On the Finite Dimensional Joint Characteristic Function of Lévy's Stochastic Area Processes

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Abstract

The goal of this paper is to derive a formula for the finite dimensional joint characteristic function (the Fourier transform of the finite dimensional distribution) of the coupled process $\{(W_t, L_t^A) : t \in [0, \infty)\}$, where $\{W_t : t \in [0, \infty)\}$ is a d-dimensional Brownian motion and $\{L_t^A : t \in [0, \infty)\}$ is the generalized d-dimensional Lévy's stochastic area process associated to a $d \times d$ matrix A. Here A need not be skew-symmetric, and in our computation we allow A to vary. The problem finally reduces to the solution of a recursive system of symmetric matrix Riccati equations and a system of independent first order linear matrix ODEs. As an example, the two dimensional Lévy's stochastic area process is studied in detail.

1 Introduction

The Lévy's stochastic area process $\{L_t : t \in [0, +\infty)\}$ associated with a two dimensional Brownian motion $\{(W_t^{(1)}, W_t^{(2)}) : t \in [0, +\infty)\}$ was introduced by P. Lévy in [10] defined as

$$L_t := \frac{1}{2} \int_0^t (W_s^{(1)} dW_s^{(2)} - W_s^{(2)} dW_s^{(1)}), \quad t \geqslant 0.$$

The geometric meaning of the process L_t is the algebraic area enclosed by the two dimensional Brownian path up to time t and the segment of the origin O and the point $(W_t^{(1)}, W_t^{(2)})$. By using stochastic Fourier expansion of Brownian motions, Lévy derived the conditional characteristic function of L_t with respect to W_t as

$$\mathbb{E}[\exp(i\lambda L_t)|W_t = x] = \frac{t\lambda}{2\sinh(t\lambda/2)} \exp\left[\frac{|x|^2}{2t}\left(1 - \frac{t\lambda}{2}\coth\frac{t\lambda}{2}\right)\right].$$

It follows that the characteristic function of L_t is given by

$$\mathbb{E}[\exp(i\lambda L_t)] = \frac{1}{\cosh(\lambda t/2)}$$

and the joint characteristic function of the coupled process $\{(W_t, L_t) : t \geqslant 0\}$ can be computed explicitly.

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To compute the finite dimensional joint characteristic function of the coupled process is harder than the one-dimensional marginal one, since it involves complicated correlations among different time spots; although the process is Markovian, it is still nontrivial to derive the transition density explicitly.

In T. Hida's paper [5], he considered the problem from a different and a more general perspective, under the framework of Wiener-Itô's chaos decomposition. He studied the law of quadratic functionals of Brownian motion on the canonical Wiener space W, which are elements in the second order chaos component. The classical Lévy's stochastic area process is a special case in his general setting. The main idea of his approach is to use the celebrated Wiener-Itô's chaos decomposition theorem to construct a one-to-one correspondance between such quadratic functionals and symmetric Hilbert-Schmidt operators on the Cameron-Martin subspace H of W. By a formal computation, he found out that for any quadratic functional F on W, if B is the corresponding symmetric Hilbert-Schmidt operator on H, then the characteristic function (Fourier transform) of F is given by

$$\int_W e^{i\lambda F(\omega)}\mu(d\omega) = \prod_{n=1}^{\infty}(1 - 2i\lambda\lambda_n)^{-1/2}e^{-i\lambda\lambda_n},$$

where $\{\lambda_n : n \geqslant 1\}$ is the family of eigenvalues of B with multiplicity. He also pointed out that the right hand side of the above identity is equal to

$$(\det{}_2(I - 2i\lambda B))^{-1/2},$$

where $\det_2$ denotes the regularized Fredholm determinant. By studying the special example of Lévy's stochastic area process and computing the eigenvalues of the corresponding operator, he recovered Lévy's formula.

After Hida's important work, different methods of computing the regularized Fredholm determinant in order to study the laws of a wider class of quadratic Wiener functionals were developed by several Japanese mathematicians: N. Ikeda, S. Manabe, S. Kusuoka, K. Hara, etc.. The fundamental ideas in their works can be summerized as two approaches: to compute the spectrum of the corresponding Hilbert-Schmidt operator directly, or to reduce the infinite dimensional case to the finite dimensional one by posing additional assumptions on the operator. The first approach was developed in Ikeda and Manabe's paper [7], in which they computed the spectrum of a variety of quadratic Wiener functionals in order to study the asymptotic behavior of stochastic oscillatory integrals on the Wiener space. The second approach was developed from different aspects based on a fundamental decomposition assumption proposed by Ikeda, Kusuoka and Manabe. They restricted themselves to the study of operators which could be divided into an operator of finite dimensional range and an operator of Volterra's type, in order to reduce the case to a finite dimensional one in a certain sense. In Ikeda, Kusuoka and Manabe's paper [6], based on such decomposition, the computation of the regularized determinant is reduced to the computation of a finite dimensional determinant, in which algebraic methods and differential equations methods can be applied. Later in [8] they developed a general method to compute the law of the corresponding type of quadratic Wiener functionls by using the Cameron-Martin transformation along critical paths. In the meanwhile, they put forward another idea for the computation by relating the problem to the Van Vleck-Pauli formula in quantum mechanics and derived a formula for the law of a certain class of quadratic Wiener functionals by using physical approach. In the paper [3] of Hara and Ikeda, they

further developed the ideas in [8] by relating the problem to the study of dynamics on Grassmannians. Due to the fact that the determinant is expressed by the solutions of the Jacobi equation with Van Vleck-Pauli's formula, they showed that the determinant could be computed in terms of Plüchker coordinates of Grassmannian manifolds.

From the works on computing the regularized determinant, one can see that the computation is very involved, even reduced to the finite dimensional case. In the series of works mentioned based on Hida's formula, they considered the marginal law of the two dimensional Lévy's stochastic area process and recovered Lévy's formula.

It should be pointed out that after Lévy's original work, several mathematicians proposed different methods to compute the marginal characteristic function of the two dimensional Lévy's stochastic area process directly. In particular, B. Gaveau [2] studied the marginal distribution of the process (inversion of the characteristic function). In the present paper, we are going to study the finite dimensional joint characteristic funtion of the d-dimensional generalized Lévy's stochastic area processes from a different angle of view by computing the multi-dimensional Fourier transform directly. Our work is based on the idea of K. Helmes and A. Schwane in their paper [4] for the computation of the marginal characteristic function of the d-dimensional Brownian motion together with the general d-dimensional Lévy's stochastic area processes.

Let $\{W_t : t \in [0, \infty)\}$ be a d-dimensional Brownian motion, and let $so(d)$ be the space of $d \times d$ skew-symmetric matrices, where $d \geqslant 2$. Fix $A \in so(d)$, the process

$$L_t^A := \int_0^t \langle AW_s, dW_s \rangle, \quad t \in [0, \infty),$$

is called the general d-dimensional Lévy's stochastic area process associated to A. In Helmes and Schwane's paper [4], they derived a formula for the marginal characteristic funcion of the coupled process $\{(W_t, L_t^A) : t \in [0, \infty)\}$, based on the Cameron-Martin-Girsanov's transformation theorem and Itô's formula. In our paper, we are going to further exploit this method to establish the finite dimensional joint characteristic function (the Fourier transform of the finite dimensional distribution) of the coupled process in a more general setting. We will see that it reduces to the solution of a recursive system of symmetric matrix Riccati equations and a system of independent first order linear matrix ODEs.

Since the computation is quite lengthy, we first present the main result in our paper and the main idea for the proof. All details and technical steps are left to section 2. As an example, we study the two dimensional Lévy's stochastic area process in detail.

Assume that $\{W_t, \mathcal{F}_t : t \geqslant 0\}$ is a d-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any A being a real $d \times d$ matrix (not necessarily skew-symmetric), we define the generalized Lévy's stochastic area process $\{L_t^A : t \geqslant 0\}$ associated to A as

$$L_t^A := \int_0^t \langle AW_s, dW_s \rangle, \quad t \geqslant 0.$$

Fix $0 = t_0 < t_1 < t_2 < \cdots < t_n < \infty$, $n \geqslant 1$. Assume that $A_1, \cdots, A_n$ are real $d \times d$ matrices. For $\gamma_1, \cdots, \gamma_n \in \mathbb{R}^d$, $\Lambda_1, \cdots, \Lambda_n \in \mathbb{R}$, we are going to compute the following functional:

$$f(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n) := \mathbb{E}\left[\exp\left(\sum_{k=1}^n i\langle \gamma_k, W_{t_k} \rangle + \sum_{k=1}^n i\Lambda_k L_{t_k}^{A_k}\right)\right].$$

where $i = \sqrt{-1}$. Notice that the matrix can vary on different time spots.

Throughout this paper, $0 < t_1 < \cdots < t_n < \infty$ and $A_1, \cdots, A_n \in M^d(\mathbb{R})$ will be fixed. The notation $*$ will denote the transpose operator. We should point out that even for complex matrices, $*$ is simply the transpose without taking conjugate. For $z = (z^1, \cdots, z^j) \in \mathbb{C}^n$, we use $\langle z \rangle^2$ to denote $\sum_{j=1}^n (z^j)^2$ which is the holomorphic extension of the real case. Tr will be denoted as the trace operator.

Our main result is the following.

Theorem 1.1 *The functional* $f(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n)$ *is determined by*

$$f(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n) = \prod_{j=1}^n \exp\{\frac{1}{2}\int_0^{t_j} Tr(K_{i\Lambda_j, \cdots, i\Lambda_n}(s))ds$$
$$-\frac{1}{2}\int_{t_{j-1}}^{t_j} \langle H_{i\Lambda_j, \cdots, i\Lambda_n}^{*-1}(s)H_{i\Lambda_j, \cdots, i\Lambda_n}^*(t_j)\mu_j\rangle^2 ds\}.$$

Here $\{K_{i\Lambda_j, \cdots, i\Lambda_n}(t) : t \in [0, t_j], j = 1, \cdots, n\}$ *is defined as the solution of the recursive system of* n *symmetric matrix Riccati equations (starting from* $j = n$ *to* $j = 1$*):*

$$\frac{d}{dt}K_{i\Lambda_j, \cdots, i\Lambda_n}(t) = C_{i\Lambda_j, \cdots, i\Lambda_n}(t) - K_{i\Lambda_j, \cdots, i\Lambda_n}(t)(\sum_{r=j+1}^n K_{i\Lambda_r, \cdots, i\Lambda_n}(t)$$
$$+ \sum_{r=j}^n (i\Lambda_r A_r)) - (\sum_{r=j+1}^n K_{i\Lambda_r, \cdots, i\Lambda_n}(t)$$
$$+ \sum_{r=j}^n (i\Lambda_r A_r^*))K_{i\Lambda_j, \cdots, i\Lambda_n}(t) - K_{i\Lambda_j, \cdots, i\Lambda_n}^2(t),$$
$$t \in [0, t_j],$$
$$K_{i\Lambda_j, \cdots, i\Lambda_n}(t_j) = 0,$$

where

$$C_{i\Lambda_j, \cdots, i\Lambda_n}(t) := \Lambda_j^2 A_j^* A_j - i\Lambda_j[(\sum_{r=j+1}^n K_{i\Lambda_r, \cdots, i\Lambda_n}(t) + \sum_{r=j+1}^n (i\Lambda_r A_r^*))A_j$$
$$+ A_j^*(\sum_{r=j+1}^n K_{i\Lambda_r, \cdots, i\Lambda_n}(t) + \sum_{r=j+1}^n (i\Lambda_r A_r))], \quad t \in [0, t_j].$$

Moreover, $\{H_{i\Lambda_j, \cdots, i\Lambda_n}(t) : t \in [0, t_j], j = 1, \cdots, n\}$ *is the solution of the system of* n *independent linear matrix ODEs:*

$$\frac{d}{dt}H_{i\Lambda_j, \cdots, i\Lambda_n}(t) = (\sum_{r=j}^n K_{i\Lambda_r, \cdots, i\Lambda_n}(t) + \sum_{r=j}^n (i\Lambda_r A_r))H_{i\Lambda_j, \cdots, i\Lambda_n}(t),$$
$$t \in [0, t_j],$$
$$H_{i\Lambda_j, \cdots, i\Lambda_n}(0) = Id,$$
$$j = 1, \cdots, n,$$

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and $\{\mu_j : j = 1, \cdots, n\}$ is defined recursively as

$$
\begin{aligned}
\mu_n &:= \gamma_n, \\
\mu_j &:= \gamma_j + H^{*-1}_{i\Lambda_{j+1}, \cdots, i\Lambda_n}(t_j) H^{*}_{i\Lambda_{j+1}, \cdots, i\Lambda_n}(t_{j+1}) \mu_{j+1}, \\
&\quad j = n-1, \cdots, 1.
\end{aligned}
$$

The subscripts of K and H are used to indicate the dependence on Λ. We do not wish to address the existence and uniqueness of the preceeding maxtrix differential system of Riccati type. Under our context, there is a unique solution to the system appearing in Theorem 1.1, for details, the reader may consult [9], [13], [14] and etc.

Now we are going to briefly express the main idea of the proof in an informal way.

The fundamental tool of the proof is Girsanov's transformation. However, it fits for real-valued processes only. Hence we focus on the real case first, that is, consider the functinal

$$
g(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) := \mathbb{E}[\exp\{\sum_{k=1}^{n} i\langle \gamma_k, W_{t_k} \rangle + \sum_{k=1}^{n} \lambda_k L^{A_k}_{t_k}\}]
$$

first for small $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$, and then try to complexify the result by standard arguments in complex analysis.

Our idea of computing the functional g is to eliminate the stochastic integrals one by one, starting from the largest time interval $[0, t_n]$, by using the Girsanov's transformation theorem, and try to track the original Brownian motion along each time of transformation in order to handle the W. When applying change of measure, a stochastic integral (with respect to a proper Brownian motion) is transformed to one-half of its quadratic variation, which is a Lebesgue integral almost surely. To handle this term, we introduce a symmetric matrix Riccati equation to split it into three terms, which will be transformed to a deterministic one by change of measure again and applying Itô's formula for a suitable process. Recursively, we will be able to cancel out all of the n stochastic integrals in g, and obtain a recursive system of symmetric matrix Riccati equations. To handle the W term, we need to track the change of W along every time of transformation. From Girsanov's theorem, we will see that the diffusion form of W under each transformation is actually invariant (it is always Gaussian by solving a linear SDE), which will enable us to do the computation easily. This procedure will lead us to a system of independent first order linear matrix ODEs.

Our interest in looking for an (as far as possible) explicit formula for the joint law of Brownian motion together with its Lévy area is motivated by the recent progress in the understanding of the solutions of Itô's stochastic differential equations revealed recently in T. Lyons' work [12], in which it has been demonstrated that a large class of Wiener functionals (including Itô solutions to SDEs) are continuous with respect to Brownian sample paths *and* their Lévy areas equipped with the law of Brownian motion together with the Lévy area process. For details about these developments, see T. Lyons [12], T. Lyons and Z. Qian [11], P.K. Friz and N.B. Victoir [1].

2   Proof of the Main Result

Now we are going to work out the idea in section 1. The procedure is to handle the real case first and then use complexification.

The following technical lemma is important (see also [4]), which gives us finiteness in the real case.

Lemma 2.1 *There exists a number $c > 0$, such that*

$$\sup\{\mathbb{E}[\exp\{\sum_{k=1}^{n}\lambda_k L_{t_k}^{A_k}\}] : \lambda_j \in (-c, c), j = 1, 2, \cdots, n\} < \infty.$$

Proof. It suffices to consider the case when $n = 1$. By Cauchy-Schwarz's inequality, we have

$$
\begin{aligned}
\mathbb{E}[\exp\{\lambda L_t^A\}] &= \mathbb{E}[\exp\{\lambda \int_0^t \langle AW_s, dW_s \rangle\}] \\
&= \mathbb{E}[\exp\{\lambda \int_0^t \langle AW_s, dW_s \rangle - \lambda^2 \int_0^t |AW_s|^2 ds\} + \lambda^2 \int_0^t |AW_s|^2 ds\}] \\
&\leqslant \mathbb{E}^{1/2}[\exp\{2\lambda \int_0^t \langle AW_s, dW_s \rangle - \frac{(2\lambda)^2}{2} \int_0^t |AW_s|^2 ds\}] \cdot \\
&\quad \mathbb{E}^{1/2}[\exp\{2\lambda^2 \int_0^t |AW_s|^2 ds\}].
\end{aligned}
$$

Notice that the first term inside the expectation $\mathbb{E}$ on the right hand side is a local martingale, the finiteness of $\mathbb{E}[\exp\{\lambda L_t^A\}]$ will follow immediately once we show that

$$\mathbb{E}[\exp\{\lambda^2 \int_0^t |AW_s|^2 ds\}] < \infty$$

when λ is small enough. By Jensen's inequality, it remains to show that

$$\sup_{s \in [0,t]} \mathbb{E}[\exp\{\lambda^2 |AW_s|^2\}] < \infty$$

when λ is small enough, which is obvious by simple calculation based on Gaussian random variables. $\blacksquare$

Now consider small $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ as in lemma 1, $\gamma_1, \cdots, \gamma_n \in \mathbb{R}$, and the functional

$$g(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) = \mathbb{E}[\exp\{\sum_{k=1}^{n} i \langle \gamma_k, W_{t_k} \rangle + \sum_{k=1}^{n} \lambda_k L_{t_k}^{A_k}\}]$$

defined in section one. The following proposition gives the formula of g.

Proposition 2.2 *The function $g(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n)$ is determined by*

$$
\begin{aligned}
&g(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) \\
&= \prod_{j=1}^{n} \exp\{\frac{1}{2} \int_0^{t_j} Tr(K_{\lambda_j, \cdots, \lambda_n}(s)) ds - \frac{1}{2} \int_{t_{j-1}}^{t_j} |H_{\lambda_j, \cdots, \lambda_n}^{*-1}(s) H_{\lambda_j, \cdots, \lambda_n}^{*}(t_j) \mu_j|^2 ds\}.
\end{aligned}
$$

Here $\{K_{\lambda_j, \cdots, \lambda_n}(t) : t \in [0, t_j], j = 1, 2, \cdots, n\}$ is defined recursively (starting from $j = n$) by the symmetric matrix Riccati equation

$$
\begin{aligned}
\frac{d}{dt} K_{\lambda_j, \cdots, \lambda_n}(t) &= C_{\lambda_j, \cdots, \lambda_n}(t) - K_{\lambda_j, \cdots, \lambda_n}(t)(\sum_{r=j+1}^{n} K_{\lambda_r, \cdots, \lambda_n}(t) + \sum_{r=j}^{n}(\lambda_r A_r)) \\
&\quad -(\sum_{r=j+1}^{n} K_{\lambda_r, \cdots, \lambda_n}(t) + \sum_{r=j}^{n}(\lambda_r A_r^*)) K_{\lambda_j, \cdots, \lambda_n}(t) - K_{\lambda_j, \cdots, \lambda_n}^{2}(t), \\
&\quad t \in [0, t_j], \\
K_{\lambda_j, \cdots, \lambda_n}(t_j) &= 0,
\end{aligned}
$$

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where

$$C_{\lambda_j,\cdots,\lambda_n}(t) : = -\lambda_j^2 A_j^* A_j - \lambda_j[(\sum_{r=j+1}^n K_{\lambda_r,\cdots,\lambda_n}(t) + \sum_{r=j+1}^n (\lambda_r A_r^*))A_j$$

$$+ A_j^*(\sum_{r=j+1}^n K_{\lambda_r,\cdots,\lambda_n}(t) + \sum_{r=j+1}^n (\lambda_r A_r))], \quad t \in [0, t_j]$$

is also defined recursively starting from $j = n$. Moreover, $\{H_{\lambda_j,\cdots,\lambda_n}(t) : t \in [0, t_j], j = 1, 2, \cdots, n\}$ is the solution of the system of n independent first order linear matrix ODEs

$$\frac{d}{dt}H_{\lambda_j,\cdots,\lambda_n}(t) = (\sum_{r=j}^n K_{\lambda_r,\cdots,\lambda_n}(t) + \sum_{r=j}^n(\lambda_j A_j))H_{\lambda_j,\cdots,\lambda_n}, \quad t \in [0, t_j],$$

$$H_{\lambda_j,\cdots,\lambda_n}(0) = Id,$$

$$j = 1, 2, \cdots, n,$$

and $\{\mu_j : j = 1, 2, \cdots, n\}$ is defined recursively by

$$\mu_n := \gamma_n,$$

$$\mu_j := \gamma_j + H_{\lambda_{j+1},\cdots,\lambda_n}^{*-1}(t_j)H_{\lambda_{j+1},\cdots,\lambda_n}^*(t_{j+1})\mu_{j+1}, \quad j = 1, 2, \cdots, n-1.$$

Proof. We divide our proof into two steps.

(1) Step one.

Consider first that $\gamma_1 = \cdots \gamma_n = 0$, and write

$$h(\lambda_1, \cdots, \lambda_n) := \mathbb{E}[\exp\{\sum_{k=1}^n \lambda_k L_{t_k}^{A_k}\}].$$

$$= \mathbb{E}[\exp\{\sum_{k=1}^n \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s\rangle\}].$$

By changing the original probability measure P on the largest time interval $[0, t_n]$, we have

$$h(\lambda_1, \cdots, \lambda_n) = \mathbb{E}[\exp\{\sum_{k=1}^{n-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s\rangle + \lambda_n \int_0^{t_n} \langle A_n W_s, dW_s\rangle$$

$$- \frac{\lambda_n^2}{2}\int_0^{t_n} |A_n W_s|^2 ds + \frac{\lambda_n^2}{2}\int_0^{t_n} |A_n W_s|^2 ds\}]$$

$$= \mathbb{E}_n[\exp\{\sum_{k=1}^{n-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s\rangle + \frac{\lambda_n^2}{2}\int_0^{t_n} |A_n W_s|^2 ds\}],$$

where $\mathbb{E}_n$ denotes the expectation under the probability measure

$$dP_n := \exp\{\lambda_n \int_0^{t_n} \langle A_n W_s, dW_s\rangle - \frac{\lambda_n^2}{2}\int_0^{t_n} |A_n W_s|^2 ds\}dP.$$

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Notice that $\{W_t : t \in [0, t_n]\}$ may not be a Brownian motion under the new measure P_n. However, by the Girsanov's theorem, under P_n, the process

$$W_t^{(n)} := W_t - \lambda_n \int_0^t A_n W_s ds, \quad t \in [0, t_n],$$

is a Brownian motion, and the original process $\{W_t : t \in [0, t_n]\}$ satisfies the following SDE:

$$dW_t = \lambda_n A_n W_t dt + dW_t^{(n)}, \quad t \in [0, t_n].$$

In order to eliminate the integral over $[0, t_n]$, let $C_n(t) := -\lambda_n^2 A_n^* A_n$ ($t \in [0, t_n]$) and introduce the following matrix Riccati equation:

$$
\begin{aligned}
\frac{d}{dt} K_{\lambda_n}(t) &= C_n(t) - \lambda_n[K_{\lambda_n}(t)A_n + A_n^* K_{\lambda_n}(t)] - K_{\lambda_n}^2(t), \quad t \in [0, t_n], \\
K_{\lambda_n}(t_n) &= 0.
\end{aligned}
$$

From now on, to simplify our notation, we will use K_n to denote K_{λ_n}, and later we will also use K_j to denote $K_{\lambda_j, \cdots, \lambda_n}$, $j = 1, \cdots, n-1$ to omit the indication on the dependence on λ. By symmetricity of the equation, the unique solution $\{K_n(t) : t \in [0, t_n]\}$ is symmetric. Hence, by substitution, we have

$$
\begin{aligned}
h(\lambda_1, \cdots, \lambda_n) &= \mathbb{E}_n[\exp\{\sum_{k=1}^{n-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_n} W_s^* C_n(s) W_s ds\}] \\
&= \mathbb{E}_n[\exp\{\sum_{k=1}^{n-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_n} W_s^* \frac{d}{ds} K_n(s) W_s ds \\
&\quad - \frac{1}{2} \int_0^{t_n} W_s^* [K_n(s)(\lambda_n A_n) + \lambda_n A_n^* K_n(s)] W_s ds \\
&\quad - \frac{1}{2} \int_0^{t_n} |K_n(s) W_s|^2 ds\}] \\
&= \mathbb{E}_n[\exp\{\sum_{k=1}^{n-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_n} W_s^* \frac{d}{ds} K_n(s) W_s ds \\
&\quad - \int_0^{t_n} \langle K_n(s) W_s, \lambda_n A_n W_s \rangle ds - \frac{1}{2} \int_0^{t_n} |K_n(s) W_s|^2 ds\}].
\end{aligned}
$$

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By changing of measure again,

$$
\begin{aligned}
h(\lambda_1, \cdots, \lambda_n) &= \widetilde{\mathbb{E}_n}[\exp\{\sum_{k=1}^{n-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_n} W_s^* \frac{d}{ds} K_n(s) W_s ds \\
&\quad - \int_0^{t_n} \langle K_n(s) W_s, \lambda_n A_n W_s \rangle ds - \int_0^{t_n} \langle K_n(s) W_s, dW_s^{(n)} \rangle \}] \\
&= \widetilde{\mathbb{E}_n}[\exp\{\sum_{k=1}^{n-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_n} W_s^* \frac{d}{ds} K_n(s) W_s ds \\
&\quad - \int_0^{t_n} \langle K_n(s) W_s, \lambda_n A_n W_s \rangle ds \\
&\quad - \int_0^{t_n} \langle K_n(s) W_s, dW_s - \lambda_n A_n W_s ds \rangle \}] \\
&= \widetilde{\mathbb{E}_n}[\exp\{\sum_{k=1}^{n-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_n} W_s^* \frac{d}{ds} K_n(s) W_s ds \\
&\quad - \int_0^{t_n} \langle K_n(s) W_s, dW_s \rangle \}].
\end{aligned}
$$

where $\widetilde{\mathbb{E}_n}$ denotes the expectation under the probability measure

$$
d\widetilde{P_n} := \exp\{\int_0^{t_n} \langle K_n(s) W_s, dW_s^{(n)} \rangle - \frac{1}{2} \int_0^{t_n} |K_n(s) W_s|^2 ds\} dP_n.
$$

Under $\widetilde{P_n}$, the process

$$
\widetilde{W_t^{(n)}} := W_t^{(n)} - \int_0^t K_n(s) W_s ds, \quad t \in [0, t_n]
$$

is a Brownian motion, and the original process $\{W_t : t \in [0, t_n]\}$ satisfies the following SDE:

$$
dW_t = (K_n(t) + \lambda_n A_n) W_t dt + d\widetilde{W_t^{(n)}}, \quad t \in [0, t_n].
$$

It should be pointed out that under $\widetilde{P_n}$, the quadratic variation process of the semi-martingale $\{W_t : t \in [0, t_n]\}$ is actually the same as that of the Brownian motion $\{\widetilde{W_t^{(n)}} : t \in [0, t_n]\}$. Now let $F(t, w) : [0, t_n] \times \mathbb{R}^d \to \mathbb{R}$ be defined as

$$
F(t, w) := w^* K_n(t) w,
$$

by applying Itô's formula to the process $\{F(t, W_t) : t \in [0, t_n]\}$, we have

$$
\int_0^{t_n} W_s^* \frac{d}{ds} K_n(s) W_s ds + 2 \int_0^{t_n} \langle K_n(s) W_s, dW_s \rangle + \int_0^{t_n} Tr(K(s)) ds = 0,
$$

where Tr denotes the trace operator. Therefore, we arrive at

$$
h(\lambda_1, \cdots, \lambda_n) = \exp\{\frac{1}{2} \int_0^{t_n} Tr(K_n(s)) ds\} \cdot \widetilde{\mathbb{E}_n}[\exp\{\sum_{k=1}^{n-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle \}].
$$

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Now we proceed a similar argument over the second largest time interval $[0, t_{n-1}]$. The difference here is that $\{W_t : t \in [0, t_{n-1}]\}$ is not a Brownian motion under the probability measure $\widetilde{P_n}$. However, still by changing of measure, we have

$$h(\lambda_1, \cdots, \lambda_n) \exp\{-\frac{1}{2} \int_0^{t_n} Tr(K_n(s)) ds\}$$

$$= \widetilde{E_n}[\exp\{\sum_{k=1}^{n-2} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle + \lambda_{n-1} \int_0^{t_{n-1}} \langle A_{n-1} W_s, dW_s \rangle\}]$$

$$= \widetilde{E_n}[\exp\{\sum_{k=1}^{n-2} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle + \lambda_{n-1} \int_0^{t_{n-1}} \langle A_{n-1} W_s, (K_n(s) + \lambda_n A_n) W_s \rangle ds$$

$$+ \lambda_{n-1} \int_0^{t_{n-1}} \langle A_{n-1} W_s, \widetilde{dW_s^{(n)}} \rangle - \frac{\lambda_{n-1}^2}{2} \int_0^{t_{n-1}} |A_{n-1} W_s|^2 ds$$

$$+ \frac{\lambda_{n-1}^2}{2} \int_0^{t_{n-1}} |A_{n-1} W_s|^2 ds\}]$$

$$= \mathbb{E}_{n-1}[\exp\{\sum_{k=1}^{n-2} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle + \lambda_{n-1} \int_0^{t_{n-1}} \langle A_{n-1} W_s, (K_n(s) + \lambda_n A_n) W_s \rangle ds$$

$$+ \frac{\lambda_{n-1}^2}{2} \int_0^{t_{n-1}} |A_{n-1} W_s|^2 ds\}].$$

Here $\mathbb{E}_{n-1}$ is the expectation under the probability measure

$$dP_{n-1} := \exp\{\lambda_{n-1} \int_0^{t_{n-1}} \langle A_{n-1} W_s, \widetilde{dW_s^{(n)}} \rangle - \frac{\lambda_{n-1}^2}{2} \int_0^{t_{n-1}} |A_{n-1} W_s|^2 ds\} d\widetilde{P_n}.$$

Under P_{n-1}, the process

$$W_t^{(n-1)} := \widetilde{W_t^{(n)}} - \lambda_{n-1} \int_0^t A_{n-1} W_s ds, \quad t \in [0, t_{n-1}]$$

is a Brownian motion, and the process $\{W_t : t \in [0, t_{n-1}]\}$ satisfies the following SDE:

$$dW_t = (K_n(t) + \lambda_n A_n + \lambda_{n-1} A_{n-1}) W_t dt + dW_t^{(n-1)}, \quad t \in [0, t_{n-1}].$$

Let

$$C_{n-1}(t) := \quad -\lambda_{n-1}^2 A_{n-1}^* A_{n-1} - \lambda_{n-1}[(K_n(t) + \lambda_n A_n^*) A_{n-1}$$
$$+ A_{n-1}^*(K_n(t) + \lambda_n A_n)], \quad t \in [0, t_{n-1}],$$

and introduce the following matrix Riccati equation:

$$\frac{d}{dt} K_{n-1}(t) = C_{n-1}(t) - K_{n-1}(t)(K_n(t) + \lambda_n A_n + \lambda_{n-1} A_{n-1})$$
$$- (K_n(t) + \lambda_n A_n^* + \lambda_{n-1} A_{n-1}^*) K_{n-1}(t) - K_{n-1}^2(t), \ t \in [0, t_{n-1}],$$
$$K_{n-1}(t_{n-1}) = 0,$$

we have

$$h(\lambda_1, \cdots, \lambda_n) \exp\{-\frac{1}{2} \int_0^{t_n} Tr(K_n(s)) ds\}$$

$$= \mathbb{E}_{n-1}[\exp\{\sum_{k=1}^{n-2} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_{n-1}} W_s^* C_{n-1}(s) W_s ds\}]$$

$$= \mathbb{E}_{n-1}[\exp\{\sum_{k=1}^{n-2} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_{n-1}} W_s^* \frac{d}{ds} K_{n-1}(s) W_s ds$$

$$- \int_0^{t_{n-1}} \langle K_{n-1}(s) W_s, (K_n(s) + \lambda_n A_n + \lambda_{n-1} A_{n-1}) W_s \rangle ds$$

$$- \frac{1}{2} \int_0^{t_{n-1}} |K_{n-1}(s) W_s|^2 ds\}]$$

$$= \widetilde{E_{n-1}}[\exp\{\sum_{k=1}^{n-2} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_{n-1}} W_s^* \frac{d}{ds} K_{n-1}(s) W_s ds$$

$$- \int_0^{t_{n-1}} \langle K_{n-1}(s) W_s, (K_n(s) + \lambda_n A_n + \lambda_{n-1} A_{n-1}) W_s \rangle ds$$

$$- \int_0^{t_{n-1}} \langle K_{n-1}(s) W_s, dW_s^{(n-1)} \rangle \}]$$

$$= \widetilde{E_{n-1}}[\exp\{\sum_{k=1}^{n-2} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle - \frac{1}{2} \int_0^{t_{n-1}} W_s^* \frac{d}{ds} K_{n-1}(s) W_s ds$$

$$- \int_0^{t_{n-1}} \langle K_{n-1}(s) W_s, dW_s \rangle \}].$$

Here we have changed the probability measure from P_{n-1} to

$$d\widetilde{P_{n-1}} := \exp\{\int_0^{t_{n-1}} \langle K_{n-1}(s) W_s, dW_s^{(n-1)} \rangle - \frac{1}{2} \int_0^{t_{n-1}} |K_{n-1}(s) W_s|^2 ds\} dP_{n-1}.$$

By applying Itô's formula to the process $\{W_t^* K_{n-1}(t) W_t : t \in [0, t_{n-1}]\}$ and noticing that the quadratic variation process of the semi-martingale $\{W_t : t \in [0, t_{n-1}]\}$ is the same as that of a Brownian motion, we again have

$$\int_0^{t_{n-1}} W_s^* \frac{d}{ds} K_{n-1}(s) W_s ds + 2 \int_0^{t_{n-1}} \langle K_{n-1}(s) W_s, dW_s \rangle + \int_0^{t_{n-1}} Tr(K_{n-1}(s)) ds = 0.$$

Therefore, we arrive at

$$h(\lambda_1, \cdots, \lambda_n) \exp\{-\frac{1}{2} \int_0^{t_n} Tr(K_n(s)) ds\}$$

$$= \exp\{\frac{1}{2} \int_0^{t_{n-1}} Tr(K_{n-1}(s)) ds\} \cdot \widetilde{\mathbb{E}_{n-1}}[\exp\{\sum_{k=1}^{n-2} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle \}].$$

Now the recursion is quite obvious from the key observation that when applying transformation of probability measures, the original process $\{W_t\}$(over the proper time interval) is always a diffusion of the same kind, namely, it satisfies an SDE of the form

$$dW_t = Q(t)W_t dt + dB_t,$$

where $\{B_t\}$ is a Brownian motion under the corresponding probability measure. To be more precise, after j steps, we will have a system of j matrix Riccati equations defined recursively,

$$
\begin{cases}
C_n(t) & := -\lambda_n^2 A_n^* A_n, \quad t \in [0, t_n], \\
\frac{d}{dt} K_n(t) & = C_n(t) - \lambda_n[K_n(t)A_n + A_n^* K_n(t)] - K_n^2(t), \quad t \in [0, t_n], \\
K_n(t_n) & = 0;
\end{cases}
$$

$$
\begin{cases}
C_{n-1}(t) & := -\lambda_{n-1}^2 A_{n-1}^* A_{n-1} - \lambda_{n-1}[(K_n(t) + \lambda_n A_n^*)A_{n-1} + A_{n-1}^*(K_n(t) \\
& \quad + \lambda_n A_n)], \quad t \in [0, t_{n-1}], \\
\frac{d}{dt} K_{n-1}(t) & = C_{n-1}(t) - K_{n-1}(t)(K_n(t) + \lambda_n A_n + \lambda_{n-1}A_{n-1}) \\
& \quad - (K_n(t) + \lambda_n A_n^* + \lambda_{n-1}A_{n-1}^*)K_{n-1}(t) - K_{n-1}^2(t), \quad t \in [0, t_{n-1}], \\
K_{n-1}(t_{n-1}) & = 0;
\end{cases}
$$

$$. \\ . \\ .$$

$$
\begin{cases}
C_{n-j+1}(t) & := -\lambda_{n-j+1}^2 A_{n-j+1}^* A_{n-j+1} - \lambda_{n-j+1}[(\sum_{r=n-j+2}^n K_r(t) \\
& \quad + \sum_{r=n-j+2}^n (\lambda_r A_r^*))A_{n-j+1} + A_{n-j+1}^*(\sum_{r=n-j+2}^n K_r(t) \\
& \quad + \sum_{r=n-j+2}^n (\lambda_r A_r))], \quad t \in [0, t_{n-j+1}], \\
\frac{d}{dt} K_{n-j+1}(t) & = C_{n-j+1}(t) - K_{n-j+1}(t)(\sum_{r=n-j+2}^n K_r(t) + \sum_{r=n-j+1}^n (\lambda_r A_r)) \\
& \quad - (\sum_{r=n-j+2}^n K_r(t) + \sum_{r=n-j+1}^n (\lambda_r A_r^*))K_{n-j+1}(t) - K_{n-j+1}^2(t), \\
& \quad t \in [0, t_{n-j+1}], \\
K_{n-j+1}(t_{n-j+1}) & = 0,
\end{cases}
$$

and we will arrive at

$$
h(\lambda_1, \cdots, \lambda_n)
$$

$$
= \exp\{\frac{1}{2} \sum_{r=n-j+1}^n \int_0^{t_r} Tr(K_r(s))ds\} \cdot \widetilde{\mathbb{E}_{n-j+1}}[\exp\{\sum_{k=1}^{n-j} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle\}].
$$

Here under the probability measure $\widetilde{P_{n-j+1}}$, $\{W_t : t \in [0, t_{n-j+1}]\}$ satisfies the SDE

$$
dW_t = (\sum_{r=n-j+1}^n K_r(t) + \sum_{r=n-j+1}^n (\lambda_r A_r))W_t dt + d\widetilde{W_t^{(n-j+1)}}, \quad t \in [0, t_{n-j+1}],
$$

where $\{\widetilde{W_t^{(n-j+1)}} : t \in [0, t_{n-j+1}]\}$ is a Brownian motion under $\widetilde{P_{n-j+1}}$. By carrying out a similar argument, that is, by changing measures and applying Itô's formula, we will obtain

$$h(\lambda_1, \cdots, \lambda_n) \exp\{-\frac{1}{2} \sum_{r=n-j+1}^{n} \int_0^{t_r} Tr(K_r(s))ds\}$$

$$= \exp\{\frac{1}{2} \int_0^{t_{n-j}} Tr(K_{n-j}(s))ds\} \widetilde{\mathbb{E}_{n-j}}[\exp\{\sum_{k=1}^{n-j-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle\}],$$

where $\{K_{n-j}(t) : t \in [0, t_{n-j}]\}$ is the solution of the matrix Riccati equation

$$\frac{d}{dt} K_{n-j}(t) = C_{n-j}(t) - K_{n-j}(t) \left(\sum_{r=n-j+1}^{n} K_r(t) + \sum_{r=n-j}^{n} (\lambda_r A_r) \right)$$

$$- \left(\sum_{r=n-j+1}^{n} K_r(t) + \sum_{r=n-j}^{n} (\lambda_r A_r^*) \right) K_{n-j}(t) - K_{n-j}^2(t),$$

$$t \in [0, t_{n-j}],$$

$$K_{n-j}(t_{n-j}) = 0,$$

in which

$$C_{n-j}(t) := -\lambda_{n-j}^2 A_{n-j}^* A_{n-j} - \lambda_{n-j} \left[\left(\sum_{r=n-j+1}^{n} K_r(t) + \sum_{r=n-j+1}^{n} (\lambda_r A_r^*) \right) A_{n-j} \right.$$

$$\left. + A_{n-j}^* \left(\sum_{r=n-j+1}^{n} K_r(t) + \sum_{r=n-j+1}^{n} (\lambda_r A_r) \right) \right], \quad t \in [0, t_{n-j}].$$

By induction on j, the proof of the case where $\gamma_1 = \cdots = \gamma_n = 0$ is now complete.

(2) Step two.

Now we consider the case with $\gamma_1, \cdots, \gamma_n \in \mathbb{R}$. In step one, the ultimate goal of applying those transformations of probability measures is to eliminate the stochastic integrals one by one, starting from the largest time interval. After each transformation, the distribution of $W_{t_k} (k = 1, \cdots, n)$ is changed. In order to work out the case involving $\gamma_1, \cdots, \gamma_n$, we need to track the original process $\{W_t : t \in [0, t_n]\}$ after each transformation. The main difficulty comes from the fact that if we apply a transformation on $[0, t_k]$, the distribution of $\{W_t\}$ over $[t_k, t_n]$ is hard to compute. However, by using the crucial observation that the diffusion form of $\{W_t\}$ is invariant, w can factor out the term over $[t_k, t_n]$.

Let's formulate the idea in detail. By using the same notation as in step one, we have

$$g(\gamma_1, \cdots, \gamma_n; \lambda_1, \lambda_n)$$

$$= \widetilde{\mathbb{E}_n}[\exp\{\sum_{k=1}^{n} i\langle \gamma_k, W_{t_k} \rangle\} \cdot \Delta_n],$$

where

$$\Delta_n := \exp\{\frac{1}{2} \int_0^{t_n} Tr(K_n(s))ds + \sum_{k=1}^{n-1} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s \rangle\}.$$

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Under $\widetilde{P_n}$, the process $\{W_t : t \in [0, t_n]\}$ is a diffusion of the form

$$dW_t = (K_n(t) + \lambda_n A_n)W_t dt + \widetilde{dW_t^{(n)}}, \quad t \in [0, t_n].$$

Let $\{H_{\lambda_n}(t) : t \in [0, t_n]\}$ be the solution of the following linear matrix ODE

$$\begin{aligned}\frac{d}{dt}H_{\lambda_n}(t) &= (K_n(t) + \lambda_n A_n)H_{\lambda_n}(t), \quad t \in [0, t_n],\\ H_{\lambda_n}(0) &= Id.\end{aligned}$$

Then by the explicit solution of linear SDE, we have

$$W_t = H_{\lambda_n}(t)\int_0^t H_{\lambda_n}^{-1}(s)\widetilde{dW_s^{(n)}}, \quad t \in [0, t_n].$$

Hence

$$\begin{aligned}&g(\gamma_1, \cdots, \gamma_n; \lambda_1, \lambda_n)\\ =&\widetilde{\mathbb{E}_n}[\exp\{\sum_{k=1}^{n-1} i\langle \gamma_k, W_{t_k}\rangle + i\langle \gamma_n, H_{\lambda_n}(t_n)\int_0^{t_n} H_{\lambda_n}^{-1}(s)\widetilde{dW_s^{(n)}}\rangle\} \cdot \Delta_n]\\ =&\widetilde{\mathbb{E}_n}[\exp\{\sum_{k=1}^{n-1} i\langle \gamma_k, W_{t_k}\rangle + i\langle \gamma_n, H_{\lambda_n}(t_n)\int_0^{t_{n-1}} H_{\lambda_n}^{-1}(s)\widetilde{dW_s^{(n)}}\rangle\\ &+ i\langle \gamma_n, H_{\lambda_n}(t_n)\int_{t_{n-1}}^{t_n} H_{\lambda_n}^{-1}(s)\widetilde{dW_s^{(n)}}\rangle\} \cdot \Delta_n].\end{aligned}$$

Notice that the stochastic integral $\int_{t_{n-1}}^{t_n} H_{\lambda_n}^{-1}(s)\widetilde{dW_s^{(n)}}$ is independent of the rest since the integrand is deterministic, we have

$$\begin{aligned}&g(\gamma_1, \cdots, \gamma_n; \lambda_1, \lambda_n)\\ =&\widetilde{\mathbb{E}_n}[\exp\{i\langle \gamma_n, H_{\lambda_n}(t_n)\int_{t_{n-1}}^{t_n} H_{\lambda_n}^{-1}(s)\widetilde{dW_s^{(n)}}\rangle\}]\cdot\\ &\widetilde{\mathbb{E}_n}[\exp\{\sum_{k=1}^{n-1} i\langle \gamma_k, W_{t_k}\rangle + i\langle H_{\lambda_n}^{*-1}(t_{n-1})H_{\lambda_n}^*(t_n)\gamma_n, W_{t_{n-1}}\rangle\} \cdot \Delta_n]\\ =&R_n \cdot \widetilde{\mathbb{E}_n}[\exp\{\sum_{k=1}^{n-2} i\langle \gamma_k, W_{t_k}\rangle + \langle \mu_{n-1}, W_{t_{n-1}}\rangle\} \cdot \Delta_n],\end{aligned}$$

where

$$\begin{aligned}R_n &:= \widetilde{\mathbb{E}_n}[\exp\{i\langle \gamma_n, H_{\lambda_n}(t_n)\int_{t_{n-1}}^{t_n} H_{\lambda_n}^{-1}(s)\widetilde{dW_s^{(n)}}\rangle\}],\\ \mu_{n-1} &:= \gamma_{n-1} + H_{\lambda_n}^{*-1}(t_{n-1})H_{\lambda_n}^*(t_n)\gamma_n.\end{aligned}$$

Now we can see that the random term over $[t_{n-1}t_n]$ is factored out.

Similarly, by applying transformations as in step one, we further have

$$g(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n)$$

$$= R_n \cdot \widetilde{\mathbb{E}_{n-1}}[\exp\{\sum_{k=1}^{n-2} i\langle \gamma_k, W_{t_k}\rangle + \langle \mu_{n-1}, W_{t_{n-1}}\rangle\} \cdot \Delta_{n-1}],$$

where

$$\Delta_{n-1} : \quad = \quad \exp\{\frac{1}{2}\int_0^{t_{n-1}} Tr(K_{n-1}(s))ds + \frac{1}{2}\int_0^{t_n} Tr(K_n(s))ds$$

$$+ \sum_{k=1}^{n-2} \lambda_k \int_0^{t_k} \langle A_k W_s, dW_s\rangle\}.$$

By a similar argument, let $\{H_{\lambda_{n-1},\lambda_n}(t) : t \in [0, t_{n-1}]\}$ be the solution of the equation

$$\frac{d}{dt}H_{\lambda_{n-1},\lambda_n}(t) \quad = \quad (K_n(t) + K_{n-1}(t) + \lambda_n A_n + \lambda_{n-1}A_{n-1})H_{\lambda_{n-1},\lambda_n}(t),$$

$$t \in [0, t_{n-1}],$$

$$H_{\lambda_{n-1},\lambda_n}(0) \quad = \quad Id,$$

we obtain that

$$g(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n)$$

$$= R_n \cdot R_{n-1} \cdot \widetilde{\mathbb{E}_{n-1}}[\exp\{\sum_{k=1}^{n-3} i\langle \gamma_k, W_{t_k}\rangle + i\langle \mu_{n-2}, W_{t_{n-2}}\rangle\} \cdot \Delta_{n-1}],$$

where

$$R_{n-1} \quad := \quad \widetilde{\mathbb{E}_{n-1}}[\exp\{i\langle \mu_{n-1}, H_{\lambda_{n-1},\lambda_n}(t_{n-1})\int_{t_{n-2}}^{t_{n-1}} H_{\lambda_{n-1},\lambda_n}^{-1}(s)\widetilde{dW_s^{(n-1)}}\rangle\}],$$

$$\mu_{n-2} \quad := \quad \gamma_{n-2} + H_{\lambda_{n-1},\lambda_n}^{*-1}(t_{n-2})H_{\lambda_{n-1},\lambda_n}^{*}(t_{n-1})\mu_{n-1}.$$

Finally, by a simple induction argument, we arrive at

$$g(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) = \prod_{j=1}^n (R_j \cdot \exp\{\frac{1}{2}\int_0^{t_j} Tr(K_j(s))ds\}).$$

Here

$$R_j : \quad = \quad \widetilde{\mathbb{E}_j}[\exp\{i\langle \mu_j, H_{\lambda_j,\cdots,\lambda_n}(t_j)\int_{t_{j-1}}^{t_j} H_{\lambda_j,\cdots,\lambda_n}^{-1}(s)\widetilde{dW_s^{(j)}}\}], \quad j = 1, 2, \cdots, n,$$

$\{H_{\lambda_j,\cdots,\lambda_n}(t) : t \in [0, t_j]\}$ is the solution of the equation

$$\frac{d}{dt}H_{\lambda_j,\cdots,\lambda_n}(t) \quad = \quad (\sum_{r=j}^n K_j(t) + \sum_{r=j}^n (\lambda_j A_j))H_{\lambda_j,\cdots,\lambda_n}(t), \quad t \in [0, t_j],$$

$$H_{\lambda_j,\cdots,\lambda_n}(0) \quad = \quad Id,$$

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and $\{\mu_j : j = 1, 2, \cdots, n\}$ is defined recursively by

$$\mu_n := \gamma_n,$$
$$\mu_j := \gamma_j + H_{\lambda_{j+1}, \cdots, \lambda_n}^{*-1}(t_j) H_{\lambda_{j+1}, \cdots, \lambda_n}^*(t_{j+1}) \mu_{j+1}, \quad j = 1, 2, \cdots, n-1.$$

It remains to compute $R_j (j = 1, 2, \cdots, n)$ explicitly, which is easy since everything here is Gaussian. Namely, we have

$$
\begin{aligned}
R_j &= \widetilde{\mathbb{E}_j}[\exp\{i\langle H_{\lambda_j, \cdots, \lambda_n}^*(t_j)\mu_j, \int_{t_{j-1}}^{t_j} H_{\lambda_j, \cdots, \lambda_n}^{-1}(s) \widetilde{dW_s^{(n)}}\}] \\
&= \widetilde{\mathbb{E}_j}[\exp\{i\int_{t_{j-1}}^{t_j} \langle H_{\lambda_j, \cdots, \lambda_n}^{*-1}(s) H_{\lambda_j, \cdots, \lambda_n}^*(t_j)\mu_j, \widetilde{dW_s^{(n)}}\rangle\}] \\
&= \exp\{-\frac{1}{2}\int_{t_{j-1}}^{t_j} |H_{\lambda_j, \cdots, \lambda_n}^{*-1}(s) H_{\lambda_j, \cdots, \lambda_n}^*(t_j)\mu_j|^2 ds\}.
\end{aligned}
$$

Therefore, the proof is now complete and we have

$$
\begin{aligned}
&g(\gamma_1, \cdots, \gamma_n; \lambda_1, \cdots, \lambda_n) \\
&= \prod_{j=1}^n \exp\{\frac{1}{2}\int_0^{t_j} Tr(K_j(s))ds - \frac{1}{2}\int_{t_{j-1}}^{t_j} |H_{\lambda_j, \cdots, \lambda_n}^{*-1}(s) H_{\lambda_j, \cdots, \lambda_n}^*(t_j)\mu_j|^2 ds\}.
\end{aligned}
$$

∎

From the proof of the above proposition, we can see that the computation of

$$\mathbb{E}[\exp\{\sum_{k=1}^n i\langle \gamma_k, W_{t_k}\rangle + \sum_{k=1}^n \lambda_k L_{t_k}^{A_k}\}]$$

for small $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ reduces to the solution of a recursive system of symmetric matrix Riccati equations and the solution of a system of independent first order linear matrix ODEs. If $\gamma_1 = \cdots = \gamma_n = 0$, then we don't need the ODE system at all. Now we are going to complexify our case before.

Lemma 2.3 *Fix $\gamma_1, \cdots, \gamma_n \in \mathbb{R}$. Then when c is small enough, the function*

$$\phi(z_1, \cdots, z_n) := \mathbb{E}[\exp\{\sum_{k=1}^n i\langle \gamma_k, W_{t_k}\rangle + \sum_{k=1}^n z_k L_{t_k}^{A_k}\}]$$

is holomorphic in the domain $D_c := \{(z_1, \cdots, z_n) \in \mathbb{C}^n : Re(z_j) \in (-c, c), j = 1, 2, \cdots, n\}$ of $\mathbb{C}^n$. Moreover, the function

$$
\begin{aligned}
\psi(\lambda_1, \cdots, \lambda_n): &= \prod_{j=1}^n \exp\{\frac{1}{2}\int_0^{t_j} Tr(K_j(s))ds \\
&\quad -\frac{1}{2}\int_{t_{j-1}}^{t_j} |H_{\lambda_j, \cdots, \lambda_n}^{*-1}(s) H_{\lambda_j, \cdots, \lambda_n}^*(t_j)\mu_j|^2 ds\}
\end{aligned}
$$

defined on $\mathbb{R}^n$ can be extended holomorphically to $\mathbb{C}^n$. Such an extension is unique, and when restricted to D_c,

$$\phi(z_1, \cdots, z_n) = \psi(z_1, \cdots, z_n).$$

Proof. By lemma 1, when c is small enough, $\phi(z_1, \cdots, z_n)$ is well defined on D_c. The continuity of $\phi(z_1, \cdots, z_n)$ follows easily from uniform integrability. Moreover, since the function

$$(z_1, \cdots, z_n) \mapsto \exp\{\sum_{k=1}^n i\langle \gamma_k, W_{t_k}\rangle + \sum_{k=1}^n z_k L_{t_k}^{A_k}\}$$

is holomorphic on $\mathbb{C}^n$ for every $\omega \in \Omega$, by Fubini's theorem and Morera's theorem, $\phi(z_1, \cdots, z_n)$ is holomorphic in D_c.

On the other hand, it is obvious that the recursive system of matrix Riccati equations and the system of independent matrix ODEs defined in proposition 2 depend analytically on $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ and extend naturally to the case where $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. Consequently, when $(\lambda_1, \cdots, \lambda_n)$ is replaced by $(z_1, \cdots, z_n) \in \mathbb{C}^n$, the two systems determine solutions depending holomorphically on $z_1, \cdots, z_n$. It follows that $\psi(\lambda_1, \cdots, \lambda_n)$ possesses a unique holomorphic extension to $\mathbb{C}^n$.

Finally, since ϕ and ψ coincide in the set $\{(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \lambda_j \in (-c, c), j = 1, 2, \cdots, n\}$, by the identity theorem, they coincide in D_c. ∎

With the preparations above, the proof of our main result on the formula for the joint characteristic function $f(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n)$ defined in section one now follows easily. In fact, the result follows immediately from proposition 2 and lemma 3 with the observation that the set

$$\{(i\Lambda_1, \cdots, i\Lambda_n) : \Lambda_j \in \mathbb{R}, j = 1, 2, \cdots, n\}$$

is contained in D_c.

3 An Example: the Two Dimensional Lévy's Stochastic Area Process

In this section, we are going to apply our result to study the two dimensional Lévy's stochastic area process, first introduced by Lévy in [10]. Namely, we consider the case where $d = 2$ and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Lévy's stochastic area process is given by

$$L_t = \int_0^t (W_s^{(1)} dW_s^{(2)} - W_s^{(2)} dW_s^{(1)}), \quad t \geqslant 0.$$

We try to derive an explicit formula for the finite dimensional joint characteristic function of the coupled process $\{(W_t, L_t) : t \geqslant 0\}$.

Throughout this section, $0 = t_0 < t_1 < \cdots < t_n$ will be fixed.

In section 2, the computation of the finite dimensional joint characteristic function of the coupled process $\{(W_t, L_t^A) : t \geqslant 0\}$ reduces to the solution of a recursive system of symmetric Riccati equations and a system of independent first order linear matrix ODEs. In the case here, we will see that the Riccati system is actually real and scalar, and the linear system is explicitly solvable. In fact, we have:

Proposition 3.1 *For the two dimensional Lévy's stochastic area process $\{L_t : t \geqslant 0\}$, by using the same notation as in theorem 1 with the assumption*

$$A_1 = \cdots = A_n = A,$$

the solution matrices of the Riccati system are real diagonal matrices with identical diagonal entries, and the Riccati system essentially reduces to a system of real scalar Riccati equations recursively defined from $j = n$ to $j = 1$ by

$$\frac{d}{dt}k_{i\Lambda_j,\cdots,i\Lambda_n}(t) = (\Lambda_j^2 + 2\Lambda_j \sum_{r=j+1}^{n} \Lambda_r) - 2(\sum_{r=j+1}^{n} k_{i\Lambda_r,\cdots,i\Lambda_n}(t))k_{i\Lambda_j,\cdots,i\Lambda_n}(t)$$

$$-k_{i\Lambda_j,\cdots,i\Lambda_n}^2(t), \quad t \in [0, t_j],$$

$$k_{i\Lambda_j,\cdots,i\Lambda_n}(t_j) = 0.$$

Moreover, the linear system in theorem 1 is explicitly solvable, namely, for $j = 1, 2, \cdots, n$,

$$H_{i\Lambda_j,\cdots,i\Lambda_n}(t) = \exp\{\int_0^t (\sum_{r=j}^{n} K_{i\Lambda_r,\cdots,i\Lambda_n}(s) + i(\sum_{r=j}^{n} \Lambda_r)A)ds\}, \quad t \in [0, t_j].$$

Proof. We first consider the Riccati system. For $j = n$,

$$C_{i\Lambda_n}(t) = \begin{pmatrix} \Lambda_n^2 & 0 \\ 0 & \Lambda_n^2 \end{pmatrix}, \quad t \in [0, t_n],$$

and the Riccati equation is defined by

$$\frac{d}{dt}K_{i\Lambda_n}(t) = \begin{pmatrix} \Lambda_n^2 & 0 \\ 0 & \Lambda_n^2 \end{pmatrix} - i\Lambda_n \cdot (K_{i\Lambda_n}(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K_{i\Lambda_n}(t)) - K_{i\Lambda_n}^2(t), \quad t \in [0, t_n],$$

$$K_{i\Lambda_n}(t_n) = 0.$$

By uniqueness, it is easy to see that the solution of the above equation is given by a matrix of the form

$$K_{i\Lambda_n}(t) = \begin{pmatrix} k_{i\Lambda_n}(t) & 0 \\ 0 & k_{i\Lambda_n}(t) \end{pmatrix}, \quad t \in [0, t_n],$$

where $\{k_{i\Lambda_n}(t) : t \in [0, t_n]\}$ solves the real scalar Riccati equation

$$\frac{d}{dt}k_{i\Lambda_n}(t) = \Lambda_n^2 - k_{i\Lambda_n}^2(t), \quad t \in [0, t_n],$$

$$k_{i\Lambda_n}(t_n) = 0.$$

For $j = n - 1$, by easy computation we have

$$C_{i\Lambda_{n-1},i\Lambda_n}(t) = \begin{pmatrix} \Lambda_n^2 + 2\Lambda_{n-1}\Lambda_n & 0 \\ 0 & \Lambda_n^2 + 2\Lambda_{n-1}\Lambda_n \end{pmatrix}, \quad t \in [0, t_{n-1}].$$

It follows from uniqueness and the case $j = n$ that $K_{i\Lambda_{n-1},i\Lambda_n}(t)$ is also a diagonal matrix of the form

$$K_{i\Lambda_{n-1},i\Lambda_n}(t) = \begin{pmatrix} k_{i\Lambda_{n-1},i\Lambda_n}(t) & 0 \\ 0 & k_{i\Lambda_{n-1},i\Lambda_n}(t) \end{pmatrix}, \quad t \in [0, t_{n-1}],$$

where $\{k_{i\Lambda_{n-1},i\Lambda_n}(t) : t \in [0, t_{n-1}]\}$ solves the real scalar Riccati equation

$$\begin{aligned} \frac{d}{dt}k_{i\Lambda_n,i\Lambda_{n-1}}(t) &= (\Lambda_n^2 + 2\Lambda_{n-1}\Lambda_n) - 2k_{i\Lambda_n}(t)k_{i\Lambda_{n-1},i\Lambda_n}(t) \\ &\quad -k_{i\Lambda_{n-1},i\Lambda_n}^2(t), \quad t \in [0, t_{n-1}], \\ k_{i\Lambda_{n-1},i\Lambda_n}(t_{n-1}) &= 0. \end{aligned}$$

The rest of the argument follows from recursion easily (the crucial observation is that $C_{i\Lambda_j,\cdots,i\Lambda_n}(t)$ is always a real diagonal matrix with identical diagonal entries).

The second part of the proposition follows immediately from the fact that for $j = 1, 2, \cdots, n$, if we denote $\Phi_j(t)$ as the coefficient matrix of the $j-$th linear ODE of the independent system, then

$$\Phi_j(s)\Phi_j(t) = \Phi_j(t)\Phi_j(s), \quad s, t \in [0, t_j].$$

∎

Now we are going to study the finite dimensional joint characteristic function

$$f(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n)$$

of $\{(W_t, L_t) : t \geqslant 0\}$. To simplify our notation, we use $\{k_j, H_j : j = 1, \cdots, n\}$ to denote the solution $\{k_{i\Lambda_j,\cdots,i\Lambda_n}, H_{i\Lambda_j,\cdots,i\Lambda_n} : j = 1, \cdots, n\}$ in proposition 5. Assume first that (nondegeneracy)

$$\Lambda_j + \cdots + \Lambda_n \neq 0. \quad j = 1, \cdots, n.$$

We first study the process $\{L_t : t \geqslant 0\}$. A crucial observation is that for $j = 1, \cdots, n$, by adding together from the $j-$th equation to the $n-$th equation in the scalar Riccati system in proposition 5, we obtain a neat scalar Riccati equation without linear terms:

$$\frac{d}{dt}(\sum_{r=j}^n k_r(t)) = (\sum_{r=j}^n \Lambda_r)^2 - (\sum_{r=j}^n k_r(t))^2, \quad t \in [0, t_j],$$

in which the unique solution is determined by the terminal data at $t = t_j$. Let

$$c_j := \sum_{r=j}^n \Lambda_r, \quad s_j(t) := \sum_{r=j}^n k_j(t), \quad t \in [0, t_j], j = 1, \cdots, n,$$

then it is not hard to derive that

$$\begin{aligned} s_j(t) &= c_j \frac{c_j \sinh(c_j(t - t_j)) + s_j(t_j)\cosh(c_j(t - t_j))}{c_j \cosh(c_j(t - t_j)) + s_j(t_j)\sinh(c_j(t - t_j))}, \\ &\quad t \in [0, t_j], j = 1, \cdots, n, \end{aligned}$$

where $\{s_j(t_j)\}_{j=1}^n$ is defined recursively by

$$
\begin{aligned}
s_n(t_n) &= 0, \\
s_{j-1}(t_{j-1}) &= c_j \frac{c_j \sinh(c_j(t_{j-1} - t_j)) + s_j(t_j) \cosh(c_j(t_{j-1} - t_j))}{c_j \cosh(c_j(t_{j-1} - t_j)) + s_j(t_j) \sinh(c_j(t_{j-1} - t_j))}, \\
& \quad j = 2, 3, \cdots, n.
\end{aligned}
$$

Now apply theorem 1, we have

$$
\begin{aligned}
& \mathbb{E}[\exp\{\sum_{k=1}^n i\Lambda_k L_{t_k}\}] \\
&= \exp\{\sum_{j=1}^n \frac{1}{2} \int_0^{t_j} Tr(K_{i\Lambda_j, \cdots, i\Lambda_n}(s))ds\} \\
&= \exp\{\sum_{j=1}^n \int_0^{t_j} k_j(s)ds\} \\
&= \exp\{\sum_{j=1}^n \int_{t_{j-1}}^{t_j} s_j(u)du\} \\
&= \prod_{j=1}^n \frac{c_j}{c_j \cosh(c_j(t_{j-1} - t_j)) + s_j(t_j) \sinh(c_j(t_{j-1} - t_j))}, \qquad (*)
\end{aligned}
$$

which seems, to our best knowledge, not appear in literatures, though it should follow from the Markov property and Lévy's formula.

It should be pointed out that the nondegeneracy assumption on $\{\Lambda_j : j = 1, \cdots, n\}$ is not important. In fact, if for some j,

$$
c_j = 0,
$$

and assume that $\{s_{j+1}(t) : t \in [0, t_{j+1}]\}$ has been solved, then $\{s_j(t) : t \in [0, t_j]\}$ can be sovled as

$$
s_j(t) = \frac{s_j(t_j)}{1 + s_j(t_j)(t - t_j)} = \frac{s_{j+1}(t_j)}{1 + s_{j+1}(t_j)(t - t_j)}, \quad t \in [0, t_j].
$$

It is easy to see that $\{s_j(t) : t \in [0, t_j]\}$ is actually the limit of the nondegenerate case as $c_j \to 0$. Moreover, the corresponding term in the product $(*)$ can be written as

$$
\frac{1}{1 + s_j(t_j)(t_{j-1} - t_j)},
$$

which is also the limit of the nondegenerate case. Therefore, we still use the same notation even in degenerate cases.

Now consider the coupled process $\{(W_t, L_t) : t \geqslant 0\}$. By proposition 5 and the above computation, for $j = 1, \cdots, n$, we can solve the linear system explicitly to obtain

$$
H_j(t) = \exp\{\begin{pmatrix} a_j(t) & -ic_jt \\ ic_jt & a_j(t) \end{pmatrix}\}, \quad t \in [0, t_j],
$$

where

$$a_j(t) : = \int_0^t s_j(u)du$$

$$= \ln \frac{c_j \cosh(c_j(t - t_j)) + s_j(t_j)\sinh(c_j(t - t_j))}{c_j \cosh(c_j t_j) - s_j(t_j)\sinh(c_j t_j)}, \quad t \in [0, t_j].$$

Here the formula for $\{a_j(t) : t \in [0, t_j]\}$ works in the degenerate case where $c_j = 0$ as well.

Finally, by theorem 1, we can write down the finite dimensional joint characteristic function of the coupled process $\{(W_t, L_t) : t \geqslant 0\}$. Namely, we have

Theorem 3.2 *For the coupled process* $\{(W_t, L_t) : t \geqslant 0\}$, *the finite dimensional joint characteristic function* $f(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n)$ *is given by*

$$f(\gamma_1, \cdots, \gamma_n; \Lambda_1, \cdots, \Lambda_n) = \prod_{j=1}^n \frac{c_j}{c_j \cosh(c_j(t_{j-1} - t_j)) + s_j(t_j)\sinh(c_j(t_{j-1} - t_j))} \cdot$$

$$\exp\{-\frac{1}{2}\int_{t_{j-1}}^{t_j} \langle H_j^{*-1}(s)H_j^*(t_j)\mu_j\rangle^2 ds\},$$

where $\{c_j, s_j(t_j), H_j : j = 1, \cdots, n\}$ *is defined previously in this section, and* $\{\mu_j : j = 1, \cdots, n\}$ *is defined recursively in terms of* $\{\gamma_j, H_j : j = 1, \cdots, n\}$ *as in theorem 1.*

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