THE MINIMAL GENUS PROBLEM FOR ELLIPTIC SURFACES

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ABSTRACT. We partly solve the minimal genus problem for embedded surfaces in the case of elliptic 4-manifolds. This involves a certain restricted transitivity property of the action of the orientation preserving diffeomorphism group on the second homology. In all cases we consider we get the minimal possible genus allowed by the adjunction inequality.

1. INTRODUCTION

Starting with the classical work of Kervaire and Milnor [8], who showed that certain second homology classes in simply-connected 4-manifolds are not represented by embedded spheres, the question arose to find for a given homology class in a 4-manifold the minimal genus of an embedded closed connected oriented surface realizing that class. This question has been solved at least partly for rational and ruled surfaces and for 4-manifolds with a free circle action [3, 4, 12, 13, 14, 15, 16, 17, 23]. On symplectic 4-manifolds the question is related to the Thom conjecture [10, 20, 21]. In particular, the adjunction inequality from Seiberg-Witten theory gives a lower bound on the genus of a surface representing a homology class in a closed, oriented 4-manifold with a basic class and we can then ask if this lower bound is indeed realized. Usually the question is more tractable for classes of positive self-intersection and is still open in most situations in the case of negative self-intersections. In particular, it is still unknown whether there exist embedded spheres in $K3$ of arbitrarily negative self-intersection.

Another interesting class of 4-manifolds are elliptic surfaces. We will restrict to relatively minimal simply-connected elliptic surfaces with $b_2^+ > 1$, but generalizations should be possible. Note that every orientation preserving diffeomorphism of a closed, oriented 4-manifold induces an isometry of the intersection form on the second homology (modulo torsion). A very useful fact is that for elliptic surfaces the image of the orientation preserving diffeomorphism group in the orthogonal group of the intersection form is known. This is due to Borcea, Donaldson and Matumoto [1, 2, 19] for the $K3$ surface and to Friedman-Morgan and Lönne in the general case [5, 18]. We will combine this knowledge with the work of Wall on the transitivity of the orthogonal groups of unimodular quadratic forms [24]. Similar to the case of rational surfaces, this will allow us to reduce the problem of representing a homology class by a minimal genus surface to certain special

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classes. We cannot treat the minimal genus problem in full generality. Instead we will concentrate on the first interesting special cases that come to mind.

We are also interested if we can realize homology classes by surfaces that are contained in certain nice neighbourhoods inside the elliptic surface. The neighbourhood we consider is the Gompf nucleus $N(2)$.

Notations. In the following X will denote a relatively minimal simply-connected elliptic surface with the complex orientation. Using the classification of such elliptic surfaces [7] X is diffeomorphic to $E(n)_{p,q}$, where the coprime indices denote logarithmic transformations. We restrict to the case $n \geq 2$ or equivalently $b_2^+ > 1$; see [12] for a discussion of Dolgachev surfaces. All self-diffeomorphisms of X are orientation preserving.

2. ACTION OF THE DIFFEOMORPHISM GROUP

Let $H_2(X)$ denote the integral second homology of X and $\mathrm{Diff}^+(X)$ the group of orientation preserving self-diffeomorphisms of X. The intersection form on second homology induces a unimodular quadratic form on $H_2(X)$. We denote by O the orthogonal group of all automorphisms of $H_2(X)$ that preserve the intersection form. The action of diffeomorphisms on homology define a group homomorphism $\mathrm{Diff}^+(X) \to O$. The *spinor norm* of an element $\phi \in O$ is defined to be ± 1 depending on whether ϕ preserves or reverses an orientation on a maximal positive definite subspace of $H_2(X;\mathbb{R})$, see Section 6.1.2 of Chapter VI in [5] for a precise definition. The subgroup of O of elements of spinor norm 1 is denoted by O'.

Definition 1. We let K denote the canonical class of X. If X is not the $K3$ surface let k denote the Poincaré dual of K divided by its divisibility. If K is the $K3$ surface let k denote the class of a general fibre. In any case, k is a primitive class of self-intersection zero. We also choose a second homology class V such that $k \cdot V = 1$. For example if X has no multiple fibres we can choose for V a section of an elliptic fibration. We denote by O_k the automorphisms fixing k and by O'_k those of spinor norm 1.

The following was proved in [18].

Theorem 2. *The image of the diffeomorphism group $\mathrm{Diff}^+(X)$ in O is equal to O' for the $K3$ surface and contains O'_k for all other elliptic surfaces X.*

We now consider integral unimodular quadratic forms in general. We let H denote the even hyperbolic form of rank 2 and E_8 the standard positive definite even form of rank 8. A *standard basis* for H is a basis e, f such that

$$e^2 = 0, f^2 = 0, e \cdot f = 1.$$

Let Q denote the quadratic form $Q = lH \oplus m(-E_8)$ with $l \geq 2$ and $m \in \mathbb{Z}$. In [24] Wall proved the following.

Theorem 3. *The orthogonal group of Q acts transitively on primitive elements of given square.*

We want to deduce the following.

Proposition 4. *The subgroup of elements of spinor norm 1 in the orthogonal group of Q acts transitively on primitive elements of given square.*

We first prove the following lemma.

Lemma 5. *For any even number $2a$ there exist primitive elements p and q of square $2a$ and automorphisms of Q of spinor norm $+1$ and -1 which map p to q.*

Proof. We consider $Q = lH \oplus m(-E_8)$ and let e, f denote a standard basis for the first H summand. Let $p = e + af$ and $q = -e - af$. Then $p^2 = q^2 = 2a$. Consider the automorphism of Q which is minus the identity on the first H summand and the identity on all other summands and the automorphism which is minus the identity on the first two H summands and the identity on all other summands. These automorphisms have spinor norm -1 and $+1$ and map p to q. □

We now prove Proposition 4.

Proof. Let x and y be arbitrary primitive elements of square $2a$ and let p and q be the elements from the lemma of the same square. By Wall's theorem there exist automorphisms in O mapping x to p and q to y. Choosing an automorphism that maps p to q of the correct spinor norm we get by composing an automorphism of spinor norm $+1$ mapping x to y. □

We now consider the elliptic surface X.

Lemma 6. *The self-intersection number V^2 is even if and only if X is spin.*

Proof. The intersection form on the span of k and V is unimodular, hence it is unimodular on the orthogonal complement. The intersection form on this complement is even, since the canonical class K is characteristic. The claim now follows because X is spin if and only if the intersection form on both summands is even. □

Let $V^2 = 2a$ in the spin case and $V^2 = 2a + 1$ in the non-spin case.

Definition 7. Define elements $e_1 = k$ and $f_1 = V - ak$. Then the intersection form on the span of e_1, f_1 is H in the spin case and H' given by

$$H' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

in the non-spin case. Note that H' is isomorphic to $\langle +1 \rangle \oplus \langle -1 \rangle$.

The complete intersection form of X is then given by

$$H \oplus lH \oplus m(-E_8) \text{ or } H' \oplus lH \oplus m(-E_8), \tag{1}$$

where $l \geq 2$ since $b_2^+ \geq 3$. We also want to choose a standard basis for the second H summand: The $K3$ surface is known to contain a rim torus R of self-intersection zero and a vanishing sphere S of self-intersection -2 such that R and S intersect transversely in one positive point (see Section 4 for an explicit model of R). Both are contained in a nucleus $N(2)$ [6], defined as the neighbourhood of a cusp fibre and a section in $K3$. Since this nucleus is disjoint from a general fibre it is still contained in an arbitrary elliptic surface X of the type above. We also choose the surface representing the class V to be disjoint from this nucleus.

Definition 8. Define $e_2 = R$ and $f_2 = S + R$. These are a standard basis for the second H summand in the intersection form of the elliptic surface X.

Using Theorem 2 and Proposition 4 we deduce the following:

Proposition 9. *Let $A \in H_2(X)$ be an arbitrary primitive element. Then there exists a self-diffeomorphism of X which maps A to*

$$A' = \alpha e_1 + \beta f_1 + \gamma e_2 + \delta f_2,$$

where $\alpha, \beta, \gamma, \delta$ are certain integers. If X is the $K3$ surface we can map A via a self-diffeomorphism to

$$A' = \alpha e_1 + f_1.$$

The self-diffeomorphisms of the $K3$ surface act transitively on primitive elements of given square.

For the proof in the non-$K3$ case we choose the identity on the first summand of the intersection form as in equation (1) and a suitable automorphism given by Proposition 4 on the rest. The result for the $K3$ surface, which is well-known [9, 12], follows similarly.

3. Minimal genus problem for the $K3$ surface

The minimal genus problem for classes of non-negative square in the $K3$ surface has already been solved [12] using the $K3$ case of Proposition 9. We want to recall this solution and also say something about realizing these surfaces in a nucleus $N(2)$. Note that the adjunction inequality for the $K3$ surface implies for the genus of a smooth surface Σ that $2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma$ if the homology class represented by this surface is non-zero.

Definition 10. By *the standard surface of genus g embedded in the nucleus $N(2)$* we mean the section of self-intersection -2 ($g = 0$), the general fibre of self-intersection 0 ($g = 1$) or the surface of genus $g \geq 2$ and self-intersection $2g - 2$ obtained by smoothing the intersection points of the section and g parallel copies of the general fibre.

Using the fact that the $K3$ surface contains a nucleus $N(2)$ and Proposition 9 we get the following.

Corollary 11. *Consider the $K3$ surface. Every primitive class of self-intersection $2c-2$ with $c \geq 0$ is represented by a surface of genus c, embedded as the standard surface in a nucleus $N(2)$ inside $K3$. This is the minimal possible genus.*

To solve the case of divisible classes with non-negative square we use Lemma 7.7 in [11] due to Kronheimer-Mrowka (see also Lemma 14 in [12]):

Lemma 12. *Let Y be a closed, connected, oriented 4-manifold. Let $a(\Sigma) = 2g(\Sigma) - 2 - \Sigma \cdot \Sigma$. If $h \in H_2(Y)$ is a homology class with $h \cdot h \geq 0$ and Σ_h is a surface of genus g representing h and $g \geq 1$ when $\Sigma_h \cdot \Sigma_h = 0$, then for all $r > 0$, the class rh can be represented by an embedded surface Σ_{rh} with*

$$a(\Sigma_{rh}) = ra(\Sigma_h).$$

Note that in our case $a(\Sigma_h)$ is zero, hence also $a(\Sigma_{rh})$ is zero and the class rh is represented by a minimal genus surface. In particular, we can apply the construction of this lemma to divisible classes of non-negative square inside the nucleus $N(2)$ to get new standard surfaces, representing these classes in the nucleus (the construction in the proof of this lemma works in a tubular neighbourhood of Σ_h and does not need the assumption that Y is closed). The transitivity of the action of the diffeomorphism group then implies that every divisible class of non-negative square in $K3$ can also be represented by such a standard surface inside a nucleus $N(2)$. Hence Corollary 11 holds without the assumption that the class is primitive.

4. Minimal genus problem for other elliptic surfaces

We now consider the general case of relatively minimal simply-connected elliptic surfaces X with $b_2^+ > 1$. Note that the adjunction inequality implies for surfaces Σ orthogonal to K again that $2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma$. The self-intersection of such a surface is even. Using Proposition 9, Lemma 12 and the discussion following it we get:

Corollary 13. *Let X be an elliptic surface. Then every class A of self-intersection $2c - 2$ with $c \geq 0$ that is orthogonal to the classes K and V is represented by a standard surface of genus c in a nucleus $N(2)$ in the 4-manifold X. This is the minimal possible genus if the class is non-zero.*

Proof. For the $K3$ surface this is a special case of what has been proved above. In the general case, the assumptions imply that A can be mapped via a diffeomorphism to $A' = \gamma e_2 + \delta f_2$. Since e_2 and f_2 are constructed in a nucleus $N(2)$ the claim follows. □

Remark 14. If we relax the assumption and only assume that A is orthogonal to K it seems that the surface is in general not contained in a nucleus $N(2)$. For example the general fibre is contained in a nucleus $N(n)_{p,q}$.

We can deal with the case $A^2 = -2$ in a slightly more general situation:

Proposition 15. *Let X be an elliptic surface. Then any class A orthogonal to K and of self-intersection -2 is represented by the standard sphere in a nucleus $N(2)$ in the 4-manifold X.*

Proof. We may assume that X is different from the $K3$ surface. The assumptions imply that there exists a self-diffeomorphism of X mapping A to

$$A' = \alpha k + S,$$

where S is the vanishing sphere. Consider the following map ϕ on $H_2(X)$ which on the first two summands of the intersection form is given by

$$\begin{aligned} k &\mapsto k \\ V &\mapsto V + \alpha R \\ R &\mapsto R \\ S &\mapsto S - \alpha k \end{aligned}$$

and is the identity on all other summands. It is easy to check that ϕ is an isometry. Letting α be a real number and taking $\alpha \to 0$ we see that ϕ has spinor norm $+1$. Hence it is an element in O_k' and therefore induced by a self-diffeomorphism. It maps A' to S. This implies the claim. □

Remark 16. This result should be compared to the fact that every class of square -2 in the complement of a general fibre in X is represented by an embedded sphere [5, 18]. Using an adjunction inequality for spheres [22] one can show that every elliptic surface X different from the $K3$ surface has a class of square -2 that intersects the canonical class and is not represented by an embedded sphere.

We now restrict to the case of elliptic surfaces without multiple fibres, i.e. $X = E(n)$. The class k is represented by a general fibre F. We also have the rim torus R. Proposition 9 implies:

Lemma 17. *If A is a class orthogonal to K and of self-intersection zero then there exists a self-diffeomorphism of X that maps A to*

$$A' = \alpha F + \gamma R.$$

We want to show that A' can be represented by an embedded torus. The construction involves the circle sum from [17]. The idea is the following: Let Σ_0 and Σ_1 denote two disjoint connected embedded oriented surfaces in a 4-manifold Y. We can tube them together in the standard way to get a surface of genus $g(\Sigma_0) + g(\Sigma_1)$. Sometimes, however, we can perform a different surgery that results in a surface of smaller genus. Let $S_i^1 \subset \Sigma_i$ denote embedded circles that represent non-trivial homology classes in the surfaces. In each surface we delete an annulus $S_i^1 \times I$. We get two disjoint surfaces whose boundaries consist of two circles for each surface. We want to connect these circles by annuli embedded in Y. There are several ways to do this: One possibility is to connect the circles from the same surface. In this way we simply get back the surfaces Σ_0 and Σ_1. Another possibility is to connect the boundary circles from different surfaces. If this is possible we get an embedded connected surface of genus $g(\Sigma_0) + g(\Sigma_1) - 1$ representing the class $[\Sigma_0] + [\Sigma_1]$.

The construction works if we can find an embedded annulus Δ in Y that intersects the surfaces Σ_0 and Σ_1 precisely in the circles S_0^1 and S_1^1. We also need a nowhere vanishing normal vector field along Δ that at the ends of Δ is tangential to the surfaces Σ_0 and Σ_1. The annuli connecting the four boundary circles are then constructed as normal push-offs of the annulus Δ.

Lemma 18. *There exists an embedded annulus Δ connecting the tori F and R that satisfies the necessary assumptions for the circle sum in* [17].

Proof. The elliptic surface $X = E(n)$ is obtained as a fibre sum of $E(n-1)$ and $E(1)$ along a general fibre. Let $S^1 \times S^1 \times D^2$ denote a tubular neighbourhood of the fibre in one of the summands. We think of D^2 as the unit disk in the complex plane and let I denote the interval $[\frac{1}{2}, 1]$ along the real axis. In forming the fibre sum we delete the open tubular neighbourhood of radius $\frac{1}{4}$ of the general fibre. The fibre F in X is realized as $S^1 \times S^1 \times \{\frac{1}{2}\}$ while the rim torus R is $S^1 \times \{*\} \times \partial D^2$. Consider the annulus $\Delta = S^1 \times \{*\} \times I$. It intersects the tori F and R precisely in the circles $S^1_F = S^1 \times \{*\} \times \{\frac{1}{2}\}$ and $S^1_R = S^1 \times \{*\} \times \{1\}$. Let v_F be a unit tangent vector to S^1 in the point $*$ and v_R a unit tangent vector to ∂D^2 in 1. Then

$$e_F = S^1 \times v_F \times \left\{ \frac{1}{2} \right\}$$

and

$$e_R = S^1 \times \{*\} \times v_R$$

are framings of the circles S^1_F and S^1_R inside the tori. Consider the normal vector field along the annulus Δ given on $S^1 \times \{*\} \times t$ by

$$e = S^1 \times (2-2t)v_F \times t \times (2t-1)v_R.$$

This is equal to the framings e_F and e_R on the boundary and is the required framing of the annulus. □

This construction allows us to circle sum F and R. A similar, but easier construction allows us to circle sum α parallel copies of F and γ parallel copies of R to get embedded tori Σ_0 and Σ_1. The torus Σ_0 contains as an open subset a copy of the torus F with an annulus deleted, and similarly for Σ_1. Circle summing Σ_0 and Σ_1 along these subsets we get an embedded torus representing the class $\alpha F + \gamma R$. This construction proves:

Theorem 19. *Let X be an elliptic surface without multiple fibres. Then any class A orthogonal to K and of self-intersection zero is represented by an embedded torus.*

This is clearly the minimal possible genus allowed by the adjunction inequality if the class A is non-zero. The same method can be used to prove the following for even self-intersection numbers greater or equal to 2:

Theorem 20. *Let X be an elliptic surface without multiple fibres. Suppose A is a class orthogonal to K such that $A^2 = 2c-2$ with $c \geq 2$. Then A is represented by a surface of genus c in X. This is the minimal possible genus.*

Proof. The assumptions imply that there exists a self-diffeomorphism of X mapping A to

$$A' = \alpha F + \gamma e_2 + \delta f_2,$$

where γ and δ are positive with $\gamma\delta = c-1$. We circle sum α parallel copies of F to get a torus Σ_0. The classes e_2 and f_2 are represented by embedded tori R and

T of self-intersection zero that intersect transversely in a single positive point, the torus T being obtained by smoothing the intersection between R and S. Taking circle sums of parallel copies we get tori representing γe_2 and δf_2 that intersect transversely in $\gamma\delta$ points. Smoothing these intersections we get a surface Σ_1 of genus $\gamma\delta + 1 = c$. This surface contains as an open subset a copy of the torus R with an annulus and δ points deleted. We circle sum the surface Σ_1 to the torus Σ_0 to get an embedded surface of genus c representing A'. □

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REFERENCES

1. C. Borcea, *Diffeomorphisms of a $K3$ surface*, Math. Ann. **275**, 1–4 (1986).
2. S. K. Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology **29**, 257–315 (1990).
3. S. Friedl, S. Vidussi, *Twisted Alexander polynomials, symplectic 4-manifolds and surfaces of minimal complexity*, Banach Center Publ., 85, Polish Acad. Sci. Inst. Math., Warsaw, 2009.
4. S. Friedl, S. Vidussi, *Minimal genus in 4-manifolds with a free circle action*, preprint arXiv:1204.3578v1.
5. R. Friedman, J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 27, Springer-Verlag, Berlin, 1994.
6. R. E. Gompf, T. S. Mrowka, *Irreducible 4-manifolds need not be complex*, Ann. of Math. (2) **138**, 61–111 (1993).
7. R. E. Gompf, A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20. Providence, Rhode Island. American Mathematical Society 1999.
8. M. A. Kervaire, J. W. Milnor, *On 2-spheres in 4-manifolds*, Proc. Nat. Acad. Sci. U.S.A. **47**, 1651–1657 (1961).
9. P. B. Kronheimer, T. S. Mrowka, *Gauge theory for embedded surfaces, I*, Topology **32**, 773–826 (1993).
10. P. B. Kronheimer, T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1**, no. 6, 797–808 (1994).
11. P. B. Kronheimer, T. S. Mrowka, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Differential Geom. **41**, 573–734 (1995).
12. T. Lawson, *The minimal genus problem*, Exposition. Math. **15**, 385–431 (1997).
13. B.-H. Li, *Representing nonnegative homology classes of $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ by minimal genus smooth embeddings*, Trans. Amer. Math. Soc. **352**, 4155–4169 (2000).
14. B.-H. Li, T.-J. Li, *Minimal genus embeddings in S^2-bundles over surfaces*, Math. Res. Lett. **4**, no. 3, 379–394 (1997).
15. B.-H. Li, T.-J. Li, *Minimal genus smooth embeddings in $S^2\times S^2$ and $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ with $n\leq 8$*, Topology **37**, 575–594 (1998).
16. B.-H. Li, T.-J. Li, *Smooth minimal genera for small negative classes in $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ with $n\leq 9$*, Topology Appl. **132**, 1–15 (2003).
17. B.-H. Li, T.-J. Li, *Circle-sum and minimal genus surfaces in ruled 4-manifolds*, Proc. Amer. Math. Soc. **135**, 3745–3753 (2007).
18. M. Lönne, *On the diffeomorphism groups of elliptic surfaces*, Math. Ann. **310**, 103–117 (1998).
19. T. Matumoto, *On diffeomorphisms of a $K3$ surface*, Algebraic and topological theories (Kinosaki, 1984), 616–621, Kinokuniya, Tokyo, 1986.
20. J. W. Morgan, Z. Szabó, C. H. Taubes, *A product formula for the Seiberg-Witten invariants and the generalized Thom conjecture*, J. Differential Geom. **44**, 706–788 (1996).
21. P. Ozsváth, Z. Szabó, *The symplectic Thom conjecture*, Ann. of Math. (2) **151**, 93–124 (2000).

22. J. Park, *An embedded 2-sphere in irreducible 4-manifolds*, Bull. Korean Math. Soc. **36**, 683–691 (1999).
23. D. Ruberman, *The minimal genus of an embedded surface of non-negative square in a rational surface*, Turkish J. Math. **20**, 129–133 (1996).
24. C. T. C. Wall, *On the orthogonal groups of unimodular quadratic forms*, Math. Ann. **147**, 328–338 (1962).

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