Visual limits of maximal flats in symmetric spaces and Euclidean buildings

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Abstract

Let X be a symmetric space of non-compact type or a locally finite, strongly transitive Euclidean building, and let $\partial_\infty X$ denote the geodesic boundary of X. We reduce the study of visual limits of maximal flats in X to the study of limits of apartments in the spherical building $\partial_\infty X$: this defines a natural, geometric compactification of the space of maximal flats of X. We then completely determine the possible degenerations of apartments when X is of rank 1, associated to a classical group of rank 2 or to PGL(4). In particular, we exhibit remarkable behaviours of visual limits of maximal flats in various symmetric spaces of small rank and surprising algebraic restrictions that occur. [1]

Introduction

Let X be a symmetric space of non-compact type, or a locally finite strongly transitive Euclidean building, with (type-preserving) isometry group G and Weyl group W. Denote by $\overline{X}^g = X \cup \partial_\infty X$ the geodesic compactification of X. This article opens a new field on geometric limits of convex subsets of X (or more generally of CAT(0) spaces), as we shall see below.

Geometric limits have been studied by Harvey in the case of Fuchsian groups (see [Har77] and [CEG87]), using the Chabauty topology on the space of closed subgroups of a locally compact topological group (see [Cha50]). Denote by Flats(X) the space of all maximal flats in X, endowed with the topology induced by the Chabauty topology on the space $\mathcal{C}(X)$ of closed subsets of X. It is a homogeneous space under the action of G of X: more specifically when X is a symmetric space, then the G-space Flats(X) is isomorphic to the homogeneous space $G/N_G(A)$, where A is a Cartan subgroup of G. For instance if $G = \mathrm{PGL}(n, \mathbb{R})$, then $G/N_G(A)$ is the space of generic n-tuples in $\mathbb{RP}^{n-1}$. Many compactifications of such spaces have been defined, from an algebraic geometry point of view: the Fulton-McPherson compactification (see [FM94]), the variety of reductions studied by Iliev, Manivel and Le Barbier Grünewald in the complex case (see [IM05a], [IM05b], [LBG11b] and [LBG11a])... In another article, inspired by the work of Guivarc'h, Ji and

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Taylor (see [GJT98] and [Hae10a]), we have defined in the real case the Chabauty compactification of $G/N_G(A)$ inside the space of closed subgroups of G, and we have studied it when G has real rank one, and for $G = \mathrm{SL}(3,\mathbb{R})$ and $\mathrm{SL}(4,\mathbb{R})$ (see [Hae12]).

A strong motivation for the study of the asymptotic geometry of the space $G/N_G(A)$ (or G/A, up to finite index if G is $\mathbb{R}$-split) comes from the dynamics of actions on homogeneous spaces. Let Γ be a lattice in a real semi-simple Lie group G, and let H be a closed subgroup of G. When H is connected and generated by unipotent elements, then Ratner's theory tells us that closures of H-orbits in G/Γ are homogeneous (see [Rat91]). And if H has semi-simple Zariski closure, then the recent work by Benoist and Quint tells us that closures of Γ-orbits in G/H are homogeneous and of finite volume (see [BQ11a], [BQ11b]). But in the case where $H = A$ is a torus, the behaviour of H-orbits in G/Γ is much less known, as we can see for instance from Margulis' conjecture (see [Mar97]) that in $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$, every orbit of the diagonal subgroup A that is relatively compact is compact. There has been recent progress towards this conjecture (see [EKL06], [ELMV09] and [ELMV11]), and Maucourant gave a counterexample if A is not the full diagonal subgroup (see [Mau10]). Our interest in studying the asymptotic geometry of G/A is to relate topological properties of A-orbits in G/Γ to geometric properties of Γ-orbits in G/A: for instance, the study of limit sets of such Γ-orbits in the geometric compactification of G/A defined in this article could bring information about the dual A-orbits in G/Γ.

Define the geometric compactification $\overline{\mathrm{Flats}(X)}^g$ of $\mathrm{Flats}(X)$ to be its closure inside the space $\mathcal{C}(\overline{X}^g)$ of closed subsets of $\overline{X}^g$, endowed with the Chabauty topology. If we have a divergent sequence of maximal flats of X, its limit in $\overline{\mathrm{Flats}(X)}^g$ represents what is asymptotically "seen" from a basepoint in X, in other words its visual limit.

Consider the structure of compact topological spherical building on $\partial_\infty X$, and denote by $\mathrm{Cham}(\partial_\infty X)$ the compact space of Weyl chambers of $\partial_\infty X$. Denote by $\mathrm{Ap}(X)$ the space of all apartments in $\partial_\infty X$, endowed with the topology induced by $\mathrm{Cham}(\partial_\infty X)^W$ obtained by considering the set of chambers of an apartment and taking the quotient by W. Define the geometric compactification $\overline{\mathrm{Ap}(X)}^g$ of $\mathrm{Ap}(X)$ to be its closure inside the compact space $\mathrm{Cham}(\partial_\infty X)^W$.

Theorem A. *The natural G-equivariant homeomorphism between* $\mathrm{Flats}(X)$ *and* $\mathrm{Ap}(X)$ *extends to a G-equivariant homeomorphism between* $\overline{\mathrm{Flats}(X)}^g$ *and* $\overline{\mathrm{Ap}(X)}^g$.

Hence, in order to describe geometric limits of maximal flats in X, we only need to understand the geometric compactification of the space of apartments in the building at infinity of X, which is more combinatorial and more tractable.

Let $\mathcal{I}$ be a compact topological spherical building, and let $\mathcal{A}$ be a fixed apartment of $\mathcal{I}$. Every apartment of $\mathcal{I}$ is the image of $\mathcal{A}$ under a type-preserving, injective morphism of simplicial complexes from $\mathcal{A}$ to $\mathcal{I}$, hence we will in fact consider the space $\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})$ of such marked apartments of $\mathcal{I}$, which is a W-principal bundle over $\mathrm{Ap}(X)$. Define the geometric compactification $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ of $\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})$ to be its closure inside the space $\mathrm{Mor}(\mathcal{A},\mathcal{I})$ of (non necessarily injective) morphisms from $\mathcal{A}$ to $\mathcal{I}$, endowed with the compact topology induced by the space $\mathrm{Cham}(\mathcal{I})^{\mathrm{Cham}(\mathcal{A})}$.

If $\mathcal{I}$ is the join of two spherical buildings $\mathcal{I}_1$ and $\mathcal{I}_2$, then the geometric compactification of the space of marked apartments of $\mathcal{I}$ is just the Cartesian product of the geometric compactifications of the spaces of marked apartments of $\mathcal{I}_1$ and $\mathcal{I}_2$. Hence we only need to study the case where $\mathcal{I}$ is an irreducible building. We are able to describe $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ entirely in the following cases.

Theorem B. *Let $\mathcal{I}$ be of type A_1 or A_2, then* $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g = \mathrm{Mor}(\mathcal{A},\mathcal{I})$.

This leads to surprising results: there exist visual limits of maximal flats in the symmetric space of unit volume ellipsoids in $\mathbb{R}^3$ which are not contained in an apartment at infinity.

If $\mathcal{I}$ is a spherical building of type C_2, call quadripod any four pairwise distinct Weyl chambers whose intersection is a vertex.

Theorem C. *Let $\mathbb{K}$ be a local field of characteristic different from 2 (or a quaternion algebra over such a local field), let V be a finite-dimensional (right) $\mathbb{K}$-vector space of dimension at least 5. Consider a (possibly trivial) involutive automorphism σ of $\mathbb{K}$, and let q be a non-degenerate Hermitian form with respect to σ on V of Witt index 2. Let $\mathcal{I}$ be the flag complex of totally isotropic subspaces of V: it is a classical thick spherical building of type C_2. Then every element of* $\mathrm{Mor}(\mathcal{A},\mathcal{I})$ *that is not a quadripod belongs to* $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.

We give a simple cross-ratio condition characterizing which quadripods belong to the compactification $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$. For instance, consider the case $\mathbb{K} = \mathbb{R}$, $V = \mathbb{R}^6$ with canonical basis $(e_1,\ldots,e_6)$ and $q(x) = x_1x_6+x_2x_5+x_3^2+x_4^2$. Consider four Weyl chambers $(\ell_i \subset \langle e_1, e_2\rangle)_{1\leqslant i\leqslant 4}$, where for all $i \in [\![1,4]\!]$, ℓ_i is a line in the plane $\langle e_1, e_2\rangle$. The intersection of these Weyl chambers is the isotropic plane $\langle e_1, e_2\rangle$, so their union is a quadripod called a plane-type quadripod. Then it is a limit of apartments if and only if the cross-ratio of $(\ell_1,\ell_2,\ell_3,\ell_4)$ is less than or equal to 1.

Let $\mathbb{K}$ be a field, and let $\mathcal{I}$ be the spherical building of complete flags of the vector space $\mathbb{K}^4$, then an apartment of $\mathcal{I}$ is simply a generic tetrahedron in $\mathbb{P}^3(\mathbb{K})$. And a morphism from an apartment to $\mathcal{I}$ is given by 4 points, 6 lines and 4 planes in $\mathbb{P}^3(\mathbb{K})$ that satisfy the incidence conditions of a tetrahedron (see Section 2). A morphism is called of type (L) if the 6 lines are equal, and it is called of type (XP) if the 4 points and the 4 planes are equal.

Theorem D. *Let $\mathbb{K}$ be a local field, and let $\mathcal{I}$ be the spherical topological building of complete flags of the vector space $\mathbb{K}^4$. Then every element of* $\mathrm{Mor}(\mathcal{A},\mathcal{I})$ *that is not of type (L) nor (XP) belongs to* $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.

A generic element of $\mathrm{Mor}(\mathcal{A},\mathcal{I})$ *of type (L) belongs to* $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ *if and only if the cross-ratio of the four points is equal to the cross-ratio of the four planes.*

A generic element of $\mathrm{Mor}(\mathcal{A},\mathcal{I})$ *of type (XP) belongs to* $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ *if and only if there is a projective involution of $\mathbb{P}^3(\mathbb{K})$ fixing the common plane and the common point, and exchanging each line with its opposite.*

In the Archimedean case, here is the corollary of these results in terms of visual limits of maximal flats: it includes the case of all real nonexceptional noncompact simple Lie groups of type C_2.

Corollary E. *(loosely stated, see Section 2 for precise statements) Let G be a connected real semi-simple Lie group with finite center, whose simple factors are of $\mathbb{R}$-rank one or* *locally isomorphic to* $\mathrm{SL}(3,\mathbb{R})$*,* $\mathrm{SL}(3,\mathbb{C})$*,* $\mathrm{SL}(3,\mathbb{H}) \simeq \mathrm{SU}^*(6)$*,* $E_{6(-26)}$*,* $\mathrm{SO}_0(2,n)$ *(where* $n \geqslant 3$*),* $\mathrm{SU}(2,n)$ *and* $\mathrm{Sp}(2,n)$ *(where* $n \geqslant 2$*),* $\mathrm{SO}(5,\mathbb{C})$*,* $\mathrm{SO}^*(10)$*,* $\mathrm{SL}(4,\mathbb{R})$ *or* $\mathrm{SL}(4,\mathbb{C})$*. Let* X *be the symmetric space of non-compact type of* G*. Then we describe all the possible visual limits of divergent sequences of maximal flats in* X*, in terms of cross-ratios.*

In the first part, we define the geometric compactifications of the spaces of flats and of apartments, and we prove Theorem A, by proving a general result on visual limits in CAT(0) spaces, Theorem 1.2, which could be used to describe visual limits in other settings.

In the second part, we compute explicitely the geometric compactifications of the space of marked apartments in each of the cases of the theorems B, C and D, and we emphasize the remarkably rich behaviours of visual limits of maximal flats that already occur in small ranks, and the surprising algebraic restrictions on their existence that occur.

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1 Geometric compactifications of spaces of flats and apartments

We begin by recalling the definition of a topological spherical building. Then we define in similar ways firstly the geometric compactification of the space of (marked) maximal flats of a symmetric space of non-compact type or a locally finite Euclidean building, and secondly the geometric compactification of the space of (marked) apartments of a compact topological spherical building. Then we show that the two compactifications are equivariantly isomorphic (Theorem 1.3), by proving a more general result on visual limits in CAT(0) spaces (Theorem 1.2).

Recall that if E is a locally compact topological space, the set $\mathcal{C}(E)$ of closed subsets of E is endowed with a natural compact topology, which is metrisable if E is metrisable, called the Chabauty topology (also called Hausdorff or Vietoris topology, see [Cha50], [dlH08], [Bou59b, §5], [CEG87, Proposition I.3.1.2, p. 59], [CDP07, Proposition 1.7, p. 58], [Hae10b], [Hae10a] or [Hae12]).

If G is a locally compact topological group acting and X is a locally compact topological G-space, a G-compactification of X is a pair (ι, K), where K is a compact G-space and $\iota : X \hookrightarrow K$ is a G-equivariant topological embedding whith open and dense image.

By spherical or Euclidean building, we mean its spherical or Euclidean geometric realisation, and we consider its maximal apartments system (see [BH99] and [AB08]).

1.1 Topological spherical buildings

Given k in $\mathbb{N}$ and C a cellular complex, we denote by $C^{\langle k\rangle}$ the set of cells of C with dimension k.

A *topological spherical building* (see [BS87] and [Ji06]) is a spherical building $\mathcal{I}$ whose set of vertices $\mathcal{I}^{\langle 0\rangle}$ is endowed with a topology such that for all $k \in [\![0, d]\!]$ (where d is the dimension of $\mathcal{I}$), the set $\mathcal{I}^{\langle k\rangle}$ of k-simplices is closed for the topology induced by $\left(\mathcal{I}^{\langle 0\rangle}\right)^{k+1}$. If the space $\mathcal{I}^{\langle 0\rangle}$ is (locally) compact, we say that the topological building $\mathcal{I}$ is (locally) compact.

Denote by $\mathcal{T}_{dist}$ the topology on $\mathcal{I}$ induced by the simplicial distance, which is not locally compact in general. The topology on the space $\mathrm{Cham}(\mathcal{I}) = \mathcal{I}^{\langle d \rangle}$ of Weyl chambers of $\mathcal{I}$ defines a new topology $\mathcal{T}_{lc}$ on $\mathcal{I}$, coarser than $\mathcal{T}_{dist}$, for which a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if

1. there exists a sequence $(C_n)_{n\in\mathbb{N}}$ of Weyl chambers such that, for all $n \in \mathbb{N}$, we have $x_n \in C_n$, and such that the sequence $(C_n)_{n\in\mathbb{N}}$ converges to a Weyl chamber C containing x in $\mathrm{Cham}(\mathcal{I})$;
2. if we denote by ϕ_n the type-preserving isometry from C_n to C for all $n \in \mathbb{N}$, then the sequence $(\phi_n(x_n))_{n\in\mathbb{N}}$ converges to x in C.

If the space $\mathcal{I}^{\langle 0 \rangle}$ is (locally) compact, then the topology $\mathcal{T}_{lc}$ is (locally) compact. An *automorphism of the topological building* $\mathcal{I}$ is a (type-preserving) automorphism of the building $\mathcal{I}$ which is also a homeomorphism for the topology $\mathcal{T}_{lc}$.

For instance, let $\mathbb{K}$ be a *local field* (i.e. a field with a non-trivial valuation, complete and locally compact, thus isomorphic to $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$, with p prime and q a power of a prime, see for instance [Ser68, Chapitre II]), and let G be the group of $\mathbb{K}$-points of an algebraic linear connected reductive $\mathbb{K}$-group $\underline{G}$. Let $\mathcal{I}$ be the spherical building of $(\underline{G}, \mathbb{K})$, then G acts transitively on the vertices of $\mathcal{I}^{\langle 0 \rangle}$ of fixed type, with stabilizer a maximal parabolic $\mathbb{K}$-subgroup. Hence the set $\mathcal{I}^{\langle 0 \rangle}$ identifies with the disjoint union $\sqcup_i G/P_i$, where the P_i's are the maximal parabolic $\mathbb{K}$-subgroups containing a fixed minimal parabolic $\mathbb{K}$-subgroup. Endow $\mathcal{I}^{\langle 0 \rangle}$ with the topology induced by this disjoint union of $\mathbb{K}$-points of projective varieties: it is a compact space. This defines a structure of compact topological spherical building on $\mathcal{I}$, and the topology $\mathcal{T}_{lc}$ is metrisable.

In our cases, the topological spherical buildings will arise as visual boundaries of some CAT(0) spaces. Let X be a product of symmetric spaces of non-compact type and locally finite strongly transitive Euclidean buildings. Then it is a complete locally compact CAT(0) space (see [AB08, Theorem 11.16, p.555] for the building case). Its visual (or geodesic, or CAT(0)) boundary at infinity $\partial_\infty X$ has a natural structure $\mathcal{I}$ of spherical building (see [BGS85] and [AB08, § 11.8]). The space $\mathcal{I}^{\langle 0 \rangle}$ of vertices of $\mathcal{I}$ is included in $\partial_\infty X$, let us endow it with the induced compact topology. This defines a natural structure of compact topological spherical building on $\mathcal{I}$, and the topology $\mathcal{T}_{lc}$ on $\mathcal{I}$ is the topology of $\partial_\infty X$. Furthermore $\mathcal{I}$ is topologically strongly transitive, which means that its group of (topological) automorphisms acts transitively on the flags $C \subset \mathcal{A}$, where C is a Weyl chamber and $\mathcal{A}$ is an apartment of $\mathcal{I}$.

1.2 Geometric compactification of the space of (marked) maximal flats

Let X be a product of symmetric spaces of non-compact type and of locally finite strongly transitive Euclidean buildings. Let G be the product of the isometry groups of the symmetric space factors and of the type-preserving automorphism groups of the building factors. Denote by $\mathrm{Flats}(X)$ the set of all maximal flats of X (i.e. the set of apartments if X is a building), endowed with the topology induced by the Chabauty topology on the space $\mathcal{C}(X)$ of closed subsets of X, and with the G-action on the left. Denote by W the Weyl group of X.

Consider $\overline{X}^g = X \cup \partial_\infty X$ the geodesic G-compactification of X. Consider the G-equivariant embedding

$$\begin{array}{rcl} \mathrm{Flats}(X) & \to & \mathcal{C}(\overline{X}^g) \\ F & \mapsto & \overline{F}, \end{array}$$

and call *geometric compactification* of the space $\mathrm{Flats}(X)$ of maximal flats of X the closure $\overline{\mathrm{Flats}(X)}^g$ of its image.

We can define as well a geometric compactification of the space of marked flats of X. Denote by $\mathrm{Cham}(\partial_\infty X)$ the space of all closed Weyl chambers in the spherical building at infinity $\partial_\infty X$, endowed with the topology induced by the Chabauty topology on the space $\mathcal{C}(\partial_\infty X)$ of closed subsets of $\partial_\infty X$. Call *marked flat* of X any $(F, C) \in \mathrm{Flats}(X) \times \mathrm{Cham}(\partial_\infty X)$ such that $C \subset \partial_\infty F$. Denote by $\mathrm{Flats}_m(X)$ the set of all marked flats of X, endowed with the compact topology induced by the product topology on $\mathrm{Flats}(X) \times \mathrm{Cham}(\partial_\infty X)$. It has a natural $(G \times W)$-action on the left. Consider the G-equivariant embedding

$$\begin{array}{rcl} \mathrm{Flats}_m(X) & \to & \mathcal{C}(\overline{X}^g) \times \mathrm{Cham}(\partial_\infty X) \\ (F, C) & \mapsto & (\overline{F}, C), \end{array}$$

and call *geometric compactification* of the space $\mathrm{Flats}_m(X)$ of marked flats of X the closure $\overline{\mathrm{Flats}_m(X)}^g$ of its image.

The forgetful map

$$\begin{array}{rcl} \mathrm{Flats}_m(X) & \to & \mathrm{Flats}(X) \\ (F, C) & \mapsto & F \end{array}$$

is continuous, surjective, G-equivariant and with finite fibers. It is the quotient map under the action of W.

If $X = \prod_{i=1}^k X_i$ is a product, where each X_i is a product of symmetric spaces of non-compact type and locally finite strongly transitive Euclidean buildings, then the natural $(G\times W)$-equivariant homeomorphism $\prod_{i=1}^k \mathrm{Flats}_m(X_i) \simeq \mathrm{Flats}_m(X)$ extends to a $(G\times W)$-equivariant homeomorphism

$$\begin{array}{rcl} \displaystyle\prod_{i=1}^k \overline{\mathrm{Flats}_m(X_i)}^g & \to & \overline{\mathrm{Flats}_m(X)}^g \\ (F_i, C_i)_{1 \leqslant i \leqslant k} & \mapsto & (\displaystyle\prod_{i=1}^k F_i, \star_{i=1}^k C_i), \end{array}$$

and similarly between $\prod_{i=1}^k \overline{\mathrm{Flats}(X_i)}^g$ and $\overline{\mathrm{Flats}(X)}^g$. Hence we only need to study the case where X is irreducible.

1.3 Geometric compactification of the space of (marked) apartments

Let $\mathcal{I}$ be a compact topological spherical building (see Subsection 1.1) of dimension d and topological automorphism group G. Fix an apartment $\mathcal{A}$ of $\mathcal{I}$, and denote by W its Weyl group. Recall that a morphism from $\mathcal{A}$ to $\mathcal{I}$ is for us a (non necessarily injective) type-preserving morphism of typed simplicial complexes. Denote by $\mathrm{Mor}(\mathcal{A},\mathcal{I})$ the set of all morphisms from $\mathcal{A}$ to $\mathcal{I}$. It has a natural $G \times W$-action on the left, given by

$$\forall g \in G, \forall w \in W, \forall f \in \mathrm{Mor}(\mathcal{A},\mathcal{I}), (g,w) \cdot f = g \circ f \circ w^{-1}.$$

Call *marked apartment* of $\mathcal{I}$ any morphism from $\mathcal{A}$ to $\mathcal{I}$ whose image is an apartment of $\mathcal{I}$. Denote by $\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})$ or $\mathrm{Ap}_m(\mathcal{I})$ the subset of all marked apartments of $\mathcal{I}$. This notation is justified by the fact that a morphism from $\mathcal{A}$ to $\mathcal{I}$ is injective if and only if its image is an apartment (see [AB08, Proposition 4.59, p.193]).

A morphism from $\mathcal{A}$ to $\mathcal{I}$ is characterized by the images of the finite number of Weyl chambers $\mathrm{Cham}(\mathcal{A})$ of $\mathcal{A}$. Denote by $\mathrm{Cham}(\mathcal{I}) = \mathcal{I}^{\langle d \rangle}$ the space of closed Weyl chambers of $\mathcal{I}$, endowed with the compact topology induced by the product topology on $(\mathcal{I}^{\langle 0 \rangle})^{d+1}$.

Endow $\mathrm{Mor}(\mathcal{A},\mathcal{I})$ with the topology induced by the product topology on $\mathrm{Cham}(\mathcal{I})^{\mathrm{Cham}(\mathcal{A})}$, which is the same as the topology induced by the compact-open topology for the topology $\mathcal{T}_{lc}$ on $\mathcal{I}$. An element f of $\mathrm{Cham}(\mathcal{I})^{\mathrm{Cham}(\mathcal{A})}$ is a morphism if and only if for every Weyl chambers C, C' of $\mathcal{A}$ whose intersection is a non-empty facet, then $f(C) \cap f(C')$ is a non-empty facet of the same type. Those are a finite number of closed conditions, so $\mathrm{Mor}(\mathcal{A},\mathcal{I})$ is compact.

Call *geometric compactification* of the space $\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I}) = \mathrm{Ap}_m(\mathcal{I})$ of marked apartments of $\mathcal{I}$ its closure $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g = \overline{\mathrm{Ap}_m(\mathcal{I})}^g$ inside $\mathrm{Mor}(\mathcal{A},\mathcal{I})$.

We can define as well a geometric compactification of the space $\mathrm{Ap}(\mathcal{I})$ of all (unmarked) apartments of $\mathcal{I}$, endowed with the quotient topology of $\mathrm{Ap}_m(\mathcal{I})$, which is the same as the compact topology induced by the Chabauty topology on the space $\mathcal{C}(\mathcal{I}, \mathcal{T}_{lc})$ of closed subsets of $\mathcal{I}$ endowed with the topology $\mathcal{T}_{lc}$. It has a natural G-action on the left. Call *geometric compactification* of the space $\mathrm{Ap}(\mathcal{I})$ of apartments of $\mathcal{I}$ its closure $\overline{\mathrm{Ap}(\mathcal{I})}^g$ inside $\mathcal{C}(\mathcal{I}, \mathcal{T}_{lc})$.

The forgetful map

$$\begin{array}{rcl} \mathrm{Ap}_m(\mathcal{I}) & \to & \mathrm{Ap}(\mathcal{I}) \\ f : \mathcal{A} \to \mathcal{I} & \mapsto & f(\mathcal{A}) \end{array}$$

is continuous, surjective, G-equivariant and with finite fibers. It is the quotient map under the action of W.

If $E_1, \dots, E_k$ are topological spaces, recall that the join $\star_{i=1}^k E_i$ is defined as a topological quotient of $\prod_{i=1}^k E_i \times \Delta^{k-1}$, where Δ^{k-1} is the $(k-1)$-simplex. If for all $i \in [\![1,k]\!]$, we have a continuous map $f_i : E_i \to F_i$ between topological spaces, the join of the maps $(f_i)_{i \in [\![1,k]\!]}$ is the map $\star_{i=1}^k f_i : \star_{i=1}^k E_i \to \star_{i=1}^k F_i$ induced by

$$\begin{array}{rcl} \prod_{i=1}^k E_i \times \Delta^{k-1} & \to & \prod_{i=1}^k F_i \times \Delta^{k-1} \\ (e_1, \dots, e_k, t) & \mapsto & (f_1(e_1), \dots, f_k(e_k), t). \end{array}$$

If $\mathcal{I} = \star_{i=1}^{k}\mathcal{I}_i$ is a join of compact topological spherical buildings, then the natural $(G \times W)$-equivariant homeomorphism $\prod_{i=1}^{k} \mathrm{Ap}_m(\mathcal{I}_i) \simeq \mathrm{Ap}_m(\mathcal{I})$ extends to a $(G \times W)$-equivariant homeomorphism

$$\begin{array}{rcl} \prod_{i=1}^{k} \overline{\mathrm{Ap}_m(\mathcal{I}_i)}^g & \to & \overline{\mathrm{Ap}_m(\mathcal{I})}^g \\ (f_i)_{1\leqslant i\leqslant k} & \mapsto & \star_{i=1}^{k} f_i, \end{array}$$

and similarly between $\prod_{i=1}^{k} \overline{\mathrm{Ap}(\mathcal{I}_i)}^g$ and $\overline{\mathrm{Ap}(\mathcal{I})}^g$. Hence we only need to study the case where $\mathcal{I}$ is irreducible.

In the classical case, there is an algebraic interpretation of this compactification. Let $\mathbb{K}$ be a local field, and let G be the group of $\mathbb{K}$-points of an algebraic linear connected reductive $\mathbb{K}$-group $\underline{G}$, with Weyl group W. Let $\mathcal{I}$ be the compact topological spherical building of $(\underline{G}, \mathbb{K})$, and let A be a maximal $\mathbb{K}$-split torus of G. Then the space $\mathrm{Ap}_m(\mathcal{I})$ is $(G \times W)$-equivariantly homeomorphic to the homogeneous space $G/Z_G(A)$. Fix P a minimal parabolic $\mathbb{K}$-subgroup of G containing A. Then the embedding

$$\begin{array}{rcl} \mathrm{Ap}_m(\mathcal{I}) \simeq G/Z_G(A) & \to & \prod_{w\in W} G/wPw^{-1} \\ gZ_G(A) & \mapsto & (gwPw^{-1})_{w\in W} \end{array}$$

extends to a $(G \times W)$-equivariant embedding of $\overline{\mathrm{Ap}_m(\mathcal{I})}^g$ into the algebraic projective $\mathbb{K}$-variety $\prod_{w\in W} G/wPw^{-1}$.

1.4 Geometric limits of closed subsets in $\mathrm{CAT}(0)$ spaces

Let X be a complete, locally compact $\mathrm{CAT}(0)$ metric space, let $\partial_\infty X$ be its geodesic (or visual, or $\mathrm{CAT}(0)$) boundary and $\overline{X}^g = X \cup \partial_\infty X$ its geodesic compactification (see [BH99] and [BGS85]). Assume there exists a non-empty set $\mathcal{F}$ of closed non-empty subsets of X satisfying the following two properties.

(1) The subspace $\{(F, x), F \in \mathcal{F}, x \in F\}$ of pointed elements of $\mathcal{F}$ is closed in the space $\mathcal{C}(X) \times X$, endowed with the product topology, and it is $\mathrm{Isom}(X)$-cocompact.

(2) For all $F \in \mathcal{F}$ and all $\xi \in \partial_\infty X$, there exists $\eta \in \partial_\infty F$ *opposed* to ξ, that is there exists a geodesic in X whose endpoints are η and ξ.

Here are families of examples where these two properties are satisfied. In each case $\mathcal{F}$ is the set of all maximal flats in X and the boundary at infinity $\partial_\infty X$ is naturally a spherical building, which implies the opposition condition (2).

- X is a symmetric space of non-compact type.
- X is a locally finite strongly transitive Euclidean building.
- X is a Gromov hyperbolic complete locally compact $\mathrm{CAT}(0)$ metric space, with extendible geodesics, whose isometry group acts cocompactly on pointed geodesics in X.

When $\partial_\infty X$ is a spherical building, we can also consider for $\mathcal{F}$ a set of closed subsets of X containing a maximal flat, which also implies the opposition condition (2).

- X is a Hermitian symmetric space of non-compact type, with $\mathcal{F}$ the set of all maximal polydiscs in X.
- X is a locally finite hyperbolic building (see [Bou97], [GP01]), whose isometry group acts strongly transitively, with $\mathcal{F}$ the set of all apartments of X.

Furthermore, any product of finitely many of the examples above (with the ℓ^2 product metric) satisfies the two properties.

Let us denote by $\sphericalangle_x$ the visual angle on $\overline{X}^g$ at $x \in X$ (see for instance [BH99, Chapter II.3]).

Lemma 1.1. *Fix $x \in X$. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in X converging to $\xi \in \partial_\infty X$, and let $\eta \in \partial_\infty X$ be opposed to ξ. Then the sequence $(\sphericalangle_{y_n}(x, \eta))_{n \in \mathbb{N}}$ converges to 0.*

Proof. Fix a geodesic (η, ξ) in X whose endpoints are η and ξ, and denote by x' the orthogonal projection of x on (η, ξ) (see Figure 1). We know that the sequence $(\sphericalangle_{x'}(y_n, \eta))_{n \in \mathbb{N}}$ converges to $\sphericalangle_{x'}(\xi, \eta) = \pi$.

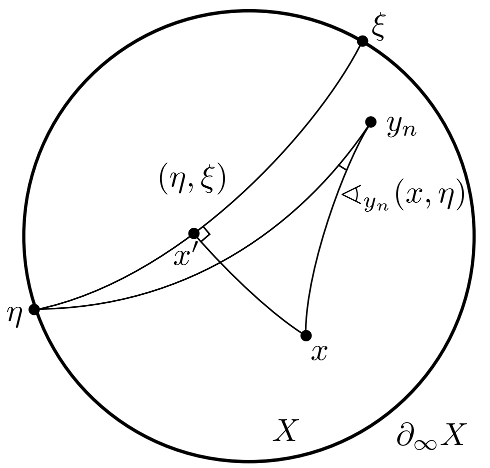

Figure 1: Lemma 1.1.

Since for all $n \in \mathbb{N}$ we have $\sphericalangle_{x'}(y_n, \eta) + \sphericalangle_{y_n}(x', \eta) \leqslant \pi$, we deduce that the sequence $(\sphericalangle_{y_n}(x', \eta))_{n \in \mathbb{N}}$ converges to 0.

As $(\sphericalangle_{y_n}(x, x'))_{n \in \mathbb{N}}$ converges to 0, we conclude that the sequence $(\sphericalangle_{y_n}(x, \eta))_{n \in \mathbb{N}}$ converges to 0. □

Theorem 1.2. *Let $(F_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F}$ leaving every compact subset of X, and such that the sequence $(\partial_\infty F_n)_{n \in \mathbb{N}}$ converges to a closed subset C of $\partial_\infty X$ in $\mathcal{C}(\partial_\infty X)$. Then the sequence $(\overline{F_n})_{n \in \mathbb{N}}$ converges to C in $\mathcal{C}(\overline{X}^g)$.*

Proof. Since the sequence $(F_n)_{n \in \mathbb{N}}$ leaves every compact subset of X, every accumulation point of the sequence $(\overline{F_n})_{n \in \mathbb{N}}$ is included in $\partial_\infty X$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points of $(F_n)_{n\in\mathbb{N}}$ converging to $\zeta \in \partial_\infty X$. Since $\mathrm{Isom}(X)$ acts cocompactly on pointed elements of $\mathcal{F}$, we may assume by the cocompactness condition (1) that there exists $F \in \mathcal{F}$, $x \in F$ and a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathrm{Isom}(X)$ such that the sequence $(g_n \cdot F_n)_{n\in\mathbb{N}}$ converges to F in $\mathcal{C}(X)$ and that the sequence $(g_n \cdot x_n)_{n\in\mathbb{N}}$ converges to x in X (see Figure 2).

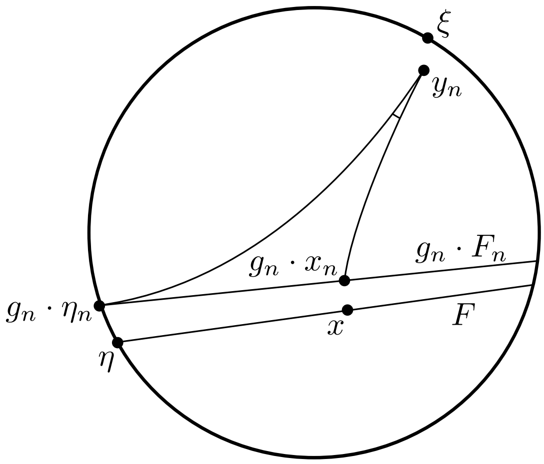

Figure 2: Theorem 1.2.

Up to passing to a subsequence, we may assume that the sequence $(y_n = g_n \cdot x)_{n\in\mathbb{N}}$ converges to a point $\xi \in \partial_\infty X$, as this sequence leaves every compact of X:

$$d(x, y_n) = d(g_n^{-1} \cdot x, x) \geqslant d(x_n, x) - d(g_n^{-1} \cdot x, x_n) \geqslant d(F_n, x) - d(x, g_n \cdot x_n) \underset{n\to+\infty}{\longrightarrow} +\infty.$$

The opposition condition (2) ensures the existence of $\eta \in \partial_\infty F$ opposed to ξ. We apply Lemma 1.1 to the basepoint $x \in X$, the sequence $(y_n)_{n\in\mathbb{N}}$ and the point $\eta \in \partial_\infty F$: the sequence $(\sphericalangle_{y_n}(x, \eta))_{n\in\mathbb{N}}$ converges to 0.

Since the sequence $(g_n \cdot F_n)_{n\in\mathbb{N}}$ converges to F, choose for all $n \in \mathbb{N}$ an element $\eta_n \in \partial_\infty F_n$ such that the sequence $(g_n \cdot \eta_n)_{n\in\mathbb{N}}$ converges to η in $\partial_\infty X$. Then we have

$$\sphericalangle_{y_n}(g_n \cdot x_n, g_n \cdot \eta_n) \leqslant \sphericalangle_{y_n}(g_n \cdot x_n, x) + \sphericalangle_{y_n}(x, \eta) + \sphericalangle_{y_n}(\eta, g_n \cdot \eta_n) \underset{n\to+\infty}{\longrightarrow} 0.$$

Since g_n^{-1} is an isometry of X for all $n \in \mathbb{N}$, we deduce that the sequence $(\sphericalangle_x(x_n, \eta_n))_{n\in\mathbb{N}}$ converges to 0. As the sequence $(x_n)_{n\in\mathbb{N}}$ converges to ζ, we conclude that the sequence $(\sphericalangle_x(\zeta, \eta_n))_{n\in\mathbb{N}}$ converges to 0, hence the sequence $(\eta_n)_{n\in\mathbb{N}}$ converges to ζ in $\partial_\infty X$. So $\zeta \in C$. The other inclusion being obvious, we conclude that the sequence $(\overline{F_n})_{n\in\mathbb{N}}$ converges to C in $\mathcal{C}(\overline{X}^g)$. □

1.5 The isomorphism between the compactifications

Let (X, G) be as in the beginning of Subsection 1.2. Denote by $\mathcal{I} = \partial_\infty X$ the compact topological spherical building at infinity of X. Denote by d the dimension of $\mathcal{I}$, it is equal $r - 1$, where r is the real rank of X if X is a symmetric space, the dimension of X minus 1 if X is a Euclidean building, or their sum if X is a product. Fix a maximal flat F_0 in X, and denote by $\mathcal{A} = \partial_\infty F_0$ its apartment at infinity, and fix a Weyl chamber C_0 of $\mathcal{A}$.

Consider the natural G-equivariant and $G \times W$-equivariant homeomorphisms

$$\begin{array}{rcl} \iota : \mathrm{Flats}(X) & \to & \mathrm{Ap}(\mathcal{I}) \\ F & \mapsto & \partial_\infty F \\ \text{and } \iota_m : \mathrm{Flats}_m(X) & \to & \mathrm{Ap}_m(\mathcal{I}) \\ (F, C) & \mapsto & f \in \mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I}) \text{ such that } f(\mathcal{A}) = \partial_\infty F \text{ and } f(C_0) = C, \end{array}$$

Consider furthermore the map $\phi : \overline{\mathrm{Ap}(\mathcal{I})}^g \to \mathcal{C}(\overline{X}^g)$ defined as follows. If $A \in \mathrm{Ap}(\mathcal{I})$, then define $\phi(A)$ to be $\overline{F}$, where F is the unique maximal flat in X such that $\partial_\infty F = A$. And if $A \in \partial\,\mathrm{Ap}(\mathcal{I})$, define $\phi(A) = A \subset \mathcal{I}$.

Consider as well the marked version of ϕ, that is the map $\phi_m : \overline{\mathrm{Ap}_m(\mathcal{I})}^g \to \mathcal{C}(\overline{X}^g) \times \mathrm{Cham}(\mathcal{I})$ defined as follows. If $f \in \mathrm{Ap}_m(\mathcal{I})$, then define $\phi_m(f)$ to be $(\overline{F}, f(C_0))$ where F is the unique maximal flat in X such that $\partial_\infty F = f(\mathcal{A})$. And if $f \in \partial\,\mathrm{Ap}_m(\mathcal{I})$, define $\phi_m(f)$ to be $(f(\mathcal{A}), f(C_0))$.

Theorem 1.3. *The geometric compactification $\overline{\mathrm{Flats}_m(X)}$ of the space of marked flats of X is $G \times W$-isomorphic to the geometric compactification $\overline{\mathrm{Ap}_m(\mathcal{I})}$ of the space of marked apartments of $\mathcal{I}$. More precisely, the following diagram of $G \times W$-equivariant embeddings is well-defined and commutes.*

$$\begin{array}{ccc} \mathrm{Flats}_m(X) & \hookrightarrow & \overline{\mathrm{Flats}_m(X)}^g \\ \iota_m \downarrow\wr & & \wr\uparrow \phi_m \\ \mathrm{Ap}_m(\mathcal{I}) & \hookrightarrow & \overline{\mathrm{Ap}_m(\mathcal{I})}^g. \end{array} \tag{1}$$

Taking the quotient with respect to W gives the same unmarked result.

Corollary 1.4. *The geometric compactification $\overline{\mathrm{Flats}(X)}$ of the space of flats of X is G-isomorphic to the geometric compactification $\overline{\mathrm{Ap}(\mathcal{I})}$ of the space of apartments of $\mathcal{I}$. More precisely, the following diagram of G-equivariant embeddings is well-defined and commutes.*

$$\begin{array}{ccc} \mathrm{Flats}(X) & \hookrightarrow & \overline{\mathrm{Flats}(X)}^g \\ \iota \downarrow\wr & & \wr\uparrow \phi \\ \mathrm{Ap}(\mathcal{I}) & \hookrightarrow & \overline{\mathrm{Ap}(\mathcal{I})}^g. \end{array} \qquad \square$$

Proof. It is clear that the diagram (1) commutes, and that each arrow is $G\times W$-equivariant and injective. So it is enough to show that ϕ_m is continuous, with values in $\overline{\mathrm{Flats}_m(X)}^g$.

The map ϕ_m restricted to $\mathrm{Ap}_m(\mathcal{I})$ is a homeomorphism onto $\mathrm{Flats}_m(X)$, since both spaces are endowed with the topology of G-homogeneous spaces. The topology on each space of closed Weyl facets $\mathcal{I}^{\langle k \rangle}$ is induced by the Chabauty topology on $\mathcal{C}(\partial_\infty X)$, hence ϕ_m restricted to $\partial\,\mathrm{Ap}_m(\mathcal{I})$ is an embedding in $\mathcal{C}(\partial_\infty X)$.

Let us show that ϕ_m is continuous on $\overline{\mathrm{Ap}_m(\mathcal{I})}^g$. Since $\overline{\mathrm{Ap}_m(\mathcal{I})}^g$ and $\mathcal{C}(\overline{X}^g) \times \mathrm{Cham}(\mathcal{I})$ are metrisable, we need only prove the sequential continuity. Let $f \in \partial\,\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})$ converging to f in $\mathrm{Mor}(\mathcal{A}, \mathcal{I})$. For all $n \in \mathbb{N}$, let $(F_n, C_n) = \phi_m(f_n)$.

The sequence $(F_n)_{n\in\mathbb{N}}$ of maximal flats of X goes to infinity in $\mathrm{Flats}(X)$, and the sequence $(\partial_\infty F_n)_{n\in\mathbb{N}}$ converges to $f(\mathcal{A})$ in $\mathcal{C}(\partial_\infty X)$ by assumption. According to Theorem 1.2, we deduce that the sequence $(\overline{F_n})_{n\in\mathbb{N}}$ converges to $f(\mathcal{A})$ in $\mathcal{C}(\overline{X}^g)$.

Furthermore by assumption we know that the sequence $(C_n = f_n(C_0))_{n\in\mathbb{N}}$ of Weyl chambers converges to $f(C_0)$. Hence the sequence $((F_n, C_n) = \phi_m(f_n))_{n\in\mathbb{N}}$ converges to $(f(\mathcal{A}), f(C_0)) = \phi_m(f)$ in $\mathcal{C}(\overline{X}^g)\times\mathrm{Cham}(\partial_\infty X)$. So ϕ_m is continuous at f, and furthermore $\phi_m(f) \in \overline{\mathrm{Flats}_m(X)}^g$.

The map ϕ_m is continuous, injective, from the compact space $\overline{\mathrm{Ap}_m(\mathcal{I})}^g$ into the Hausdorff space $\overline{\mathrm{Flats}_m(X)}^g$: it is an embedding. Its image is a compact subspace of $\overline{\mathrm{Flats}_m(X)}^g$, which contains the dense subspace $\phi_m(\mathrm{Ap}_m(\mathcal{I})) = \mathrm{Flats}_m(X)$, hence ϕ_m is surjective. So ϕ_m is a $G \times W$-equivariant homeomorphism from $\overline{\mathrm{Ap}_m(\mathcal{I})}^g$ onto $\overline{\mathrm{Flats}_m(X)}^g$. $\square$

2 Geometric limits of marked apartments in topological spherical buildings

In this Section, we compute explicitely the geometric compactifications of the space of marked apartments in each of the cases of the theorems B, C and D. First we look at the rank 1 case. Then in the rank 2 case (or the Moufang polygon case), we study completely the type A_2, and for the type C_2 we get a complete description in the orthogonal and unitary classical cases. Finally, we study the case of PGL(4) over a field.

2.1 Rank 1

A compact topological spherical building $\mathcal{I}$ of rank 1 is just a compact space $\mathcal{I}^{\langle 0\rangle}$ with the set of pairs of distinct points of $\mathcal{I}^{\langle 0\rangle}$ as apartment system. So the space of marked apartments of $\mathcal{I}$ is the set of couples of distinct points of $\mathcal{I}^{\langle 0\rangle}$.

Assume that $\mathcal{I}^{\langle 0\rangle}$ has no isolated point, then the geometric compactification $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ of the space of marked apartments of $\mathcal{I}$ is just $\mathrm{Mor}(\mathcal{A},\mathcal{I}) = (\mathcal{I}^{\langle 0\rangle})^2$. And the geometric compactification $\overline{\mathrm{Ap}(\mathcal{I})}^g$ of the space of unmarked apartments of $\mathcal{I}$ is the quotient of $(\mathcal{I}^{\langle 0\rangle})^2$ by the diagonal involution. This is the first part of Theorem B.

For instance, the space of oriented geodesics of the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ is homeomorphic to the one-sheeted hyperboloid (in the hyperboloid model), or to the space of couples of distinct points of $\mathbb{S}^1$ (in the disc model), and the geometric compactification is homeomorphic to the torus $(\mathbb{S}^1)^2$. And the geometric compactification of the space of non-oriented geodesics of $\mathbb{H}^2_{\mathbb{R}}$ is homeomorphic to the quotient of $(\mathbb{S}^1)^2$ by the diagonal involution.

If we apply Theorem 1.3, we get the following well-known result.

Corollary 2.1. *Let X be a $\mathbb{R}$-rank 1 symmetric space of non-compact type or a locally finite, strongly transitive tree. Then any visual limit of a divergent sequence of geodesics in X is a single point in the visual boundary $\partial_\infty X$.*

2.2 Rank 2 : Moufang polygons

2.2.1 Notations

Let $\mathcal{I}$ be a topological spherical building of rank 2 : it is a bipartite graph, with a topology on its set of edges. Assume that this topology is locally compact and has no isolated point. Let G be the group of automorphisms of the topological spherical building $\mathcal{I}$, assume that

$\mathcal{I}$ is topologically strongly transitive. Let $\mathcal{I}^{\langle 0\rangle}$ be the space of vertices of $\mathcal{I}$, and let $\mathcal{I}^{\langle 1\rangle}$ be the space of edges of $\mathcal{I}$.

Fix an apartment $\mathcal{A}$ of $\mathcal{I}$: it is the boundary of a $2p$-gon, with $p \in \mathbb{N}$ at least 2. We will be interested in the cases $p = 2$, $p = 3$ and $p = 4$. Let $(C_i)_{i\in\mathbb{Z}/2p\mathbb{Z}}$ be the edges (Weyl chambers) of $\mathcal{A}$, ordered in such a way that for all $i \in \mathbb{Z}/2p\mathbb{Z}$, the edges C_i and C_{i+1} are adjacent. For all $i \in \mathbb{Z}/2p\mathbb{Z}$, let $x_{i,i+1}$ be the intersection of C_i and C_{i+1} ; the parity of i determines the type of $x_{i,i+1}$.

If α is a root (or half-apartment) of $\mathcal{I}$, let U_α be its root group :

$$U_\alpha = \{g \in G \,:\, g(\alpha) = \alpha\}.$$

The topological building $\mathcal{I}$ is said to be *topologically Moufang* if, for every root α of $\mathcal{I}$, the root group U_α acts transitively on the set of apartments containing α. This condition is stable under product, and it is true for the topological spherical building at infinity of a symmetric space of non-compact type or of a locally finite strongly transitive Euclidean building. In Subsection 2.2, we will assume that $\mathcal{I}$ is topologically Moufang.

Since by assumption $\mathcal{I}^{\langle 1\rangle}$ has no isolated point, each root group of $\mathcal{I}$ is not compact.

For all $i \in \mathbb{Z}/2p\mathbb{Z}$, let α_i be the root (or half-apartment) $\{C_i, C_{i+1}, \dots, C_{i+p-1}\}$, and let U_i be its root group U_{α_i}. The Weyl group W of $\mathcal{I}$ is isomorphic to the dihedral group D_{2p} of order $2p$.

If x is a vertex in $\mathcal{I}$, let $\mathrm{st}(x)$ be the *star* of x:

$$\mathrm{st}\, x = \{C \in \mathcal{I}^{\langle 1\rangle} \,:\, x \in C\},$$

it is a closed subset of $\mathcal{I}$ for $\mathcal{T}_{lc}$.

Fix a root α_i of $\mathcal{A}$, and consider the folding $p_i : \mathcal{A} \to \alpha_i$ on α_i. The following lemma is immediate, but quite useful.

Lemma 2.2. *Let $(g_n)_{n\in\mathbb{N}}$ be a divergent sequence in U_i. Then the sequence $(g_n|_{\mathcal{A}})_{n\in\mathbb{N}}$ converges to p_i in $\mathrm{Mor}(\mathcal{A},\mathcal{I})$. Consequently, if $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ and if $f(\mathcal{A}) \subset \mathcal{A}$, then $p_i \circ f = \lim_{n\to+\infty} g_n \circ f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ for all $i \in \mathbb{Z}/2p\mathbb{Z}$.* □

2.2.2 Type A_1^2 : Moufang squares

Assume here that $p = 2$. Then the building $\mathcal{I}$ is not irreducible, so we can deduce the following proposition from the A_1 case, but its direct proof is easy.

Proposition 2.3. *If $\mathcal{I}$ has type A_1^2, then the space $\mathrm{Mor}(\mathcal{A},\mathcal{I})$ has 4 orbits under the action of $G \times W$, whose representatives are*

$$\mathrm{id} : \mathcal{A} \to \mathcal{A}, p_1 : \mathcal{A} \to \alpha_1, p_2 : \mathcal{A} \to \alpha_2 \text{ and } p_1 \circ p_2 = p_2 \circ p_1 : \mathcal{A} \to \alpha_1 \cap \alpha_2 = C_2.$$

Hence $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g = \mathrm{Mor}(\mathcal{A},\mathcal{I})$.

If we apply Theorem 1.3, we get the following result.

Corollary 2.4. *Let X be a product of two $\mathbb{R}$-rank 1 symmetric spaces of non-compact type or locally finite, strongly transitive trees. Then the visual limits of divergent sequences of maximal flats in X are the two types of half-apartments at infinity, or a Weyl chamber in the visual boundary $\partial_\infty X$.*

2.2.3 Type A_2 : Moufang hexagons

Assume here that $p = 3$. For instance, we may take $\mathbb{K}$ a finite-dimensional division algebra over a local field, and $\mathcal{I}$ the topological spherical building of complete flags of the right vector space $\mathbb{K}^3$.

Call (marked) *tripod* of $\mathcal{I}$ any morphism $f : \mathcal{A} \to \mathcal{I}$ whose image consists of three pairwise intersecting edges. Call *type* of a tripod the type of this common intersection.

For example, fix C an edge in $\mathcal{I}$ containing x_{12}, different from C_1 and C_2. Then $t_{12} : \mathcal{A} \to \mathcal{I}$, defined by $t_{12}(C_6) = t_{12}(C_1) = C_1$, $t_{12}(C_2) = t_{12}(C_3) = C_2$ and $t_{12}(C_4) = t_{12}(C_5) = C$, is a tripod of one type.

Similarly, fix C' an edge in $\mathcal{I}$ containing x_{23}, different from C_2 and C_3. Then $t_{23} : \mathcal{A} \to \mathcal{I}$, defined by $t_{23}(C_1) = t_{23}(C_2) = C_2$, $t_{23}(C_3) = t_{23}(C_4) = C_3$ and $t_{23}(C_5) = t_{23}(C_6) = C'$, is a tripod of the other type (see Figure 3).

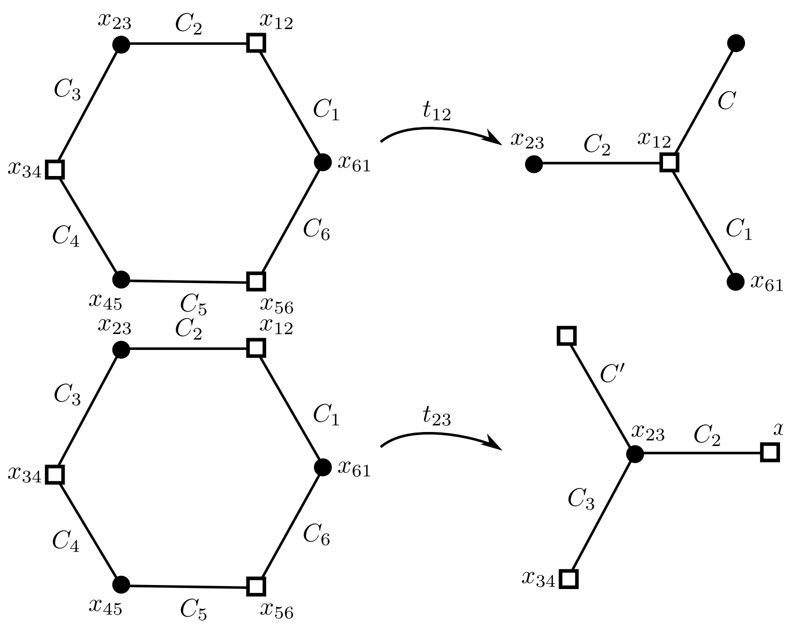

Figure 3: The marked tripods of the two types

Theorem 2.5. *If $\mathcal{I}$ has type A_2, then the space* $\mathrm{Mor}(\mathcal{A}, \mathcal{I})$ *has 7 orbits under the action of $G \times W$, whose representatives are*

$$\mathrm{id} : \mathcal{A} \to \mathcal{A},\ p_1 : \mathcal{A} \to \alpha_1,\ p_2 \circ p_1 : \mathcal{A} \to \alpha_1 \cap \alpha_2 = \{C_2, C_3\},$$
$$p_6 \circ p_1 : \mathcal{A} \to \alpha_1 \cap \alpha_6 = \{C_1, C_2\},\ p_3 \circ p_2 \circ p_1 : \mathcal{A} \to \{C_3\},\ t_{12} \text{ and } t_{23}.$$

Furthermore $\overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g = \mathrm{Mor}(\mathcal{A}, \mathcal{I})$.

Proof. Let $f \in \mathrm{Mor}(\mathcal{A}, \mathcal{I}) \setminus \mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})$.

- If $f(\mathcal{A})$ is included in an apartment, since $\mathcal{I}$ is topologically transitive, up to post-composing with an element of G, we can assume that $f(\mathcal{A}) \subset \mathcal{A}$. Since f is not injective, assume that $f(\mathcal{A}) \subset \alpha_1$.

- If $f(\mathcal{A}) = \alpha_1$, up to precomposing with an element of W, we can assume that $f = p_1$. Applying Lemma 2.2 to $\mathrm{id} : \mathcal{A} \hookrightarrow \mathcal{I}$, we get $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.
- If $f(\mathcal{A})$ consists of two edges, then $f(\mathcal{A}) = \{C_2, C_3\}$ or $f(\mathcal{A}) = \{C_1, C_2\}$. For instance, assume $f(\mathcal{A}) = \{C_2, C_3\}$: up to precomposing with an element of W, we can assume that $\mathrm{Card}\, f^{-1}(C_2) = 4$, and that $f = p_2 \circ p_1$. Applying Lemma 2.2 to p_1, we get $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.
- If $f(\mathcal{A})$ is just one edge, assume $f(\mathcal{A}) = \{C_3\}$. Then $f = p_3 \circ p_2 \circ p_1$, and applying Lemma 2.2 to $p_2 \circ p_1$, we get $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.

- If $f(\mathcal{A})$ is not included in an apartment, then f is a marked tripod. Assume for instance that f has the same type as t_{12}, and assume up to postcomposing by an element of G that $f = t_{12}$: that is $f(C_6) = f(C_1) = C_1$, $f(C_2) = f(C_3) = C_2$ and $f(C_4) = f(C_5) = C$, where C is an edge in $\mathcal{I}$ containing x_{12}, different from C_1 and C_2.

  Since $\mathcal{I}$ is topologically Moufang and has no isolated edge, let $(g_n)_{n\in\mathbb{N}}$ be a sequence in the root group $U_1 \subset G$ such that the sequence of edges $(g_n \cdot C_6)_{n\in\mathbb{N}}$ converges to C_1. Then the sequence of edges $(g_n \cdot C_5)_{n\in\mathbb{N}}$ converges to C_2. So the sequence of vertices $(g_n \cdot x_{56})_{n\in\mathbb{N}}$ converges to x_{12}.

  Since $\mathcal{I}$ is topologically strongly transitive, the compact topology on the space of vertices of $\mathcal{I}$ with the same type as x_{12} is the same as the topology induced by the topology of G-homogeneous space. Hence there exists a sequence $(g'_n)_{n\in\mathbb{N}}$ in G, converging to e, such that for every $n \in \mathbb{N}$ we have $g_n \cdot x_{56} = g'_n \cdot x_{12}$. Hence the sequence of stars $(\mathrm{st}(g_n \cdot x_{56}) = \mathrm{st}(g'_n \cdot x_{12}))_{n\in\mathbb{N}}$ converges to $\mathrm{st}(x_{12})$ in the space of closed subsets of $\mathcal{I}^{<1>}$.

  In particular, for every $n \in \mathbb{N}$, there exists an edge $C'_n \in \mathrm{st}(g_n \cdot x_{56})$ such that the sequence of edges $(C'_n)_{n\in\mathbb{N}}$ converges to $C \in \mathrm{st}(x_{12})$. Since $\mathcal{I}$ is topologically Moufang, for every $n \in \mathbb{N}$, let $h_n \in U_6$ be such that $g_n h_n \cdot C_5 = C'_n$. It follows that the sequence of marked apartments $(g_n h_n : \mathcal{A} \to \mathcal{I})_{n\in\mathbb{N}}$ converges to f in $\mathrm{Mor}(\mathcal{A}, \mathcal{I})$, hence $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$. □

Applying Theorem 1.3, we get the following result.

Corollary 2.6. *Let X be the symmetric space of non-compact type or the Bruhat-Tits building of the group* PGL(3) *over a local field. Then the visual limits of divergent sequences of maximal flats in X are the connected unions of* 1, 2 *or* 3 *Weyl chambers in an apartment, and the tripods of the two types.*

2.2.4 Type C_2 : Moufang octogons

Assume here that $p = 4$. Call (marked) *quadripod* of $\mathcal{I}$ any morphism $f : \mathcal{A} \to \mathcal{I}$ whose image consists of four pairwise intersecting edges. Call *type* of a quadripod the type of this common intersection (see Figure 4).

For example, let C, C' be two edges in $\mathcal{I}$ containing x_{12}, both different from C_1 and C_2. Then $f : \mathcal{A} \to \mathcal{I}$, defined by $f(C_8) = f(C_1) = C_1$, $f(C_2) = f(C_3) = C_2$, $f(C_4) = f(C_5) = C$ and $f(C_6) = f(C_7) = C'$, is a quadripod.

Call (marked) *T-shape* of $\mathcal{I}$ any morphism $f : \mathcal{A} \to \mathcal{I}$ whose image consists of four pairwise different edges $\{D_1, D'_1, D_2, D_3\}$, which are pairwise adjacent except that D_3 is not adjacent to D_1 nor D'_1 (see Figure 4). Call *type* of a T-shape the type of the intersection of D_1, D'_1 and D_2.

For example, fix $C'_1 \in \mathrm{st}(x_{12})\backslash\{C_1, C_2\}$, and consider $t_{12} : \mathcal{A} \to \mathcal{I}$ defined by $t_{12}(C_8) = t_{12}(C_1) = C_1$, $t_{12}(C_2) = t_{12}(C_5) = C_2$, $t_{12}(C_3) = t_{12}(C_4) = C_3$ and $t_{12}(C_6) = t_{12}(C_7) = C'_1$, it is a T-shape of one type. Similarly, fix $C'_2 \in \mathrm{st}(x_{23})\backslash\{C_2, C_3\}$, and consider $t_{23} : \mathcal{A} \to \mathcal{I}$ defined by $t_{23}(C_1) = t_{23}(C_2) = C_2$, $t_{23}(C_3) = t_{23}(C_6) = C_3$, $t_{23}(C_4) = t_{23}(C_5) = C_4$ and $t_{23}(C_7) = t_{23}(C_8) = C'_2$, it is a T-shape of the other type.

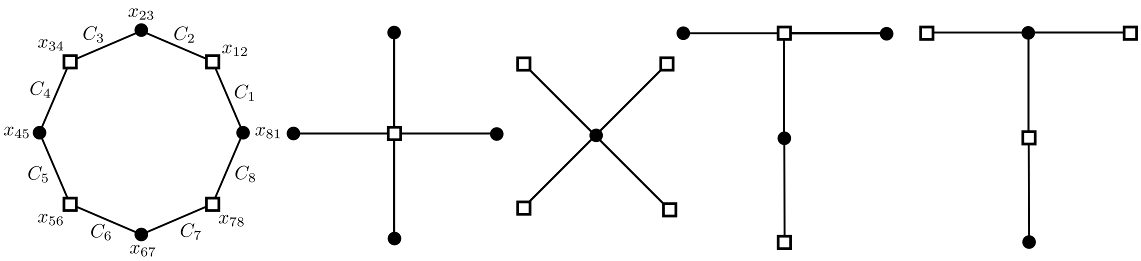

Figure 4: The type C_2 apartment, the quadripods and the T-shapes

Consider also degenerate quadripods and T-shapes, whose support are tripods. Define $t'_{12} : \mathcal{A} \to \{C_1, C'_1, C_2\}$ and $t''_{12} : \mathcal{A} \to \{C_1, C'_1, C_2\}$ by

$$t'_{12}(C_8) = t'_{12}(C_1) = C_1, t'_{12}(C_2) = t'_{12}(C_3) = C_2, t'_{12}(C_4) = t'_{12}(C_5) = t'_{12}(C_6) = t'_{12}(C_7) = C'_1,$$

$$t''_{12}(C_8) = t''_{12}(C_1) = C_1, t''_{12}(C_2) = t''_{12}(C_3) = t''_{12}(C_6) = t''_{12}(C_7) = C'_1, t''_{12}(C_4) = t''_{12}(C_5) = C_2.$$

Similarly, define $t'_{23} : \mathcal{A} \to \{C_2, C'_2, C_3\}$ and $t''_{23} : \mathcal{A} \to \{C_2, C'_2, C_3\}$ by

$$t'_{23}(C_1) = t'_{23}(C_2) = C_2, t'_{23}(C_3) = t'_{23}(C_4) = C_3, t'_{23}(C_5) = t'_{23}(C_6) = t'_{23}(C_7) = t'_{23}(C_8) = C'_2,$$

$$t''_{23}(C_1) = t''_{23}(C_2) = C_2, t''_{23}(C_3) = t''_{23}(C_4) = t''_{23}(C_7) = t''_{23}(C_8) = C'_2, t''_{23}(C_5) = t''_{23}(C_6) = C_3.$$

Finally, define the morphism $f_0 : \mathcal{A} \to \{C_1, C_2, C_3\}$ by

$$f_0(C_8) = f_0(C_1) = C_1, f_0(C_2) = f_0(C_3) = f_0(C_4) = f_0(C_7) = C_2, f_0(C_5) = f_0(C_6) = C_3.$$

Theorem 2.7. *If $\mathcal{I}$ has type C_2, then the space $\mathrm{Mor}(\mathcal{A}, \mathcal{I})\backslash\{quadripods\}$ has 19 orbits under the action of $G \times W$, whose representatives are*

- $\mathrm{id} : \mathcal{A} \to \mathcal{A}$,
- $p_1 : \mathcal{A} \to \alpha_1$, $p_2 : \mathcal{A} \to \alpha_2$,
- $p_2 \circ p_1 : \mathcal{A} \to \alpha_1 \cap \alpha_2 = \{C_2, C_3, C_4\}$, $p_8 \circ p_1 : \mathcal{A} \to \alpha_1 \cap \alpha_8 = \{C_1, C_2, C_3\}$,
- f_0,
- $p_3 \circ p_2 \circ p_1 : \mathcal{A} \to \{C_3, C_4\}$, $p_4 \circ p_3 \circ p_2 : \mathcal{A} \to \{C_4, C_5\}$,
- $p_3 \circ p_1 : \mathcal{A} \to \{C_3, C_4\}$, $p_4 \circ p_2 : \mathcal{A} \to \{C_4, C_5\}$,
- $p_8 \circ p_2 \circ p_1 : \mathcal{A} \to \{C_2, C_3\}$, $p_1 \circ p_3 \circ p_2 : \mathcal{A} \to \{C_3, C_4\}$,

- $p_4 \circ p_3 \circ p_2 \circ p_1 : \mathcal{A} \to \{C_4\}$,
- *the T-shapes* t_{12} *and* t_{23},
- t'_{12}, t'_{23},
- t''_{12} *and* t''_{23}.

The space $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ *contains all these orbits, except maybe the two orbits of* t''_{12} *and* t''_{23}.

Proof. Let $f \in \mathrm{Mor}(\mathcal{A},\mathcal{I}) \backslash \mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})$, which is not a quadripod.

- If $f(\mathcal{A})$ is not included in an apartment, then the Tits diameter of $f(\mathcal{A})$ is equal to $\frac{\pi}{2}$ or $\frac{3\pi}{4}$. Assume first that the diameter is $\frac{3\pi}{4}$, then f is a T-shape. Up to the action of $G \times W$, we may assume that $f = t_{12}$ or $f = t_{23}$. For instance, assume that $f = t_{12}$.

  Since $\mathcal{I}$ is topologically Moufang and has no isolated edge, let $(g_n)_{n\in\mathbb{N}}$ be a sequence in the root group $U_1 \subset G$ such that the sequence of edges $(g_n \cdot C_8)_{n\in\mathbb{N}}$ converges to C_1. Then the sequence of edges $(g_n \cdot C_7)_{n\in\mathbb{N}}$ converges to C_2. So the sequence of vertices $(g_n \cdot x_{78})_{n\in\mathbb{N}}$ converges to x_{12}.

  Since $\mathcal{I}$ is topologically strongly transitive, the sequence of stars $(\mathrm{st}(g_n \cdot x_{78}))_{n\in\mathbb{N}}$ converges to $\mathrm{st}(x_{12})$ in the space of closed subsets of $\mathcal{I}^{<1>}$. In particular, for every $n \in \mathbb{N}$, there exists an edge $D'_n \in \mathrm{st}(g_n \cdot x_{78})$ such that the sequence of edges $(D_n)_{n\in\mathbb{N}}$ converges to $C'_1 \in \mathrm{st}(x_{12})$. Since $\mathcal{I}$ is topologically Moufang, for every $n \in \mathbb{N}$, let $h_n \in U_8$ be such that $g_n h_n \cdot C_7 = D_n$. It follows that the sequence of marked apartments $(g_n h_n : \mathcal{A} \to \mathcal{I})_{n\in\mathbb{N}}$ converges to t_{12} in $\mathrm{Mor}(\mathcal{A},\mathcal{I})$, hence $t_{12} \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.

- If the Tits diameter of $f(\mathcal{A})$ is $\frac{\pi}{2}$, since f is not assumed to be a quadripod, then $f(\mathcal{A})$ is a tripod. Up to postcomposing with an element of G, assume that $f(\mathcal{A}) = \{C_1, C_2, C'_1\}$ or $f(\mathcal{A}) = \{C_2, C'_2, C_3\}$: assume the former. Then up to precomposing with an element of W, we may assume that $f = t'_{12}$ or $f = t''_{12}$. The theorem does not say anything about t''_{12}, so assume that $f = t'_{12}$.

  Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of edges in $\mathrm{st}(x_{23}) \backslash \{C_2\}$ converging to C_2. For all $n \in \mathbb{N}$, let t_n be the T-shape with image $\{C_1, C_2, D_n, C'_1\}$. The sequence $(t_n)_{n\in\mathbb{N}}$ converges to t'_{12} in $\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})$, and $t_n \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ by the previous case, so $t'_{12} \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.

- If $f(\mathcal{A})$ is included in an apartment, up to postcomposing with an element of G, assume that $f(\mathcal{A}) \subset \mathcal{A}$. Since f is not injective, assume that $f(\mathcal{A}) \subset \alpha_1$ or $f(\mathcal{A}) \subset \alpha_2$.

  - If $f(\mathcal{A}) = \alpha_1$ or $f(\mathcal{A}) = \alpha_2$, up to precomposing with an element of W, we can assume that $f = p_1$ or $f = p_2$. According to lemma 2.2, $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.
  - If $f(\mathcal{A})$ consists of three consecutive edges, we may assume that $f(\mathcal{A}) = \{C_1, C_2, C_3\}$. If $\mathrm{Card}\, f^{-1}(C_3) = 4$, then $f = p_8 \circ p_1$ and according to lemma 2.2, $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$. If $\mathrm{Card}\, f^{-1}(C_1) = 4$, then up to precomposing with an element w of W, we can assume that $\mathrm{Card}(fw)^{-1}(C_1) = 4$. And if $\mathrm{Card}\, f^{-1}(C_2) = 4$, then $f = f_0$.

Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of edges in $\mathrm{st}(x_{12})\backslash\{C_1, C_2\}$ converging to C_2. For all $n \in \mathbb{N}$, let t_n be the T-shape with image $\{C_1, C_2, C_3, D_n\}$. The sequence $(t_n)_{n\in\mathbb{N}}$ converges to f_0 in $\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})$, and $t_n \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$ so $f_0 \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.

- If $f(\mathcal{A})$ consists of two adjacent edges, we may assume that $f(\mathcal{A}) = \{C_3, C_4\}$ or $f(\mathcal{A}) = \{C_2, C_3\}$. For instance, assume $f(\mathcal{A}) = \{C_3, C_4\}$. If $\mathrm{Card}\, f^{-1}(C_3) = \mathrm{Card}\, f^{-1}(C_4) = 4$, then $f = p_3 \circ p_1$ or $f = p_1 \circ p_3 \circ p_2$, so according to Lemma 2.2, $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$. If $\mathrm{Card}\, f^{-1}(C_3) = 8$ then $f = p_3 \circ p_2 \circ p_1$, so according to Lemma 2.2, $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.
- If $f(\mathcal{A})$ is just one edge, assume $f(\mathcal{A}) = \{C_4\}$. Then $f = p_4 \circ p_3 \circ p_2 \circ p_1$, and according to Lemma 2.2, $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$. □

2.2.5 Type C_2 : the classical case

We will describe completely $\overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$ in the classical case (see [Tit74] and [TW02]).

Let $\mathbb{K}$ be a local field of characteristic different from 2, or a quaternion algebra over a local field of charactistic different from 2, and denote its center by $\mathbb{Z}(\mathbb{K})$. Denote by $\sigma : x \mapsto \overline{x}$ an involutive $\mathbb{Z}(\mathbb{K})$-antiautomorphism of $\mathbb{K}$, possibly trivial. Denote by $\mathbb{K}^\sigma = \{x \in \mathbb{K} : \overline{x} = x\}$ the $\mathbb{Z}(\mathbb{K})$-vector subspace of fixed points of σ in $\mathbb{K}$.

If $\mathbb{K}$ is a quaternion algebra different from the Hamilton quaternion division algebra $\mathbb{H}$, we will assume that σ is equal to $\mathrm{id}_{\mathbb{K}}$ or to the standard quaternionic involution σ_0. If we denote by $(1, i, j, k = ij)$ a standard $\mathbb{Z}(\mathbb{K})$-basis of the quaternion algebra $\mathbb{K}$, then σ_0 is defined by

$$\sigma_0(i) = -i, \sigma_0(j) = -j \text{ and } \sigma_0(k) = -k.$$

If $\mathbb{K} = \mathbb{H}$ is the Hamilton quaternion division algebra, then up to automorphism there is only one involution different from the standard one: we will denote by τ such an involution, defined by

$$\tau(i) = i, \tau(j) = -j \text{ and } \tau(k) = k.$$

So if $\mathbb{K} = \mathbb{H}$ is the Hamilton quaternion division algebra, we will assume (without loss of generality) that σ is equal to $\mathrm{id}_{\mathbb{K}}$, to the standard quaternionic involution σ_0 or to τ.

Let V be a finite-dimensional right vector space over $\mathbb{K}$, and let q be a nondegenerate σ-Hermitian form on V. It means that the associated form $\varphi : V \times V \to \mathbb{K}$ is σ-sesquilinear, and σ-Hermitian symmetric (see [Bou59a] and [Tit74]):

$$\forall v, w \in V, \forall x, y \in \mathbb{K},\ \varphi(vx, wy) = \sigma(x)\varphi(v, w)y \text{ and } \varphi(w, v) = \sigma\left(\varphi(v, w)\right).$$

Assume further that the Witt index of q is 2, which means that all the maximal totally isotropic subspaces of V have the same dimension 2.

Remark that if $\sigma = \mathrm{id}_{\mathbb{K}}$, the existence of q implies that $\mathbb{K}$ is commutative.

Let $\mathcal{I}$ be the flag complex of totally isotropic subspaces of V: it is a spherical building of type C_2, called *classical*. It is always thick, except when $\sigma = \mathrm{id}_{\mathbb{K}}$ and $\dim V = 4$. Since $\mathrm{PU}(2, 2)$ is locally isomorphic to $\mathrm{PO}(2, 4)$, we will assume that $\dim V \geqslant 5$.

The projective space $\mathbb{P}(V)$ and the Grassmannian of 2-planes $\mathcal{G}r_2(V)$ are naturally endowed with compact, non-discrete topologies. Consider the topology on $\mathcal{I}^{<1>}$ induced by the product topology on $\mathbb{P}(V) \times \mathcal{G}r_2(V)$: it turns $\mathcal{I}$ into a compact non-discrete topological spherical building. Let $G = \mathrm{PU}(q)$ be the projective isotropy group of the Hermitian form q, it acts strongly transitively on $\mathcal{I}$.

In the Archimedean case, this setting includes all real nonexceptional noncompact simple Lie groups of real rank 2, more precisely:

- $\mathrm{PO}(2,n)$ for $n \geqslant 3$, when $\mathbb{K} = \mathbb{R}$ and $\dim V = 2+n$.
- $\mathrm{PO}(5,\mathbb{C})$, when $\mathbb{K} = \mathbb{C}$, $\sigma = \mathrm{id}_{\mathbb{C}}$ and $\dim V = 5$.
- $\mathrm{PU}(2,n)$ for $n \geqslant 3$, when $\mathbb{K} = \mathbb{C}$, $\sigma \neq \mathrm{id}_{\mathbb{C}}$ and $\dim V = 2+n$.
- $\mathrm{PSp}(2,n)$ for $n \geqslant 3$, when $\mathbb{K} = \mathbb{H}$, $\sigma = \sigma_0$ and $\dim V = 2+n$.
- $\mathrm{PO}^*(10)$, when $\mathbb{K} = \mathbb{H}$, $\sigma = \tau$ and $\dim V = 5$.

Let us now describe the standard apartment $\mathcal{A}$ of $\mathcal{I}$. If E is a subset of V, denote by $\langle E\rangle$ the right $\mathbb{K}$-vector subspace spanned by E. Fix (e_1, e_2, e_3, e_4) a free family of isotropic vectors of V such that $\langle e_1, e_3\rangle$ and $\langle e_2, e_4\rangle$ are orthogonal hyperbolic planes, normalised in such a way that

$$\varphi(e_1, e_3) = \varphi(e_2, e_4) = 1.$$

Let $\mathcal{A}$ denote the apartment of $\mathcal{I}$ whose *line-type* vertices are the 4 isotropic lines $x_{81} = \langle e_1\rangle$, $x_{23} = \langle e_2\rangle$, $x_{45} = \langle e_3\rangle$ and $x_{67} = \langle e_4\rangle$, and whose *plane-type* vertices are the 4 isotropic planes $x_{12} = \langle e_1, e_2\rangle$, $x_{34} = \langle e_2, e_3\rangle$, $x_{56} = \langle e_3, e_4\rangle$ and $x_{78} = \langle e_4, e_1\rangle$.

Let q_{an} denote the restriction of q to the orthogonal V' of $\langle e_1, e_2, e_3, e_4\rangle$ in V: since the Witt index of q is 2, the Hermitian form q_{an} is anisotropic.

If f is a (non-degenerated) line-type quadripod, call *dimension* of f the dimension of the right $\mathbb{K}$-vector space generated by the 4 plane-type vertices. Assume the dimension of f is at most 4, and let p_1, p_2, p_3, p_4 denote the four cyclically ordered isotropic 2-planes of f, with $p_1 \cap p_2 \cap p_3 \cap p_4 = \ell$, an isotropic line. Say f is *symmetric* if there exists a σ-semilinear semi-isometric involution s of V, such that $s(\ell) = \ell$ and $\forall i \in [\![1,4]\!]$, $s(p_i) = p_{i+2}$.

Note that f is symmetric if and only if for all $i \in [\![1,4]\!]$, there exists $v_i \in p_i \backslash \ell$ such that $v_1 + v_2 + v_3 + v_4 = 0$ and $\varphi(v_2, v_3) = \varphi(v_1, v_4)$, or equivalently to $\varphi(v_1, v_2) = \varphi(v_4, v_3)$.

If f is a plane-type quadripod and $\mathbb{K}$ is commutative, call *cross-ratio* of f the cross-ratio of the 4 line-type vertices of f, which are all included in the central plane-type vertex of f.

Theorem 2.8. *If $\mathcal{I}$ is of type C_2 and classical, and fix a morphism $f : \mathcal{A} \to \mathcal{I}$.*

- *If f is not a quadripod, then $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.*
- *If f is a line-type quadripod, then $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$ if and only if the dimension of f is at most 4, and $\sigma = \mathrm{id}_{\mathbb{K}}$ or f is symmetric.*
- *If f is a plane-type quadripod, then $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$ if and only if $\sigma \neq \mathrm{id}_{\mathbb{K}}$, or $\sigma = \mathrm{id}_{\mathbb{K}}$ and $1 - c \in \{q_{an}(v) q_{an}(v')^{-1} : v, v' \in V' \backslash \{0\}\}$, where c is the cross-ratio of f.*

Assume $\sigma = \mathrm{id}_{\mathbb{K}}$. If $\dim V = 5$, then all line-type quadripods belong to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$, and a plane-type quadripod with cross-ratio c belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ if and only if $1-c$ is a square in $\mathbb{K}$. So if $\mathbb{K}=\mathbb{R}$, then the condition on c reduces to $c \leqslant 1$, and if further $K=\mathbb{C}$ then the condition on c is void. In particular if $\mathbb{K}=C$, we have

$$\mathrm{Mor}(\mathcal{A},\mathcal{I}) = \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g.$$

Proof. • Consider the orthogonal case $\sigma = \mathrm{id}_{\mathbb{K}}$. Then $\mathbb{K}$ is commutative, hence $\mathbb{K}$ is a local field.

Let us first focus on quadripods.

– Let f be a line-type quadripod. If $\mathcal{A}'$ is an apartment in $\mathcal{I}$, the $\mathbb{K}$-vector subspace generated by the 4 plane-type vertices and the 4 line-type vertices of $\mathcal{A}'$ has dimension 4. Hence if $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$, the dimension of f is at most 4 (by lower semi-continuity of the dimension). Conversely, let us assume that the dimension of f is at most 4: assume first it is equal to 4. Then, according to Witt's Theorem and up to postcomposing by G, we may assume that the central line-type vertex of f is $\langle e_1\rangle$, $f(\langle e_1,e_2\rangle) = \langle e_1,e_2\rangle$ and $f(\langle e_1,e_4\rangle) = \langle e_1,e_4\rangle$. The $\mathbb{K}$-vector subspace E of V generated by the plane-type vertices of f has dimension 4, contains $\langle e_1,e_2,e_4\rangle$ and is included in the orthogonal of $\langle e_1\rangle$, hence it is equal to $E=\langle e_1,e_2,e_4,e_5\rangle$, where e_5 is orthogonal to $\langle e_1,e_2,e_3,e_4\rangle$. Let $r \in \mathbb{K}\backslash\{0\}$ such that $\varphi(e_5,e_5) = -2r$. Up to multiplying e_2, e_4 and e_5 by scalars, we may further assume that $f(\langle e_2,e_3\rangle) = \langle e_1, e_2+e_4r+e_5\rangle$. And there exists $a \in \mathbb{K}\backslash\{0,1\}$ such that $f(\langle e_3,e_4\rangle) = \langle e_1, e_2+e_4a^2r+e_5a\rangle$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ going to infinity. Let us define the marked apartment $f_n \in \mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})$ by $f_n(\langle e_i\rangle) = \langle \widetilde{f_n}(e_i)\rangle$ for all $i \in [\![1,4]\!]$, where

$$\begin{aligned}\widetilde{f_n}(e_1) = e_1, \widetilde{f_n}(e_2) = -e_1arx_n + e_2, \widetilde{f_n}(e_4) = -e_1x_n + e_4\\ \text{and } \widetilde{f_n}(e_3) = e_1ar(a-1)x_n^2 + e_2x_n + e_3 + e_4arx_n + e_5ax_n.\end{aligned}$$

Each of these vectors is isotropic :

$$\begin{aligned}q(\widetilde{f_n}(e_1)) = 0, q(\widetilde{f_n}(e_2)) = 0, q(\widetilde{f_n}(e_4)) = 0\\ \text{and } q(\widetilde{f_n}(e_3)) = 2ar(a-1)x_n^2 + 2arx_n^2 + (-2r)a^2x_n^2 = 0.\end{aligned}$$

Furthermore $\varphi(\widetilde{f_n}(e_1),\widetilde{f_n}(e_2)) = \varphi(\widetilde{f_n}(e_4),\widetilde{f_n}(e_1)) = 0$, and

$$\begin{aligned}\varphi(\widetilde{f_n}(e_2),\widetilde{f_n}(e_3)) &= -arx_n + arx_n = 0\\ \varphi(\widetilde{f_n}(e_3),\widetilde{f_n}(e_4)) &= x_n - x_n = 0.\end{aligned}$$

Hence f_n is a marked apartment of $\mathcal{I}$. Let us show that $(f_n)_{n\in\mathbb{N}}$ converges to f.

First of all, for all $i \in [\![1,4]\!]$, the sequence $(f_n(\langle e_i\rangle))_{n\in\mathbb{N}}$ converges to $\langle e_1\rangle = f(\langle e_1\rangle)$. For all $n \in \mathbb{N}$, we have $f_n(\langle e_1,e_2\rangle) = \langle e_1,e_2\rangle = f(\langle e_1,e_2\rangle)$ and $f_n(\langle e_1,e_4\rangle) = \langle e_1,e_4\rangle = f(\langle e_1,e_4\rangle)$.

Every accumulation point of the sequence of planes $(f_n(\langle e_2, e_3\rangle))_{n\in\mathbb{N}}$ contains the line

$$\begin{aligned}\lim_{n\to+\infty}\langle \widetilde{f_n}(e_3)+\widetilde{f_n}(e_2)(arx_n)^{-1}ar(a-1)x_n^2\rangle &= \\ \lim_{n\to+\infty}\langle e_2(x_n+(a-1)x_n)+e_3+e_4arx_n+e_5ax_n\rangle &= \\ \langle e_2+e_4r+e_5\rangle \subset f(\langle e_2,e_3\rangle).\end{aligned}$$

Hence the sequence of planes $(f_n(\langle e_2, e_3\rangle))_{n\in\mathbb{N}}$ converges to $f(\langle e_2, e_3\rangle)$.
Similarly, every accumulation point of the sequence of planes $(f_n(\langle e_3, e_4\rangle))_{n\in\mathbb{N}}$ contains the line

$$\begin{aligned}\lim_{n\to+\infty}\langle f_n(e_3)+f_n(e_4)x_n^{-1}ar(a-1)x_n^2\rangle &= \\ \lim_{n\to+\infty}\langle e_2x_n+e_3+e_4(arx_n+ar(a-1)x_n)+e_5ax_n\rangle &= \\ \langle e_2+e_4a^2r+e_5a)\rangle \subset f(\langle e_3,e_4\rangle).\end{aligned}$$

Hence the sequence of planes $(f_n(\langle e_3, e_4\rangle))_{n\in\mathbb{N}}$ converges to $f(\langle e_3, e_4\rangle)$.
To conclude this case, the sequence of marked apartments $(f_n)_{n\in\mathbb{N}}$ converges to f, so $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.
If the dimension of f is less than 4, then f is a limit of line-type quadripods whose dimension is 4, hence according to the previous case f also belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.

– Let f be a plane-type quadripod. Then, according to Witt's Theorem and up to postcomposing by G, we may assume that the central plane-type vertex of f is $\langle e_1, e_2\rangle$, $f(\langle e_1\rangle) = \langle e_1\rangle$ and $f(\langle e_2\rangle) = \langle e_2\rangle$. Up to multiplying e_1 and e_2 by scalars, we may further assume that if $c \in \mathbb{K}\backslash\{0,1\}$ is the cross-ratio of f, then

$$f(\langle e_1\rangle) = \langle e_1\rangle, f(\langle e_2\rangle) = \langle e_2\rangle, f(\langle e_3\rangle) = \langle e_1+e_2\rangle \text{ and } f(\langle e_4\rangle) = \langle e_1c+e_2\rangle.$$

Assume first that there exists $v, v' \in V'\backslash\{0\}$ such that $1-c = q(v)q(v')^{-1}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ going to infinity. Let us define the marked apartment $f_n \in \mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})$ by $f_n(\langle e_i\rangle) = \langle \widetilde{f_n}(e_i)\rangle$ for all $i \in [\![1,4]\!]$, where

$$\begin{gathered}\widetilde{f_n}(e_1) = e_1, \widetilde{f_n}(e_2) = e_2, \widetilde{f_n}(e_3) = (e_1+e_2)(\tfrac{-1}{2}q(v+v')x_n^2)+e_3+(v+v')x_n \\ \text{and } \widetilde{f_n}(e_4) = (e_1c+e_2)(\tfrac{-1}{2}q(v')x_n^2)+e_4+v'x_n.\end{gathered}$$

Each of these vectors is isotropic :

$$\begin{gathered}q(\widetilde{f_n}(e_1)) = 0, q(\widetilde{f_n}(e_2)) = 0, q(\widetilde{f_n}(e_3)) = 2(\tfrac{-1}{2}q(v+v')x_n^2)+q(v+v')x_n^2 = 0 \\ \text{and } q(\widetilde{f_n}(e_4)) = 2(\tfrac{-1}{2}q(v')x_n^2)+q(v')x_n^2 = 0.\end{gathered}$$

Furthermore $\varphi(\widetilde{f_n}(e_1), \widetilde{f_n}(e_2)) = \varphi(\widetilde{f_n}(e_2), \widetilde{f_n}(e_3)) = \varphi(\widetilde{f_n}(e_4), \widetilde{f_n}(e_1)) = 0$, and

$$\begin{aligned}\varphi(\widetilde{f_n}(e_3), \widetilde{f_n}(e_4)) &= \frac{-1}{2}q(v+v')x_n^2 + c\frac{-1}{2}q(v')x_n^2 + \varphi(v+v', v')x_n^2 \\ &= \frac{-1}{2}(q(v+v')+cq(v')-2\varphi(v+v',v'))x_n^2 \\ &= \frac{-1}{2}(q(v+v')+q(v')-q(v)-2\varphi(v+v',v'))x_n^2 = 0.\end{aligned}$$

Hence f_n is a marked apartment of $\mathcal{I}$. And it is clear that $(f_n)_{n\in\mathbb{N}}$ converges to f, so $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.

Conversely, assume that $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$: let $(f_n)_{n\in\mathbb{N}}$ be a sequence of marked apartments converging to f. Then, according to Witt's Theorem and up to postcomposing by G, we may assume that, for all $n \in \mathbb{N}$, we have $f_n(\langle e_1\rangle) = \langle e_1\rangle$ and $f_n(\langle e_2\rangle) = \langle e_2\rangle$. For all $n \in \mathbb{N}$, since f_n is a marked apartment we know there exists $x_n, y_n, z_n, t_n \in \mathbb{K}$ and $u_n, v_n \in V'$ such that

$$f_n(\langle e_3\rangle) = \langle e_1 + e_2x_n + e_3z_n + u_n\rangle \text{ and } f_n(\langle e_4\rangle) = \langle e_1c + e_2y_n + e_4t_n + v_n\rangle.$$

As the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f, we deduce that $x_n, y_n \underset{n\to+\infty}{\longrightarrow} 1$, $z_n, t_n \underset{n\to+\infty}{\longrightarrow} 0$, and $u_n, v_n \underset{n\to+\infty}{\longrightarrow} 0$. Furthermore $f_n(\langle e_3\rangle)$ and $f_n(\langle e_4\rangle)$ are isotropic and orthogonal, so $2z_n + q(u_n) = 0$, $2y_nt_n + q(v_n) = 0$ and $x_nt_n + cz_n + \varphi(u_n, v_n) = 0$. Hence

$$c = (-x_nt_n - \varphi(u_n, v_n))z_n^{-1} = (2^{-1}x_ny_n^{-1}q(v_n) - \varphi(u_n, v_n))(-\frac{1}{2}q(u_n))^{-1}.$$

Assume first that, up to passing to a subsequence, the sequence $(q(v_n)q(u_n)^{-1})_{n\in\mathbb{N}}$ is bounded. Then $c = \lim_{n\to+\infty} (q(v_n) - 2\varphi(u_n, v_n))(-q(u_n))^{-1}$, and

$$1 - c = \lim_{n\to+\infty} q(u_n - v_n)q(u_n)^{-1}.$$

If not, then up to passing to a subsequence we may assume that the sequence $(q(u_n)q(v_n)^{-1})_{n\in\mathbb{N}}$ converges to 0. Since q_{an} is anisotropic, $\sqrt{|q_{an}|}$ is equivalent to a norm on V', hence $\varphi(u_n, v_n))(q(u_n)q(v_n))^{-1} \underset{n\to+\infty}{\longrightarrow} 0$ and

$$1 - c = \lim_{n\to+\infty} (q(u_n) + q(v_n))q(u_n)^{-1} = \lim_{n\to+\infty} q(u_n - v_n)q(u_n)^{-1}.$$

Since the set $\{q_{an}(v)q_{an}(v')^{-1} : v, v' \in V'\backslash\{0\}\}$ is stable under multiplication by squares in $\mathbb{K}\backslash\{0\}$, it is closed in $\mathbb{K}\backslash\{0\}$ since the subgroup of squares in the multiplicative group $\mathbb{K}\backslash\{0\}$ is of finite index. Hence $1 - c \in \{q_{an}(v)q_{an}(v')^{-1} : v, v' \in V'\backslash\{0\}\}$.

By Theorem 2.7, to prove Theorem 2.8 in the case $\sigma = \mathrm{id}_{\mathbb{K}}$ it remains to prove that the morphisms t''_{12} and t''_{23} belong to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.

Firstly, the morphism t''_{12} can be interpreted as a degenerated plane-type quadripod, whose cross-ratio is 0. The set $\{q_{an}(v)q_{an}(v')^{-1} : v, v' \in V'\backslash\{0\}\}$ is stable under multiplication by squares in $\mathbb{K}\backslash\{0\}$, and since the subgroup of squares in the multiplicative group $\mathbb{K}\backslash\{0\}$ is of finite index, then 1 is the limit of a sequence $(c_n)_{n\in\mathbb{N}}$ in $\{q_{an}(v)q_{an}(v')^{-1} : v, v' \in V'\backslash\{0\}\}\backslash\{1\}$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of plane-type quadripods, whose cross-ratios are $(c_n)_{n\in\mathbb{N}}$, which converges to t''_{12}. For all $n \in \mathbb{N}$, since $c_n \notin \{0, 1\}$ we know that f_n is a nondegenerate plane-type quadripod, which then belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$. Hence $t''_{12} \in \overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.

Secondly, the morphism t''_{23} can be interpreted as a degenerated line-type quadripod, for which two opposite plane-type vertices coincide. Let E denote the vector space generated by the plane-type vertices of t''_{23}, it has dimension at most 4. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of line-type quadripods, for which the vector space generated by the plane-type vertices is E, which converges to t''_{23}. But we know that each f_n has dimension equal to $\dim E \leqslant 4$ and belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$, and so t''_{23} belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.

Assume now that $\dim V = 5$. Then since q is nondegenerate, the dimension of any line-type quadripod is at most 4. And $V' = \langle e_1, e_2, e_3, e_4\rangle^\perp$ is a line, and so the set $\{q_{an}(v)q_{an}(v')^{-1} : v, v' \in V'\backslash\{0\}\}$ is just the set of squares in $\mathbb{K}\backslash\{0\}$.

- Consider now the unitary case $\sigma \neq \mathrm{id}_{\mathbb{K}}$.

  Let us first focus on quadripods.

  - Let f be a line-type quadripod. Let us assume that the dimension of f is at most 4 and that f is symmetric. Since f is non-degenerated and symmetric, then its dimension is exactly 4. Then, according to Witt's Theorem and up to postcomposing by G, we may assume that the central line-type vertex of f is $\langle e_1\rangle$, $f(\langle e_1, e_2\rangle) = \langle e_1, e_2\rangle$ and $f(\langle e_1, e_4\rangle) = \langle e_1, e_4\rangle$. The $\mathbb{K}$-vector subspace E of V generated by the plane-type vertices of f has dimension 4, contains $\langle e_1, e_2, e_4\rangle$ and is included in the orthogonal of $\langle e_1\rangle$, hence it is equal to $E = \langle e_1, e_2, e_4, e_5\rangle$, where e_5 is orthogonal to $\langle e_1, e_2, e_3, e_4\rangle$. Denote $\varphi(e_5, e_5) = r \in \mathbb{K}^\sigma\backslash\{0\}$. There exists $a, b, c, d \in \mathbb{K}$ such that

    $$f(\langle e_2, e_3\rangle) = \langle e_1, e_2 + e_4 a + e_5 b\rangle \text{ and } f(\langle e_3, e_4\rangle) = \langle e_1, e_2 + e_4 c + e_5 d\rangle,$$

    where $a + \overline{a} + \overline{b} r b = c + \overline{c} + \overline{d} r d = 0$.
    Since the dimension of f is 4, b and d are not both zero. And since f is symmetric, neither b nor d can be zero, hence a and c are non-zero as well.
    Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Z}(\mathbb{K})^\sigma$ going to infinity. Let us define the marked apartment $f_n \in \mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})$ by $f_n(\langle e_i\rangle) = \langle \widetilde{f_n}(e_i)\rangle$ for all $i \in [\![1,4]\!]$, where

    $$\begin{gathered}\widetilde{f_n}(e_1) = e_1, \widetilde{f_n}(e_2) = e_1(-x_n) + e_2, \widetilde{f_n}(e_4) = e_1(-\overline{a^{-1}}\,\overline{b}\,\overline{d^{-1}} x_n) + e_4 \\ \text{and } \widetilde{f_n}(e_3) = e_1(a^{-1} - d^{-1}ba^{-1})x_n^2 + e_2 d^{-1}ba^{-1}x_n + e_3 + e_4 x_n + e_5 ba^{-1}x_n.\end{gathered}$$

    Each of these vectors is isotropic $q(\widetilde{f_n}(e_1)) = q(\widetilde{f_n}(e_2)) = q(\widetilde{f_n}(e_4)) = 0$ and

    $$\begin{aligned} q(\widetilde{f_n}(e_3)) &= (\overline{a^{-1}} - \overline{a^{-1}}\,\overline{b}\,\overline{d^{-1}})x_n^2 + (a^{-1} - d^{-1}ba^{-1})x_n^2 \\ &\quad + \overline{a^{-1}}\,\overline{b}\,\overline{d^{-1}} x_n^2 + d^{-1}ba^{-1}x_n^2 + \overline{a^{-1}}\,\overline{b} r b a^{-1} x_n^2 = 0. \end{aligned}$$

    Furthermore $\varphi(\widetilde{f_n}(e_1), \widetilde{f_n}(e_2)) = \varphi(\widetilde{f_n}(e_4), \widetilde{f_n}(e_1)) = 0$, and

    $$\begin{aligned} \varphi(\widetilde{f_n}(e_2), \widetilde{f_n}(e_3)) &= -x_n + x_n = 0 \\ \varphi(\widetilde{f_n}(e_3), \widetilde{f_n}(e_4)) &= \overline{d^{-1}ba^{-1}} x_n + (-\overline{a^{-1}}\,\overline{b}\,\overline{d^{-1}} x_n) = 0. \end{aligned}$$

    Hence f_n is a marked apartment of $\mathcal{I}$. Let us show that $(f_n)_{n\in\mathbb{N}}$ converges to f.

First of all, for all $i \in [\![1,4]\!]$, the sequence $(f_n(\langle e_i \rangle))_{n\in\mathbb{N}}$ converges to $\langle e_1 \rangle = f(\langle e_1 \rangle)$. For all $n \in \mathbb{N}$, we have $f_n(\langle e_1, e_2 \rangle) = \langle e_1, e_2 \rangle = f(\langle e_1, e_2 \rangle)$ and $f_n(\langle e_1, e_4 \rangle) = \langle e_1, e_4 \rangle = f(\langle e_1, e_4 \rangle)$.
Every accumulation point of the sequence of planes $(f_n(\langle e_2, e_3 \rangle))_{n\in\mathbb{N}}$ contains the line

$$\begin{aligned} \lim_{n\to+\infty} \langle \widetilde{f_n}(e_3) + \widetilde{f_n}(e_2)(a^{-1} - d^{-1}ba^{-1})x_n \rangle &= \\ \lim_{n\to+\infty} \langle e_2(d^{-1}ba^{-1} + a^{-1} - d^{-1}ba^{-1})x_n + e_3 + e_4x_n + e_5ba^{-1}x_n \rangle &= \\ \langle e_2 + e_4a + e_5b \rangle \subset f(\langle e_2, e_3 \rangle). \end{aligned}$$

Hence the sequence of planes $(f_n(\langle e_2, e_3 \rangle))_{n\in\mathbb{N}}$ converges to $f(\langle e_2, e_3 \rangle)$.
Similarly, every accumulation point of the sequence of planes $(f_n(\langle e_3, e_4 \rangle))_{n\in\mathbb{N}}$ contains the line

$$\begin{aligned} \lim_{n\to+\infty} \langle \widetilde{f_n}(e_3) + \widetilde{f_n}(e_4)(\overline{a^{-1}}\overline{bd^{-1}})^{-1}(a^{-1} - d^{-1}ba^{-1})x_n \rangle &= \\ \lim_{n\to+\infty} \langle e_2d^{-1}ba^{-1}x_n + e_3 + e_4(1 + \overline{d}\overline{b^{-1}}\overline{a}(a^{-1} - d^{-1}ba^{-1}))x_n + e_5ba^{-1}x_n \rangle &= \\ \langle e_2 + e_4(ab^{-1}d + \overline{d}\overline{b^{-1}}\overline{a}(b^{-1}d - 1)) + e_5d) \rangle. \end{aligned}$$

So it remains to show that $ab^{-1}d + \overline{d}\overline{b^{-1}}\overline{a}(b^{-1}d - 1) = c$. Let

$$\begin{aligned} v_{12} &= e_2(d^{-1} - b^{-1}) \in f(\langle e_1, e_2 \rangle)\backslash\langle e_1 \rangle, \\ v_{23} &= e_2b^{-1} + e_4ab^{-1} + e_5 \in f(\langle e_2, e_3 \rangle)\backslash\langle e_1 \rangle \\ v_{34} &= -e_2d^{-1} - e_4cd^{-1} - e_5 \in f(\langle e_3, e_4 \rangle)\backslash\langle e_1 \rangle \\ \text{and } v_{41} &= e_4(cd^{-1} - ab^{-1}) \in f(\langle e_4, e_1 \rangle)\backslash\langle e_1 \rangle, \end{aligned}$$

they are such that $v_{12} + v_{23} + v_{34} + v_{41} = 0$, so the symmetry assumption about f tells us that

$$\begin{aligned} \varphi(v_{12}, v_{23}) &= \varphi(v_{41}, v_{34}) \\ (\overline{d}^{-1} - \overline{b}^{-1})ab^{-1} &= (\overline{d}^{-1}\overline{c} - \overline{b}^{-1}\overline{a})(-d^{-1}) \\ \overline{d}^{-1}\overline{c}d^{-1} &= \overline{b}^{-1}\overline{a}d^{-1} - \overline{d}^{-1}ab^{-1} + \overline{b}^{-1}ab^{-1} \\ c &= ab^{-1}d + \overline{d}\overline{b}^{-1}\overline{a}b^{-1}d - \overline{d}\overline{b}^{-1}\overline{a}. \end{aligned}$$

Hence the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f.

Conversely, assume that $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$. If $\mathcal{A}'$ is an apartment in $\mathcal{I}$, the $\mathbb{K}$-vector space generated by the 4 plane-type vertices of $\mathcal{A}'$ has dimension 4. Hence the dimension of f is at most 4, and so we can choose $v_{12} \in f(\langle e_1, e_2 \rangle)\backslash f(\langle e_1 \rangle)$, $v_{23} \in f(\langle e_2, e_3 \rangle)\backslash f(\langle e_1 \rangle)$, $v_{34} \in f(\langle e_3, e_4 \rangle)\backslash f(\langle e_1 \rangle)$ and $v_{41} \in f(\langle e_4, e_1 \rangle)\backslash f(\langle e_1 \rangle)$ such that $v_{12} + v_{23} + v_{34} + v_{41} = 0$.
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of marked apartments converging to f. For $i \in \mathbb{Z}/4\mathbb{Z}$, choose a sequence $(v^n_{i,i+1})_{n\in\mathbb{N}}$ in $(f_n(\langle e_i, e_{i+1} \rangle))_{n\in\mathbb{N}}$ converging to $v_{i,i+1}$, such that for all $n \in \mathbb{N}$ we have $v^n_{12} + v^n_{23} + v^n_{34} + v^n_{41} = 0$. Hence for all $n \in \mathbb{N}$, we have

$$\varphi(v^n_{23}, v^n_{43}) = \varphi(-v^n_{12}, -v^n_{41}) = \varphi(v^n_{12}, v^n_{41}).$$

Taking the limit $n \to +\infty$, we get $\varphi(v_{23}, v_{43}) = \varphi(v_{12}, v_{41})$, so f is symmetric.

– Let f be a plane-type quadripod. Then, according to Witt's Theorem and up to postcomposing by G, we may assume that the central plane-type vertex of f is $\langle e_1, e_2\rangle$, $f(\langle e_1\rangle) = \langle e_1\rangle$ and $f(\langle e_2\rangle) = \langle e_2\rangle$. Up to multiplying e_1 and e_2 by scalars, we may further assume that there exists $c \in \mathbb{K}\backslash\{0,1\}$ such that

$$f(\langle e_1\rangle) = \langle e_1\rangle, f(\langle e_2\rangle) = \langle e_2\rangle, f(\langle e_3\rangle) = \langle e_1 + e_2\rangle \text{ and } f(\langle e_4\rangle) = \langle e_1 c + e_2\rangle.$$

In fact, the scalar c is the cross-ratio of f (or an element of the cross-ratio if $\mathbb{K}$ is not commutative: see [Bae52] for instance). If $\mathbb{K}$ is a quaternion division algebra and $\sigma = \sigma_0$ is the standard quaternionic involution, then $c\overline{c} \in \mathbb{K}^\sigma = \mathbb{Z}(\mathbb{K})$, hence $\overline{c}$ commutes with c. If $\mathbb{K} = \mathbb{H}$ is the Hamilton quaternion division algebra and $\sigma = \tau$, then since c is well-defined up to conjugation we may assume that $c \in \mathbb{R} \oplus \mathbb{R}j \subset \mathbb{H}$, hence $\overline{c}$ commutes with c.

Fix $e_5 \in \langle e_1, e_2, e_3, e_4\rangle^\perp\backslash\{0\}$, and denote $r = q(e_5) \in \mathbb{K}^\sigma\backslash\{0\}$. If $\mathbb{K}^\sigma$ is not included in the center of $\mathbb{K}$, then by assumption $\mathbb{K} = \mathbb{H}$ and $\sigma = \tau$. Then r is a square in $\mathbb{H}^\tau\backslash\{0\}$, hence up to multiplying e_5 by a scalar we may assume that $r = 1$, so that in any case we have $r \in \mathbb{Z}(\mathbb{K})$.

Assume first that $\overline{c} \neq c$. Let $a = (c+\overline{c}-2)(\overline{c}-c)^{-1}$, it commutes with c and is such that $\overline{a} = -a$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Z}(\mathbb{K})$ going to infinity. Let us define the marked apartment $f_n \in \mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})$ by $f_n(\langle e_i\rangle) = \langle \widetilde{f_n}(e_i)\rangle$ for all $i \in [\![1, 4]\!]$, where

$$\widetilde{f_n}(e_1) = e_1, \widetilde{f_n}(e_2) = e_2, \widetilde{f_n}(e_3) = (e_1 + e_2)(2^{-1}(1-a)\overline{c} - 1)rx_n^2 + e_3 + e_5 x_n$$
$$\text{and } \widetilde{f_n}(e_4) = -(e_1 c + e_2)2^{-1}(1+a)rx_n^2 + e_4 + e_5 x_n.$$

Each of these vectors is isotropic : $q(\widetilde{f_n}(e_1)) = 0$, $q(\widetilde{f_n}(e_2)) = 0$,

$$\begin{aligned} q(\widetilde{f_n}(e_3)) &= (2^{-1}(1-a)\overline{c} - 1)rx_n^2 + (2^{-1}(1+a)c - 1)rx_n^2 + rx_n^2 \\ &= 2^{-1}(\overline{c} + c - a(\overline{c} - c) - 2)rx_n^2 = 0 \text{ and} \\ q(\widetilde{f_n}(e_4)) &= -2^{-1}(1+a)rx_n^2 - 2^{-1}(1-a)rx_n^2 + rx_n^2 = 0. \end{aligned}$$

Furthermore $\varphi(\widetilde{f_n}(e_1), \widetilde{f_n}(e_2)) = \varphi(\widetilde{f_n}(e_2), \widetilde{f_n}(e_3)) = \varphi(\widetilde{f_n}(e_4), \widetilde{f_n}(e_1)) = 0$, and

$$\varphi(\widetilde{f_n}(e_3), \widetilde{f_n}(e_4)) = (2^{-1}(1+a)c - 1)rx_n^2 - c2^{-1}(1+a)rx_n^2 + rx_n^2 = 0.$$

Hence f_n is a marked apartment of $\mathcal{I}$. And it is clear that $(f_n)_{n\in\mathbb{N}}$ converges to f, so $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.

If $\overline{c} = c$, then f is a limit of plane-type quadripods whose cross-ratios do not belong to $\mathbb{K}^\sigma$, hence according to the previous case f also belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.

It remains to show that the morphisms t''_{12} and t''_{23} belong to $\overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.

Firstly, the morphism t''_{12} can be interpreted as a degenerate plane-type quadripod, and it is a limit of non-degenerate plane-type quadripods, so $t''_{12} \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.

Secondly, the morphism t''_{23} can be interpreted as a degenerated line-type quadripod, for which two opposite plane-type vertices coincide: it is symmetric. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of symmetric non-degenerated line-type quadripods which converges to t''_{23}. Hence each f_n belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$, and so does t''_{23}. □

This concludes the proof of Theorem C.

2.3 Rank 3 : PGL(4)

Let $\mathbb{K}$ be a local field, and let $\mathcal{I}$ be the topological spherical building of complete flags of the vector space $V = \mathbb{K}^4$: it is the spherical building of $(\mathrm{PGL}_4, \mathbb{K})$. Let $\mathcal{A}$ be the standard apartment in $\mathcal{I}$ which corresponds to the four canonical points $\{\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle\}$ of $\mathbb{P}(V)$. Denote by $[\![1,4]\!]_2$ the set of subsets of $[\![1,4]\!]$ of cardinal 2.

If $f : \mathcal{A} \to \mathcal{I}$ is a type-preserving simplicial morphism, we will denote

$$\begin{aligned} \forall i \in [\![1,4]\!], \quad & x_i = f(\langle e_i\rangle) \\ \forall \{i,j\} \in [\![1,4]\!]_2, \quad & \ell_{ij} = f(\langle e_i, e_j\rangle) \\ \forall i \in [\![1,4]\!], \quad & p_i = f(\langle e_j \,:\, j \neq i\rangle), \end{aligned}$$

where $[\![1,4]\!]_2$ denote the set of all subsets of $[\![1,4]\!]$ with 2 elements. Choosing a marked apartment of $\mathcal{I}$ is the same thing as choosing 4 points, 6 lines and 4 planes in $\mathbb{P}(V)$,

$$((x_i)_{i\in[\![1,4]\!]}, (\ell_{ij})_{\{i,j\}\in[\![1,4]\!]_2}, (p_i)_{i\in[\![1,4]\!]}) \in \mathbb{P}(V)^4 \times \mathcal{G}r_2(V)^6 \times \mathbb{P}(V^*)^4,$$

which satisfy the incidence conditions of a tetrahedron (see Figure 5),

$$\forall i,j \in [\![1,4]\!] \text{ distinct}, \quad x_i \subset \ell_{ij}$$

$$\forall i,j,k \in [\![1,4]\!] \text{ pairwise distinct}, \quad \ell_{ij} \subset p_k$$

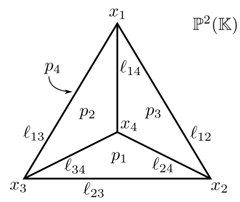

Figure 5: An apartment: a tetrahedron in $\mathbb{P}^2(\mathbb{K})$

If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{Mor}(\mathcal{A},\mathcal{I})$, we will denote similarly

$$\begin{aligned} \forall i \in [\![1,4]\!], \quad & x_i^n = f_n(\langle e_i\rangle) \\ \forall \{i,j\} \in [\![1,4]\!]_2, \quad & \ell_{ij}^n = f_n(\langle e_i, e_j\rangle) \\ \forall i \in [\![1,4]\!], \quad & p_i^n = f_n(\langle e_j \,:\, j \neq i\rangle). \end{aligned}$$

If $f \in \mathrm{Mor}(\mathcal{A},\mathcal{I})$ is such that there exists $\ell \in \mathcal{G}r_2(V)$ such that $\forall \{i,j\} \in [\![1,4]\!]_2, \ell_{ij} = \ell$, we will say that f is of type (L) (see Figure 6). In that case, we will say that f is *symmetrical* if $\mathrm{Card}\{x_1,x_2,x_3,x_4\} \leqslant 2$, or if $\mathrm{Card}\{p_1,p_2,p_3,p_4\} \leqslant 2$, or otherwise if there exists a projective isomorphism $s : \mathbb{P}(V) \simeq \mathbb{P}(V^*)$ such that for all $i \in [\![1,4]\!]$, we have $s(x_i) = p_i$. This last condition is equivalent to asking that the cross-ratio of the four points $x_1,\dots,x_4$ on the projective line ℓ is equal to the cross-ratio of the four planes $p_1,\dots,p_4$ on the projective line $\ell^\perp$ of $\mathbb{P}(V^*)$.

Recall that if x_1,x_2,x_3,x_4 are points of the projective line $\mathbb{KP}^1 \simeq \mathbb{K}\cup\{\infty\}$ such that x_1, x_2 and x_3 are pairwise distinct, the cross-ratio of (x_1,x_2,x_3,x_4) is defined as $f(x_4) \in \mathbb{KP}^1 \simeq \mathbb{K}\cup\{\infty\}$, where f is the unique homography such that $f(x_1) = \infty$, $f(x_2) = 0$ and $f(x_3) = 1$. If x_1,x_2,x_3,x_4 are points of a projective line such that $\mathrm{Card}\{x_1,x_2,x_3,x_4\} \geqslant 3$, choose a double transposition σ of $[\![1,4]\!]$ such that $x_{\sigma(1)}$, $x_{\sigma(2)}$ and $x_{\sigma(3)}$ are pairwise distinct, then define the cross-ratio of (x_1,x_2,x_3,x_4) to be the cross-ratio of $(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)})$ (note that it is independent on the choice of σ). Hence the cross-ratio is a continous map from $\{(x_1,x_2,x_3,x_4) \in (\mathbb{KP}^1)^4 \,:\, \mathrm{Card}\{x_1,x_2,x_3,x_4\} \geqslant 3\}$ to $\mathbb{K}\cup\{\infty\}$.

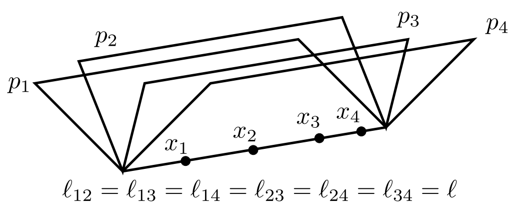

Figure 6: A morphism of type (L)

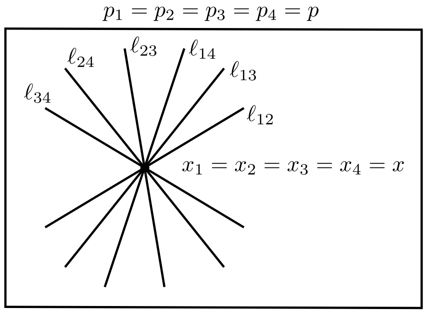

Figure 7: A morphism of type (XP)

If $f \in \mathrm{Mor}(\mathcal{A}, \mathcal{I})$ is such that there exists $x \in \mathbb{P}(V)$ and $p \in \mathbb{P}(V^*)$ with for all $i \in [\![1,4]\!]$, $x_i = x$ and $p_i = p$, we will say that f is of type (XP) (see Figure 7). In that case, we will say that f is *symmetrical* if there exists three pairwise intersecting $a, b, c \in [\![1,4]\!]_2$ such that $\ell_a = \ell_b = \ell_c$, or if there exists an involution s of the projective line $\{\ell \in \mathcal{G}r_2(V) : x \subset \ell \subset p\}$ such that for all $a \in [\![1,4]\!]_2$, we have $s(\ell_a) = \ell_{[\![1,4]\!] \setminus a}$.

Notice that if f is both of types (L) and (XP) (that is, if the image of f is a complete flag), then f is symmetrical for both definitions.

Lemma 2.9. *Let $f \in \mathrm{Mor}(\mathcal{A}, \mathcal{I})$ be of type (L). Then f belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$ if and only if f is symmetrical.*

Proof. Fix $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$ of type (L), and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of marked apartments which converges to f. Call ℓ the common line of f. Let us show that f is symmetrical. Fix a line ℓ_∞ in $\mathbb{P}(V)$ generic for f and every $(f_n)_{n \in \mathbb{N}}$, that is, ℓ is such that $\ell \cap \ell_\infty = \emptyset$ and $\forall n \in \mathbb{N}, \forall \{i,j\} \in [\![1,4]\!]_2, \ell_{ij}^n \cap \ell_\infty = \emptyset$.

Fix $n \in \mathbb{N}$. Since $(x_i^n)_{i \in [\![1,4]\!]}$ and $(p_i^n)_{i \in [\![1,4]\!]}$ are projective frames of $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$ respectively, the space of projective isomorphisms $s : \mathbb{P}(V) \simeq \mathbb{P}(V^*)$ such that for all $i \in [\![1,4]\!]$ we have $s(x_i^n) = p_i^n$ has dimension 3. The condition that $s(\ell_\infty) = \ell_\infty$ is given by 3 independent linear homogeneous equations, hence there exists a unique isomorphism s_n satisfying both properties.

Hence for all $n \in \mathbb{N}$, we have the equality between the two cross-ratios, the first one being in $\ell_\infty^\perp$ and the second one in ℓ_∞

$$[\langle x_i^n, \ell_\infty \rangle]_{1 \leqslant i \leqslant 4} = [\langle p_i^n \cap \ell_\infty \rangle]_{1 \leqslant i \leqslant 4}.$$

If $\mathrm{Card}\{x_1, x_2, x_3, x_4\} \leqslant 2$, or if $\mathrm{Card}\{p_1, p_2, p_3, p_4\} \leqslant 2$, then f is symmetrical. Otherwise, since the cross-ratio is continuous, taking the limit as n goes on infinity gives

$$[\langle x_i, \ell_\infty \rangle]_{1 \leqslant i \leqslant 4} = [\langle p_i \cap \ell_\infty \rangle]_{1 \leqslant i \leqslant 4}.$$

Since the maps $\ell_\infty^\perp \to \ell : q \mapsto q \cap \ell$ and $\ell_\infty \to \ell^\perp : y \mapsto \langle y, \ell \rangle$ are isomorphisms of projective lines, we deduce that

$$[x_i]_{1 \leqslant i \leqslant 4} = [p_i]_{1 \leqslant i \leqslant 4},$$

hence f is symmetrical.

Conversely, fix $f \in \mathrm{Mor}(\mathcal{A}, \mathcal{I})$ of type (L) and symmetrical. Let us show that f belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.

1. Assume first that $\mathrm{Card}\{p_1,p_2,p_3,p_4\}=2$, for instance $p_1=p_2=p_3$. Assume first that $\mathrm{Card}\{x_1,x_2,x_3,x_4\}=4$. Up to precomposing f by an element of the Weyl group and postcomposing f by an element of $\mathrm{PGL}(V)$, we may assume that there exists $a\in\mathbb{K}\backslash\{0,1\}$ such that

$$p_1=p_2=p_3=\langle e_1,e_2,e_3\rangle, p_4=\langle e_1,e_2,e_4\rangle,$$
$$x_1=\langle e_1\rangle, x_2=\langle e_2\rangle, x_3=\langle e_1+e_2\rangle \text{ and } x_4=\langle e_1a+e_2\rangle.$$

Fix a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{K}\backslash\{0\}$ converging to 0. For all $n\in\mathbb{N}$, let f_n be the marked apartment defined by

$$x_1^n=\langle e_1\rangle, x_2^n=\langle e_2\rangle, x_3^n=\langle e_1+e_2+e_4\alpha_n^2\rangle \text{ and } x_4^n=\langle e_1a+e_2+e_3\alpha_n\rangle.$$

The sequence $(f_n)_{n\in\mathbb{N}}$ converges to f. And if $\mathrm{Card}\{x_1,x_2,x_3,x_4\}<4$, then f is a limit of cases where the four points are distinct.

2. Assume $\mathrm{Card}\{x_1,x_2,x_3,x_4\}=2$, then by duality we are in the previous case.

3. Assume that $\mathrm{Card}\{x_1,x_2,x_3,x_4\}=\mathrm{Card}\{p_1,p_2,p_3,p_4\}=1$. Then the image of f is simply a complete flag in V, and it belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$.

4. Assume that we are in none of the previous cases, then $\mathrm{Card}\{x_1,x_2,x_3,x_4\}\geqslant 3$ and $\mathrm{Card}\{p_1,p_2,p_3,p_4\}\geqslant 3$. Up to precomposing f by an element of the Weyl group and postcomposing f by an element of $\mathrm{PSL}(V)$, we may assume that there exists $a\in\mathbb{K}\cup\{\infty\}$ (the cross-ratio of (x_1,x_2,x_3,x_4)) such that

$$x_1=\langle e_1\rangle, x_2=\langle e_2\rangle, x_3=\langle e_1+e_2\rangle, x_4=\langle e_1a+e_2\rangle,$$
$$p_1=\langle e_1,e_2,e_3\rangle, p_2=\langle e_1,e_2,e_4\rangle, p_3=\langle e_1,e_2,e_3+e_4\rangle \text{ and } p_4=\langle e_1,e_2,e_3a+e_4\rangle.$$

Fix a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{K}\backslash\{0\}$ converging to 0. For all $n\in\mathbb{N}$, let f_n be the marked apartment defined by

$$x_1^n=\langle e_1\rangle, x_2^n=\langle e_2\rangle, x_3^n=\langle e_1+e_2+e_3a\alpha_n+e_4\alpha_n\rangle \text{ and } x_4^n=\langle e_1a+e_2+e_3a\alpha_n+e_4a\alpha_n\rangle.$$

Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f. □

Lemma 2.10. *Let $f\in\mathrm{Mor}(\mathcal{A},\mathcal{I})$ be of type (XP). Then f belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ if and only if f is symmetrical.*

Proof. Fix $f\in\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ of type (XP), and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of marked apartments which converges to f. Call x and p the common point and plane of f. Let us show that f is symmetrical. Fix a point $x_\infty\in\mathbb{P}(V)$ generic for f and every $(f_n)_{n\in\mathbb{N}}$, that is, x_∞ is such that $x_\infty\notin p$ and $\forall n\in\mathbb{N},\forall i\in[\![1,4]\!], x_\infty\notin p_i^n$. Fix a line ℓ_∞ in p that does not contain x.

Consider the continuous projection

$$\begin{array}{rcl}\pi:\mathbb{P}(V)\backslash p & \to & p\\ x' & \mapsto & \langle x',x_\infty\rangle\cap p.\end{array}$$

Then the sequence $(g_n=\pi\circ f_n)_{n\in\mathbb{N}}$ converges to $\pi\circ f=f$.

Fix $n \in \mathbb{N}$ such that $\forall i \in [\![1,4]\!], \pi(x_i^n) \notin \ell_\infty$. Then according to Desargues' involution theorem (see [Sam88, Chap. II, Theorem 31] or [Per12, Chap. 1, Théorème 3.5.6]), there exists a (unique, projective) involution s_n of ℓ_∞ such that for all $a \in [\![1,4]\!]_2$, we have $s_n(\pi(\ell_a^n) \cap \ell_\infty) = \pi(\ell^n_{[\![1,4]\!]\setminus a}) \cap \ell_\infty$.

If there exists three pairwise intersecting $a, b, c \in [\![1,4]\!]_2$ such that $\ell_a = \ell_b = \ell_c$, then f is symmetrical. Assume this is not the case, then there exists a unique involution s of ℓ_∞ such that $s(\ell_{12} \cap \ell_\infty) = \ell_{34} \cap \ell_\infty$ and $s(\ell_{13} \cap \ell_\infty) = \ell_{24} \cap \ell_\infty$. Since for all $a \in [\![1,4]\!]_2$, the sequence of points $(\pi(\ell_a^n))_{n\in\mathbb{N}}$ converges to ℓ_a, we know that the sequence of involutions $(s_n)_{n\in\mathbb{N}}$ of ℓ_∞ converges to s, hence $s(\ell_{14} \cap \ell_\infty) = \ell_{23} \cap \ell_\infty$. So f is symmetrical.

Conversely, let $f \in \mathrm{Mor}(\mathcal{A}, \mathcal{I})$ be of type (XP) and symmetrical. Let us show that f belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$. Let us call x and p the common point and plane of f.

Assume first that there does not exist three pairwise intersecting $a, b, c \in [\![1,4]\!]_2$ such that $\ell_a = \ell_b = \ell_c$. Further assume that for all $a, b \in [\![1,4]\!]_2$ such that $a \notin \{b, \bar{b}\}$, we have $\{\ell_a, \ell_{\bar{a}}\} \cap \{\ell_b, \ell_{\bar{b}}\} \neq \emptyset$: up to precomposing f by an element of the Weyl group, we may assume that $\ell_{12} = \ell_{13}$, $\ell_{34} = \ell_{23}$ and $\ell_{24} = \ell_{14}$. Since f is symmetrical, there exists a projective involution of the projective line $\{\ell \in \mathcal{G}r_2(V) : x \subset \ell \subset p\}$ exchanging each pair of these three lines, so we have $\ell_{12} = \ell_{13} = \ell_{34} = \ell_{23} = \ell_{24} = \ell_{14}$: this contradicts the assumption.

So we can assume that there exist $a, b \in [\![1,4]\!]_2$ such that $a \notin \{b, \bar{b}\}$ and $\{\ell_a, \ell_{\bar{a}}\} \cap \{\ell_b, \ell_{\bar{b}}\} = \emptyset$. Up to precomposing f by an element of the Weyl group, we may assume that $a = \{12\}$ and $b = \{13\}$. Assume further that $\ell_{12} \neq \ell_{34}$. Up to postcomposing f by an element of $\mathrm{PGL}(V)$, we may assume that there exist $u, v, w \in \mathbb{K} \cup \{\infty\}$ such that $x = \langle e_3 \rangle$, $p = \langle e_1, e_2, e_3 \rangle$ and

$$\ell_{12} = \langle e_1, e_3 \rangle, \ell_{13} = \langle e_1 + e_2, e_3 \rangle, \ell_{14} = \langle e_1 u + e_2, e_3 \rangle,$$
$$\ell_{23} = \langle e_1 v + e_2, e_3 \rangle, \ell_{24} = \langle e_1 w + e_2, e_3 \rangle \text{ and } \ell_{34} = \langle e_2, e_3 \rangle.$$

The assumption we made tells us that $w \notin \{0, \infty\}$, and the symmetry condition tells us that there exists a projective involution s of the projective line $\{\ell \in \mathcal{G}r_2(V) : x \subset \ell \subset p\}$ such that for all $c \in [\![1,4]\!]_2$ we have $s(\ell_c) = \ell_{\bar{c}}$. This involution is unique, it is defined by

$$\begin{aligned} s : \{\ell \in \mathcal{G}r_2(V) : x \subset \ell \subset p\} &\to \{\ell \in \mathcal{G}r_2(V) : x \subset \ell \subset p\} \\ \ell = \langle e_1 z + e_2, e_3 \rangle &\mapsto \langle e_1 \frac{w}{z} + e_2, e_3 \rangle. \end{aligned}$$

Since $s(\ell_{14}) = \ell_{23}$, we deduce that $\frac{w}{u} = v$.

Assume further that $u, v \notin \{0, \infty\}$. Fix a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{K}\setminus\{0\}$ going to ∞. For all $n \in \mathbb{N}$, let f_n be the marked apartment defined by

$$x_1^n = \langle e_3 \rangle, x_2^n = \langle e_1(1-v) + e_2 \alpha_n^{-1} + e_3 \alpha_n \rangle, x_3^n = \langle e_1 + e_2 + e_3 \alpha_n \rangle \text{ and } x_4^n = \langle e_1 + e_2 u^{-1} + e_3 \alpha_n + e_4 \alpha_n^{-1} \rangle.$$

Then if $a \in \{12, 13, 14\}$, it is easy to see that the sequence of lines $(\ell_a^n)_{n\in\mathbb{N}}$ converges to ℓ_a. Every accumulation point of the sequence of lines $(\ell_{23}^n)_{n\in\mathbb{N}}$ contains the vector

$$\lim_{n\to+\infty} e_1 v + e_2(1 - \alpha_n^{-1}) = e_1 v + e_2,$$

so it converges to ℓ_{23}. And every accumulation point of the sequence of lines $(\ell_{24}^n)_{n\in\mathbb{N}}$ contains the vector

$$\lim_{n\to+\infty} e_1 v + e_2(u^{-1} - \alpha_n^{-1}) + e_4 \alpha_n^{-1} = e_1 v + e_2 u^{-1} = (e_1 w + e_2) u^{-1},$$

so it converges to ℓ_{24}. And every accumulation point of the sequence of lines $(\ell_{34}^n)_{n\in\mathbb{N}}$ contains the vector

$$\lim_{n\to+\infty} e_2(1-u^{-1}) - e_4\alpha_n^{-1} = e_2(1-u^{-1}),$$

so it converges to ℓ_{34}.

Hence the sequence of marked apartments $(f_n)_{n\in\mathbb{N}}$ converges to f.

If for instance $u=0$ and $v=\infty$, then f is a limit of previous cases with u going to 0 and $v=\frac{w}{u}$ going to infinity, so f is a limit of marked apartments as well.

And if $\ell_{12}=\ell_{34}$, then f is a limit of previous cases with $\ell_{12}\neq\ell_{34}$, so f is a limit of marked apartments as well.

Assume now that there exist three pairwise intersecting $a,b,c\in[\![1,4]\!]_2$ such that $\ell_a=\ell_b=\ell_c$. Up to precomposing f by an element of the Weyl group, we may assume that $\ell_{14}=\ell_{24}=\ell_{34}$. Fix $v\in\mathbb{K}\backslash\{0\}$, and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{K}\backslash\{0\}$ going to ∞. For all $n\in\mathbb{N}$, let f_n be the morphism of type (XP) with common point $x=\langle e_3\rangle$, common plane $p=\langle e_1,e_2,e_3\rangle$ and the following lines:

$$\ell_{12}=\langle e_1,e_3\rangle,\ell_{13}=\langle e_1+e_2,e_3\rangle,\ell_{14}=\langle e_1\alpha_n+e_2,e_3\rangle,$$
$$\ell_{23}=\langle e_1v+e_2,e_3\rangle,\ell_{24}=\langle e_1v\alpha_n+e_2,e_3\rangle \text{ and } \ell_{34}=\langle e_2,e_3\rangle.$$

According to the previous case, f_n is a limit of marked apartments. And the sequence $(f_n)_{n\in\mathbb{N}}$ converges to the morphism of type (XP) with the following lines:

$$\ell_{12}=\langle e_1,e_3\rangle,\ell_{13}=\langle e_1+e_2,e_3\rangle,\ell_{14}=\langle e_2,e_3\rangle,$$
$$\ell_{23}=\langle e_1v+e_2,e_3\rangle,\ell_{24}=\langle e_2,e_3\rangle \text{ and } \ell_{34}=\langle e_2,e_3\rangle.$$

Hence we have shown that a generic morphism of type (XP) such that $\ell_a=\ell_b=\ell_c$ is a limit of marked apartments, hence any morphism of type (XP) with $\ell_a=\ell_b=\ell_c$ is also a limit of marked apartments. □

Theorem 2.11. *Let $\mathcal{I}$ be the topological spherical building of $(\mathrm{PGL}_4,\mathbb{K})$, and let $\mathcal{A}$ be an apartment of $\mathcal{I}$, then any morhism of $\mathrm{Mor}(\mathcal{A},\mathcal{I})$ which is not of type (XP) or (L) belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$. And a morphism of type (XP) or (L) belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})}^g$ if and only if it is symmetrical.*

Proof. Fix $f\in\mathrm{Mor}(\mathcal{A},\mathcal{I})$.

- Assume that there exists $i\in[\![1,4]\!]$ such that $x_i\notin p_i$, then f is not of type (L) nor (XP). Let us assume that $i=1$. According to Theorem 2.5 applied to p_1, there exists three sequences of points $(x_2^n)_{n\in\mathbb{N}}$, $(x_3^n)_{n\in\mathbb{N}}$ and $(x_4^n)_{n\in\mathbb{N}}$ of p_1 (which are not aligned for any $n\in\mathbb{N}$), which converge to x_2, x_3 and x_4 respectively, and such that the three sequence of lines $(\ell_{23}^n)_{n\in\mathbb{N}}$, $(\ell_{34}^n)_{n\in\mathbb{N}}$ and $(\ell_{42}^n)_{n\in\mathbb{N}}$ they define converge to ℓ_{23}, ℓ_{34} and ℓ_{42} respectively. For all $n\in\mathbb{N}$, let $f_n\in\mathrm{Mor}_{inj}(\mathcal{A},\mathcal{I})$ be the marked apartment whose vertices are x_1, x_2^n, x_3^n and x_4^n. Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f.

- Assume that there exists a partition $[\![1,4]\!]=\{i,j\}\sqcup\{k,l\}$ such that $\ell_{ij}\cap\ell_{kl}=\emptyset$, then f is not of type (L) nor (XP). Let us assume that $\{i,j\}=\{1,2\}$ and $\{k,l\}=\{3,4\}$. According to Proposition 2.3 applied to $\ell_{12}\oplus\ell_{34}$, there exists four sequences of pairwise distinct points $(x_1^n)_{n\in\mathbb{N}}$, $(x_2^n)_{n\in\mathbb{N}}$, $(x_3^n)_{n\in\mathbb{N}}$ and $(x_4^n)_{n\in\mathbb{N}}$, which converge to

x_1, x_2, x_3 and x_4 respectively, such that $x_1^n, x_2^n \in \ell_{12}$ and $x_3^n, x_4^n \in \ell_{34}$. For all $n \in \mathbb{N}$, let $f_n \in \mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})$ be the marked apartment whose vertices are x_1^n, x_2^n, x_3^n and x_4^n. Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f.

- Assume that there exists $p \in \mathbb{P}(V^*)$ such that for all $i \in [\![1,4]\!]$ we have $p_i = p$, and that there is a point of $\{x_1, \dots, x_4\}$ distinct from the three others. For instance $x_1 \notin \{x_2, x_3, x_4\}$. According to Theorem 2.5 applied to p_1, there exist three sequences of points $(x_2^n)_{n\in\mathbb{N}}$, $(x_3^n)_{n\in\mathbb{N}}$ and $(x_4^n)_{n\in\mathbb{N}}$ of p_1 (which are not aligned for any $n \in \mathbb{N}$), which converge to x_2, x_3 and x_4 respectively, and such that the three sequence of lines $(\ell_{23}^n)_{n\in\mathbb{N}}$, $(\ell_{34}^n)_{n\in\mathbb{N}}$ and $(\ell_{42}^n)_{n\in\mathbb{N}}$ converge to ℓ_{23}, ℓ_{34} and ℓ_{42} respectively. Choose a sequence $(x_1^n)_{n\in\mathbb{N}}$ in $\mathbb{P}(V)\backslash p$ converging to x_1. For all $n \in \mathbb{N}$, let $f_n \in \mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})$ be the marked apartment whose vertices are x_1^n, x_2^n, x_3^n and x_4^n. Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f.

- Assume that there exists $p \in \mathbb{P}(V^*)$ such that for all $i \in [\![1,4]\!]$ we have $p_i = p$, and that for instance $x_1 = x_2 \neq x_3 = x_4$. Choose a sequence $(x_1^n)_{n\in\mathbb{N}}$ in $\ell_{12}\backslash\{x_1\}$ converging to x_1. For all $n \in \mathbb{N}$, let $f_n \in \mathrm{Mor}(\mathcal{A}, \mathcal{I})$ be the morphism which differs from f only by $f_n(\langle e_1 \rangle) = x_1^n$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f. According to the previous case, we know that $f_n \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$ for all $n \in \mathbb{N}$, so $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.

- Assume that there exists $p \in \mathbb{P}(V^*)$ such that for all $i \in [\![1,4]\!]$ we have $p_i = p$, and that $x_1 = x_2 = x_3 = x_4$. Then f is of type (XP), and this case is covered by Lemma 2.10.

- Assume that there exists $x \in \mathbb{P}(V)$ such that for all $i \in [\![1,4]\!]$ we have $x_i = x$, then by duality between $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, we are in one of the three previous cases.

- Assume that we are in none of the previous cases. Consider the intersection $\cap_{i\in[\![1,4]\!]} p_i$ of the 4 planes : since f is not a marked apartment, it is not empty. And since we are in none of the previous cases, the intersection is not a plane nor a point, so it is a line $\ell = \cap_{i\in[\![1,4]\!]} p_i$. By duality between $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, we also know that the projective subspace generated by the $(x_i)_{i\in[\![1,4]\!]}$ is ℓ. If for all $\{i,j\} \in [\![1,4]\!]_2$ we have $\ell_{ij} = \ell$, then f is of type (L), and this case is covered by Lemma 2.9. Otherwise assume that $\ell_{12} \neq \ell$, and let $x = \cap_{\{i,j\}\in[\![1,4]\!]_2} \ell_{ij}$: it is strictly included in $\cap_{i\in[\![1,4]\!]} p_i = \ell$ and non-empty by the previous cases, so it is a point $x \in \mathbb{P}(V)$. Similarly, let $h = \langle \ell_{ij} \,:\, \{i,j\} \in [\![1,4]\!]_2 \rangle \in \mathbb{P}(V^*)$.
  - If $x_1 = x_2 = x \notin \{x_3, x_4\}$ for instance, choose a sequence $(x_2^n)_{n\in\mathbb{N}}$ in $\ell_{12}\backslash\{x\}$ which converges to x. For all $n \in \mathbb{N}$, let $f_n \in \mathrm{Mor}(\mathcal{A}, \mathcal{I})$ be the morphism which differs from f only by $f_n(\langle e_2 \rangle) = x_2^n$. Since $x_2^n \notin \ell$, according to the previous cases we know that $f_n \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$. As the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f, we know that $f \in \overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.
  - If $x_1 = x_2 = x_3 = x \neq x_4$ for instance, then by duality we may assume that $p_1 = p_3 = p_4 = p \neq p_2$ as well, and up to precomposing f by an element of the Weyl group and postcomposing f by an element of $\mathrm{PGL}(V)$, this case is covered by the following Lemma 2.12 (see Figure 8).

$\square$

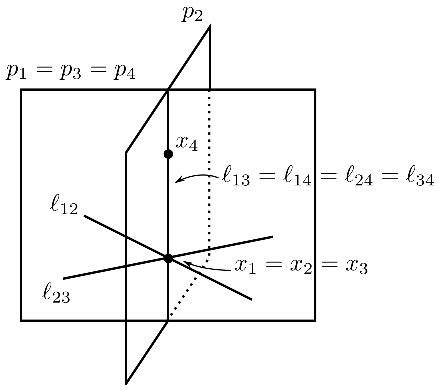

Figure 8: The morphism in Lemma 2.12

Lemma 2.12. *Let $f \in \mathrm{Mor}(\mathcal{A}, \mathcal{I})$ be defined by $x_1 = x_2 = x_3 = \langle e_1 \rangle$, $x_4 = \langle e_2 \rangle$, $p_1 = p_3 = p_4 = \langle e_1, e_2, e_3 \rangle$, $p_2 = \langle e_1, e_2, e_4 \rangle$, $\ell_{13} = \ell_{14} = \ell_{24} = \ell_{34} = \langle e_1, e_2 \rangle$, $\ell_{12} = \langle e_1, e_3 \rangle$ and $\ell_{23} = \langle e_1, ae_2 + be_3 \rangle$, where $(a, b) \in \mathbb{K}^2 \backslash \{(0, 0)\}$. Then f belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$.*

Proof. Assume first that a and b are both non-zero. Fix a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{K}\backslash\{0\}$ going to 0. For all $n \in \mathbb{N}$, let f_n be the marked apartment defined by

$$x_1^n = \langle e_1 \rangle, x_2^n = \langle e_1 + e_3 b\alpha_n \rangle, x_3^n = \langle e_1 - e_2 a\alpha_n + e_4 \alpha_n^2 \rangle \text{ and } x_4^n = \langle e_2 \rangle.$$

Then it is easy to see that for all $i \in [\![1, 4]\!]$, the sequence of points $(x_i^n)_{n\in\mathbb{N}}$ converges to x_i and the sequence of planes $(p_i^n)_{n\in\mathbb{N}}$ converges to p_i. And if $a \in \{12, 13, 14, 24, 34\}$, it is easy to see that the sequence of lines $(\ell_a^n)_{n\in\mathbb{N}}$ converges to ℓ_a. Every accumulation point of the sequence of lines $(\ell_{23}^n)_{n\in\mathbb{N}}$ contains the vector

$$\lim_{n\to+\infty} e_2 a\alpha_n + e_3 b\alpha_n - e_4\alpha_n^2 = e_2 a + e_3 b,$$

so it converges to ℓ_{23}. Hence the sequence of marked apartments $(f_n)_{n\in\mathbb{N}}$ converges to f.

If $a = 0$ or $b = 0$, then f is a limit of similar morphism for which we have a and b both non-zero, so f belongs to $\overline{\mathrm{Mor}_{inj}(\mathcal{A}, \mathcal{I})}^g$. □

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