New examples of K-monotone weighted Banach couples

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Abstract

Some new examples of K-monotone couples of the type $(X, X(w))$, where X is a symmetric space on $[0, 1]$ and w is a weight on $[0, 1]$, are presented. Based on the property of the w-decomposability of a symmetric space we show that, if a weight w changes sufficiently fast, all symmetric spaces X with non-trivial Boyd indices such that the Banach couple $(X, X(w))$ is K-monotone belong to the class of ultrasymmetric Orlicz spaces. If, in addition, the fundamental function of X is $t^{1/p}$ for some $p \in [1, \infty]$, then $X = L_p$. At the same time a Banach couple $(X, X(w))$ may be K-monotone for some non-trivial w in the case when X is not ultrasymmetric. In each of the cases where X is a Lorentz, Marcinkiewicz or Orlicz space we have found conditions which guarantee that $(X, X(w))$ is K-monotone.

1 Introduction

One of the fundamental problems in interpolation theory is to find a description of all interpolation spaces between two fixed Banach spaces X_0 and X_1, which form a Banach couple $\bar{X} = (X_0, X_1)$, i.e., the description of all intermediate Banach spaces X with respect to $\bar{X}$ such that every linear operator $T\colon \bar{X} \to \bar{X}$ maps X into X boundedly.

An important role in the interpolation theory is played by the *K-monotone spaces* between fixed Banach spaces X_0 and X_1, which are defined as follows: if $x \in X$, $y \in X_0 + X_1$, and the inequality

$$K(t, y; X_0, X_1) \leq K(t, x; X_0, X_1) \text{ holds for all } t > 0,$$

then $y \in X$ and $\|y\|_X \leq C\|x\|_X$ for some constant $C \geq 1$ independent of x and y. Here

$$K(t, x; X_0, X_1) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

is the classical K-functional of Peetre.

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A couple $\bar{X} = (X_0, X_1)$ is called *K-monotone* (or *Calderón-Mityagin couple*) if all interpolation spaces between X_0 and X_1 are K-monotone.

By a theorem due to Brudnyĭ and Krugljak [9, Theorem 4.4.5] all interpolation spaces with respect to a K-monotone Banach couple (X_0, X_1) can be represented in the form $X = (X_0, X_1)^K_\Phi$, where Φ is a Banach lattice of measurable functions on $(0, \infty)$ and

$$\|x\|_{(X_0,X_1)^K_\Phi} = \|K(\cdot, x; X_0, X_1)\|_\Phi.$$

Moreover, even if (X_0, X_1) is not K-monotone, every interpolation space X with respect to (X_0, X_1) which happens to be a K-monotone space satisfies $X = (X_0, X_1)^K_\Phi$ for some suitable Φ, and of course this is only up to equivalence of norms (Brudnyĭ and Krugljak [9, Theorem 3.3.20]). Therefore, the problem of finding new examples of K-monotone couples or K-monotone spaces becomes very important.

Calderón [10] and independently Mitjagin [26] proved that the couple (L_1, L_∞) is K-monotone. Several years later Sedaev and Semenov [33] proved that a weighted couple $(L_1(w_0), L_1(w_1))$ is K-monotone (cf. also Cwikel-Kozlov [13] for another proof) and then Sedaev [32] generalized this result to the couples of the form $(L_p(w_0), L_p(w_1))$ $(1 \leq p \leq \infty)$. Finally, Sparr [35], [36] showed that $(L_p(w_0), L_q(w_1))$ is a K-monotone couple for $0 < p, q \leq \infty$. There are other proofs of Sparr's result, for example, in papers of Dmitriev [17], Cwikel [11] and of Arazy-Cwikel [2].

In [15], Cwikel and Nilsson considered the problem of K-monotonicity from a somewhat different point of view. Namely, they studied the problem when a weighted Banach couple $(X(w_0), Y(w_1))$, with X, Y being separable Banach lattices with the Fatou property on a measure space (Ω, Σ, μ), is K-monotone for all weights w_0, w_1 on Ω. They proved that this can happen if and only if $X = L_p(v_0)$ and $Y = L_q(v_1)$ for some weights v_0, v_1 and some numbers $1 \leq p, q < \infty$. In their proof the concept of a decomposable Banach lattice on a measure space is essentially used. A Banach lattice X is called *decomposable* if for any convergent series $\sum_{n=1}^\infty f_n$ in X with pairwise disjoint f_n $(n = 1, 2, \dots)$ and any (formal) series $\sum_{n=1}^\infty g_n$, $g_n \in X$, $\|g_n\|_X \leq \|f_n\|_X$ $(n = 1, 2, \dots)$, such that all g_n are pairwise disjoint, we have $\sum_{n=1}^\infty g_n \in X$ and $\|\sum_{n=1}^\infty g_n\|_X \leq C \|\sum_{n=1}^\infty f_n\|_X$ with a constant C independent of f_n, g_n. This notion or some variants of it were introduced earlier by Cwikel [12] and Cwikel-Nilsson [14].

Note that the problem of K-monotonicity of weighted couples $(X(w_0), Y(w_1))$ can be reduced to considering couples of the form $(X, Y(w))$. Therefore, in what follows, we will examine couples with one weight only. We will say that a weight w is non-trivial if either w or $1/w$ is unbounded.

In [37], the concept of w-decomposability of a Banach lattice, which generalizes in a sense the previous one due to Cwikel, was introduced. A theorem proved in [37] states that, whenever X is a Banach lattice with the Fatou property, the couple $(X, X(w))$ is K-monotone if and only if X is w-decomposable (see Theorem 3.1 below in Section 3). Earlier Kalton [18] showed that in the case of symmetric sequence spaces with the Fatou property the K-monotonicity of a couple $(X, Y(w))$ for some non-trivial weight w implies that $X = l_p$ and $Y = l_q$ for some $1 \leq p, q \leq \infty$ (note, however, that there exist examples of shift-invariant sequence spaces X with the Fatou property, such that $(X, X(2^{-k}))$ is K-monotone but X is not isomorphic to l_p for any $1 \leq p \leq \infty$ [5], [6]).

Tikhomirov's theorem from [37] allows us to examine whether the result of Kalton extends to symmetric function spaces. We will see that this is not the case and the situation here will be essentially different.

The paper is organized as follows. After the introduction, in Section 2, some necessary definitions and notations are collected. In the first part, we recall necessary information about symmetric spaces on $[0,1]$ and then, in the second part, regularly varying convex Orlicz functions on $[0,\infty)$ and regularly varying quasi-concave functions on $[0,1]$ are discussed.

In Section 3 we consider the notion of a w-decomposable Banach lattice, which plays a central role in these investigations. Using the Krivine theorem we show that it can be essentially simplified in the case of symmetric function spaces. Namely, we prove condition (9) which means that for any w-decomposable symmetric space X there exists $p \in [1,\infty]$ (depending on X) such that X has, roughly speaking, both "restricted lower and upper p-estimates". In particular, its fundamental function φ satisfies condition (13) for some p, which means that the function φ^p is "almost additive" near zero.

Section 4 contains results on the w-decomposability of Lorentz and Marcinkiewicz spaces on $[0,1]$. If φ is a concave increasing function on $[0,1]$ with $\gamma_\varphi > 0$ and $1 \le p < \infty$, then the couple $(X, X(w))$ with $X = \Lambda_{p,\varphi}([0,1])$ and a given non-trivial weight w is K-monotone if and only if condition (10) holds. This couple is K-monotone for some weight w if and only if φ is equivalent to a regularly varying function at 0 of order p. Moreover, for any weight w on $[0,1]$ we can construct a concave function φ on $[0,1]$ such that the couple $(X, X(w))$ with $X = \Lambda_{1,\varphi}([0,1])$ is K-monotone and $\Lambda_{1,\varphi}([0,1]) \ne L_1[0,1]$.

We obtain analogous results for Marcinkiewicz spaces, as a consequence of a new duality theorem which is of independent interest. It states that under suitable mild conditions on a Banach lattice X, the weighted couple $(X, X(w))$ is K-monotone if and only if the couple $(X', X'(w))$ is K-monotone, where X' means the Köthe dual to X.

Section 5 deals with conditions of w-decomposability of Orlicz spaces $L_F[0,1]$. It is shown, in Theorem 6, that if an Orlicz function F satisfies the Δ_2-condition for large arguments, then $L_F[0,1]$ is w-decomposable if and only if it satisfies some restricted p-upper and p-lower estimates (see condition (32)). Moreover, it is proved, in Theorem 7, that if an Orlicz function F is equivalent to an Orlicz function which is regularly varying at ∞ of order $p \in [1,\infty)$, then the Orlicz space $L_F = L_F[0,1]$ is w-decomposable for some weight w on $[0,1]$ and therefore the couple $(L_F, L_F(w))$ is K-monotone.

Finally, in Section 6, we prove that if a symmetric space X on $[0,1]$ with non-trivial Boyd indices is w-decomposable with respect to a weight changing sufficiently fast, then X is an ultrasymmetric Orlicz space. The result implies that, for such a weight w, every K-monotone couple $(X, X(w))$ with X having the Fatou property must be an ultrasymmetric Orlicz space. Moreover, if its fundamental function is of the form $\varphi_X(t) = t^{1/p}$ for some $1 \le p \le \infty$, then $X = L_p$.

2 Preliminaries

Let us collect necessary information and results, in two parts, on symmetric (rearrangement invariant) spaces and regularly varying functions.

2a. Symmetric spaces. Let (Ω, Σ, μ) be a complete σ-finite measure space and $L^0 = L^0(\Omega)$ be the space of all classes of μ-measurable real-valued functions defined on Ω. A Banach space $X = (X, \|\cdot\|_X)$ is said to be a *Banach lattice* on Ω if X is a linear subspace of $L^0(\Omega)$ and satisfies the so-called ideal property, which means that if $y \in X, x \in L^0$ and $|x(t)| \leq |y(t)|$ for μ-almost all $t \in \Omega$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$. We also assume that the support of the space X is Ω (supp $X = \Omega$), that is, there is an element $x_0 \in X$ such that $x_0(t) > 0$ μ-a.e. on Ω.

We will say that X has the *Fatou property* if $0 \leq x_n \uparrow x \in L^0$ with $x_n \in X$ and $\sup_{n \in \mathbb{N}} \|x_n\|_X < \infty$ imply that $x \in X$ and $\|x_n\|_X \uparrow \|x\|_X$.

A Banach lattice X is said to be *p-convex* $(1 \leq p < \infty)$, respectively *q-concave* $(1 \leq q < \infty)$, if there is a constant $C > 0$ such that

$$\|(\sum_{k=1}^n |x_k|^p)^{1/p}\|_X \leq C(\sum_{k=1}^n \|x_k\|_X^p)^{1/p},$$

respectively,

$$(\sum_{k=1}^n \|x_k\|_X^q)^{1/q} \leq C\|(\sum_{k=1}^n |x_k|^q)^{1/q}\|_X,$$

for any choice of vectors $x_1, x_2, \ldots, x_n$ in X and any $n \in \mathbb{N}$. If in the above definitions vectors $x_1, x_2, \ldots, x_n \in X$ are pairwise disjoint, then X is said to satisfy an upper p-estimate and lower q-estimate, respectively. Of course, p-convexity implies upper p-estimate and q-concavity implies lower q-estimate of a Banach lattice X. More properties can be found in the book [22].

Let w be a weight on (Ω, Σ, μ), i.e., positive finite a.e. function, and let X be a Banach lattice on (Ω, Σ, μ). Then the weighted space $X(w)$ on (Ω, Σ, μ) is defined by $X(w) = \{x \in \Omega : xw \in X\}$ with the norm $\|x\|_{X(w)} = \|xw\|_X$. In what follows, we will always suppose that the weight w is *non-trivial*, that is, w or $1/w$ is an unbounded function on (Ω, Σ, μ).

For two Banach spaces E and F the symbol $E \overset{C}{\hookrightarrow} F$ means that the embedding $E \subset F$ is continuous with the norm which is not bigger than C, i.e., $\|x\|_F \leq C\|x\|_E$ for all $x \in E$.

By a *symmetric space* (symmetric Banach function space), we mean a Banach lattice $X = (X, \|\cdot\|_X)$ on $I = [0, 1]$ with the Lebesgue measure m satisfying the following additional property: for any two equimeasurable functions $x, y \in L^0(I)$ (that is, they have the same distribution functions $d_x(\lambda) = d_y(\lambda)$, where $d_x(\lambda) = m(\{t \in I : |x(t)| > \lambda\}), \lambda \geq 0$) the condition $x \in X$ implies that $y \in X$ and $\|x\|_X = \|y\|_X$. In particular, $\|x\|_X = \|x^*\|_X$, where $x^*(t) = \inf\{\lambda > 0 \colon\ d_x(\lambda) \leq t\},\ t \geq 0$.

Recall that a non-negative function $\varphi : [0, 1] \to [0, \infty)$ is called *quasi-concave* if it is non-decreasing on $[0, 1]$ with $\varphi(0) = 0$ and if $\frac{\varphi(t)}{t}$ is non-increasing on $(0, 1]$. The *fundamental function* φ_X of a symmetric space X on I is defined by the formula $\varphi_X(t) = \|\chi_{[0,\, t]}\|_X, t \in I$. It is well known that every fundamental function is quasi-concave on I. Taking $\tilde{\varphi}_X(t) := \inf_{s \in (0,1)}(1 + \frac{t}{s})\varphi_X(s)$ we obtain a concave function $\tilde{\varphi}_X$ satisfying $\varphi_X(t) \leq \tilde{\varphi}_X(t) \leq 2\varphi_X(t)$ for all $t \in I$. For any quasi-concave function φ on I the

Marcinkiewicz space M_φ is defined by the norm

$$\|x\|_{M_\varphi} = \sup_{t\in I, t>0} \varphi(t)x^{**}(t), \quad x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)ds.$$

This is a symmetric space on I with the fundamental function $\varphi_{M_\varphi}(t) = \varphi(t)$ and $X \overset{1}{\hookrightarrow} M_{\varphi_X}$. The fundamental function of a symmetric space $X = (X, \|\cdot\|_X)$ is not necessarily concave but we can introduce an equivalent norm on X in such a way that the fundamental function will be concave (take $\|x\|_X^1 = \max(\|x\|_X, \|x\|_{M_{\tilde{\varphi}_X}}),\ x \in X)$.

For any symmetric function space X with a concave fundamental function $\varphi = \varphi_X$ there is also the smallest symmetric space with the same fundamental function. This space is the *Lorentz space* given by the norm

$$\|x\|_{\Lambda_\varphi} = \int_I x^*(t)d\varphi(t) := \varphi(0^+)\|x\|_{L_\infty(I)} + \int_I x^*(t)\varphi'(t)dt.$$

We have then embeddings $\Lambda_{\varphi_X} \overset{1}{\hookrightarrow} X \overset{1}{\hookrightarrow} M_{\varphi_X}$. A non-trivial symmetric function space X on $I = [0,1]$ is an intermediate space between the spaces $L_1(I)$ and $L_\infty(I)$ and $L_\infty(I) \overset{C_1}{\hookrightarrow} X \overset{C_2}{\hookrightarrow} L_1(I)$, where $C_1 = \varphi_X(1), C_2 = 1/\varphi_X(1)$ (see [7], Corollary 6.7 on page 78 or Theorem 4.1 on page 91 of [21] for a similar result when the underlying measure space is $(0,\infty)$.)

The lower and upper *Boyd indices* α_X resp. β_X and the *dilation indices* γ_X resp. δ_X of a symmetric space X on $I = [0,1]$ with the fundamental function $\varphi_X = \varphi$ are defined as follows:

$$\alpha_X := \lim_{t\to 0^+} \frac{\ln \|\sigma_t\|_{X\to X}}{\ln t}, \ \beta_X := \lim_{t\to\infty} \frac{\ln \|\sigma_t\|_{X\to X}}{\ln t}, \ \sigma_t x(s) = x(s/t)\chi_I(s/t)$$

and

$$\gamma_X := \gamma_\varphi = \lim_{t\to 0^+} \frac{\ln \bar{\varphi}(t)}{\ln t}, \ \delta_X := \delta_\varphi = \lim_{t\to\infty} \frac{\ln \bar{\varphi}(t)}{\ln t}, \ \bar{\varphi}(t) = \sup_{s,st\in I} \frac{\varphi(st)}{\varphi(s)}.$$

We have the relations $0 \le \alpha_X \le \gamma_X \le \delta_X \le \beta_X \le 1$ (see [21], pp. 101-102 and [24], p. 28).

A function $F : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is convex and increasing with $F(0) = 0$. For a given Orlicz function F the *Orlicz space* $L_F = L_F(I)$ on $I = [0,1]$ is defined as

$$L_F(I) = \{x \in L^0(I) : I_F(cx) < \infty \text{ for some } c = c(x) > 0\},$$

where $I_F(x) := \int_I F(|x(t)|)dt$. The Orlicz space L_F is a symmetric space on I with the so-called *Luxemburg-Nakano norm* defined by

$$\|x\|_{L_F} = \inf\{\lambda > 0 : I_F(x/\lambda) \le 1\}.$$

An Orlicz function F satisfies the Δ_2-*condition for large* u if there exist constants $C \ge 1, u_0 \ge 0$ such that $F(2u) \le CF(u)$ for all $u \ge u_0$.

The following notation will be used throughout the text: $f \overset{C}{\approx} g$ means that the functions f and g are equivalent with the constant $C > 0$, that is, $C^{-1} f(t) \le g(t) \le C f(t)$ for all points t of the whole set on which these functions are defined, or at all points of some explicitly designated subset of that set. In the case when the constant of equivalence is not important we will write just $f \approx g$. By $[r]$ we will denote the integer part of a real number r.

More information about Banach lattices and symmetric spaces can be found, for example, in [7], [21] and [22]; about Orlicz spaces one can read e.g. in [20] and [25].

2b. Regularly varying convex and concave functions. An Orlicz function F on $[0, \infty)$ is called *regularly varying at* ∞ *of order* p $(1 \le p < \infty)$ if

$$\lim_{t \to \infty} \frac{F(tu)}{F(t)} = u^p \text{ for all } u > 0. \tag{1}$$

The following result is due to Kalton [18, Lemma 6.1].

Lemma 2.1. *Let F be an Orlicz function. The following three conditions are equivalent:*

(a) F is equivalent to a regularly varying Orlicz function at ∞ of order $p \in [1, \infty)$.

(b) There exists a constant $C > 0$ such that for any $u \in (0, 1]$ we can find $t_0 = t_0(u)$ with

$$\frac{F(tu)}{F(t)} \overset{C}{\approx} u^p \quad \textit{for all} \quad t \ge t_0.$$

Although we do not need it here, there is an analogous definition to the one above for Orlicz functions which are regularly varying of order p at 0 instead of at ∞ (see e.g. [18]). However, we do need to consider quasi-concave functions which are regularly varying of order p at 0. Before recalling the definition of these we should point out that it is not quite analogous to the definitions for regularly varying Orlicz functions, because the power p which appeared in (1) and in the corresponding definition in [18] will be replaced in (2) by the power $1/p$.

A function $\varphi : [0, 1] \to [0, \infty)$ which is quasi-concave and satisfies $\varphi(0) = 0$ is said to be *regularly varying at zero of order* p $(1 \le p \le \infty)$ if

$$\lim_{t \to 0^+} \frac{\varphi(tu)}{\varphi(t)} = u^{1/p} \text{ for all } u > 0. \tag{2}$$

Abakumov and Mekler [1, Theorem 5] proved that a quasi-concave function φ is equivalent to a quasi-concave regularly varying function at zero of order $p \in [1, \infty]$ if and only if

$$\limsup_{t \to 0^+} \frac{\varphi(tu)}{\varphi(t)} \approx u^{1/p} \text{ for all } u > 0.$$

The following lemma is an immediate consequence of this result (see also the proof of Theorem 5 in [1]).

Lemma 2.2. *A quasi-concave function φ on $[0,1]$ is equivalent to a quasi-concave function which is regularly varying at zero of order $p \in [1,\infty]$ if and only if for some $C > 0$ and any $N \in \mathbb{N}$ there exists $\tau(N) \in (0,1]$ such that for all $0 < t \le \tau(N), 0 < tN \le 1$ we have*

$$\frac{\varphi(Nt)}{\varphi(t)} \overset{C}{\approx} N^{1/p}. \tag{3}$$

Recall that the fundamental function of an Orlicz space L_F on $[0,1]$ with the Luxemburg-Nakano norm is $\varphi_{L_F}(t) = \frac{1}{F^{-1}(1/t)}$ for $0 < t \le 1$ and $\varphi_{L_F}(0) = 0$, where F^{-1} is the inverse of F (see formula (9.23) in [20] on page 79 of the English version or Corollary 5 in [25] on page 58). The function φ_{L_F} is quasi-concave but not necessarily concave on $[0,1]$ (see [20] or [25]).

The notions of regularly varying Orlicz and quasi-concave functions are closely interrelated. Using Lemmas 2.1 and 2.2 and routine arguments we establish the following quantitative result showing a connection between an regularly varying Orlicz function F and the fundamental function of the corresponding Orlicz space L_F.

Proposition 2.3. *Suppose that $p \in [1,\infty)$ and let F be an Orlicz function such that both F and its complementary function F^* satisfy the Δ_2-condition for large u. Then the following conditions are equivalent:*

(a) *There exists a constant $C' > 0$ such that for any $N \in \mathbb{N}$ there exists $\tau(N) \in (0,1]$ with*
$$\frac{F(u)}{F(uN^{-1/p})} \overset{C'}{\approx} N \text{ for all } u \ge F^{-1}(1/\tau(N)). \tag{4}$$

(b) *There exists a constant $C > 0$ such that for any $N \in \mathbb{N}$ the fundamental function φ_{L_F} satisfies condition* (3) *with the same $\tau(N)$.*

3 w–decomposable Banach lattices

Later on C will denote a constant whose value may be different in its different appearances.

The following notion was introduced in paper [37] and it will be very important for us. Let X be a Banach lattice on (Ω, Σ, μ) and w be a weight on Ω. We say that X is *w-decomposable* if there exists $C > 0$ such that for any $n \in \mathbb{N}$ and for all $x_1, \dots, x_n$, $y_1, \dots, y_n$ in X satisfying the conditions:

$$\|x_i\|_X = \|y_i\|_X, \; i = 1, 2, \dots, n, \tag{5}$$

and

$$\inf w(\operatorname{supp} x_i \cup \operatorname{supp} y_i) \ge 2 \sup w(\operatorname{supp} x_{i+1} \cup \operatorname{supp} y_{i+1}), \; i = 1, 2, \dots, n-1, \tag{6}$$

we have that

$$\|\sum_{i=1}^n x_i\|_X \overset{C}{\approx} \|\sum_{i=1}^n y_i\|_X. \tag{7}$$

To clarify the meaning of condition (6), consider the following example: let X be a Banach lattice of Lebesgue measurable functions on $[0,1]$ and $w(t) = 1/t$ $(0 < t \le 1)$. Then (6) is equivalent to the following inequality

$$2 \sup(\operatorname{supp} x_i \cup \operatorname{supp} y_i) \le \inf(\operatorname{supp} x_{i+1} \cup \operatorname{supp} y_{i+1}),\ i = 1, 2, \ldots, n-1.$$

In other words, there are some intervals $[a_i, b_i] \subset [0,1]$ (depending on x_i, y_i) such that $2b_i \le a_{i+1}$ $(i = 1, 2, \ldots, n-1)$, $\operatorname{supp} x_i \subset [a_i, b_i]$ and $\operatorname{supp} y_i \subset [a_i, b_i]$ $(i = 1, 2, \ldots, n)$.

It is not hard to see that $1/t$-decomposability is equivalent to $1/t^q$-decomposability and, more generally, w-decomposability and w^q-decomposability are equivalent for any weight w and any $q > 0$ (see [38], Corollary 2.2 on page 61).

It turns out that the w-decomposability of a Banach lattice X guarantees the K–monotonicity of the weighted couple $(X, X(w))$. More precisely, Tikhomirov in [37] obtained the following generalization of Kalton's results from [18].

Theorem 3.1. *Suppose X is a Banach lattice on a σ–finite measure space (Ω, Σ, μ) with $\operatorname{supp} X = \Omega$ which has the Fatou property and w is a (non-trivial) weight on Ω. Then the Banach couple $(X, X(w))$ is K–monotone if and only if X is w-decomposable.*

In the case of symmetric spaces on $[0,1]$ the notion of w-decomposability can be clarified by using the well–known Krivine theorem.

Proposition 3.2. *Let w be a weight on $[0,1]$. A symmetric space X on $[0,1]$ is w-decomposable if and only if there exist $C > 0$ and $1 \le p \le \infty$ such that for any $n \in \mathbb{N}$ and for all $x_1, x_2, \ldots, x_n \in X$ satisfying the conditions*

$$\inf w(\operatorname{supp} x_i) \ge 2 \sup w(\operatorname{supp} x_{i+1}),\ 1 \le i \le n-1, \tag{8}$$

we have that

$$\Big\| \sum_{i=1}^n x_i \Big\|_X \overset{C}{\approx} \left(\sum_{i=1}^n \|x_i\|_X^p \right)^{1/p}, \tag{9}$$

where, as usual, in the case $p = \infty$ the right hand side should be replaced by $\max_{1 \le i \le n} \|x_i\|_X$.

Proof. By Krivine's theorem (see [22, Theorem 2.b.6] or [31]), there exists $p \in [1/\beta_X, 1/\alpha_X]$ such that for every $m \in \mathbb{N}$ there are pairwise disjoint equimeasurable functions $y_1, y_2, \ldots, y_m \in X$, $\|y_k\|_X = 1$ $(k = 1, 2, \ldots, m)$, such that for any $\alpha_k \in \mathbb{R}$ $(k = 1, 2, \ldots, m)$ we have

$$\frac{1}{2} \|(\alpha_k)\|_p \le \Big\| \sum_{k=1}^m \alpha_k y_k \Big\|_X \le 2 \|(\alpha_k)\|_p. \tag{10}$$

Obviously, the support of each function y_k has measure not greater than $1/m$.

Suppose that a symmetric space X is w-decomposable and that, for some $n \in \mathbb{N}$, functions $x_1, \ldots, x_n$ in X satisfy condition (8). Without loss of generality we may assume that $x_i \ne 0$ for each $i = 1, 2, \ldots, n$. We choose $m \in \mathbb{N}$ sufficiently large so that the support of each x_i has measure greater than $1/m$ (and so of course we also have $m \ge n$). For this choice of m we consider the disjoint measurable functions $y_1, y_2, \ldots, y_m$, $\|y_k\|_X = 1$ $(k = 1, 2, \ldots, m)$, obtained as it is described in the previous paragraph. In fact, we will only need the first n of these functions, and we will only need special case of (10) for sequences (α_k) which satisfy $\alpha_k = 0$ for $k > n$. We may assume without loss of generality, that the support of y_i is contained in the support of x_i for each $i = 1, 2, \ldots, n$. (If not, since X is symmetric, we can simply replace each y_i by an equimeasurable function which has this property and the above mentioned special case of (10) will remain valid.) Thus condition (8) implies that condition (6) is satisfied and therefore, applying w-decomposability (see (7)) and then the special case of (10), we obtain that

$$\Big\| \sum_{i=1}^{n} \alpha_i \frac{x_i}{\|x_i\|_X} \Big\|_X \approx \Big\| \sum_{i=1}^{n} \alpha_i y_i \Big\|_X \approx \|(\alpha_k)_{i=1}^{n}\|_p$$

for all choices of real numbers α_i. In particular, when $\alpha_i = \|x_i\|_X$ we obtain (9). Since the reverse implication is obvious, the proof is complete. $\square$

For a given weight w consider the sets

$$M_k := \{t \in [0,1] : w(t) \in [2^k, 2^{k+1})\}, k \in \mathbb{Z}.$$

Let $(w_r)_{r=1}^{\infty}$ be the non-increasing rearrangement of the sequence $(m(M_k))_{k=-\infty}^{+\infty}$. Since the weight w is non-trivial it follows that $w_r > 0$ for all $r = 1, 2, \ldots$.

For some fixed $n \in \mathbb{N}$, let $x_1, x_2, \ldots, x_n$ be functions in X. Suppose first that these n functions satisfy condition (8). Then it is easy to see that

$$\mathrm{card}\{i : M_k \cap \mathrm{supp}\, x_i \neq \emptyset\} \leq 1 \quad \text{for each} \quad k \in \mathbb{Z}.$$

Alternatively, more or less conversely, suppose that the functions x_i satisfy

$$\mathrm{card}\{k : M_k \cap \mathrm{supp}\, x_i \neq \emptyset\} \leq 1 \quad \text{for each} \quad i \in \{1, 2, \ldots, n\},$$

i.e., for each i, there exists a unique $k_i \in \mathbb{Z}$ for which $\mathrm{supp}\, x_i \subset M_{k_i}$. Furthermore, suppose $k_1 < k_2 < \ldots < k_n$. While this is not sufficient to imply that the collection of functions $x_1, x_2, \ldots, x_n$ satisfies condition (8), it does imply that (after relabelling) the collection of functions $x_1, x_3, x_5, \ldots$ satisfies (8) and so does the collection $x_2, x_4, \ldots$.

By $\{\overline{M}_r\}_{r=1}^{\infty}$ we will denote any rearrangement of the sets M_k $(k = 0, \pm 1, \pm 2, \ldots)$ such that $m(\overline{M}_r) = w_r, r = 1, 2, \ldots$. Thus, by Proposition 3.2, we obtain the following result.

Theorem 3.3. *Suppose w is a non-trivial weight on $[0,1]$. A symmetric space X on $[0,1]$ is w-decomposable if and only if there exist $C > 0$ and $1 \leq p \leq \infty$ such that for any $n \in \mathbb{N}$ and for all $x_1, x_2, \ldots, x_n \in X$ satisfying the condition*

$$\mathrm{supp}\, x_i \subset \overline{M}_i, \quad 1 \leq i \leq n, \tag{11}$$

we have (9).

Next, we will need some corollaries of Theorem 3.3. Firstly, using the symmetry of the norm in X, we get

Corollary 3.4. *Let w be a non-trivial weight on $[0,1]$. A symmetric space X on $[0,1]$ is w-decomposable if and only if there exist $C > 0$ and $1 \le p \le \infty$ such that for any $n \in \mathbb{N}$ and for all pairwise disjoint $x_1, x_2, \dots, x_n \in X$ satisfying the condition*

$$m(\operatorname{supp} x_i) \le w_i, \quad 1 \le i \le n, \tag{12}$$

we have (9).

Corollary 3.5. *A symmetric space X on $[0,1]$ is w-decomposable for some non-trivial weight w on $[0,1]$ if and only if there exist $C > 0$, $1 \le p \le \infty$, and a sequence of disjoint intervals $\{\Delta_k\}_{k=1}^{\infty}$ from $[0,1]$ such that for any $n \in \mathbb{N}$ and for all $x_1, x_2, \dots, x_n \in X$ satisfying the condition $\operatorname{supp} x_i \subset \Delta_i$ $(1 \le i \le n)$ we have* (9).

Corollary 3.6. *Let w be a non-trivial weight on $[0,1]$ and let the sequence $(w_r)_{r=1}^{\infty}$ be as above. Suppose that X is a w-decomposable symmetric space X on $[0,1]$ with the fundamental function φ. Then there exist some $C > 0$ and $p \in [1,\infty]$ such that, for every sequence of reals $(\tau_r)_{r=1}^{\infty}$ satisfying $0 < \tau_r \le w_r$ $(r \in \mathbb{N})$, we have*

$$\varphi\Big(\sum_{r=1}^{\infty} \tau_r\Big) \overset{C}{\approx} \left(\sum_{r=1}^{\infty} \varphi^p(\tau_r)\right)^{1/p} \tag{13}$$

with the natural modification for $p = \infty$.

Corollary 3.7. *Let w be a non-trivial weight on $[0,1]$ such that a symmetric space X on $[0,1]$ is w-decomposable. Then there exist $C > 0$ and $1 \le p \le \infty$ such that condition* (3) *is fulfilled with $\tau(N) = w_N$ $(N \in \mathbb{N})$. In particular, the fundamental function φ of X is equivalent to a regularly varying function at zero of order p and $\alpha_X = \gamma_\varphi = \delta_\varphi = \beta_X = 1/p$.*

Proof. First we note that condition (3) is an immediate consequence of (13). Moreover, it is well known that the assertion of Krivine's theorem holds for both $p = 1/\alpha_X$ and $p = 1/\beta_X$ (see [22, p. 141], [31] and [3]). Therefore, coincidence of the Boyd indices and dilation indices follows from an inspection of the proof of Proposition 3.2 and the inequalities $\alpha_X \le \gamma_\varphi \le \delta_\varphi \le \beta_X$ (cf. [21, p. 102] and [24, p. 28]). ☐

Let us show that, conversely, (13) can be derived from (3) with $\tau(N) = w_N$ for a large class of weights w.

Theorem 3.8. *Let w be a weight on $[0,1]$ such that $qw_{r+1} \le w_r$ $(r = 1, 2, \dots)$ for some $q > 1$ and let φ be a quasi–concave function on $[0,1]$. Suppose there exist $C > 0$ and $1 \le p \le \infty$ such that φ satisfies* (3) *with $\tau(N) = w_N$ $(N = 1, 2, \dots)$. Then, for any sequence of reals $(\tau_r)_{r=1}^{\infty}$ such that $0 < \tau_r \le w_r$ $(r = 1, 2, \dots)$, estimate* (13) *holds.*

Proof. We present the proof for $1 \le p < \infty$ since the case $p = \infty$ needs only minor changes.

Firstly, it is easy to see that condition (3) can be extended as follows: we can find a (possibly different) constant $C > 0$ such that for every real $z \ge 1$ and $\tau(z) := \tau([z])$ we have

$$\frac{\varphi(zt)}{\varphi(t)} \overset{C}{\approx} z^{1/p} \ \text{ if } \ 0 < t \le \tau(z). \tag{14}$$

Let us show that for every $m \in \mathbb{N}$ there is a constant $C(m) > 0$ such that for all even $N \in \mathbb{N}$ satisfying the inequality $N^m \le q^{N/2}$ and all $z \in [1, N]$ we have

$$\frac{\varphi(z^m t)}{\varphi(t)} \overset{C(m)}{\approx} z^{m/p} \ \text{ if } \ 0 < t \le \tau(N). \tag{15}$$

In fact, by the assumption, $\tau(N/2) \ge q^{N/2}\tau(N)$, whence

$$z^k t \le z^m t \le N^m \tau(N) \le q^{N/2}\tau(N) \le \tau(N/2) \le 1 \ (k = 0, 1, \dots, m)$$

provided that $t \le \tau(N)$. Therefore, using the quasi–concavity of φ and equivalence (14) for $\max(1, z/2)$ we obtain that

$$\frac{\varphi(z^k t)}{\varphi(z^{k-1}t)} \approx \frac{\varphi(\max(1, z/2) z^{k-1} t)}{\varphi(z^{k-1}t)} \approx z^{1/p} \ \text{ if } \ 0 < t \le \tau(N),$$

with a constant of equivalence depending on p. Multiplying these relations for all $k = 1, 2, \dots, m$, we come to (15).

Next, let

$$\overline{\varphi}^0(s) = \limsup_{t\to 0^+} \frac{\varphi(ts)}{\varphi(t)} \ \text{for} \ \ s > 0.$$

Clearly, condition (14) implies $\overline{\varphi}^0(s) \approx s^{1/p} (s > 0)$. On the other hand, in view of Boyd's result [8] (see also [24, Theorem 2.2]) $\overline{\varphi}^0(s) \ge s^{\gamma_\varphi}$ if $0 < s \le 1$ and $\overline{\varphi}^0(s) \ge s^{\delta_\varphi}$ if $s > 1$. Since $\gamma_\varphi \le \delta_\varphi$ it follows that $\gamma_\varphi = \delta_\varphi = \frac{1}{p} > 0$. Therefore, there exist $A > 0$ and $\kappa > 0$ such that

$$\sup_{0<s\le 1} \frac{\varphi(st)}{\varphi(s)} \le A t^\kappa \ \text{ for all } \ 0 \le t \le 1. \tag{16}$$

Let us prove that (13) is a consequence of (15) and (16). Take a natural number $m_0 \ge 2$ such that $\kappa m_0 > 1$ and consider an arbitrary sequence $(\tau_r)_{r=1}^\infty$ satisfying $\tau_r \le w_r, r = 1, 2, \dots$. Since the non-increasing rearrangement $(\tau_r^*)_{r=1}^\infty$ of this sequence also satisfies $\tau_r^* \le w_r$ for $r = 1, 2, \dots$ we can assume without loss of generality that the sequence $(\tau_r)_{r=1}^\infty$ is itself non-increasing. Further, set $I = \{r \in \mathbb{N} : \tau_r r^{m_0} \ge \tau_1\}$, $J = \mathbb{N} \setminus I$. Clearly, $1 \in I$. By (16) and the choice of m_0,

$$\varphi\Big(\sum_{r\in J} \tau_r\Big) \le \varphi\Big(\sum_{r=2}^\infty \frac{\tau_1}{r^{m_0}}\Big) \le A\Big(\sum_{r=2}^\infty r^{-m_0}\Big)^\kappa \varphi(\tau_1) \le C_1 \varphi(\tau_1).$$

Analogously,

$$\sum_{r\in J} \varphi^p(\tau_r) \le \sum_{r=2}^\infty \varphi^p(\tau_1/r^{m_0}) \le A^p \sum_{r=2}^\infty r^{-p\kappa m_0} \varphi^p(\tau_1) \le C_2 \varphi^p(\tau_1).$$

Thus, it is sufficient to prove equivalence (13) for $(\tau_r)_{r\in I}$.

If $\operatorname{card} I < \infty$ then there is nothing to prove. So, assume that $\operatorname{card} I = \infty$. Choose a positive integer $i_0 \in I$, $i_0 \ge 2$ such that for $N = 2\,[i_0/2]$ we have $N^{m_0} \le q^{N/2}$. Denote $\delta_r = (\tau_r/\tau_{i_0})^{1/m_0}$ for $r \in I \cap \{1, 2, \dots, i_0\}$. Then, by the definition of I, $\delta_r \le (\tau_1/\tau_{i_0})^{1/m_0} \le$

$i_0 \le 2N$. Applying (15) in the case $m = m_0, z = \max(1, \delta_r/2)$ for all $r \in I, r \le i_0$, we get

$$\frac{\varphi(\tau_r)}{\varphi(\tau_{i_0})} = \frac{\varphi(\delta_r^{m_0}\tau_{i_0})}{\varphi(\tau_{i_0})} \approx \delta_r^{m_0/p} = \left(\frac{\tau_r}{\tau_{i_0}}\right)^{1/p},$$

with a constant of equivalence depending on m_0 and p. The last formula implies that

$$\sum_{r\in I\cap\{1,2,\ldots,i_0\}} \varphi^p(\tau_r) \approx \frac{\varphi^p(\tau_{i_0})}{\tau_{i_0}} \sum_{r\in I\cap\{1,2,\ldots,i_0\}} \tau_r.$$

On the other hand, setting $\delta := \left(\sum_{r\in I\cap\{1,2,\ldots,i_0\}} \tau_r/\tau_{i_0}\right)^{1/(m_0+1)}$ we get

$$\delta \le \Big(\sum_{r\in I\cap\{1,2,\ldots,i_0\}} \tau_1/\tau_{i_0}\Big)^{1/(m_0+1)} \le i_0.$$

Therefore, again by (15), we obtain

$$\frac{\varphi^p(\tau_{i_0})}{\tau_{i_0}} \sum_{r\in I\cap\{1,2,\ldots,i_0\}} \tau_r = \delta^{m_0+1}\varphi^p(\tau_{i_0}) \approx \varphi^p(\delta^{m_0+1}\tau_{i_0}) = \varphi^p\Big(\sum_{r\in I\cap\{1,2,\ldots,i_0\}} \tau_r\Big),$$

with a constant depending on m_0 and p. Combining the above formulas and noting that i_0 can be arbitrarily large, we conclude that equivalence (13) holds and the proof is complete. $\square$

Theorem 3.8 allows us to construct non-trivial quasi–concave functions satisfying condition (13), for a large class of weights. For example, let $w(t) = 1/t$ $(0 < t \le 1)$. In this case $w_r = 2^{-r}$, $r = 1, 2, \ldots$ Define $\varphi(t) = t \log\frac{e}{t}$ $(0 < t \le 1)$. Obviously, φ is quasi–concave. Elementary calculations show that (3) is fulfilled for φ with $p = 1$ and $\tau(N) = w_N = 2^{-N}$ $(N = 1, 2, \ldots)$. Thus, by Theorem 3.8, φ satisfies (13).

4 w–decomposable Lorentz and Marcinkiewicz spaces

For $1 \le p < \infty$ and any increasing concave function φ, $\varphi(0) = 0$, the Lorentz space $\Lambda_{p,\varphi}$ consists of all classes of measurable functions x on $[0, 1]$ such that

$$\|x\|_{\Lambda_{p,\varphi}} = \left(\int_0^1 [x^*(t)\varphi(t)]^p \,\frac{dt}{t}\right)^{1/p} < \infty.$$

The space $\Lambda_{p,\varphi}$ was investigated by Sharpley [34] and Raynaud [30], who proved that if $0 < \gamma_\varphi \le \delta_\varphi < 1$, then $\Lambda_{p,\varphi}$ is a symmetric space on $[0, 1]$ with an equivalent norm

$$\|x\|^\star_{\Lambda_{p,\varphi}} = \left(\int_0^1 [x^{**}(t)\varphi(t)]^p \,\frac{dt}{t}\right)^{1/p},$$

where $x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)\,ds$ (cf. [34], Lemma 3.1). Moreover, if $\gamma_\varphi > 0$, then applying Corollary 3 on page 57 of [21] to the function $\psi = \varphi^p$ $(1 \le p < \infty)$ (see also [24, Theorem 6.4(a)]), we obtain that there exists a constant $K = K(p) \ge 1$ such that

$$K^{-1}\varphi^p(t) \le \int_0^t \frac{\varphi^p(s)}{s}\,ds \le K\varphi^p(t) \ \ (0 < t \le 1). \tag{17}$$

Therefore, the fundamental function $\varphi_{\Lambda_{p,\varphi}}(t)$ is equivalent to $\varphi(t)$. Inequalities (17) imply also that, if $\gamma_\varphi > 0$, then the space $\Lambda_{1,\varphi}$ coincides with the Lorentz space Λ_φ with the norm

$$\|x\|_{\Lambda_\varphi} := \int_0^1 x^*(t)d\varphi(t).$$

Recall also that the Köthe dual of the Lorentz space Λ_φ is isometric to the Marcinkiewicz space $M_{\tilde{\varphi}}$ with $\tilde{\varphi}(t) = \frac{t}{\varphi(t)}$ and its norm is

$$\|x\|_{M_{\tilde{\varphi}}} = \sup_{0<t\le 1} \tilde{\varphi}(t)x^{**}(t) = \sup_{0<t\le 1} \frac{1}{\varphi(t)} \int_0^t x^*(s)ds$$

(cf. [21], Theorem 5.2 on page 112).

We will prove that condition (13) is necessary and sufficient for Lorentz and Marcinkiewicz spaces to be w-decomposable. We start by proving a specific geometric property of Lorentz spaces.

Proposition 4.1. *Let φ be an increasing non-negative concave function on $[0,1]$ such that $\gamma_\varphi > 0$, and let $1 \le p < \infty$. Then for arbitrary $b > 1$ there exists a constant $C = C(b,\varphi,p) > 0$ with the following property: for any two-sided non-decreasing sequence $(a_j)_{j=-\infty}^{+\infty}$ of reals from $[0,1]$ such that the function $x = \sum_{j=-\infty}^{+\infty} b^{-j}\chi_{(a_{j-1},a_j]}$ belongs to $\Lambda_{p,\varphi}$, we have*

$$\|x\|_{\Lambda_{p,\varphi}}^p \overset{C}{\approx} \sum_{j=-\infty}^{+\infty} b^{-pj}\varphi^p(a_j - a_{j-1}). \tag{18}$$

Proof. Since $\gamma_\varphi > 0$, there exist $\kappa > 0$ and $A > 0$ such that inequality (16) holds. Choose a constant $C_1 = C_1(\varphi) > 1$ satisfying the inequality

$$\frac{(C_1+1)^\kappa}{A} \ge 2\,K^2, \tag{19}$$

where K is the constant from (17), and denote by I the set of all indices $j \in \mathbb{Z}$ such that $a_j - a_{j-1} \ge C_1\, a_{j-1}$. We prove the following equivalences:

$$\int_{a_{j-1}}^{a_j} \frac{\varphi^p(t)}{t}\,dt \approx \varphi^p(a_j - a_{j-1}),\ j \in I \tag{20}$$

and, if $b > C_1 + 1$,

$$\|x\|_{\Lambda_{p,\varphi}}^p \approx \sum_{j\in I} b^{-pj} \int_{a_{j-1}}^{a_j} \frac{\varphi^p(t)}{t}\,dt, \tag{21}$$

$$\sum_{j=-\infty}^{+\infty} b^{-pj}\varphi^p(a_j - a_{j-1}) \approx \sum_{j\in I} b^{-pj}\varphi^p(a_j - a_{j-1}), \tag{22}$$

with constants which depend only on b, φ and p.

At first, if $j \in I$ then, by (16) and (19),

$$\varphi(a_j) \ge \varphi((C_1+1)a_{j-1}) \ge \frac{(C_1+1)^\kappa}{A}\varphi(a_{j-1}) \ge 2K^2\,\varphi(a_{j-1}).$$

Combining this with (17) and the inequality

$$\varphi(a_j) \le \varphi(a_j - a_{j-1}) + \varphi(a_{j-1}) \le 2\,\varphi(a_j - a_{j-1}), \tag{23}$$

we obtain

$$\begin{aligned}
\frac{1}{2K}\,\varphi^p(a_j &- a_{j-1}) \le \frac{1}{2K}\,\varphi^p(a_j) \le \frac{1}{2K}[2\,\varphi^p(a_j) - (2K^2)^p\,\varphi^p(a_{j-1})] \\
&\le \frac{1}{2K}[2\,\varphi^p(a_j) - 2K^2\,\varphi^p(a_{j-1})] = \frac{\varphi^p(a_j)}{K} - K\,\varphi^p(a_{j-1}) \\
&\le \int_0^{a_j} \frac{\varphi^p(t)}{t}\,dt - \int_0^{a_{j-1}} \frac{\varphi^p(t)}{t}\,dt = \int_{a_{j-1}}^{a_j} \frac{\varphi^p(t)}{t}\,dt \\
&\le \int_0^{a_j} \frac{\varphi^p(t)}{t}\,dt \le K\varphi^p(a_j) \le 2^p K\varphi^p(a_j - a_{j-1}),
\end{aligned}$$

which implies (20).

Now, assuming $b > C_1 + 1$, we show that the set I is unbounded from below. In fact, otherwise there is $j_0 \in \mathbb{Z}$ such that $a_j - a_{j-1} < C_1 a_{j-1}$ for all $j \le j_0$. Then, we have $a_{j_0} \le (C_1+1)^{j_0-j}\,a_j$ ($j \le j_0$) and by (17) and the concavity of φ,

$$\begin{aligned}
\|x\|^p_{\Lambda_{p,\varphi}} &\ge \sup_{j\le j_0} b^{-pj}\int_0^{a_j} \frac{\varphi^p(t)}{t}\,dt \ge \frac{1}{K}\sup_{j\le j_0} b^{-pj}\varphi^p(a_j) \\
&\ge \frac{1}{K}\sup_{j\le j_0} \frac{(C_1+1)^{p(j-j_0)}}{b^{pj}}\,\varphi^p(a_{j_0}) = \infty.
\end{aligned}$$

Therefore, for a given $i \notin I$ we can find $k = \max\{j < i : j \in I\}$. Further, from the definition of I it follows that $a_i < (C_1+1)^{i-k}\,a_k$. Since φ is concave and $2a_{k-1} \le a_k$, we get

$$\begin{aligned}
\int_{a_{i-1}}^{a_i} \frac{\varphi^p(t)}{t}\,dt &\le \int_{a_k}^{(C_1+1)^{i-k}a_k} \frac{\varphi^p(t)}{t}\,dt \\
&\le \varphi^{p-1}((C_1+1)^{i-k}a_k)\int_{a_k}^{(C_1+1)^{i-k}a_k} \frac{\varphi(t)}{t}\,dt \\
&\le 2^{p-1}(C_1+1)^{(p-1)(i-k)}\,\varphi^{p-1}(\frac{a_k}{2})\int_{a_k}^{(C_1+1)^{i-k}a_k} \frac{\varphi(t)}{t}\,dt \\
&\le 2^{p-1}(C_1+1)^{p(i-k)}\,\varphi^{p-1}(\frac{a_k}{2})\,a_k\frac{\varphi(a_k)}{a_k} \\
&\le 2^{p}(C_1+1)^{p(i-k)}\varphi^{p-1}(\frac{a_k}{2})\int_{a_k/2}^{a_k} \frac{\varphi(t)}{t}\,dt \\
&\le 2^{p}(C_1+1)^{p(i-k)}\int_{a_k/2}^{a_k} \frac{\varphi^p(t)}{t}\,dt \\
&\le 2^{p}(C_1+1)^{p(i-k)}\int_{a_{k-1}}^{a_k} \frac{\varphi^p(t)}{t}\,dt
\end{aligned}$$

and so

$$b^{-pi}\int_{a_{i-1}}^{a_i} \frac{\varphi^p(t)}{t}\,dt \le 2^p\Big(\frac{C_1+1}{b}\Big)^{p(i-k)} b^{-pk}\int_{a_{k-1}}^{a_k} \frac{\varphi^p(t)}{t}\,dt.$$

Since $b > C_1 + 1$ we obtain (21).

In a similar way, applying (23) for $j = k$, we get

$$\begin{aligned} b^{-pi}\varphi^p(a_i - a_{i-1}) &\le b^{-pi}(C_1+1)^{p(i-k)}\varphi^p(a_k) \\ &\le 2^p\left(\frac{C_1+1}{b}\right)^{p(i-k)} b^{-pk}\varphi^p(a_k - a_{k-1}), \end{aligned}$$

which implies (22).

Relations (20)–(22) imply (18), so we proved the statement for $b > C_1 + 1$. To extend this result to all $b > 1$ it suffices to prove the following: whenever (18) holds for some $b > 1$ and arbitrary non-decreasing sequence $(a_j)_{j=-\infty}^{\infty}$ with a constant C, it is automatically fulfilled for $b^{1/2}$ with a constant not exceeding $2^p b^p C$. Indeed, if

$$y = \sum_{j=-\infty}^{+\infty} b^{-j/2}\chi_{(a_{j-1},a_j]} \text{ and } z = \sum_{j=-\infty}^{+\infty} b^{-j}\chi_{(a_{2j-2},a_{2j}]},$$

then

$$\begin{aligned} \|y\|^p_{\Lambda_{p,\varphi}} &= \sum_{j=-\infty}^{+\infty} b^{-pj/2}\int_{a_{j-1}}^{a_j}\varphi^p(t)\frac{dt}{t} \\ &= \sum_{j=-\infty}^{+\infty} b^{-p(2j-1)/2}\int_{a_{2j-2}}^{a_{2j-1}}\varphi^p(t)\frac{dt}{t} + \sum_{j=-\infty}^{+\infty} b^{-pj}\int_{a_{2j-1}}^{a_{2j}}\varphi^p(t)\frac{dt}{t} \\ &\overset{b^{p/2}}{\approx} \sum_{j=-\infty}^{+\infty} b^{-pj}\int_{a_{2j-2}}^{a_{2j}}\varphi^p(t)\frac{dt}{t} = \|z\|^p_{\Lambda_{p,\varphi}}. \end{aligned}$$

On the other hand,

$$b^{-pj}\varphi^p(a_{2j} - a_{2j-2}) \overset{2^p b^{p/2}}{\approx} b^{-pj}\varphi^p(a_{2j} - a_{2j-1}) + b^{-p(2j-1)/2}\varphi^p(a_{2j-1} - a_{2j-2}),$$

so we get an analog of (18) for y and $b^{1/2}$ and the proof is complete. $\square$

Remark 4.2. *For the space $\Lambda_{1,\varphi} = \Lambda_\varphi$ the result can be proved also by using the following well-known formula* (cf. formula 5.1 in [21] on page 108)

$$\|x\|_{\Lambda_\varphi} = \sum_{j=-\infty}^{+\infty}(b^{-j} - b^{-j-1})\varphi(a_j).$$

Let, as above, for a given weight w, $M_k = \{t \in [0,1] : w(t) \in [2^k, 2^{k+1})\}$ $(k \in \mathbb{Z})$ and $(w_r)_{r=1}^{\infty}$ be the non-increasing rearrangement of the sequence $(m(M_k))_{k=-\infty}^{+\infty}$.

Theorem 4.3. *Let φ be an increasing concave function on $[0,1]$ such that $\gamma_\varphi > 0$, $1 \le p < \infty$ and let w be a weight on $[0,1]$. Then the Lorentz space $X := \Lambda_{p,\varphi}$ is w-decomposable if and only if φ satisfies condition* (13).

Proof. If $X = \Lambda_{p,\varphi}$ is w-decomposable then, by Corollary 3.6, the relation (13) holds for the fundamental function φ_X. Since, as it was mentioned above, $\varphi \approx \varphi_X$, then (13) is fulfilled for φ as well.

Conversely, suppose that φ satisfies (13). Let $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n$ be non-negative functions from X satisfying (12). Evidently, there exist $x_1', x_2', \dots, x_n' \in X$ taking their values from the set $\{2^{-k}\}_{k=-\infty}^{\infty} \cup \{0\}$ and such that $x_i(t) \overset{2}{\approx} x_i'(t)$ $(0 < t \leq 1)$. Clearly, $m(\operatorname{supp} x_i') = m(\operatorname{supp} x_i) \leq w_i$ $(1 \leq i \leq n)$ and

$$m\Big\{t : \sum_{i=1}^{n} x_i'(t) = 2^{-k}\Big\} = \sum_{i=1}^{n} m\{t : x_i'(t) = 2^{-k}\}$$

for all integer k. Therefore, applying (13), we get that

$$\sum_{i=1}^{n} \varphi^p(m\{t : x_i'(t) = 2^{-k}\}) \approx \varphi^p(m\{t : \sum_{i=1}^{n} x_i'(t) = 2^{-k}\}) \ \ (k \in \mathbb{Z}). \tag{24}$$

On the other hand, Proposition 4.1 yields

$$\|x_i'\|_X^p \approx \sum_{k=-\infty}^{+\infty} 2^{-pk} \varphi^p(m\{t : x_i'(t) = 2^{-k}\}) \ \ (1 \leq i \leq n) \tag{25}$$

and

$$\|\sum_{i=1}^{n} x_i'\|_X^p \approx \sum_{k=-\infty}^{+\infty} 2^{-pk} \varphi^p(m\{t : \sum_{i=1}^{n} x_i'(t) = 2^{-k}\}) \tag{26}$$

with a constant which depends only on φ and p. Combining relations (25) and (26) with (24), we obtain (9) for x_i' and so for x_i. The proof is complete. □

In particular, from the above theorem and a remark after Theorem 3.8 it follows that the Lorentz space Λ_φ generated by the function $\varphi(t) = t \log \frac{e}{t}$ is $1/t$-decomposable and therefore the Banach couple $(\Lambda(\varphi), \Lambda(\varphi)(\frac{1}{t}))$ is K-monotone.

Theorem 4.4. *Suppose that φ is an increasing concave function on $[0,1]$ such that $\gamma_\varphi > 0$ and $1 \leq p < \infty$. The following conditions are equivalent:*

(a) there exists a weight w on $[0,1]$ such that the Lorentz space $\Lambda_{p,\varphi}$ is w-decomposable;

(b) φ is equivalent to a regularly varying function at zero of order p.

Proof. First, if $X := \Lambda_{p,\varphi}$ is w-decomposable for some weight w on $[0,1]$, then, by Corollary 3.7, as in the proof of the previous theorem, we conclude that φ is equivalent to a regularly varying function at zero of order p.

Conversely, suppose that φ is equivalent to a function that varies regularly at zero with order p, that is, φ satisfies (3) for some $\tau(N)$ $(N = 1, 2, \dots)$. Consider a family $(M_N)_{N=1}^{\infty}$ of pairwise disjoint measurable subsets of $[0,1]$ such that $m(M_2) = \min(\tau(2), 1/4)$,

$$m(M_N) = \min(\tau(N), \frac{m(M_{N-1})}{2}), \ \ N > 2,$$

and let $M_1 := [0,1] \setminus \bigcup_{N=2}^{\infty} M_N$. Set $w(t) := 2^N$ for all $t \in M_N$ and $N \in \mathbb{N}$. Clearly, $m(M_{N+1}) \leq m(M_N)/2$ $(N \in \mathbb{N})$. Therefore, by Theorem 3.8, φ satisfies (13) for any sequence $(\tau_N)_{N=1}^{\infty}$ majorized by the sequence $(m(M_N))_{N=1}^{\infty}$. To complete the proof it remains to apply Theorem 4.3. □

It is obvious that L_p-spaces ($1 \le p \le \infty$) are w-decomposable for every weight w. On the other hand, we show that for an arbitrary weight w there exist w-decomposable Lorentz spaces Λ_φ different from L_1.

Theorem 4.5. *Let w be an arbitrary weight on $[0,1]$. Then there exists an increasing concave function φ such that the space Λ_φ is w-decomposable and $\Lambda_\varphi \neq L_1$.*

Proof. As above, $M_k = \{t \in [0,1] : w(t) \in [2^k, 2^{k+1})\}$ for $k \in \mathbb{Z}$ and $(w_r)_{r=1}^\infty$ is the non-increasing rearrangement of the sequence $(m(M_k))_{k=-\infty}^{+\infty}$. Define

$$G(\alpha) := \sum_{r=1}^{\infty} \min\{\alpha, w_r\}, \quad \alpha \ge 0.$$

Then $G(1) = 1, G(0) = 0$ and G is increasing and continuous at zero.

Let $(t_k)_{k=0}^\infty$ be a sequence from $(0,1]$ such that $t_0 = 1, 0 < t_k < t_{k-1}/3$ for $k \ge 1$ and

$$G(t_{k+1}) \le 2^{-k}\, t_k, \;\; k = 0, 1, \dots. \tag{27}$$

Then we set $\varphi_k'(t) = \max_{i=0,1,\dots,k}\{2^i \chi_{[0,t_i]}(t)\}, k = 0, 1, \dots$ and $\varphi'(t) = \lim_{k\to\infty} \varphi_k'(t)$ ($0 < t \le 1$). It is easy to see that φ_k' and φ' are non-increasing functions on$(0,1]$. Moreover, since

$$t_k \varphi'(t_k) = t_k\, 2^k \le \frac{2}{3} t_{k-1}\, 2^{k-1} = \frac{2}{3} t_{k-1} \varphi'(t_{k-1})$$

it follows that

$$\int_0^1 \varphi'(t)dt \le \sum_{k=0}^{\infty} \varphi'(t_k) t_k \le \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k < \infty.$$

Therefore, the function $\varphi(t) := \int_0^t \varphi'(s)\, ds$ is well-defined, increasing and concave on $(0,1]$. We shall prove that the Lorentz space Λ_φ is w-decomposable.

In view of Theorem 4.3, it suffices to show that for some constant $C \ge 1$ and for any sequence of reals $(d_r)_{r=1}^\infty$ such that $0 < d_r \le w_r$ ($r = 1, 2, \dots$) we have

$$\varphi\left(\sum_{r=1}^{\infty} d_r\right) \le \sum_{r=1}^{\infty} \varphi(d_r) \le C\varphi\left(\sum_{r=1}^{\infty} d_r\right).$$

Note that the left hand side of this inequality is an immediate consequence of the concavity of φ. Further, since $\varphi_k(t) := \int_0^t \varphi_k'(s)\, ds \uparrow \varphi(t)$, then $\lim_{k\to\infty} \sum_{r=1}^\infty \varphi_k(d_r) = \sum_{r=1}^\infty \varphi(d_r)$. Therefore, it is enough to prove that

$$\frac{\sum_{r=1}^\infty \varphi_k(d_r)}{\varphi_k\left(\sum_{r=1}^\infty d_r\right)} \le 3, \quad k \ge 0. \tag{28}$$

Noting that $\sum_{r=1}^\infty d_r \le t_0 = 1$, we set

$$k_0 := \max\Big\{k = 0, 1, 2, \dots : \sum_{r=1}^{\infty} d_r \le t_k\Big\}.$$

From the definition of φ_k it follows that

$$\varphi_k\left(\sum_{r=1}^{\infty} d_r\right) = 2^k \sum_{r=1}^{\infty} d_r = \sum_{r=1}^{\infty} \varphi_k(d_r) \ \text{ if } \ 0 \le k \le k_0. \tag{29}$$

Since $t_{k_0+1} < \sum\limits_{r=1}^{\infty} d_r \le t_{k_0}$, then, again by the definition of φ_k,

$$\sum_{r=1}^{\infty} \varphi_{k_0+1}(d_r) \le 2^{k_0+1} \sum_{r=1}^{\infty} d_r \le 2\varphi_{k_0+1}\left(\sum_{r=1}^{\infty} d_r\right). \tag{30}$$

Let $k > k_0$ be arbitrary. The inequality $\sum\limits_{r=1}^{\infty} d_r > t_k$ implies that

$$\varphi_k\left(\sum_{r=1}^{\infty} d_r\right) > \varphi_k(t_k) = 2^k t_k. \tag{31}$$

Moreover, since

$$\varphi_{k+1}(d_r) = \begin{cases} 2^{k+1} d_r = 2\,\varphi_k(d_r), & \text{if } d_r \le t_{k+1}, \\ 2^k t_{k+1} + \varphi_k(d_r), & \text{if } d_r > t_{k+1}, \end{cases}$$

we obtain

$$\sum_{r=1}^{\infty} \varphi_{k+1}(d_r) - \sum_{r=1}^{\infty} \varphi_k(d_r) = \sum_{r=1}^{\infty} \min(2^k t_{k+1}, 2^k d_r) \le 2^k G(t_{k+1}).$$

Hence, for any $k > k_0$, by (31) and (27), we obtain

$$\begin{aligned} \frac{\sum_{r=1}^{\infty} \varphi_{k+1}(d_r)}{\varphi_{k+1}\left(\sum_{r=1}^{\infty} d_r\right)} &\le \frac{\sum_{r=1}^{\infty} \varphi_k(d_r)}{\varphi_k\left(\sum_{r=1}^{\infty} d_r\right)} + \frac{\sum_{r=1}^{\infty} \varphi_{k+1}(d_r) - \sum_{r=1}^{\infty} \varphi_k(d_r)}{\varphi_k\left(\sum_{r=1}^{\infty} d_r\right)} \\ &\le \frac{\sum_{r=1}^{\infty} \varphi_k(d_r)}{\varphi_k\left(\sum_{r=1}^{\infty} d_r\right)} + \frac{G(t_{k+1})}{t_k} \le \frac{\sum_{r=1}^{\infty} \varphi_k(d_r)}{\varphi_k\left(\sum_{r=1}^{\infty} d_r\right)} + 2^{-k}. \end{aligned}$$

Applying the last estimate together with (29) and (30), we obtain (28). It is easy to see that $\varphi(t)$ is not equivalent to t, and therefore $\Lambda_\varphi \ne L_1$. The proof is complete. □

Remark 4.6. *Theorem 4.5 can be easily extended to the spaces $\Lambda_{p,\psi}$ with $p \in (1,\infty)$. Indeed, let w be an arbitrary weight on $[0,1]$ and φ be the function from the proof of Theorem 4.5. Set $\psi := \varphi^{1/p}$. Clearly, ψ is an increasing concave function not equivalent to the function $t^{1/p}$. Therefore, $\Lambda_{p,\psi} \ne L_p$. Since relation (13) is fulfilled for ψ as well, then, by Theorem 4.3, the space $\Lambda_{p,\psi}$ is w-decomposable.*

Our next goal is to prove analogous results for Marcinkiewicz spaces M_φ. To make use of the duality of Lorentz and Marcinkiewicz spaces we will need the following statement which is of interest in its own right.

Theorem 4.7. *Let X be a Banach lattice on a σ-finite measure space (Ω, Σ, μ) with* $\operatorname{supp} X = \Omega$ *which has the Fatou property and w be a non-trivial weight on Ω. Then the couple $(X, X(w))$ is K-monotone if and only if $(X', X'(w))$ is K-monotone, where X' is the Köthe dual of X.*

The proof follows from Theorem 3.1 proved in [37] and the following result.

Theorem 4.8. *Let X be a Banach lattice on a σ-finite measure space (Ω, Σ, μ) with* $\operatorname{supp} X = \Omega$ *which has the Fatou property and w be a non-trivial weight on Ω. Then X is w-decomposable if and only if its Köthe dual X' is w-decomposable.*

Proof. Suppose that X is w-decomposable. Let $n \in \mathbb{N}$ and the functions $x'_1, x'_2, \ldots, x'_n$, $y'_1, y'_2, \ldots, y'_n \in X'$ satisfy (5) (with the norm from X') and (6). Take a function $x \in X$, $\|x\|_X = 1$, such that $\operatorname{supp} x \subset \bigcup_{i=1}^n \operatorname{supp} x'_i$ and

$$\|\sum_{i=1}^n x'_i\|_{X'} \leq 2 \int_\Omega |\sum_{i=1}^n x'_i(t)x(t)| \, d\mu.$$

Now, consider $y_i \in X$ such that $\operatorname{supp} y_i \subset \operatorname{supp} y'_i$, $\|y_i\|_X = \|x\chi_{\operatorname{supp} x'_i}\|_X$ and

$$\|y'_i\|_{X'} \leq \frac{2}{\|y_i\|_X} \int_\Omega |y'_i(t)y_i(t)| \, d\mu, \ \ 1 \leq i \leq n.$$

Then, according to the hypothesis,

$$\|\sum_{i=1}^n y_i\|_X \leq C \|\sum_{i=1}^n x\chi_{\operatorname{supp} x'_i}\|_X = C,$$

and, therefore,

$$\begin{aligned} \|\sum_{i=1}^n y'_i\|_{X'} &\geq \frac{1}{C} \int_\Omega |\sum_{i=1}^n y_i(t) \sum_{j=1}^n y'_j(t)| \, d\mu = \frac{1}{C} \sum_{i=1}^n \int_\Omega |y'_i(t)y_i(t)| \, d\mu \\ &\geq \frac{1}{2C} \sum_{i=1}^n \|y'_i\|_{X'} \|y_i\|_X = \frac{1}{2C} \sum_{i=1}^n \|x'_i\|_{X'} \|x\chi_{\operatorname{supp} x'_i}\|_X \\ &\geq \frac{1}{2C} \sum_{i=1}^n \int_\Omega |x'_i(t)x(t)\chi_{\operatorname{supp} x'_i}(t)| \, d\mu \geq \frac{1}{4C} \|\sum_{i=1}^n x'_i\|_{X'}. \end{aligned}$$

Certainly, the same argument can be applied to get the opposite estimate. The proof is complete. □

Since $M'_\varphi = \Lambda_{\tilde{\varphi}}$ (cf. [21], p. 117) and $\delta_\varphi + \gamma_{\tilde{\varphi}} = 1$ for any increasing concave function φ on $[0, 1]$ (cf. [21], Theorem 4.12 on page 107 or [24], p. 28), then by Theorems 4.3, 4.4 and 4.8 we immediately obtain the following statements.

Corollary 4.9. *Let φ be an increasing concave function on $[0,1]$ such that $\delta_\varphi < 1$ and let w be a weight on $[0,1]$. Then the Marcinkiewicz space M_φ is w-decomposable if and only if $\tilde{\varphi}(t) = t/\varphi(t)$ satisfies* (13) *with $p = 1$.*

Corollary 4.10. *If φ is an increasing concave function on $[0,1]$ such that $\delta_\varphi < 1$, then the space M_φ is w-decomposable for some weight w on $[0,1]$ if and only if φ is equivalent to a regularly varying function at zero of order ∞.*

In the paper [18], Kalton proved that if X and Y are symmetric sequence spaces with the Fatou property such that the couple $(X, Y(w))$ is K-monotone for some non-trivial weight w, then $X = l_p$ and $Y = l_q$ with $1 \le p, q \le \infty$. The results in this section and Theorem 3.1 show that in the case of symmetric function spaces on $[0,1]$ the situation is completely different. The following theorems present new examples of K-monotone Banach couples of weighted Lorentz and Marcinkiewicz function spaces. The first theorem follows from Theorem 3.1, Theorem 4.4, Theorem 4.5 and Remark 2 and the second one from Theorem 3.1, Theorem 4.8 on the duality and Corollary 4.10.

Theorem 4.11. *If φ is an increasing concave function on $[0,1]$ such that $\gamma_\varphi > 0$ and $1 \le p < \infty$, then the weighted couple $(\Lambda_{p,\varphi}, \Lambda_{p,\varphi}(w))$ is K-monotone for some (non-trivial) weight w on $[0,1]$ if and only if φ is equivalent to a regularly varying function at zero of order p. On the other hand, for arbitrary weight w on $[0,1]$ and $1 \le p < \infty$ there exists an increasing concave function φ on $[0,1]$ such that the couple $(\Lambda_{p,\varphi}, \Lambda_{p,\varphi}(w))$ is K-monotone and $\Lambda_{p,\varphi} \ne L_p$.*

Theorem 4.12. *If φ is an increasing concave function on $[0,1]$ such that $\delta_\varphi < 1$, then the weighted couple $(M_\varphi, M_\varphi(w))$ is K-monotone for some (non-trivial) weight w on $[0,1]$ if and only if φ is equivalent to a regularly varying function at zero of order ∞.*

5 w–decomposable Orlicz spaces

As we have seen in the previous section, in order to check the property of w-decomposability for Lorentz spaces, it is enough to consider only characteristic functions (Theorem 4.3). In this section we will prove that in the case of Orlicz spaces it is sufficient to examine scalar multiples of characteristic functions.

As above, for a weight w on $[0,1]$ let $M_k := \{t \in [0,1] : w(t) \in [2^k, 2^{k+1})\}$ $(k \in \mathbb{Z})$, $(w_r)_{r=1}^{\infty}$ be the non-increasing rearrangement of the sequence $(m(M_k))_{k=-\infty}^{+\infty}$ and $\{\bar{M}_r\}_{r=1}^{\infty}$ denote any rearrangement of the sets M_k such that $m(\bar{M}_r) = w_r, r = 1, 2, \ldots$

Theorem 5.1. *Let an Orlicz function F satisfy the Δ_2-condition for large u and let w be a weight on $[0,1]$. Then, the Orlicz space $L_F = L_F[0,1]$ is w-decomposable if and only if there exists $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, all measurable sets $A_k \subset \overline{M}_k$ and reals c_k $(1 \le k \le n)$ we have*

$$\| \sum_{k=1}^{n} c_k \chi_{A_k} \|_{L_F}^p \approx \sum_{k=1}^{n} \| c_k \chi_{A_k} \|_{L_F}^p \tag{32}$$

with a constant independent of c_k, A_k $(1 \le k \le n)$ and $n \in \mathbb{N}$. If, in addition, the complementary function F^* satisfies the Δ_2-condition for large u, then the w-decomposability of L_F implies that F is equivalent to a regularly varying Orlicz function at ∞ of order p.

Proof. Suppose, first, that L_F is w-decomposable. By Proposition 3.2, there is $p \in [1, \infty]$ such that (9) holds for $X = L_F$, which implies (32). Since F satisfies the Δ_2-condition for large $u > 0$, then $\alpha_X > 0$. Therefore, by Corollary 3.7, $p < \infty$.

Conversely, let $n \in \mathbb{N}$ and $y_k \in L_F$, $\operatorname{supp} y_k \subset \bar{M}_k$, $1 \leq k \leq n$. We may (and will) assume that y_k are positive bounded functions and

$$\sum_{k=1}^{n} \|y_k\|_{L_F}^p = 1. \tag{33}$$

Taking into account Theorem 3.3, we need to show that

$$\|\sum_{k=1}^{n} y_k\|_{L_F}^p \approx 1, \tag{34}$$

with a constant independent from n and y_k. For each $1 \leq k \leq n$ we set

$$c_k = \frac{\|y_k\|_{L_F}}{2\varphi_{L_F}(m(\operatorname{supp} y_k))}$$

and

$$\tilde{y}_k(t) := \begin{cases} y_k(t), & \text{if } y_k(t) \geq c_k, \\ 0, & \text{if } y_k(t) < c_k. \end{cases}$$

Applying (32) to the functions $c_k \chi_{\mathrm{supp} y_k}$ and taking into account the definition of c_k and (33) we get

$$\begin{aligned} \|\sum_{k=1}^{n} c_k \chi_{\mathrm{supp} y_k}\|_{L_F}^p &\leq C_1 \sum_{k=1}^{n} c_k^p \varphi_{L_F}(m(\mathrm{supp} y_k))^p \\ &= C_1 \sum_{k=1}^{n} 2^{-p} \|y_k\|_{L_F}^p = 2^{-p} C_1. \end{aligned}$$

Up to equivalence of norms the Orlicz space $L_F = L_F[0,1]$ depends only on the behaviour of F for large enough $u > 0$. Therefore, we may assume that $F(2u) \leq C_2 F(u)$ for all $u > 0$. Then, from the last inequality it follows that

$$\sum_{k=1}^{n} m(\mathrm{supp} y_k) F(c_k) \leq C_3, \tag{35}$$

where C_3 is a constant independent of n and y_k. Moreover, from the definition of c_k and $\tilde{y}_k$ we have

$$\|\tilde{y}_k\|_{L_F} \leq \|y_k\|_{L_F} \text{ and } \|\tilde{y}_k\|_{L_F} \geq \|y_k\|_{L_F} - \|c_k \chi_{\operatorname{supp} y_k}\|_{L_F} = \frac{1}{2}\|y_k\|_{L_F}. \tag{36}$$

Next, let us show that there is $r_k \in [c_k, \sup_t \tilde{y}_k(t)]$ such that

$$F(r_k) = F\left(\frac{r_k}{\|\tilde{y}_k\|_{L_F}}\right) \int_0^1 F(\tilde{y}_k(t))dt. \tag{37}$$

In fact, consider the function

$$H_k(t) := \frac{F(\tilde{y}_k(t))}{F\left(\frac{\tilde{y}_k(t)}{\|\tilde{y}_k(t)\|_{L_F}}\right)},\ t \in \operatorname{supp} \tilde{y}_k.$$

From the equality $\int_0^1 F(\frac{\tilde{y}_k(t)}{\|\tilde{y}_k(t)\|_{L_F}})dt = 1$ it follows that

$$\inf_{t \in \operatorname{supp} \tilde{y}_k} H_k(t) \le \int_0^1 F[\tilde{y}_k(t)]dt \le \sup_{t \in \operatorname{supp} \tilde{y}_k} H_k(t).$$

Thus, since $\inf_{t \in \operatorname{supp} \tilde{y}_k} \tilde{y}_k(t) \ge c_k$, by the continuity of F, equality (37) holds for some r_k from the interval $[c_k, \sup_t \tilde{y}_k(t)]$.

Next, define $d_k \in [0,1]$ $(k = 1, 2, \dots, n)$ as follows:

$$d_k = \begin{cases} \varphi_{L_F}^{-1}\left(\frac{\|\tilde{y}_k\|_{L_F}}{r_k}\right), & \text{if } \|\tilde{y}_k\|_{L_F} \le r_k \varphi_{L_F}(m(\operatorname{supp} y_k)), \\ m(\operatorname{supp} y_k), & \text{if } \|\tilde{y}_k\|_{L_F} > r_k \varphi_{L_F}(m(\operatorname{supp} y_k)). \end{cases}$$

Clearly, by the definition of d_k,

$$r_k \varphi_{L_F}(d_k) \le \|\tilde{y}_k\|_{L_F}. \tag{38}$$

On the other hand, since $r_k \ge c_k$, we obtain

$$r_k \varphi_{L_F}(d_k) \ge \frac{1}{2}\|\tilde{y}_k\|_{L_F}, \tag{39}$$

whence $d_k \ge \varphi_{L_F}^{-1}(\|\tilde{y}_k\|_{L_F}/(2r_k))$. Hence, taking into account that F satisfies the Δ_2-condition with constant C_2 for all $u > 0$, the formula $\varphi_{L_F}(t) = 1/F^{-1}(1/t)$ (see formula (9.23) in [20] on page 79 of the English version or Corollary 5 in [25] on page 58) and (37), we have

$$d_k F(r_k) \ge \frac{F(r_k)}{F\left(\frac{2r_k}{\|\tilde{y}_k\|_{L_F}}\right)} \ge \frac{1}{C_2} \frac{F(r_k)}{M\left(\frac{r_k}{\|\tilde{y}_k\|_{L_F}}\right)} = \frac{1}{C_2} \int_0^1 F[\tilde{y}_k(t)]\, dt. \tag{40}$$

Conversely, from the equality $1/d_k = F\left(1/\varphi_{L_F}(d_k)\right)$, (38) and (37) it follows that

$$d_k F(r_k) = \frac{F(r_k)}{F(\frac{1}{\varphi_{L_F}(d_k)})} \le \frac{F(r_k)}{F\left(\frac{r_k}{\|\tilde{y}_k\|_{L_F}}\right)} = \int_0^1 F[\tilde{y}_k(t)]\, dt. \tag{41}$$

Now, by the definition of d_k, we have $d_k \le m(\operatorname{supp} y_k)$. Therefore, we can define the scalar multiples of characteristic functions $f_k(t) := r_k \chi_{B_k}(t)$, where $B_k \subset \operatorname{supp} y_k$ and $m(B_k) = d_k$. According to (38), (39) and (36), we have

$$\frac{1}{4}\|y_k\|_{L_F} \le \|f_k\|_{L_F} \le \|y_k\|_{L_F}, k = 1, 2, \dots, n.$$

Therefore, in view of (32) and (33), we obtain

$$\|\sum_{k=1}^n f_k\|_{L_F}^p \approx \sum_{k=1}^n \|f_k\|_{L_F}^p \ \approx \ \sum_{k=1}^n \|y_k\|_{L_F}^p = 1,$$

with constants which depend only on p. Hence, taking into account that F satisfies the Δ_2-condition, we conclude that (34) will be proved once we show that

$$\|\sum_{k=1}^n y_k\|_{L_F} \ \approx \ \|\sum_{k=1}^n f_k\|_{L_F}$$

with constants independent of n and y_k. Since the functions f_k (respectively, y_k) are pairwise disjoint, in view of estimate (41), we find that

$$\begin{aligned}\int_0^1 F[\sum_{k=1}^n f_k(t)]\,dt \ &= \ \sum_{k=1}^n d_k F(r_k) \le \sum_{k=1}^n \int_0^1 F(\tilde{y}_k(t))\,dt \\ &\le \ \int_0^1 F[\sum_{k=1}^n y_k(t)]\,dt.\end{aligned}$$

Conversely, by (40) and (35), we get

$$\begin{aligned}\int_0^1 F[\sum_{k=1}^n y_k(t)]\,dt \ &\le \ \sum_{k=1}^n \int_0^1 F[\tilde{y}_k(t)]\,dt + \sum_{k=1}^n m(\operatorname{supp} y_k)F(c_k) \\ &\le \ C_2 \int_0^1 F[\sum_{k=1}^n f_k(t)]\,dt + C_3,\end{aligned}$$

and we come to the desired result.

In order to obtain the second assertion of the theorem it is sufficient to apply Corollary 3.6, Lemmas 2.1 and 2.2, Proposition 2.3 and the elementary observation that condition (a) in that proposition implies the equivalence of F to an Orlicz function which is regularly varying at ∞ of order p. $\square$

Remark 5.2. Arguing in the same way as in the proof of Theorem 5.1 we may obtain the following result: *Let an Orlicz function F satisfy the Δ_2-condition for large u and $1 < p, q < \infty$. The Orlicz space $L_F[0,1]$ satisfies the upper p-estimate, respectively the lower q-estimate, if and only if there exsists a constant $C > 0$ such that for any $n \in \mathbb{N}$, all pairwise disjoint measurable sets A_k and reals c_k we have*

$$\|\sum_{k=1}^n c_k\chi_{A_k}\|_{L_F} \le C(\sum_{k=1}^n \|c_k\chi_{A_k}\|_{L_F}^p)^{1/p},$$

respectively,

$$(\sum_{k=1}^n \|c_k\chi_{A_k}\|_{L_F}^q)^{1/q} \le C\|\sum_{k=1}^n c_k\chi_{A_k}\|_{L_F}.$$

However, an inspection of the proof of results from [19] (pages 120-121 and 124) shows that the first of these inequalities is equivalent to either of the following conditions: the Orlicz space $L_F[0,1]$ is p-convex or $L_F[0,1]$ satisfies the upper p-estimate or there exists an Orlicz function F_1 equivalent to F for large arguments such that $F_1(u^{1/p})$ is a convex function on $[0,\infty)$. At the same time, the second of them is equivalent to either of the following conditions: the Orlicz space $L_F[0,1]$ is q-concave or $L_F[0,1]$ satisfies the lower q-estimate or there exists an Orlicz function F_1 equivalent to F for large arguments such that $F_1(u^{1/q})$ is a concave function on $[0,\infty)$.

The following result is analogous to Theorem 4.4 for Lorentz spaces.

Theorem 5.3. *Let F be an Orlicz function equivalent to an Orlicz function which is regularly varying at ∞ of order $p \in [1,\infty)$. Then there is a weight w on $[0,1]$ such that the Orlicz space L_F is w-decomposable and, consequently, the couple $(L_F, L_F(w))$ is K-monotone.*

Proof. By Corollary 3.5, it is sufficient to find a sequence of pairwise disjoint intervals $\{\Delta_k\}_{k=1}^{\infty}$ from $[0,1]$ such that for any $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$ satisfying the condition $\operatorname{supp} x_i \subset \Delta_i$ $(1 \le i \le n)$, relation (9) holds.

First, since F is equivalent to a regularly varying Orlicz function at ∞ of order p, then Lemma 1 and a simple compactness argument (see also [18, Lemma 6.1]) show that there exists a constant $C_1 > 1$ such that for every $k \in \mathbb{N}$ there is $v_k > 0$ such that for all $v \ge v_k$ and $u \in [k^{-2}/8, 1]$ we have that

$$F(uv) \overset{C_1}{\approx} u^p F(v). \tag{42}$$

Let $v > 0$, $\varepsilon > 0$ be arbitrary and Δ be an interval from $[0,1]$ such that $m(\Delta) \le \varepsilon/F(v)$. Moreover, suppose that $z \in L_F$, $z \ge 0$ and $\operatorname{supp} z \subset \Delta$. Then

$$\int_{\{t\in\Delta:\, z(t)\le v\}} F[z(t)]\,dt \le F(v)\,m(\Delta) \le \varepsilon.$$

Let $\{\Delta_k\}_{k=1}^{\infty}$ be a sequence of disjoint intervals from $[0,1]$ such that

$$m(\Delta_k) \le 2^{-k-1}(F(v_k))^{-1} \ \ (k = 1,2,\ldots).$$

Then, as it was noted above, for every $z \in L_F$ such that $z \ge 0$ and $\operatorname{supp} z \subset \Delta_k$, we have

$$\int_{\{t\in\Delta_k:\, z(t)\le v_k\}} F[z(t)]\,dt \le 2^{-k-1} \ \ (k = 1,2,\ldots). \tag{43}$$

Suppose that $\{x_k\}_{k=1}^{\infty}$ is an arbitrary sequence from L_F such that $x_k \ge 0$ and $\operatorname{supp} x_k \subset \Delta_k$ $(k = 1,2,\ldots)$. To prove (9) we assume that

$$\Big\| \sum_{i=1}^{n} x_i \Big\|_{L_F} = 1,$$

or, equivalently,

$$\sum_{i=1}^{n} \int_{\Delta_i} F[x_i(t)]\,dt = 1. \tag{44}$$

If $\lambda_i := \|x_i\|_{L_F}$ $(i=1,2,\dots)$, then $0 \le \lambda_i \le 1$ and

$$\int_{\Delta_i} F\Big[\frac{x_i(t)}{\lambda_i}\Big]\,dt = 1 \quad (i=1,2,\dots). \tag{45}$$

Denote $I_1 := \{i = 1,2,\dots,n : \lambda_i \le i^{-2}/8\}$, $I_2 := \{1,2,\dots,n\} \setminus I_1$. Then

$$\sum_{i\in I_1} \lambda_i^p \le \frac{1}{8}\sum_{i\in I_1} i^{-2p} \le \frac{1}{4}. \tag{46}$$

Now, let $i \in I_2$, i.e., $\lambda_i \ge i^{-2}/8$. Then, if $x_i(t) \ge \lambda_i v_i$, from (42) it follows that

$$C_1^{-1}\lambda_i^p F\Big[\frac{x_i(t)}{\lambda_i}\Big] \le F[x_i(t)] \le C_1\lambda_i^p F\Big[\frac{x_i(t)}{\lambda_i}\Big]. \tag{47}$$

Moreover, by (43) and (45), we have

$$\begin{aligned}\int_{\{t\in\Delta_i:\, x_i(t)>\lambda_i v_i\}} F\Big[\frac{x_i(t)}{\lambda_i}\Big]\,dt &= 1 - \int_{\{t\in\Delta_i:\, x_i(t)\le\lambda_i v_i\}} F\Big[\frac{x_i(t)}{\lambda_i}\Big]\,dt \\ &\ge 1 - 2^{-i-1} \ge \frac{3}{4},\end{aligned}$$

whence, taking into account the left hand side of (47), we obtain

$$\int_{\Delta_i} F[x_i(t)]\,dt \ge C_1^{-1}\lambda_i^p \int_{\{t\in\Delta_i:\, x_i(t)>\lambda_i v_i\}} F\Big[\frac{x_i(t)}{\lambda_i}\Big]\,dt \ge \frac{3}{4}C_1^{-1}\lambda_i^p, \quad i \in I_2.$$

Combining this with (44) and (46), we get

$$\sum_{i=1}^n \lambda_i^p = \sum_{i\in I_1}\lambda_i^p + \sum_{i\in I_2}\lambda_i^p \le \frac{1}{4} + \frac{4}{3}C_1 \sum_{i=1}^n \int_{\Delta_i} F[x_i(t)]\,dt \le 2C_1,$$

and the first inequality in (9) is proved.

On the other hand, using the right hand side of (47) and (45), we infer that

$$\begin{aligned}\sum_{i\in I_2}\int_{\{t\in\Delta_i:\, x_i(t)>\lambda_i v_i\}} F[x_i(t)]\,dt &\le C_1 \sum_{i\in I_2}\lambda_i^p \int_{\{t\in\Delta_i:\, x_i(t)>\lambda_i v_i\}} F\Big[\frac{x_i(t)}{\lambda_i}\Big]\,dt \\ &\le C_1 \sum_{i=1}^n \lambda_i^p.\end{aligned} \tag{48}$$

At the same time, by (43) and the convexity of F, we obtain

$$\begin{aligned}\sum_{i\in I_2}\int_{\{t\in\Delta_i:\, x_i(t)\le\lambda_i v_i\}} F[x_i(t)]\,dt &\le \sum_{i\in I_2}\lambda_i \int_{\{t\in\Delta_i:\, x_i(t)\le\lambda_i v_i\}} F\Big[\frac{x_i(t)}{\lambda_i}\Big]\,dt \\ &\le \sum_{i=1}^\infty 2^{-i-1} = \frac{1}{2}\end{aligned}$$

and, by (45) and the definition of I_1,

$$\sum_{i\in I_1}\int_{\Delta_i} F[x_i(t)]\,dt \le \sum_{i\in I_1}\lambda_i\int_{\Delta_i} F\Big[\frac{x_i(t)}{\lambda_i}\Big]\,dt \le \frac{1}{4}.$$

Hence, taking into account (44), we get

$$\begin{aligned}\sum_{i\in I_2}\int_{\{t\in\Delta_i:\,x_i(t)>\lambda_i v_i\}} F[x_i(t)]\,dt &= 1-\sum_{i\in I_2}\int_{\{t\in\Delta_i:\,x_i(t)\le\lambda_i v_i\}} F[x_i(t)]\,dt\\ &- \sum_{i\in I_1}\int_{\Delta_i} F[x_i(t)]\,dt \ge \frac{1}{4}.\end{aligned}$$

From this and (48) it follows that $\sum_{i=1}^n \lambda_i^p \ge 1/(4C_1)$, and so the proof of (9) is complete. □

6 Ultrasymmetric Orlicz spaces and w–decomposability

In the previous sections we have examined the problem of the K-monotonicity of weighted couples generated by Lorentz, Marcinkiewicz and Orlicz spaces. We have seen that the central role in the question is played by the notion of w-decomposibility. It turns out that studying the last property in a natural way leads to the so-called ultrasymmetric Orlicz spaces.

Recall that a symmetric space X on $[0,1]$ is *ultrasymmetric* if X is an interpolation space between the Lorentz space Λ_{φ_X} and the Marcinkiewicz space M_{φ_X}. These spaces were studied by Pustylnik [27], who proved that they embrace all possible generalizations of Lorentz-Zygmund spaces and have a simple analytical description. Moreover, one could substitute ultrasymmetric spaces into almost all results concerning classical spaces such as Lorentz-Zygmund spaces, and so they are very useful in many applications (see, for example, Pustylnik [28] and [29]).

Pustylnik asked about a description of ultrasymmetric Orlicz spaces (see [27], p. 172). In the case of reflexive Orlicz spaces this problem was solved in [4]: such a space is ultrasymmetric if and only if it coincides (up to equivalence of norms) with a Lorentz space $\Lambda_{p,\varphi}$ for some $1<p<\infty$ and some increasing concave function φ on $[0,1]$.

As it was said above, the class of w–decomposable symmetric spaces is closely related to the class of ultrasymmetric Orlicz spaces. Our next theorem shows that in the case when a weight w changes sufficiently fast any w–decomposable symmetric space with non-trivial Boyd indices is an ultrasymmetric Orlicz space.

Again, as above, for a weight w defined on $[0,1]$, let $M_k := \{t\in[0,1] : w(t)\in[2^k,2^{k+1})\}$ $(k\in\mathbb{Z})$ and $(w_k)_{k=1}^{\infty}$ be the non-increasing rearrangement of the sequence $(m(M_k))_{k=-\infty}^{+\infty}$

Theorem 6.1. *Let X be a symmetric space on $[0,1]$ with non-trivial Boyd indices and w be a weight on $[0,1]$ satisfying the condition:*

$$\text{there are } k_0\in\mathbb{N} \text{ and } c_0>0 \text{ such that } w_k 2^k \ge c_0 \text{ for } k\ge k_0. \qquad (49)$$

(a) *If X is w–decomposable, then X is an ultrasymmetric Orlicz space.*

(b) *If X has the Fatou property and $(X, X(w))$ is a K-monotone couple, then X is an ultrasymmetric Orlicz space.*

Proof. (a) Firstly, taking into account the boundedness of the dilation operator and Theorem 3.3, a symmetric space X is w–decomposable if and only if it is v–decomposable, where $v(u) = w(cu)$ for some $c > 0$. Therefore, we may assume that $c_0 = 1$. Denote $I_k := [2^{-k}, 2^{-k+1})$, $\bar{\chi}_{I_k} := \chi_{I_k}/\varphi(2^{-k})$ $(k = 1, 2, \dots)$, where φ is the fundamental function of X. From (49) it follows that $m(\text{supp}\bar{\chi}_{I_k}) \le w_k$ for all $k \ge k_0$. Applying Corollary 3.4 to scalar multiples of $\bar{\chi}_{I_k}$ $(k \ge k_0)$, we get that $(\bar{\chi}_{I_k})_{k=k_0}^{\infty}$ spans l_p for some $p \in [1, \infty)$ $(p \ne \infty$ because the Boyd indices of X are non-trivial). Obviously, replacing $(\bar{\chi}_{I_k})_{k=k_0}^{\infty}$ with $(\bar{\chi}_{I_k})_{k=1}^{\infty}$ does not change this property, so for all $a_k \in \mathbb{R}$ $(k = 1, 2, \dots)$

$$\Big\| \sum_{k=1}^{\infty} a_k \bar{\chi}_{I_k} \Big\|_X \approx \|(a_k)\|_{l_p}.$$

Then, taking into account [4, Proposition 2], we get

$$X = (L_1, L_\infty)^K_{l_p((\varphi(2^{-k})2^{-k})_{k=1}^{\infty})}.$$

By Corollary 3.7, $\delta_\varphi = \beta_X < 1$. Therefore, $\lim_{t\to\infty} \|\sigma_t\|_{X\to X}/t = 0$, and we can apply [21, Theorem II.6.6, p. 137] in the case when A is the identity operator, to obtain

$$\begin{aligned} \|x\|_X \approx \|(\varphi(2^{-k})\, x^{**}(2^{-k}))_{k=1}^{\infty}\|_{l_p} &\approx \|(\varphi(2^{-k})\, x^{*}(2^{-k}))_{k=1}^{\infty}\|_{l_p} \\ &\approx \left(\int_0^1 [x^*(t)\varphi(t)]^p \, \frac{dt}{t} \right)^{1/p}, \end{aligned}$$

and we conclude that

$$X = \Lambda_{p,\varphi}. \tag{50}$$

Next, denote

$$F(u) = \int_0^u \frac{\tilde{F}(t)}{t}\, dt, \text{ where } \tilde{F}(t) = \begin{cases} \frac{t}{\varphi^{-1}(1)} & \text{if } 0 \le t \le 1, \\ \frac{1}{\varphi^{-1}(\frac{1}{t})} & \text{if } t \ge 1. \end{cases}$$

Since $\tilde{F}(t)/t$ is increasing on $(0, \infty)$, then $F(u)$ is a convex function and for $u > 0$ we have that

$$\tilde{F}(u/2) \le \int_{u/2}^{u} \frac{\tilde{F}(t)}{t}\, dt \le F(u) \le \tilde{F}(u).$$

Moreover, by Corollary 3.7, we have that $\gamma_\varphi = \alpha_X > 0$, which implies that $\tilde{F}$ satisfies the Δ_2–condition for all $u > 0$. Therefore, for all $u > 0$

$$F(u) \ge \tilde{F}(u/2) \ge c\tilde{F}(u),$$

that is, the functions F and $\tilde{F}$ are equivalent on $(0, \infty)$.

Now, we recall the following definition due to Kalton [18] (see also [4], where the notion is used): For an Orlicz function F and $1 \le p < \infty$, define the function $\Psi^{\infty}_{F,p}(u, C)$ for $0 < u \le 1, C > 1$ to be the supremum (possibly ∞) of all N such that there exist $1 \le a_1 < a_2 < \ldots < a_N$, $\frac{a_k}{a_{k-1}} \ge 2$ for $k = 2, \ldots, N$ such that for all k either $F_{a_k}(u) \ge Cu^p$ or $u^p \ge CF_{a_k}(u)$, where $F_a(u) := \frac{F(au)}{F(a)}$ for $a, u > 0$.

To complete the proof it suffices to verify that for some $C_0 > 0, C_1 > 0$ and $r > 0$ we have that

$$\Psi^{\infty}_{F,p}(u, C_0) \le C_1 u^{-r} \quad \text{for all} \quad u \in (0, 1].$$

Indeed, once it is done, we can apply Theorem 1 from [4] to conclude that the Orlicz space L_F is ultrasymmetric and that it coincides with a Lorentz space $\Lambda_{p,\psi}$ generated by some increasing concave function ψ. Since the fundamental function of L_F is equivalent to φ, then $L_F = \Lambda_{p,\varphi}$, and, in view of (50), the proof is complete.

Since the functions F and $\tilde{F}$ are equivalent, then, by [4, Lemma 1], it is sufficient to prove the inequality for $\tilde{F}$, i.e., to prove that for some $C_0 > 0, C_1 > 0$ and $r > 0$ we have

$$\Psi^{\infty}_{\tilde{F},p}(u, C_0) \le C_1 u^{-r} \quad \text{for all} \quad u \in (0, 1]. \tag{51}$$

In view of w-decomposability, Corollary 3.7, Lemma 2.2 and the inequality $w_k \ge 2^{-k}$, there is a constant $C > 0$ such that for any $l = 1, 2, \ldots$

$$\frac{\varphi(lt)}{\varphi(t)} \overset{C}{\approx} l^{1/p} \quad \text{if} \quad 0 < t \le 2^{-l}.$$

Since $0 < \alpha_X \le \beta_X < 1$ it follows that $0 < \gamma_\varphi \le \delta_\varphi < 1$. Therefore, from the definition of $\tilde{F}$ it follows that both $\tilde{F}$ and its complementary function satisfy the Δ_2-condition. Hence, by Proposition 2.3 and by the definition of $\tilde{F}$ once more, we obtain that there exists a constant $C_1 > 0$ such that, for any $l \in \mathbb{N}$ and for all $x \ge \tilde{F}^{-1}(2^l)$, we have

$$\frac{1}{C_1 l} \le \frac{\tilde{F}(x l^{-1/p})}{\tilde{F}(x)} \le \frac{C_1}{l}.$$

By standard arguments, there are constants $C_2 > 0$ and $C_3 > 0$ such that

$$C_2^{-1} u^p \le \frac{\tilde{F}(ua)}{\tilde{F}(a)} \le C_2 u^p, \tag{52}$$

for all $0 < u \le 1$ and any a satisfying $\tilde{F}(a) \ge C_3 2^{u^{-p}}$.

Suppose that $1 \le a_1 < a_2 < \ldots < a_N$, $\frac{a_k}{a_{k-1}} \ge 2$ for $k = 2, \ldots, N$ such that for all k

$$\text{either} \quad \frac{\tilde{F}(ua_k)}{\tilde{F}(a_k)} \ge 2C_2 u^p \quad \text{or} \quad \frac{\tilde{F}(ua_k)}{\tilde{F}(a_k)} \le \frac{1}{2C_2} u^p.$$

Then, by (52), we have that $\tilde{F}(a_N) \le C_3 2^{u^{-p}}$, which implies $\tilde{F}(a_1 2^{N-1}) \le C_3 2^{u^{-p}}$. Hence, $N \le C_4 u^{-p}$, that is, $\Psi^{\infty}_{\tilde{F},p}(u, 2C_2) \le C_4 u^{-p}$ $(0 < u \le 1)$, and (51) is proved.

(b) This part follows immediately from (a) and Theorem 3.1. $\square$

Using equality (50) from the proof of Theorem 6.1, we obtain the following corollary.

Corollary 6.2. *Let X be a symmetric space on $[0,1]$ and w be a weight on $[0,1]$ satisfying the condition (49). Assume that either X is w–decomposable or X has the Fatou property and $(X, X(w))$ is K-monotone couple. If $\varphi_X(t) = t^{1/p}$ for some $1 < p < \infty$, then $X = L_p$.*

Remark 6.3. *Using Krivine's theorem and the arguments from the beginning of the proof of Theorem 6.1, the last assertion can be proved for $p = 1$ and $p = \infty$ as well.*

Remark 6.4. *It is well known that there is a regularly varying at ∞ Orlicz function F such that the corresponding Orlicz space L_F is not ultrasymmetric (see [18]). Thus, Theorems 4.4 and 5.3 show that condition (49) on the weight w from Theorem 6.1 and Corollary 6.2 is essential.*

Remark 6.5. *Conversely, if L_F is an ultrasymmetric reflexive Orlicz space on $[0,1]$, then there is a weight w on $[0,1]$ such that L_F is w-decomposable and, equivalently, the Banach couple $(L_F, L_F(w))$ is K-monotone. In fact, in that case F is regularly varying at ∞ of order $p \in (1,\infty)$ (cf. [4]) and we can apply Theorem 5.3.*

Examples. Theorem 5.3 guarantees that a weighted couple of Orlicz spaces $(L_F, L_F(w))$ on $[0,1]$ is K-monotone for some weight w on $[0,1]$ if F is equivalent to an Orlicz function which is regularly varying at ∞ of order $p \in [1,\infty)$. We present some examples of such Orlicz functions below.

1. The function $F(u) = u^p(1 + |\ln u|)$ for $p \geq (3+\sqrt{5})/2$ is an Orlicz function on $(0,\infty)$ which is regularly varying at ∞ of order p (cf. [24, Example 4]).

2. The function $F(u) = u^p[1 + c\sin(p\ln u)]$ for $0 < c < 1/\sqrt{2}$ and $p \geq (1 - \frac{\sqrt{2}c}{\sqrt{1-2c^2}})^{-1}$ is an Orlicz function on $(0,\infty)$ which is not regularly varying but it is equivalent to u^p and $\frac{1}{4}u^p \leq F(u) \leq 2u^p$ for all $u > 0$ (cf. [24, Example 10] and [25], Example 5 on p. 93 with $c = 1/\sqrt{5}$ and $p \geq 6$).

3. Let an Orlicz function F be equivalent for large u to the function

$$\tilde{F}(u) = u^p(\ln u)^{q_1}(\ln\ln u)^{q_2}\dots(\ln\dots\ln u)^{q_n},$$

where $p \in (1,\infty)$ and $q_1,\dots,q_n$ are arbitrary real numbers. It is easy to see that F is equivalent to a regularly varying function at ∞ of order p (in fact, the corresponding Orlicz space L_F is even ultrasymmetric [4]).

4. Some more examples of Orlicz functions that are equivalent to some regularly varying functions at ∞ of order p are given by Kalton [18].

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