran} h_m$ and $|h_m(x_k)| > \varepsilon$ for $k = 3, \dots, m-2$. Choose h a w^*-limit point of $\{h_m\}_m$. Then $|h(x_k)| \geqslant \varepsilon$ for every $k > 2$. Corollary 2.7 yields a contradiction.

Hence, we may assume that $[\mathbb{N} \setminus \{1\}]^2 \subset C$, and that $\{j_k\}_k$ is strictly increasing. Lemma 4.10 and (15) yield that there exists $b = \{f_q, g_q\}_{q=1}^\infty \in \mathcal{B}$, such that $\{j_k : k \in \mathbb{N}\} \subset \{w(f_q) : q \in \mathbb{N}\}$.

We will show that b ε'-norms $\{x_k\}_k$, which will complete the proof. Let $1 < k < m \in \mathbb{N}$. Arguing as previously, there exists $t_{k,m} \in F_{k,m}$, such that $|(f_{t_{k,m}}^{k,m} + g_{t_{k,m}}^{k,m})| > \varepsilon'$. Evidently, $q_{k,m} \leqslant t_{k,m} \leqslant r_{k,m}$ Set

$$D = \{\{k,m\} \in [\mathbb{N} \setminus \{1\}]^2 : t_{k,m} < r_{k,m}\}$$

Applying Ramsey theorem one last time, we may assume that either $[\mathbb{N} \setminus \{1\}]^2 \subset D$, or $[\mathbb{N} \setminus \{1\}]^2 \subset D^c$.

If $[\mathbb{N} \setminus \{1\}]^2 \subset D^c$, then for $m > 3$, by (15) we have that $t_{m-2,m} = r_{m-2,m} = r_m + 1$ and $\{f_1^m, g_1^m, \dots, f_{r_m}^m, g_{r_m}^m, f_{t_{m-2,m}}^{m-2,m}, g_{t_{m-2,m}}^{m-2,m}\}$ is a special sequence.

Similarly, by (15) we have that $t_{m-1,m} = r_{m-1,m} = r_m + 1$ and that $\{f_1^m, g_1^m, \dots, f_{r_m}^m, g_{r_m}^m, f_{t_{m-1,m}}^{m-1,m}, g_{t_{m-1,m}}^{m-1,m}\}$ is a special sequence.

Since $q_{m-1,m} < r_{m-1,m} = t_{m-1,m}$, we have that there exists $q \leqslant r_m$, such that $\{f_{q_{m-1,m}}^{m-1,m}, g_{q_{m-1,m}}^{m-1,m}\} = \{f_q^m, g_q^m\}$.

This means the following.

$$\begin{aligned} \max\operatorname{supp} x_{m-2} &< \min\operatorname{supp} x_{m-1} \leqslant \max\operatorname{supp} g_{q_{m-1,m}} \\ &= \max\operatorname{supp} g_q^m < \min\operatorname{supp} f_{t_{m-2,m}}^{m-2,m} \end{aligned}$$

We conclude that $\operatorname{ran}(f^{m-2,m}_{t_{m-2,m}} + g^{m-2,m}_{t_{m-2,m}}) \cap \operatorname{ran} x_{m-2} = \varnothing$. This cannot be the case and hence we conclude that $[\mathbb{N} \setminus \{1\}]^2 \subset D$.

Let $k \in \mathbb{N}$. We will show that $f^{k,k+3}_{t_{k,k+3}} + g^{k,k+3}_{t_{k,k+3}} \in b_+$. First, observe that by (15) and the fact that $t_{k,k+3} \leqslant r_{k,k+3} - 1 = r_{k+3}$, we have that

$$\begin{aligned}
(f^{k,k+3}_{t_{k,k+3}} + g^{k,k+3}_{t_{k,k+3}}) &\in \{f^{k+3}_q + g^{k+3}_q : q = 1, \dots, r_{k+3}\}\\
(f^{k+1,k+3}_{q_{k+1,k+3}} + g^{k+1,k+3}_{q_{k+1,k+3}}) &\in \{f^{k+3}_q + g^{k+3}_q : q = 1, \dots, r_{k+3}\}\\
(f^{k+2,k+3}_{q_{k+2,k+3}} + g^{k+2,k+3}_{q_{k+2,k+3}}) &\in \{f^{k+3}_q + g^{k+3}_q : q = 1, \dots, r_{k+3}\}
\end{aligned}$$

Thus, we moreover have that

$$(f^{k,k+3}_{t_{k,k+3}} + g^{k,k+3}_{t_{k,k+3}}) \leqslant (f^{k+1,k+3}_{q_{k+1}-1} + g^{k+1,k+3}_{q_{k+1}}) < (f^{k+2,k+3}_{q_{k+2}} + g^{k+2,k+3}_{q_{k+2}})$$

By the fact that σ is one to one, we conclude that $\{f^{k,k+3}_{t_{k,k+3}}, g^{k,k+3}_{t_{k,k+3}}\} \in \sigma^{-1}(\{j_{k+2}\}) \subset \big\{\{f_q, g_q\} : q \in \mathbb{N}\big\}$.

□

5. c_0 SPREADING MODELS

In this section we prove that a sequence $\{x_k\}_k$ satisfying $\mathcal{B} \otimes \{x_k\}_k = 0$, $\alpha\big(\{x_k\}_k\big) = 0$ as well as $\beta\big(\{x_k\}_k\big) = 0$ has a subsequence generating a c_0 spreading model. This is crucial, as a spreading model universal sequence is constructed on a sequence generating a c_0 spreading model.

Proposition 5.1. Let $x_1 < \cdots < x_n$ be a seminormalized block sequence in $\mathfrak{X}_{\text{usm}}$, such that $\|x_k\| \leqslant 1$ for $k = 1, \dots, n$, $n \geqslant 3$ and there exist $n+3 \leqslant j_1 < \cdots < j_n$ strictly increasing naturals, such that the following are satisfied.

(i) For any $k_0 \in \{1, \dots, n\}$, for any $k \geqslant k_0, k \in \{1, \dots, n\}$, for any $\{\alpha_q\}_{q=1}^d$ very fast growing and $\mathcal{S}_j$-admissible sequence of α-averages, with $j < j_{k_0}$ and $s(\alpha_1) > \min\operatorname{supp} x_{k_0}$, we have that $\sum_{q=1}^d |\alpha_q(x_k)| < \frac{1}{n\cdot 2^n}$.

(ii) For any $k_0 \in \{1, \dots, n\}$, for any $k \geqslant k_0, k \in \{1, \dots, n\}$, for any $\{\beta_q\}_{q=1}^d$ very fast growing and $\mathcal{S}_j$-admissible sequence of β-averages, with $j < j_{k_0}$ and $s(\beta_1) > \min\operatorname{supp} x_{k_0}$, we have that $\sum_{q=1}^d |\beta_q(x_k)| < \frac{1}{n\cdot 2^n}$.

(iii) For $k = 1, \dots, n-1$, the following holds: $\frac{1}{2^{j_{k+1}}} \max\operatorname{supp} x_k < \frac{1}{2^n}$.

(iv) For any $1 \leqslant k_1 < k_2 < k_3 \leqslant n$, for any functional $f \in W$ of type II that separates $x_{k_1}, x_{k_2}, x_{k_3}$, we have that $|f(x_{k_i})| < \frac{1}{n\cdot 2^n}$, for some $i \in \{1, 2, 3\}$.

Then $\{x_k\}_{k=1}^n$ is equivalent to ℓ_∞^n basis, with an upper constant $4 + \frac{5}{2^n}$. Moreover, for any functional $f \in W$ of type I_α with weight $w(f) = j < j_1$, we have that $|f(\sum_{k=1}^n x_k)| < \frac{4+\frac{6}{2^n}}{2^j}$.

Proof. As in the proof of Proposition 4.7 from [3], we will inductively prove, that for any $\{c_k\}_{k=1}^n \subset [-1, 1]$ the following hold.

(i) For any $f \in W$, we have that $|f(\sum_{k=1}^{n} c_k x_k)| < (4 + \frac{5}{2^n}) \max\{|c_k| : k = 1, \dots, n\}$.
(ii) If f is of type I_α and $w(f) \geqslant 3$, then $|f(\sum_{k=1}^{n} c_k x_k)| < (1 + \frac{2}{2^n}) \max\{|c_k| : k = 1, \dots, n\}$.
(iii) If f is of type I_α and $w(f) = j < j_1$, then $|f(\sum_{k=1}^{n} c_k x_k)| < \frac{4+\frac{6}{2^n}}{2^j} \max\{|c_k| : k = 1, \dots, n\}$.

For any functional $f \in W_0$ the inductive assumption holds. Assume that it holds for any $f \in W_m$ and let $f \in W_{m+1}$. If f is a convex combination, then there is nothing to prove.

Assume that f is of type I_α, $f = \frac{1}{2^j} \sum_{q=1}^{d} \alpha_q$, where $\{\alpha_q\}_{q=1}^{d}$ is a very fast growing and $\mathcal{S}_j$-admissible sequence of α-averages in W_m.

Set $k_1 = \min\{k : \operatorname{ran} f \cap \operatorname{ran} x_k \neq \varnothing\}$ and $q_1 = \min\{q : \operatorname{ran} \alpha_q \cap \operatorname{ran} x_{k_1} \neq \varnothing\}$.

We distinguish 3 cases.

Case 1: $j < j_1$.

For $q > q_1$, we have that $s(\alpha_q) > \min\operatorname{supp} x_{k_1}$, therefore we conclude that

$$\sum_{q > q_1} |\alpha_q(\sum_{k=1}^{n} c_k x_k)| < \frac{1}{2^n} \max\{|c_k| : k = 1, \dots, n\} \tag{16}$$

while the inductive assumption yields that

$$|\alpha_{q_1}(\sum_{k=1}^{n} c_k x_k)| < (4 + \frac{5}{2^n}) \max\{|c_k| : k = 1, \dots, n\} \tag{17}$$

Then (16) and (17) allow us to conclude that

$$|f(\sum_{k=1}^{n} c_k x_k)| < \frac{4 + \frac{6}{2^n}}{2^j} \max\{|c_k| : k = 1, \dots, n\} \tag{18}$$

Hence, (iii) from the inductive assumption is satisfied.

Case 2: There exists $k_0 < n$, such that $j_{k_0} \leqslant j < j_{k_0+1}$.

Arguing as previously we get that

$$|f(\sum_{k > k_0} c_k x_k)| < \frac{4 + \frac{6}{2^n}}{2^{j_{k_0}}} \max\{|c_k| : k = 1, \dots, n\} < \frac{1}{2^n} \max\{|c_k| : k = 1, \dots, n\} \tag{19}$$

and

$$|f(\sum_{k < k_0} c_k x_k)| < \frac{1}{2^n} \max\{|c_k| : k = 1, \dots, n\} \tag{20}$$

Using (19), (20), the fact that $|f(x_{k_0})| \leqslant 1$, we conclude that

$$|f(\sum_{k=1}^{n} c_k x_k)| < (1 + \frac{2}{2^n}) \max\{|c_k| : k = 1, \dots, n\} \tag{21}$$

Case 3: $j \geqslant j_n$

By using the same arguments, we conclude that

$$|f(\sum_{k=1}^{n} c_k x_k)| < (1+\frac{1}{2^n})\max\{|c_k| : k=1,\dots,n\} \tag{22}$$

Then (18), (21) and (22) yield that (ii) from the inductive assumption is satisfied.

If f is of type I_β, then the proof is exactly the same, therefore assume that f is of type II_+, $f=\frac{1}{2}\sum_{q\in F}^{d}(f_q+g_q)$, where $\{f_q,g_q\}_{q\in F}$ are functionals of type I_α. Set

$$\begin{aligned} E &= \{k : |f(x_k)| \geqslant \frac{1}{n\cdot 2^n}\} \\ E_1 &= \{k\in E : \text{ there exist at least two } q \text{ such that } \operatorname{ran}(f_q+g_q)\cap \operatorname{ran} x_k \neq \varnothing\} \end{aligned}$$

Then $\#E_1 \leqslant 2$. Indeed, if $k_1<k_2<k_3\in E_1$, then f separates x_{k_1}, x_{k_2} and x_{k_3} which contradicts our initial assumptions.

If moreover we set $J=\{q : \text{there exists } k\in E\setminus E_1 \text{ such that } \operatorname{ran}(f_q+g_q)\cap\operatorname{ran} x_k\neq\varnothing\}$, then for the same reasons we get that $\#J\leqslant 2$.

Since for any j, we have that $w(f_q), w(g_q)\in L_0$, we get that $w(f_j)>9$, therefore:

$$\begin{aligned} |f(\sum_{k\in E\setminus E_1}^{n} c_k x_k)| &< (2+\frac{4}{2^n})\max\{|c_k| : k=1,\dots,n\} && (23)\\ |f(\sum_{k\in E_1}^{n} c_k x_k)| &\leqslant 2\max\{|c_k| : k=1,\dots,n\} && (24)\\ |f(\sum_{k\notin E}^{n} c_k x_k)| &\leqslant n\cdot\frac{1}{n\cdot 2^n}\max\{|c_k| : k=1,\dots,n\} && (25) \end{aligned}$$

Finally, (23) to (25) yield the following.

$$|f(\sum_{k=1}^{n} c_k x_k)| < (4+\frac{5}{2^n})\max\{|c_k| : k=1,\dots,n\}$$

If f is of type II_-, the proof is exactly the same. This means that (i) from the inductive assumption is satisfied an this completes the proof.

□

Proposition 5.2. Let $\{x_k\}_{k\in\mathbb{N}}$ be a seminormalized block sequence in $\mathfrak{X}_{\mathrm{usm}}$, such that $\|x_k\|\leqslant 1$ for all $k\in\mathbb{N}$, $\alpha\big(\{x_k\}_k\big)=0$ as well as $\beta\big(\{x_k\}_k\big)=0$ and $\mathcal{B}\otimes\{x_k\}_k=0$. Then it has a subsequence, again denoted by $\{x_k\}_{k\in\mathbb{N}}$ satisfying the following.

(i) $\{x_k\}_{k\in\mathbb{N}}$ generates a c_0 spreading model. More precisely, for any $n\leqslant k_1<\dots<k_n$, we have that $\|\sum_{i=1}^{n} x_{k_i}\|\leqslant 5$.

(i) There exists a strictly increasing sequence of naturals $\{j_n\}_{n\in\mathbb{N}}$, such that for any $n \leqslant k_1 < \cdots < k_n$, for any functional f of type I_α with $w(f) = j < j_n$, we have that

$$|f(\sum_{i=1}^{n} x_{k_i})| < \frac{5}{2^j}$$

6. Spreading model universal block sequences

In this section we define exact pairs and exact nodes in $\mathfrak{X}_{\text{usm}}$. Then, using a sequence generating a c_0 spreading model, we pass to a sequence of exact nodes $\{x_k, y_k, f_k, g_k\}$, such that $\{f_k, g_k\}_{k=1}^{\infty}$ defines a special branch. Setting $z_k = x_k - y_k$, we prove that $\{z_k\}_k$ is a spreading model universal sequence. Using the structure of such sequences, we also prove that the space $\mathfrak{X}_{\text{usm}}$ is hereditarily indecomposable.

Definition 6.1. A pair $\{x, f\}$, where $x \in \mathfrak{X}_{\text{usm}}$, $f \in W$ is called an n-exact pair if the following hold.

(i) f is a functional of type I_α with $w(f) = n$, $\min\operatorname{supp} x \leqslant \min\operatorname{supp} f$ and $\max\operatorname{supp} x \leqslant \max\operatorname{supp} f$.

(ii) There exists $x' \in \mathfrak{X}_{\text{usm}}$ a $(5,1,n)$ exact vector such that $1 \geqslant f(x') > \frac{35}{36}$ and $x = \frac{x'}{f(x')}$.

Remark 6.2. If $\{x, f\}$ is a n-exact pair, then $f(x) = 1$ and by Remark 2.12 we have that $1 \leqslant \|x\| \leqslant 36$.

Proposition 6.3. Let $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ and $n \in \mathbb{N}$. Then there exists x supported by $\{x_k\}_k$ and $f \in W$ such that $\{x, f\}$ is an n-exact pair.

Proof. By Proposition 4.8 there exists $\{y_k\}_k$ a further normalized block sequence satisfying the assumptions of Proposition 5.2. Therefore we may choose $\{n_k\}_k$ a strictly increasing sequence of natural numbers and $\{F_k\}_k$ an increasing sequence of subsets of the naturals satisfying the following.

(i) $\#F_k \leqslant \min F_k$, therefore $1 \leqslant \|\sum_{i\in F_k} y_i\| \leqslant 5$, for all $k \in \mathbb{N}$.

(ii) $\#F_{k+1} \geqslant 2^{\max\operatorname{supp} y_{\max F_k}}$, for all $k \in \mathbb{N}$.

(iii) For any $j, k \in \mathbb{N}$ with $j < n_k$ and f a functional of type I_α in W with $w(f) = j$, we have that $|f(\sum_{i\in F_k} y_i)| < \frac{5}{2^j}$.

Setting $z_k = \sum_{i\in F_k} y_i$, by (i) and (iii) we conclude that $\{z_k\}_k$ is $(5, \{n_k\}_k)$ α-RIS. By Proposition 2.2, for $0 < \varepsilon < \frac{1}{32\cdot 5\cdot 2^{3n}}$, there exists G a subset of the natural numbers with $\min\operatorname{supp} z_{\min G} \geqslant 8\cdot 5\cdot 2^{2n}$, $n_{\min G} > 2^{2n}$ and $\{c_k\}_{k\in G} \subset [0,1]$, such that $\sum_{k\in G} c'_k z_k$ is a $(n, \varepsilon(1-\varepsilon))$ s.c.c.

Setting $c_k = \frac{c'_k}{1 - c_{\max G}}$, it is straightforward to check that $\sum_{k\in G\setminus\{\max G\}} c_k z_k$ is a (n, ε) s.c.c.

Set $x' = 2^n \sum_{k\in G\setminus\{\max G\}} c_k z_k$. In order for x' to be a $(5,1,n)$ exact vector, it remains to be shown that $\|x'\| \geqslant 1$.

We shall prove that for any $\eta > 0$, there exists f_η a functional of type I_α in W with $\min\operatorname{supp} x' \leqslant \min\operatorname{supp} f_\eta$, $\max\operatorname{supp} x' \leqslant \max\operatorname{supp} f_\eta$ and $w(f_\eta) = n$, such that $1 \geqslant f_\eta(x') > 1 - \eta$.

Observe that for $k \in G$, there exists α_k an α-average in W with $s(\alpha_k) = \#F_k$, such that $\operatorname{ran}\alpha_k \subset \operatorname{ran} z_k$ and $1 \geqslant \alpha_k(z_k) > 1 - \eta$.

By (ii) we conclude that $\{\alpha_k\}_{k\in G}$ is very fast growing and since $\operatorname{ran}\alpha_k \subset \operatorname{ran} z_k$, it is $\mathcal{S}_n$ admissible. Therefore $f_\eta = \frac{1}{2^n}\sum_{k\in G}\alpha_k$ is of type I_α in W with $\min\operatorname{supp} x' \leqslant \min\operatorname{supp} f_\eta$, $\max\operatorname{supp} x' \leqslant \max\operatorname{supp} f_\eta$ and $w(f_\eta) = n$. By doing some easy calculations we conclude that it is the desired functional, hence $\|x'\| \geqslant 1$.

Moreover, for $0 < \eta < 1/36, f = f_\eta$ and $x = \frac{x'}{f(x')}$, we have that $\{x, f\}$ is the desired exact pair.

$\square$

Definition 6.4. A quadruple $\{x, y, f, g\}$ is called an n-exact node if $\{x, f\}$ and $\{y, g\}$ are both n-exact pairs and $\max\operatorname{supp} f < \min\operatorname{supp} y$.

A sequence of quadruples $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ is called a dependent sequence, if $\{x_k, y_k, f_k, g_k\}$ is an n_k exact node for all $k \in \mathbb{N}$, $\max\operatorname{supp} g_k < \min\operatorname{supp} x_{k+1}$ for all $k \in \mathbb{N}$ and $\{f_k, g_k\}_{k=1}^\infty$ is a special branch.

Remarks 6.5. If $\{x, y, f, g\}$ is an n-exact node, then $(f + g)(x + y) = 2, (f - g)(x - y) = 2, (f + g)(x - y) = 0, (f + g)(x) = 1, (f + g)(y) = 1$ and $1 \leqslant \|x \pm y\| \leqslant 72$.

If $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ is a dependent sequence, by the above and Proposition 4.3, we conclude that any spreading model admitted by $\{x_k+y_k\}_k$, $\{x_k\}_k$ or $\{y_k\}_k$, is ℓ_1.

Moreover, for $k_0 \in \mathbb{N}$ and $k \geqslant k_0$ by Lemma 3.10 and the fact that $\min\operatorname{supp} x_{k_0} \geqslant 8 \cdot 5 \cdot 2^{2n_{k_0}}$, we have that for any very fast growing and $\mathcal{S}_j$-admissible sequence of α-averages $\{\alpha_q\}_{q=1}^d$ with $j < n_{k_0}$ and $s(\alpha_1) \geqslant \min\operatorname{supp} x_{k_0}$, we have that

$$\sum_{q=1}^{d} |\alpha_q(x_k \pm y_k)| < \frac{5}{2^{n_{k_0}}} \tag{26}$$

Similarly, by Lemma 3.15, for any very fast growing and $\mathcal{S}_j$-admissible sequence of β-averages $\{\beta_q\}_{q=1}^d$ with $j < n_{k_0} - 2$ and $s(\beta_1) \geqslant \min\operatorname{supp} x_{k_0}$, we have that

$$\sum_{q=1}^{d} |\beta_q(x \pm y)| < \frac{5}{2^{n_{k_0}}} \tag{27}$$

Lemma 6.6. Let $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ be a dependent sequence. Then for every $k \in \mathbb{N}$, if $n_k = w(f_k)$ and $n_{k+1} = w(f_{k+1})$, the following holds.

$$\frac{1}{2^{n_{k+1}-3}} \max\operatorname{supp} y_k < \frac{1}{2^{n_k}} \tag{28}$$

Proof. By the definition of the coding function σ, we have that $n_{k+1} > 2^{n_k}\max\operatorname{supp} g_k \geqslant 2^{n_k}\max\operatorname{supp} y_k$.

Since $n_{k+1}\in L$, we have that $n_{k+1} > 9$. It easily follows that $2^{n_{k+1}-3} > n_{k+1}$. Combining this with the above, we conclude the desired result. $\square$

Proposition 6.7. Let Y be a block subspace of $\mathfrak{X}_{\mathrm{usm}}$. Then there exist $\{x_k\}_k, \{y_k\}_k$ block sequences in Y and $b = \{f_k, g_k\}_{k=1}^\infty \in \mathcal{B}$, such that $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ is a dependent sequence.

Proof. Choose $n_1 \in L_1$. By Proposition 6.3 there exists $\{x_1, y_1, f_1, g_1\}$ an n_1-exact node in Y.

Suppose that we have chosen $\{x_k, y_k, f_k, g_k\}$ n_k-exact nodes for $k = 1,\ldots,m$ such that $\{f_k, g_k\}_{k=1}^m$ is a special sequence and $\max\operatorname{supp} g_k < \min\operatorname{supp} x_{k+1}$ for $k = 1,\ldots,m-1$.

Set $n_{m+1} = \sigma(f_1, g_1,\ldots, f_m, g_m)$. Then applying Proposition 6.3 once more, there exists $\{x_{m+1}, y_{m+1}, f_{m+1}, g_{m+1}\}$ an n_{m+1}-exact node in Y, such that $\max\operatorname{supp} g_m < \min\operatorname{supp} x_{m+1}$.

The inductive construction is complete and $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ is a dependent sequence.

$\square$

An easy modification of the above proof yields the following.

Corollary 6.8. If X, Y are block subspaces of $\mathfrak{X}_{\mathrm{usm}}$, then a dependent sequence $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ can be chosen, such that $x_k \in X$ and $y_k \in Y$ for all $k \in \mathbb{N}$.

Proposition 6.9. Let $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ be a dependent sequence and set $z_k = x_k - y_k$. Then for every $m \leqslant k_1 < \cdots < k_m$ natural numbers and $c_1,\ldots,c_m$ real numbers, the following holds.

$$\|\sum_{i=1}^m c_i u_{k_i}\|_u \leqslant \|\sum_{i=1}^m c_i z_{k_i}\| \leqslant 146\|\sum_{i=1}^m c_i u_{k_i}\|_u \tag{29}$$

Proof. Set $n_k = w(f_k)$ for all $k\in\mathbb{N}$. Choose $m \leqslant k_1 < \cdots < k_m$ natural numbers and $c_1,\ldots,c_m \subset [-1,1]$, such that $\|\sum_{i=1}^m c_i u_{k_i}\|_u = 1$.

We first prove that $\|\sum_{i=1}^m c_i z_{k_i}\| \geqslant 1$.

Since $\min\operatorname{supp} z_{k_1} = \min\operatorname{supp} x_{k_1} \geqslant \min\operatorname{supp} x_m \geqslant 40\cdot 2^{2n_m} \geqslant 40\cdot 2^m > 2m$ and $\min\operatorname{supp} f_{k_1} \geqslant \min\operatorname{supp} x_{k_1}$, by the definition of the norming set W, it follows that for every $\lambda_1,\ldots,\lambda_m$ rational numbers such that $\|\sum_{i=1}^m \lambda_i u_{k_i}^*\|_u \leqslant 1$, the functional $f = \frac{1}{2}\sum_{i=1}^m \lambda_i(f_{k_i} - g_{k_i})$ is a functional of type II$_-$ in W. We conclude that

$$\|\sum_{i=1}^m c_i z_{k_i}\| \geqslant \sup\Big\{\sum_{i=1}^m \frac{1}{2}\lambda_i(f_{k_i} - g_{k_i})(c_i z_{k_i}) : \{\lambda_i\}_{i=1}^m \subset \mathbb{Q},\ \|\sum_{i=1}^m \lambda_i u_{k_i}^*\|_u \leqslant 1\Big\}$$

By Remark 6.5, for $\lambda_1,\ldots,\lambda_q$ as above, we have that $\sum_{i=1}^m \frac{1}{2}\lambda_i(f_{k_i} - g_{k_i})(c_i z_{k_i}) = \sum_{i=1}^m \lambda_i c_i$. This yields the following.

$$\begin{aligned}\|\sum_{i=1}^m c_i z_{k_i}\| &\geqslant \sup\big\{\sum_{i=1}^m \lambda_i c_i : \{\lambda_i\}_{i=1}^m \subset \mathbb{Q},\ \|\sum_{i=1}^m \lambda_i u_{k_i}^*\|_u \leqslant 1\big\}\\ &= \|\sum_{i=1}^m c_i u_{k_i}\|_u = 1\end{aligned}$$

To prove the inverse inequality, we will follow similar steps, as in the proof of Proposition 5.1. We shall inductively prove the following.

(i) For any $f \in W$, we have that $|f(\sum_{k=1}^n c_k x_k)| < 146$.
(ii) If f is of type I_α or type I_β and $w(f) \geqslant 9$, then $|f(\sum_{k=1}^n c_k x_k)| < 72 + 1/4$.

For any functional in W_0 the inductive assumption holds.Assume that it holds for any $f \in W_p$ and let $f \in W_{p+1}$. If f is a convex combination, then there is nothing to prove.

Assume that f is of type I_β, $f = \frac{1}{2^j}\sum_{q=1}^d \beta_q$, where $\{\beta_q\}_{q=1}^d$ is a very fast growing and $\mathcal{S}_j$-admissible sequence of β-averages in W_p.

Set $q_1 = \min\big\{q : \operatorname{ran}\beta_q \cap \operatorname{ran} z_{k_i} \neq \varnothing \text{ for some } i \in \{1,\dots,m\}\big\}$.

We distinguish 3 cases.

Case 1: $j + 2 < n_{k_1}$.

For $q > q_1$, we have that $s(\beta_q) > \min\operatorname{supp} x_{k_1}$, therefore, using (27) we conclude that

$$\sum_{q>q_1} |\beta_q(\sum_{i=1}^m c_i z_{k_i})| < \frac{m}{2^{n_{k_1}}} < \frac{m}{2^m} < 1 \tag{30}$$

while the inductive assumption yields that

$$|\beta_{q_1}(\sum_{i=1}^m c_i z_{k_i})| < 146 \tag{31}$$

Then (30) and (31) allow us to conclude that

$$|f(\sum_{i=1}^m c_i z_{k_i})| < \frac{147}{2^j} \tag{32}$$

Case 2: There exists $i_0 < m$, such that $n_{k_{i_0}} \leqslant j + 2 < n_{k_{i_0}+1}$.

Arguing as previously we get that

$$|f(\sum_{i>i_0} c_i z_{k_i})| < \frac{147}{2^{n_{k_{i_0}+1}}} < \frac{147}{2^{11}} < \frac{1}{8} \tag{33}$$

and by Lemma 6.6

$$|f(\sum_{i<i_0} c_i z_{k_i})| < \frac{1}{2^{n_{k_1}}} < \frac{1}{8} \tag{34}$$

Using (33), (34) and the fact that $|f(z_{k_{i_0}})| \leqslant 72$, we conclude that

$$|f(\sum_{k=1}^{m} c_k x_k)| < 72 + \frac{1}{4} \tag{35}$$

Case 3: $j + 2 \geqslant n_{k_m}$

By using the same arguments, we conclude that

$$|f(\sum_{i=1}^{m} c_i z_{k_i})| < 72 + \frac{1}{4} \tag{36}$$

Then (32), (35) and (36) yield that (i) and (ii) from the inductive assumption are satisfied.

If f is of type I_α, using (26) and the exact same arguments one can prove that (i) and (ii) from the inductive assumption are again satisfied.

Assume now that f is of type II_- (or f is of type II_+), $f = E\big(\frac{1}{2}\sum_{j=1}^{d} \lambda_j (f'_{q_j} - g'_{q_j})\big)$ (or $f = E\big(\frac{1}{2}\sum_{j=1}^{d}(f'_{q_j} + g'_{q_j})\big)$), where E is an interval of the natural numbers, $\{f'_q, g'_q\}_{q=1}^{\infty} \in \mathcal{B}$, $q_1 < \cdots < q_d$ and $2q_d \leqslant \min\operatorname{supp} f'_{q_1}$.

We may clearly assume that $\operatorname{ran}(f'_{q_1} \pm g'_{q_1}) \cap \operatorname{ran}(\sum_{i=1}^{m} c_i z_{k_i}) \neq \varnothing$ and $\min E \geqslant \min\operatorname{supp} f'_{q_1}$.

Similarly, we assume that $\operatorname{ran}(f'_{q_d} \pm g'_{q_d}) \cap \operatorname{ran}(\sum_{i=1}^{m} c_i z_{k_i}) \neq \varnothing$ and $\max E \leqslant \max\operatorname{supp} g'_{q_d}$.

The inductive assumption yields the following.

$$|E\big(\frac{1}{2}(f'_{q_1} \pm g'_{q_1})\big)(\sum_{i=1}^{m} c_i z_{k_i})| < 72 + \frac{1}{4} \tag{37}$$

Set $t_j = w(f'_{q_j})$ for $j = 1, \ldots, d$. By the definition of the coding function, we have that $t_j > 2^{t_1} \min\operatorname{supp} x_{k_1} > \min\operatorname{supp} x_m > 40 \cdot 2^m$, for $j = 2, \ldots, d$. We conclude the following.

$$\sum_{j>1} \frac{72m}{2^{t_j}} \leqslant \frac{144m}{2^{t_2}} < \frac{144m}{2^{40} \cdot 2^m} < \frac{1}{4} \tag{38}$$

We distinguish two cases.

Case 1: There exist $2 \leqslant j_0 \leqslant d$ and $k \in \mathbb{N}$ such that $t_j = n_k$.

In this case, the fact that σ is one to one, yields that $f'_{q_j} \pm g'_{q_j} = f_{q_j} \pm g_{g_j}$ for $2 \leqslant j < j_0$ and hence

$$\begin{aligned} |E\big(\frac{1}{2}\sum_{j=2}^{j_0-1} \lambda_j (f'_{q_j} - g'_{q_j})\big)(\sum_{i=1}^{m} c_i z_{k_i})| &= |\frac{1}{2}\sum_{j=2}^{j_0-1} \lambda_j (f'_{q_j} - g'_{q_j})(\sum_{i=1}^{m} c_i z_{k_i})| \\ &= |\frac{1}{2}\sum_{j=2}^{j_0-1} \lambda_j (f_{q_j} - g_{q_j})(\sum_{i=1}^{m} c_i z_{k_i})| \leqslant 1 \end{aligned} \tag{39}$$

if f is of type II_- and

$$\begin{aligned}|E\big(\frac{1}{2}\sum_{j=2}^{j_0-1}(f'_{q_j}+g'_{q_j})\big)(\sum_{i=1}^{m}c_iz_{k_i})| &= |\frac{1}{2}\sum_{j=2}^{j_0-1}(f'_{q_j}+g'_{q_j})(\sum_{i=1}^{m}c_iz_{k_i})| \\ &= |\frac{1}{2}\sum_{j=2}^{j_0-1}(f_{q_j}+g_{q_j})(\sum_{i=1}^{m}c_iz_{k_i})|=0 \qquad (40)\end{aligned}$$

if f is of type II_+.

The inductive assumption yields that

$$|E\big(\frac{1}{2}(f'_{q_{j_0}}-g'_{q_{j_0}})\big)(\sum_{i=1}^{m}c_iz_{k_i})|<72+\frac{1}{4} \qquad (41)$$

Moreover, using Corollary 3.14, for $i=1,\dots,m$ we have that

$$|E\big(\frac{1}{2}\sum_{j=j_0+1}^{d}\lambda_j(f'_{q_j}-g'_{q_j})\big)(z_{k_i})|<\sum_{j>j_0}\frac{72}{2^{t_j}}+\frac{22}{2^{n_{k_i}}}$$

Combining this with (38)

$$\begin{aligned}|E\big(\frac{1}{2}\sum_{j=j_0+1}^{d}\lambda_j(f'_{q_j}-g'_{q_j})\big)(\sum_{i=1}^{m}c_iz_{k_i})| &< \sum_{j>1}\frac{72m}{2^{t_j}}+\sum_{i=1}^{m}\frac{22}{2^{n_{k_i}}} \\ &< \frac{1}{4}+\frac{22}{1000}<\frac{1}{2} \qquad (42)\end{aligned}$$

Similarly,

$$|E\big(\frac{1}{2}\sum_{j=j_0+1}^{d}(f'_{q_j}+g'_{q_j})\big)(\sum_{i=1}^{m}c_iz_{k_i})|<\frac{1}{2} \qquad (43)$$

If f is of type II_- Combining (37), (39) and (42), we conclude that $|f(\sum_{i=1}^{m}c_iz_{k_i})|<146$, while if f is of type II_+ combining (37), (40) and (43), we conclude that $|f(\sum_{i=1}^{m}c_iz_{k_i})|<145$

Case 1: $t_j\neq n_k$, for all $j=2,\dots,d$ and $k\in\mathbb{N}$.

Arguing as previously, we conclude that

$$|E\big(\frac{1}{2}\sum_{j=2}^{d}\lambda_j(f'_{q_j}-g'_{q_j})\big)(\sum_{i=1}^{m}c_iz_{k_i})|<\frac{1}{2} \quad \text{and} \qquad (44)$$

$$|E\big(\frac{1}{2}\sum_{j=2}^{d}(f'_{q_j}+g'_{q_j})\big)(\sum_{i=1}^{m}c_iz_{k_i})|<\frac{1}{2}$$

Therefore, (37) and (45) yield that $|f(x)|<73$. The induction is complete and so is the proof.

$\square$

Proposition 6.10. Let Y be a block subspace of $\mathfrak{X}_{\text{usm}}$. Then there exist $\{z_k\}_k$ a seminormalized block sequence in Y and $\{z_k^*\}_k$ a seminormalized block sequence in $\mathfrak{X}_{\text{usm}}^*$ satisfying the following.

(i) $z_k^*(z_n) = \delta_{k,n}$

(ii) For every unconditional and spreading sequence $\{w_n\}_n$, there exists $\{k_n\}_n$ a strictly increasing sequence of natural numbers, such that $\{z_{k_n}\}_n$ generates a spreading model which is 146-equivalent to $\{w_n\}_n$ and $\{z_{k_n}^*\}_n$ generates a spreading model which is 146-equivalent to $\{w_n^*\}_n$

Proof. By Proposition 6.7, there exists $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ a dependent sequence in Y. Set $z_k = x_k - y_k$ and $z_k^* = \frac{1}{2}(f_k - g_k)$. Then $z_k^*(z_n) = \delta_{k,n}$.

Let $\{w_n\}_n$ be an unconditional and spreading sequence, which also yields that it is suppression unconditional and hence there exists $\{k_n\}_n$ a strictly increasing sequence of natural numbers, such that $\{u_{k_n}\}_{n\geqslant j}$ is $1+\varepsilon_j$ equivalent to $\{w_n\}_{n\geqslant j}$, where $\{\varepsilon_j\}_j$ is null sequence of positive reals.

Moreover, due to unconditionality, $\{u_{k_n}^*\}_{n\geqslant j}$ is $1+\varepsilon_j$ equivalent to $\{w_n^*\}_{n\geqslant j}$.

Proposition 6.9 yields that for every $m \leqslant n_1 < \cdots < n_m$ natural numbers and $c_1, \cdots, c_m$ real numbers, we have that

$$\frac{1}{1+\varepsilon_m}\|\sum_{i=1}^m c_i w_i\| \leqslant \|\sum_{i=1}^m c_i z_{n_{k_i}}\| \leqslant (1+\varepsilon_m)146\|\sum_{i=1}^m c_i w_i\| \tag{45}$$

This yields that any spreading model admitted by $\{z_{k_n}\}_n$ is 146-equivalent to $\{w_n\}_n$.

Moreover, by the definition of the norming set, for every $m \leqslant n_1 < \cdots < n_m$ natural numbers and $c_1, \cdots, c_m$ real numbers, we have that

$$\|\sum_{i=1}^m c_i z_{n_{k_i}}^*\| \leqslant \|\sum_{i=1}^m c_i u_{n_{k_i}}^*\|_u \leqslant (1+\varepsilon_m)\|\sum_{i=1}^m c_i w_i^*\| \tag{46}$$

Property (i) and (45) yield the following.

$$\frac{1}{146(1+\varepsilon_m)}\|\sum_{i=1}^m c_i w_i^*\| \leqslant \|\sum_{i=1}^m c_i z_{n_{k_i}}^*\| \tag{47}$$

Combining (46) and (46), we conclude that any spreading model admitted by $\{z_{k_n}^*\}_n$ is 146-equivalent to $\{w_n^*\}_n$. □

Proposition 6.11. The space $\mathfrak{X}_{\text{usm}}$ is hereditarily indecomposable.

Proof. It is enough to show that for X, Y block subspaces of $\mathfrak{X}_{\text{usm}}$ and $\varepsilon > 0$, there exist $x \in X$ and $y \in Y$ such that $\|x+y\| \geqslant 1$ and $\|x-y\| < \varepsilon$.

By Corollary 6.8, there exists $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ a dependent sequence, with $x_k \in X$ and $y_k \in Y$ for all $k \in \mathbb{N}$.

By Remark 6.5 and Proposition 6.9, there exists $\{k_n\}_n$ a strictly increasing sequence of natural numbers, such that $\{x_{k_n} + y_{k_n}\}_n$ generates an ℓ_1 spreading model and $\{x_{k_n} - y_{k_n}\}_n$ generates a c_0 spreading model.

Fix $c > 0$ such that for any $m \leqslant n_1 < \cdots < n_m$ natural numbers the following holds.

$$\frac{1}{m}\|\sum_{i=1}^{m}(x_{k_{n_i}} - y_{k_{n_i}})\| \leqslant \frac{1}{c\cdot m}$$
$$\frac{1}{m}\|\sum_{i=1}^{m}(x_{k_{n_i}} + y_{k_{n_i}})\| \geqslant c$$

Fix $m \leqslant n_1 < \cdots < n_m$ natural numbers such that $\frac{1}{c^2 m} < \varepsilon$ and set $x = \frac{1}{c\cdot m}\sum_{i=1}^{m} x_{k_{n_i}}$ and $y = \frac{1}{c\cdot m}\sum_{i=1}^{m} y_{k_{n_i}}$.

Then $\|x+y\| \geqslant 1$ and $\|x-y\| \leqslant \frac{1}{c^2 m} < \varepsilon$.

□

7. Bounded operators on $\mathfrak{X}_{\text{usm}}$

This section is devoted to operators on $\mathfrak{X}_{\text{usm}}$. We prove that in every block subspace of $\mathfrak{X}_{\text{usm}}$ there exist equivalent intertwined block sequences $\{x_k\}_k, \{y_k\}_k$ and an onto isomorphism $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$, such that $Tx_k = y_k$. This yields that $\mathfrak{X}_{\text{usm}}$ does not contain a block subspace that is tight by range and hence, $\mathfrak{X}_{\text{usm}}$ is saturated with sequentially minimal subspaces (see [7]). We then proceed to identify block sequences witnessing this fact. We moreover construct a strictly singular operator $S : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ which is not polynomially compact. All the above properties of $\mathfrak{X}_{\text{usm}}$ are based on the way type II functionals are constructed in the norming set W and the rich spreading model structure of $\mathfrak{X}_{\text{usm}}$.

The following result is proven in a similar manner as Theorem 5.6 from [3] and therefore its proof is omitted.

Proposition 7.1. Let Y be an infinite dimensional closed subspace of $\mathfrak{X}_{\text{usm}}$ and $T : Y \to \mathfrak{X}_{\text{usm}}$ be a bounded linear operator. Then there exists $\lambda \in \mathbb{R}$, such that $T - \lambda I_{Y,\mathfrak{X}_{\text{usm}}} : Y \to \mathfrak{X}_{\text{usm}}$ is strictly singular.

The following result follows from Proposition 3.1 from [2], see also [14].

Proposition 7.2. Let $\{x_m^*\}_m$ be a block sequence in $\mathfrak{X}_{\text{usm}}^*$ generating a c_0 spreading model and $\{x_k\}_k$ be a block sequence in $\mathfrak{X}_{\text{usm}}$ generating a spreading model which is not equivalent to ℓ_1. Then there exists a strictly increasing sequence of natural numbers $\{t_j\}_j$, such that the following is satisfied. For every strictly increasing sequence of natural numbers $\{m_k\}_k$ with $m_k \geqslant t_k$ for all $k \in \mathbb{N}$, the map $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ with $Tx = \sum_{k=1}^{\infty} x_{m_k}^*(x)x_k$ is bounded and non compact.

The proof of the following result uses an argument, which first appeared in [8], namely the following. If $\{x_k\}_k$, $\{y_k\}_k$ are basic sequences in a space X, such that the maps $x_k \to x_k - y_k$ and $y_k \to x_k - y_k$ extend to bounded linear operators, then $\{x_k\}_k$ is equivalent to $\{y_k\}_k$.

Proposition 7.3. Let $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ be a dependent sequence. Then there exists $\{k_n\}_n$ a strictly increasing sequence of natural numbers, such that $\{x_{k_n}\}_n$ is equivalent to $\{y_{k_n}\}_n$. More precisely, there exists $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ an onto isomorphism, with $Tx_{k_n} = y_{k_n}$ for all $n \in \mathbb{N}$.

Proof. First observe the following, for any $k \in \mathbb{N}$, we have that

$$2 \geqslant \|f_k + g_k\| \geqslant (f_k + g_k)\big(\frac{x_k + y_k}{\|x_k + y_k\|}\big) \geqslant \frac{2}{72}$$

Hence $\{f_k + g_k\}_k$ is seminormalized and by the definition of the norming set W, any spreading model admitted by it, is c_0.

By Proposition 6.9, $\{x_k - y_k\}_k$ admits a c_0 spreading model. Proposition 7.2, yields that there exists $\{k_n\}_n$ a strictly increasing sequence of natural numbers, such that the operator $S : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ with

$$Sx = \sum_{n=1}^{\infty}(f_{k_n} + g_{k_n})(x)(x_{k_n} - y_{k_n})$$

is bounded.

Then, for every $n \in \mathbb{N}$ we have that $Sx_{k_n} = x_{k_n} - y_{k_n}$. Setting $T = I - S$, we evidently have that $Tx_{k_n} = y_{k_n}$, hence $\{x_k\}_k$ is dominated by $\{y_k\}_k$.

Similarly, for every $n \in \mathbb{N}$ we have that $Sy_{k_n} = x_{k_n} - y_{k_n}$. Setting $Q = I + S$, we evidently have that $Qy_{k_n} = x_{k_n}$. Therefore $\{y_k\}_k$ is dominated by $\{x_k\}_k$, which yields that they are actually equivalent.

We shall moreover prove that T is invertible, in fact $Q = T^{-1}$. Notice that $TQ = QT = I - S^2$. It remains to be shown that $S^2 = 0$.

Since $Sx_{k_n} = x_{k_n} - y_{k_n} = Sy_{k_n}$ for all $n \in \mathbb{N}$, we evidently have that $S(x_{k_n} - y_{k_n}) = 0$ for all $n \in \mathbb{N}$. This yields that $[\{x_{k_n} - y_{k_n}\}_n] \subset \ker S$. Evidently, we have that $S[\mathfrak{X}_{\text{usm}}] \subset [\{x_{k_n} - y_{k_n}\}_n]$, therefore $S[\mathfrak{X}_{\text{usm}}] \subset \ker S$. We conclude that $S^2 = 0$ and this completes the proof. □

Before the statement of the next result, we remind the notion of even-odd sequences and intertwined block sequences. A Schauder basic sequence $\{x_k\}_k$ is called even-odd, if $\{x_{2k}\}_k$ is equivalent to $\{x_{2k-1}\}_k$ (see [10]).

Two block sequences $\{x_k\}_k$, $\{y_k\}_k$ are called intertwined, if $x_k < y_k < x_{k+1}$ for all $k \in \mathbb{N}$.

Evidently, two intertwined block sequences $\{x_k\}_k$, $\{y_k\}_k$ are equivalent, if and only if the sequence $\{z_k\}_k$ with $z_{2k-1} = x_k$ and $z_{2k} = y_k$ for all $k \in \mathbb{N}$, is an even-odd sequence.

Proposition 7.4. Every block subspace of $\mathfrak{X}_{\text{usm}}$ contains an even-odd block sequence. More precisely, in every block subspace Y of $\mathfrak{X}_{\text{usm}}$, there exists a block sequence $\{z_k\}_k$ and $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ an onto isomorphism, such that $Tz_{2k-1} = z_{2k}$, for all $k \in \mathbb{N}$.

Proof. By Proposition 6.7, there exists $\{x_k, y_k, f_k, g_k\}_{k=1}^\infty$ a dependent sequence in Y and by Proposition 7.3 there exist $\{k_n\}_n$ a strictly increasing sequence of natural numbers and $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ an onto isomorphism, such that $Tx_{n_k} = y_{n_k}$ for all $k \in \mathbb{N}$. Setting $z_{2k-1} = x_{n_k}$ and $z_{2k} = y_{n_k}$ for all $k \in \mathbb{N}$, we have that $\{z_k\}_k$ is the desired even-odd block sequence and T the desired operator. $\square$

Corollary 7.5. The space $\mathfrak{X}_{\text{usm}}$ does not contain a block subspace which is tight by range.

Theorem 1.4 from [7] yields that $\mathfrak{X}_{\text{usm}}$ is saturated with sequentially minimal block subspaces. The next result identifies block subspaces of $\mathfrak{X}_{\text{usm}}$ with the aforementioned property.

Proposition 7.6. There exists a set of block sequences $\big\{\{x_k^{(Y)}\}_k : Y$ is a block subspace of $\mathfrak{X}_{\text{usm}}\big\}$, with $\{x_k^{(Y)}\}_k \subset Y$ for every Y block subspace of $\mathfrak{X}_{\text{usm}}$, satisfying the following. For every Y, Z block subspaces of $\mathfrak{X}_{\text{usm}}$, there exist $\{k_n\}_n, \{m_n\}_n$ strictly increasing sequences of natural numbers, such that $\{x_{k_n}^{(Y)}\}_n$ and $\{x_{m_n}^{(Z)}\}_n$ are intertwined and equivalent. More precisely, there exists $T : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ an onto isomorphism, such that $Tx_{k_n}^{(Y)} = x_{m_n}^{(Z)}$ for all $n \in \mathbb{N}$.

Proof. Let Y be a block subspace of $\mathfrak{X}_{\text{usm}}$. By Proposition 6.3, we may choose a block sequence $\{x_k\}_k$ in Y, satisfying the following.

(i) There exists $\{f_k\}_k$ a sequence of type I_α functionals in W, such that $\{x_k, f_k\}$ is a $w(f_k)$-exact pair for all $k \in \mathbb{N}$.
(ii) For every $n \in \mathbb{N}$, the set $\{k \in \mathbb{N} : w(f_k) = n\}$ is infinite.

For every Y block subspace of $\mathfrak{X}_{\text{usm}}$, choose $\{x_k^{(Y)}\}_k$ satisfying properties (i) and (ii).

Let now Y, Z be block subspaces of $\mathfrak{X}_{\text{usm}}$. We shall recursively choose $\{k_n\}_n, \{m_n\}_n$ strictly increasing sequences of natural numbers and $\{f_n\}_n, \{g_n\}_n$ sequences of type I_α functionals, such that $\{x_{k_n}^{(Y)}, x_{m_n}^{(Z)}, f_n, g_n\}_{n=1}^\infty$ is an exact sequence.

Choose $p_1 \in L_1$ and $k_1 \in \mathbb{N}, f_1 \in W$ a functional of type I_α, such that $\{x_{k_1}^{(Y)}, f_1\}$ is a p_1 exact pair.

Similarly, choose $m_1 \in \mathbb{N}, g_1 \in W$ a functional of type I_α, such that $\{x_{m_1}^{(z)}, f_1\}$ is a p_1 exact pair and $\max\operatorname{supp} f_1 < \min\operatorname{supp} x_{m_1}^{(z)}$.

Suppose that we have chosen $\{k_n\}_{n=1}^\ell, \{m_n\}_{n=1}^\ell$ strictly increasing sequences of natural numbers and $\{f_n\}_{n=1}^\ell, \{g_n\}_{n=1}^\ell$, sequences of type I_α functionals, such that $\{x_{k_n}^{(Y)}, x_{m_n}^{(Z)}, f_n, g_n\}$ are p_n-exact nodes for $k = 1, \dots, \ell$ n_ℓ, $\{f_n, g_n\}_{n=1}^\ell$ is a special sequence and $\max\operatorname{supp} g_n < \min\operatorname{supp} x_{n+1}^{(Y)}$ for $k = 1, \dots, m-1$.

Set $p_{\ell+1} = \sigma(f_1, g_1, \dots, f_\ell, g_\ell)$. Then arguing as previously, we may choose $k_{\ell+1} > k_\ell, m_{\ell+1} > m_\ell$ and $f_{\ell+1}, g_{\ell+1}$ functionals of type I_α, such that $\{x_{k_{\ell+1}}^{(Y)}, x_{m_{\ell+1}}^{(Z)}, f_{\ell+1}, g_{\ell+1}\}$ is an $p_{\ell+1}$-exact node and $\max\operatorname{supp} g_\ell < \min\operatorname{supp} x_{m_{\ell+1}}^{(Y)}$.

The inductive construction is complete and $\{x_{k_n}^{(Y)}, x_{m_n}^{(Z)}, f_n, g_n\}_{n=1}^\infty$ is a dependent sequence.

Proposition 7.3 yields the desired result. □

A related result to the following can be found in [14], Proposition 2.1.

Proposition 7.7. Let $1 < q < \infty$, q' be its conjugate and set $t_j = \lceil (4 \cdot 2^{j+1})^{q'} \rceil$. Then the following holds.

If $\{m_j\}_j$ is a strictly increasing sequence of natural numbers with $m_j \geqslant t_j$ for all $j \in \mathbb{N}$, $\{x_m^*\}_m$ is a block sequence in $\mathfrak{X}_{\mathrm{usm}}^*$ and $\{x_k\}_k$ is a block sequence in $\mathfrak{X}_{\mathrm{usm}}$ satisfying the following,

(i) $\{x_m^*\}_m$ is either generating an ℓ_p spreading model, with $p > q'$, or a c_0 spreading model
(ii) $\{x_k\}_k$ is either generating an ℓ_r spreading model with $r \geqslant q$, or a c_0 spreading model

then the map $T : \mathfrak{X}_{\mathrm{usm}} \to \mathfrak{X}_{\mathrm{usm}}$ with $Tx = \sum_{k=1}^\infty x_{m_k}^*(x)x_k$ is bounded and non compact.

If moreover $\dim(Y/[\{x_k\}_k]) = \infty$, then T is strictly singular.

Proof. If $\{x_m^*\}_m$ generates a c_0 spreading model, fix $q' < p < \infty$. Note that by the choice of t_j, we have that

$$\begin{aligned}\frac{t_j^{1/p}}{2^j} &\leqslant \frac{\big((4\cdot 2^{j+1})^{q'}+1\big)^{1/p}}{2^j} \leqslant \frac{(4\cdot 2^{j+1})^{q'/p}}{2^j} + \frac{1}{2^j}\\ &= 8^{q'/p}\frac{1}{(2^{1-q'/p})^j} + \frac{1}{2^j}\end{aligned}$$

Since $p > q'$, we have that $\sum_{j=1}^\infty \frac{1}{(2^{1-q'/p})^j} < \infty$. We conclude that if we set

$$\alpha = 8^{q'/p}\sum_{j=1}^\infty \frac{1}{(2^{1-q'/p})^j} + 1$$

Then

$$\sum_{j=1}^\infty \frac{t_j^{1/p}}{2^j} \leqslant \alpha \tag{48}$$

Fix $C > 0$ such that for any $n \leqslant m_1 < \cdots < m_n$ natural numbers and $c_1, \ldots, c_m$ real numbers the following holds.

$$\|\sum_{i=1}^n c_i x_{m_i}^*\| \leqslant C(\sum_{i=1}^n |c_i|^p)^{1/p} \tag{49}$$

By multiplying the x_k with an appropriate scalar, we may assume that $\|x_k\| \leqslant 1/2$ for all $k \in \mathbb{N}$ and that for any $n \leqslant m_1 < \cdots < m_n$ natural numbers and $c_1, \ldots, c_m$ real numbers the following holds.

$$\|\sum_{i=1}^n c_i x_{m_i}\| \leqslant (\sum_{i=1}^n |c_i|^q)^{1/q} \tag{50}$$

Let $x \in X, \|x\| = 1$, $x^* \in Y^*, \|x^*\| = 1$. For $j \in \mathbb{N}$, set

$$B_j = \{k \in \mathbb{N} : \frac{1}{2^{j+1}} < |x^*(x_k)| \leqslant \frac{1}{2^j}\}$$

Then $\{B_j\}_j$ is a partition of the natural numbers and

$$|x^*(Tx)| \leqslant \sum_{j=1}^{\infty} |\sum_{k\in B_j} x^*(x)x^*_{m_k}(x)| \tag{51}$$

We will show that $\#B_j \leqslant t_j$.

Assume that this is not the case. Then we may choose $F \subset B_j$ with $\#F > t_j/2$ and $\#F \leqslant \min F$.

Set

$$\begin{aligned} F_1 &= \{k \in B_j : x^*(x_k) \geqslant 0\} \\ F_2 &= \{k \in B_j : x^*(x_k) < 0\} \end{aligned}$$

Then either $\#F_1 > t_j/4$, or $\#F_2 > t_j/4$ and we shall assume the first. Choose $G \subset F_1$ with $\#G = \lceil t_j/4 \rceil$.

Then, by (50) and the choice of G, we have the following.

$$t_j^{1/q} \geqslant \|\sum_{k\in G} x_k\| \geqslant x^*(\sum_{k\in G} x_k) > \frac{t_j}{4\cdot 2^{j+1}}$$

We conclude that $t_j < (4 \cdot 2^{j+1})^{q'}$, which contradicts the choice of t_j.

Set

$$C_j = \{k \in B_j : k \geqslant j\}, \quad D_j = B_j \setminus C_j$$

Evidently $\#D_j \leqslant j - 1$, hence

$$|\sum_{k\in D_j} x^*(x_k)x^*_{m_k}(x)| \leqslant \frac{j-1}{2^j} \tag{52}$$

Moreover,

$$\#\{m_k : k \in C_j\} \leqslant t_j \leqslant \min\{t_k : k \in C_j\} \leqslant \min\{m_k : k \in C_j\}$$

Therefore, using (49) and the definition of C_j,

$$\begin{aligned} |\sum_{k\in C_j} x^*(x_k)x^*_{m_k}(x)| &\leqslant \|\sum_{k\in C_j} x^*(x_k)x^*_{m_k}\| \\ &\leqslant C(\sum_{k\in C_j} |x^*(x_k)|^p)^{1/p} \leqslant \frac{C\cdot t_j^{1/p}}{2^j} \end{aligned}$$

The above, combined with (48), (51) and (52) yields the following.

$$\begin{aligned} |x^*(Tx)| &\leqslant \sum_{j=1}^{\infty} \Big| \sum_{k\in B_j} x^*(x) x^*_{m_k}(x) \Big| \\ &\leqslant \sum_{j=1}^{\infty} \Big| \sum_{k\in C_j} x^*(x_k) x^*_{m_k}(x) \Big| + \sum_{j=1}^{\infty} \Big| \sum_{k\in D_j} x^*(x_k) x^*_{m_k}(x) \Big| \\ &\leqslant C\sum_{j=1}^{\infty} \frac{t_j^{1/p}}{2^j} + \sum_{j=1}^{\infty} \frac{j-1}{2^j} \\ &\leqslant C\cdot\alpha + 1 \end{aligned}$$

We conclude that $\|T\| \leqslant C\cdot\alpha + 1$. The non compactness of T follows easily, if we consider $\{z_k\}_k$ the biorthogonals of $\{x^*_{m_k}\}_k$. Then $\{z_k\}_k$ is seminormalized and $\{Tz_k\}_k = \{x_k\}_k$, therefore it is not norm convergent.

We now prove that T is strictly singular. Suppose that it is not, then by Proposition 7.1, there exists $\lambda \neq 0$ such that $Q = T - \lambda I$ is strictly singular. Since λI is a Fredholm operator and Q is strictly singular, it follows that $T = Q + \lambda I$ is also a Fredholm operator, therefore $\dim(\mathfrak{X}_{\text{usm}}/T[\mathfrak{X}_{\text{usm}}]) < \infty$. The fact that $T[\mathfrak{X}_{\text{usm}}] \subset [\{x_k\}_k]$ and $\dim(\mathfrak{X}_{\text{usm}}/[\{x_k\}_k]) = \infty$ yields a contradiction.

□

Proposition 7.8. There exists $S : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ a strictly singular operator which is not polynomially compact.

Proof. Choose $\{p_n\}_n$ a strictly increasing sequence of real numbers, with $p_1 > 2$ and let p_n' be the conjugate of p_n for all $n \in \mathbb{N}$.

By Proposition 6.10, for every $n \in \mathbb{N}$ there exist $\{x_k^n\}_k$ a seminormalized block sequence in $\mathfrak{X}_{\text{usm}}$, with $\|x_k^n\| \geqslant 1$ for all $k, n \in \mathbb{N}$ and $\{x_k^{n*}\}_k$ a seminormalized block sequence in $\mathfrak{X}^*_{\text{usm}}$, satisfying the following.

(i) $x_k^{n*}(x_m^n) = \delta_{k,m}$
(ii) $\{x_k^n\}_k$ generates an ℓ_{p_n} spreading model and $\{x_k^{n*}\}_k$ generates an $\ell_{p_n'}$ spreading model.

If we set $E_k^n = \operatorname{ran}(\operatorname{ran} x_k^n \cup \operatorname{ran} x_k^{n*})$, using a diagonal argument we may assume that the intervals $\{E_k^n\}_{k,n}$ are pairwise disjoint.

Set $m_k = \lceil (4\cdot 2^{k+1})^2 \rceil$ and $S_n : \mathfrak{X}_{\text{usm}} \to \mathfrak{X}_{\text{usm}}$ with

$$S_n x = \sum_{k=1}^{\infty} x^{n*}_{m_k}(x) x^{n+1}_{m_k}$$

Proposition 7.7 (for $q = p_{n+1}$), yields that S_n is bounded and strictly singular. Moreover the following holds.

(a) For every $k, n \in \mathbb{N}$, $S_n x^n_{m_k} = x^{n+1}_{m_k}$
(b) For every $n \neq l \in \mathbb{N}$ and $k \in \mathbb{N}$, $S_n x^l_{m_k} = 0$.

Set $S = \sum_{n=1}^{\infty} \frac{1}{2^n \|S_n\|} S_n$. Then S is strictly singular and we shall prove that that it is not polynomially compact.

Properties (a) and (b), yield that for every $k, n \in \mathbb{N}$ we have that $S x^n_{m_k} = \frac{1}{2^n \|S_n\|} x^{n+1}_{m_k}$.

Using an easy induction we conclude the following.

$$S^n x^1_{m_k} = \Big(\prod_{j=1}^{n} \frac{1}{2^j \|S_j\|}\Big) x^{n+1}_{m_k}, \quad \text{for every } k, n \in \mathbb{N} \tag{53}$$

Set $a_n = \prod_{j=1}^{n} \frac{1}{2^j \|S_j\|}$ for $n \in \mathbb{N}$ and $a_0 = 1$.

Let now $T = \sum_{n=0}^{d} b_n S^n$ be a non zero polynomial of S. Then, using (53), for every $k \in \mathbb{N}$, we have that

$$T x^1_{m_k} = \sum_{n=0}^{d} b_n a_n x^{n+1}_{m_k}$$

The fact that the basis of $\mathfrak{X}_{\text{usm}}$ is bimonotone, the $x^1_{m_k}, \dots, x^{d+1}_{m_k}$ are disjointly ranged and $\|x^n_{m_k}\| \geqslant 1$, for all $k, n \in \mathbb{N}$, yields that $\|T x^1_{m_k}\| \geqslant \max\{|a_n b_n| : n = 0, \dots, d\}$, for all $k \in \mathbb{N}$. We conclude that $\{T x^1_{m_k}\}_k$ has no norm convergent subsequence, therefore T is not compact. □

Remark 7.9. A slight modification of the above yields that in every block subspace of $\mathfrak{X}_{\text{usm}}$ there exists a strictly singular operator which is not polynomially compact.

We close the paper with the following two problems, which are open to us.

Problem 1. Does there exist a reflexive Banach space with an unconditional basis, which is hereditarily unconditional spreading model universal?

Although it does not seem necessary to use conditional structure in order to construct a hereditarily unconditional spreading model universal space, in our approach the conditional structure of the type II_+ functionals cannot be avoided, resulting in an HI space.

Problem 2. Does there exist a Banach space hereditarily spreading model universal, for both conditional and unconditional spreading sequences?

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