A SHORT NOTE ON MAPPING CYLINDERS

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Abstract. Given a homotopy equivalence $f : X \to Y$ we obtain an explicit formula for a strong deformation retraction of the mapping cylinder of f onto its top.

1. Setup

The mapping cylinder M_f of a map $f : X \to Y$ between two topological spaces is the quotient space of the disjoint union $X \times I + Y$ by the equivalence relation arising from the identifications $(x, 0) \sim f(x)$. The quotient map $q : X \times I + Y \to M_f$ embeds $X \times \{1\}$ and Y as closed subsets of M_f. Let $\widetilde{X} = q(X \times \{1\})$, and $\widetilde{Y} = q(Y)$. We will denote by $[x, t]$ and $[y]$ the equivalence classes of $(x, t) \in X \times I$, and $y \in Y$, respectively.

Theorem. *A map $f : X \to Y$ between two topological spaces is a homotopy equivalence iff $\widetilde{X}$ is a strong deformation retract of M_f.*

This theorem is well known, and its origins can be traced back to the results presented by Fox [1] and Fuchs [2]. Modern proofs are usually presented within the contexts of cofibrations or the homotopy extension property (e.g. Corollary 0.21 in [3]).

The backward implication is usually dealt with by giving an explicit formula for a strong deformation retraction of M_f onto its bottom $\widetilde{Y}$ (e.g. the map H_1 given below).

The forward implication has a conceptual proof that we believe is better than a proof by a formula. However, we also believe that it is desirable to have a formula because of computer modeling, which has become an important research tool across the scientific community. Mathematical concepts are generally easier to implement on a computer if a formula or algorithm describing the concept is found.

2. Tools

Suppose that $f : X \to Y$ is a homotopy equivalence with homotopy inverse $g : Y \to X$. Then we have homotopies $F : X \times I \to X$ from $g \circ f$ to $\mathbf{1}_X$, and $G : Y \times I \to Y$ from $f \circ g$ to $\mathbf{1}_Y$.

We want to construct a strong deformation retraction of M_f onto its top $\widetilde{X}$ in terms of f, g, F and G. The following are the tools we will need:

2.1. A deformation retraction of M_f onto its top $\widetilde{X}$:

(1) Define a map $H_1 : M_f \times I \to M_f$ by

$$\big([x, t], s\big) \longmapsto [x, t(1 - s)] \quad \text{and} \quad \big([y], s\big) \longmapsto [y]$$

This is a well defined strong deformation retraction from $\mathbf{1}_{M_f}$ to the retraction $r : M_f \to M_f$ given by the canonical projection of the mapping cylinder onto its bottom $\widetilde{Y}$. That is, r is defined by $[x, t] \mapsto [x, 0]$, and $[y] \mapsto [y]$.

(2) Use f and G to define a map $H_2 : M_f \times I \to M_f$ by

$$\big([x, t], s\big) \longmapsto [G(f(x), 1 - s)] \quad \text{and} \quad \big([y], s\big) \longmapsto [G(y, 1 - s)]$$

This is a well defined homotopy from r to a map $h : M_f \to M_f$ defined by $[x, t] \mapsto [g \circ f(x), 0]$ and $[y] \mapsto [g(y), 0]$.

(3) Use g and F to define a map $H_3 : M_f \times I \longrightarrow M_f$ by

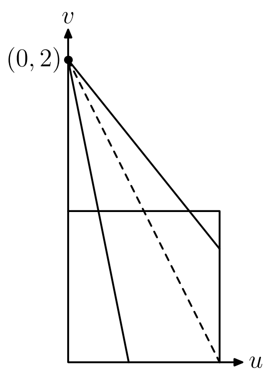
$$\big([x,t],s\big) \longmapsto [F(x,st),s] \;\text{ and }\; \big([y],s\big) \longmapsto [g(y),s]$$

This is a well defined homotopy from the map h to a retraction $r' : M_f \longrightarrow M_f$ of the mapping cylinder onto its top $\widetilde{X}$ defined by

$$[x,t] \longmapsto [F(x,t),1] \;\text{ and }\; [y] \longmapsto [g(y),1].$$

The concatenation $H = H_1 * H_2 * H_3$ is the desired deformation retraction from $\mathbf{1}_{M_f}$ to a retraction r' of M_f onto its top.

2.2. **The Homotopy extension property of $(M_f \times I, \widetilde{X} \times I)$ made explicit:** We need to construct a retraction $R : M_f \times I \times I \longrightarrow M_f \times I \times I$ of $M_f \times I \times I$ onto $M_f \times I \times \{0\} \cup \widetilde{X} \times I \times I$. This is achieved by defining first a retraction $\varphi : I^2 \longrightarrow I^2$ of I^2 onto $I \times \{0\} \cup \{1\} \times I$ via radial projection from the point $(0,2)$, as the figure below illustrates.



$$\varphi(u,v) = \begin{cases} \left(\dfrac{2u}{2-v}, 0\right) & \text{if } v \leq 2-2u \\ \left(1, \dfrac{2u+v-2}{u}\right) & \text{if } v \geq 2-2u. \end{cases}$$

If we let φ_1 and φ_2 denote the components of φ the retraction R is defined by

$$\big([x,t],s,l\big) \longmapsto \big([x,\varphi_1(t,l)],s,\varphi_2(t,l)\big) \quad\text{and}\quad \big([y],s,l\big) \longmapsto \big([y],s,0\big).$$

3. Construction

In order to facilitate clarity a point in M_f will be denoted by the letter p. If the point is in $\widetilde{X}$ (i.e. $p = [x,1]$) we will denote it by $\tilde{p}$.

Our goal is to modify the deformation retraction H constructed earlier so that it leaves $\widetilde{X}$ fixed.

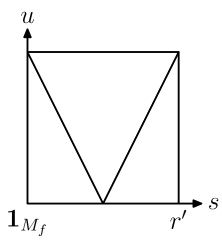
We begin by defining a homotopy $K : M_f \times I \longrightarrow M_f$ by

$$K(p,s) = \begin{cases} H^{-1}(p, 1-2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ H^{-1}(r'(p), 2s-1) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}$$

where H^{-1} denotes the *inverse* of H (i.e. H with s running backward). Roughly speaking, the first part of K is simply H^{-1} twice as fast, while the second half is a map that begins and ends with r', and is homotopic[1] to H^{-1}. It easy to check that K is a well defined homotopy from $\mathbf{1}_{M_f}$ to r', and that its restriction to $\widetilde{X} \times I$ is a homotopy beginning and ending with $\mathbf{1}_{\widetilde{X}}$. Moreover, it behaves well with respect to time reversal in the sense that $K(\tilde{p}, 1-s) = K(\tilde{p}, s)$.

The next step will be to define a homotopy $L : (\widetilde{X} \times I) \times I \longrightarrow M_f$. We use K and the diagram below to accomplish this.

[1]Because $\mathbf{1}_{M_f} \sim r'$ implies that $\mathbf{1}_{M_f} \times \mathbf{1}_I \sim r' \times \mathbf{1}_I$, and therefore $H^{-1} \sim H^{-1} \circ (r' \times \mathbf{1}_I)$.



Roughly speaking, we want $L(\tilde{p}, s, u)$ to be independent of u below the 'V', and independent of s above the 'V'. More formally:

$$L(\tilde{p}, s, u) = \begin{cases} K(\tilde{p}, s) & \text{if } u \leq |2s - 1| \\ \\ K\left(\tilde{p}, \dfrac{1-u}{2}\right) & \text{if } 2s - u \leq 1 \leq 2s + u. \end{cases}$$

This is a well defined[2] homotopy from $K\Big|_{\widetilde{X}\times I}$ to a map sending $(\tilde{p}, s)$ to $\tilde{p}$. Moreover, $L(\tilde{p}, s, u) = \tilde{p}$ for $(s, u) \in \partial I \times I \cup I \times \{1\}$. Therefore, looking at the diagram above we would like to extend L to the whole mapping cylinder and then follow this extension from the lower left corner $\mathbf{1}_{M_f}$ along the left, top and right edges to the right corner r'. Indeed, define a map $K_0 : M_f \times I \times \{0\} \longrightarrow M_f$ by $K_0(p, s, 0) = K(p, s)$, and combine it with L to give a map[3] $(K_0, L) : M_f \times I \times \{0\} \cup \widetilde{X} \times I \times I \longrightarrow M_f$. Then, define L' by the composition

$$M_f \times I \times I \xrightarrow{\ R\ } M_f \times I \times \{0\} \cup \widetilde{X} \times I \times I \xrightarrow{(K_0,L)} M_f,$$

This map clearly extends L and satisfies $L'(p, s, 0) = K(p, s)$. Finally, define a map $\Gamma : M_f \times I \longrightarrow M_f$ by

$$\Gamma(p, s) = \begin{cases} L'(p, 0, 3s) & \text{if } 0 \leq s \leq \dfrac{1}{3} \\ \\ L'(p, 3s-1, 1) & \text{if } \dfrac{1}{3} \leq s \leq \dfrac{2}{3} \\ \\ L'(p, 1, 3-3s) & \text{if } \dfrac{2}{3} \leq s \leq 1. \end{cases}$$

It is easy to check that this map verifies the following:

- $\Gamma(p, 0) = L'(p, 0, 0) = K(p, 0) = p$,
- $\Gamma(p, 1) = L'(p, 1, 0) = K(p, 1) = r'(p)$,
- $\Gamma(\tilde{p}, s) = \tilde{p}$ because L' is an extension of L.

Therefore, Γ is the desired strong deformation retraction from the identity $\mathbf{1}_{M_f}$ to a retraction r' of the mapping cylinder onto its top $\widetilde{X}$.

4. Formula

If we unravel Γ we will obtain the desired formula in terms of f, g, F and G. It is given in the next two pages.

[2]Because $K(\tilde{p}, 1-s) = K(\tilde{p}, s)$ implies that $K(\tilde{p}, \frac{1-u}{2}) = K(\tilde{p}, \frac{1+u}{2})$.

[3]This map is unambiguously defined because $K_0 = L$ on the overlap $\widetilde{X} \times I \times \{0\}$. Continuity follows from the gluing lemma because $\widetilde{X}$ is a closed subset of in M_f.

$$
\Gamma\big([x,t],s\big) = \begin{cases}
\text{if } 0 \le s \le \dfrac{1}{3} & \begin{cases} \left[x, \dfrac{2t}{2-3s}\right] & \text{if } s \le \dfrac{2-2t}{3} \\[2ex] [x,1] & \text{if } s \ge \dfrac{2-2t}{3} \end{cases} \\[6ex]
\text{if } \dfrac{1}{3} \le s \le \dfrac{2}{3} & \begin{cases} \text{if } t \le \dfrac{1}{2} & \begin{cases} [x, 2t(7-18s)] & \text{if } \dfrac{1}{3} \le s \le \dfrac{7}{18} \\[1ex] [G(f(x), 8-18s)] & \text{if } \dfrac{7}{18} \le s \le \dfrac{8}{18} \\[1ex] [F\left(x, 2t(18s-8)\right), 18s-8] & \text{if } \dfrac{8}{18} \le s \le \dfrac{1}{2} \\[1ex] [F\left(F(x,2t), 10-18s\right), 10-18s] & \text{if } \dfrac{1}{2} \le s \le \dfrac{10}{18} \\[1ex] [G\left(f \circ F(x,2t), 18s-10\right)] & \text{if } \dfrac{10}{18} \le s \le \dfrac{11}{18} \\[1ex] [F(x,2t), 18s-11] & \text{if } \dfrac{11}{18} \le s \le \dfrac{2}{3} \end{cases} \\[12ex]
\text{if } t \ge \dfrac{1}{2} & \begin{cases} [x, 7-18s] & \text{if } \dfrac{1}{3} \le s \le \dfrac{7}{18} \\[1ex] [G(f(x), 8-18s)] & \text{if } \dfrac{7}{18} \le s \le \dfrac{8}{18} \\[1ex] [F\left(x, 18s-8\right), 18s-8] & \text{if } \dfrac{8}{18} \le s \le \dfrac{1}{2} \\[1ex] [F\left(x, 10-18s\right), 10-18s] & \text{if } \dfrac{1}{2} \le s \le \dfrac{10}{18} \\[1ex] [G(f(x), 18s-10)] & \text{if } \dfrac{10}{18} \le s \le \dfrac{11}{18} \\[1ex] [x, 18s-11] & \text{if } \dfrac{11}{18} \le s \le \dfrac{2}{3} \end{cases} \end{cases} \\[6ex]
\text{if } \dfrac{2}{3} \le s \le 1 & \begin{cases} \left[F\left(x, \dfrac{2t}{3s-1}\right), 1\right] & \text{if } s \ge \dfrac{1+2t}{3} \\[2ex] [x,1] & \text{if } s \le \dfrac{1+2t}{3} \end{cases}
\end{cases}
$$

$$\Gamma\big([y],s\big)=\begin{cases}[y] & \text{if } 0\le s\le \frac{7}{18}\\ [G(y,8-18s)] & \text{if } \frac{7}{18}\le s\le \frac{8}{18}\\ [g(y),18s-8] & \text{if } \frac{8}{18}\le s\le \frac{1}{2}\\ [F\left(g(y),10-18s\right)] & \text{if } \frac{1}{2}\le s\le \frac{10}{18}\\ [G\left(f\circ g(y),18s-10\right)] & \text{if } \frac{10}{18}\le s\le \frac{11}{18}\\ [g(y),18s-11] & \text{if } \frac{11}{18}\le s\le \frac{2}{3}\\ [g(y),1] & \text{if } \frac{2}{3}\le s\le 1\end{cases}$$

References

[1] R.H. Fox, On homotopy type and deformation retracts, *Ann. of Math.*, Vol. 44, 1 (1943), 40-50.

[2] M. Fuchs, A note on mapping cylinders, *Michigan Math. J.*, Vol. 18, 4 (1971), 289-290.

[3] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

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