ON SOME INEQUALITIES OF SIMPSON'S TYPE VIA $h-$CONVEX FUNCTIONS

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ABSTRACT. In this paper, we prove some new inequalities of Simpson's type for functions whose derivatives of absolute values are $h-$convex and $h-$concave functions. Some new estimations are obtained. Also we give some sophisticated results for some different kinds of convex functions.

1. INTRODUCTION

The following inequality is well known in the literature as Simpson's inequality;

(1.1)
$$\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4,$$

where the mapping $f:[a,b]\to\mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $f^{(4)}$ to be bounded on (a,b), that is,

$$\left\|f^{(4)}\right\|_\infty=\sup_{t\in(a,b)}\left|f^{(4)}(t)\right|<\infty.$$

For some results which generalize, improve and extend the inequality (1.1) see the papers [1]-[3].

Definition 1. [5] *We say that $f:I\to\mathbb{R}$ is Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x,y\in I$ and $t\in(0,1)$ we have*

$$f(tx+(1-t)y)\leq\frac{f(x)}{t}+\frac{f(y)}{1-t}. \tag{1.2}$$

The class $Q(I)$ was firstly described in [5] by Godunova-Levin. Among others, it is noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

Definition 2. [4] *We say that $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a $P-$function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x,y\in I$ and $t\in[0,1]$, we have*

$$f(tx+(1-t)y)\leq f(x)+f(y). \tag{1.3}$$

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Definition 3. [8] *Let $s \in (0,1]$ be a fixed real number. A function $f : [0,\infty) \to [0,\infty)$ is said to be $s-$convex (in the second sense) if*

$$f\left(tx+(1-t)\,y\right) \leq t^{s} f\left(x\right) + (1-t)^{s} f\left(y\right), \tag{1.4}$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$. This class of s-convex functions is usually denoted by K_s^2.

In 1978, Breckner introduced $s-$convex functions as a generalization of convex functions [8]. Also, in that one work Breckner proved the important fact that the setvalued map is $s-$convex only if the associated support function is s-convex function [9]. Of course, $s-$convexity means just convexity when $s = 1$.

Definition 4. [7] *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function . We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $h-$convex function, or that f belongs to the class $SX\left(h, I\right)$, if f is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f\left(tx+(1-t)\,y\right) \leq h\left(t\right) f\left(x\right) + h\left(1-t\right) f\left(y\right). \tag{1.5}$$

If inequality (1.5) is reversed, then f is said to be $h-$concave, i.e. $f \in SV\left(h, I\right)$. Obviously, if $h\left(t\right) = t$, then all nonnegative convex functions belong to $SX\left(h, I\right)$ and all nonnegative concave functions belong to $SV\left(h, I\right)$; if $h\left(t\right) = \frac{1}{t}$, then $SX\left(h, I\right) = Q\left(I\right)$; if $h\left(t\right) = 1$, then $SX\left(h, I\right) \supseteq P\left(I\right)$; and if $h\left(t\right) = t^s$, where $s \in (0,1)$, then $SX\left(h, I\right) \supseteq K_s^2$.

Remark 1. [7] *Let h be a non-negative function such that*

$$h\left(\alpha\right) \geq \alpha$$

for all $\alpha \in (0,1)$. If f is a non-negative convex function on I , then for $x, y \in I$, $\alpha \in (0,1)$ we have

$$f\left(\alpha x+(1-\alpha)y\right) \leq \alpha f(x) + (1-\alpha)f(y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

So, $f \in SX(h, I)$. Similarly, if the function h has the property: $h(\alpha) \leq \alpha$ for all $\alpha \in (0,1)$, then any non-negative concave function f belongs to the class $SV(h, I)$.

Definition 5. [7] *A function $h : J \to \mathbb{R}$ is said to be a supermultiplicative function if*

$$h\left(xy\right) \geq h\left(x\right) h\left(y\right) \tag{1.6}$$

for all $x, y \in J$.

If inequality (1.6) is reversed, then h is said to be a submultiplicative function. If equality held in (1.6), then h is said to be a multiplicative function.

In [1], Sarıkaya *et.al* established the following Simpson-type inequality for convex functions:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is a convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{6}\left[f\left(a\right)+4f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\ \leq\;& \frac{5(b-a)}{72}\left[|f'(a)|+|f'(b)|\right] \end{aligned} \tag{1.7}$$

In [10], Sarıkaya *et.al* established the following Hadmard-type inequality for $h-$convex functions:

Theorem 2. *Let $f \in SX(h, I)$, $a, b \in I$, with $a < b$ and $f \in L_1([a, b])$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq [f(a) + f(b)]\int_0^1 h(t)\,dt. \tag{1.8}$$

For recent results and generalizations concerning $h-$convex functions see [7], [10].

The aim of this paper is to establish new inequalities for functions whose derivatives in absolute value are $h-$convex and $h-$concave functions.

2. Inequalities for $h-$convex and $h-$concave functions

To prove our new result we need the following lemma (see [3]).

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I^0 where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\begin{aligned} &\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx \\ = \;& (b-a)\int_0^1 k(t) f'(ta + (1-t)b)\,dt, \end{aligned}$$

where

$$k(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right) \\ t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Theorem 3. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function, $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on I° such that $h^q, f' \in L[a, b]$ where $a, b \in I^\circ$ with $a < b$ and $h(\alpha) \geq \alpha$. If $|f'|$ is $h-$convex on I, then*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \\ \leq \;& \frac{(b-a)}{3}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}} \times \left\{|f'(a)|\left[\left(\int_0^{\frac{1}{2}} h^q(t)\,dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 h^q(t)\,dt\right)^{\frac{1}{q}}\right]\right. \\ &\left. + |f'(b)|\left[\left(\int_0^{\frac{1}{2}} h^q(1-t)\,dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 h^q(1-t)\,dt\right)^{\frac{1}{q}}\right]\right\}. \end{aligned} \tag{2.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1, $h-$convexity of $|f'|$ and properties of absolute value, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
=\;& (b-a)\left|\int_0^1 k(t)\,f'(ta+(1-t)b)\,dt\right| \\
\leq\;& (b-a)\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(ta+(1-t)b)\right|dt\right) \\
\leq\;& (b-a)\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left(h(t)\left|f'(a)\right|+h(1-t)\left|f'(b)\right|\right)dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left(h(t)\left|f'(a)\right|+h(1-t)\left|f'(b)\right|\right)dt\right\} \\
=\;& (b-a)\left|f'(a)\right|\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|h(t)\,dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|h(t)\,dt\right\} \\
&+(b-a)\left|f'(b)\right|\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|h(1-t)\,dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|h(1-t)\,dt\right\}.
\end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\;& (b-a)\left|f'(a)\right|\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}h^q(t)\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 h^q(t)\,dt\right)^{\frac{1}{q}}\right\} \\
&+(b-a)\left|f'(b)\right|\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}h^q(1-t)\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 h^q(1-t)\,dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

Since

$$\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt=\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^p dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^p dt=\frac{1}{p+1}\left(\frac{1}{6^{p+1}}+\frac{1}{3^{p+1}}\right)$$

and

$$\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt=\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^p dt+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)^p dt=\frac{1}{p+1}\left(\frac{1}{6^{p+1}}+\frac{1}{3^{p+1}}\right),$$

we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\quad & \frac{(b-a)}{3}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}}\times\left\{|f'(a)|\left[\left(\int_0^{\frac{1}{2}}h^q(t)\,dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 h^q(t)\,dt\right)^{\frac{1}{q}}\right]\right. \\
&\left.+|f'(b)|\left[\left(\int_0^{\frac{1}{2}}h^q(1-t)\,dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1 h^q(1-t)\,dt\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

which completes the proof. □

Corollary 1. *In Theorem 3, if we choose $p=q=2$, we obtain*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\quad & \frac{b-a}{3\sqrt{2}}\left\{|f'(a)|\left[\left(\int_0^{\frac{1}{2}}h\left(t^2\right)dt\right)^{\frac{1}{2}}+\left(\int_{\frac{1}{2}}^1 h\left(t^2\right)dt\right)^{\frac{1}{2}}\right]\right. \\
&\left.+|f'(b)|\left[\left(\int_0^{\frac{1}{2}}h\left((1-t)^2\right)dt\right)^{\frac{1}{2}}+\left(\int_{\frac{1}{2}}^1 h\left((1-t)^2\right)dt\right)^{\frac{1}{2}}\right]\right\}
\end{aligned}$$

where h is supermultiplicative.

Corollary 2. *In Theorem 3, if $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$ and $h(t)=1$, then we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right| \\
\leq\quad & \frac{2(b-a)}{3}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\{|f'(a)|+|f'(b)|\} \\
=\quad & \frac{b-a}{3}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\{|f'(a)|+|f'(b)|\}
\end{aligned}$$

Corollary 3. *In Theorem 3, if we choose $h(t)=t$, we obtain*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\quad & \frac{(b-a)}{3}\left(\frac{1+2^{p+1}}{6(p+1)}\right)^{\frac{1}{p}}\left\{|f'(a)|\left[\left(\frac{\left(\frac{1}{2}\right)^{q+1}}{q+1}\right)^{\frac{1}{q}}+\left(\frac{1}{2q+2}\left(2-\left(\frac{1}{2}\right)^q\right)\right)^{\frac{1}{q}}\right]\right. \\
&\left.+|f'(b)|\left[\left(\frac{1}{2q+2}\left(2-\left(\frac{1}{2}\right)^q\right)\right)^{\frac{1}{q}}+\left(\frac{\left(\frac{1}{2}\right)^{q+1}}{q+1}\right)^{\frac{1}{q}}\right]\right\} \\
=\quad & \frac{(b-a)}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[\frac{|f'(a)|}{2}+|f'(b)|\left(2-\left(\frac{1}{2}\right)^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 4. *Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative supermultiplicative functions, $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$ where $a, b \in I^{\circ}$ with $a < b$ and $h(\alpha) \geq \alpha$. If $|f'|$ is $h-$convex on I, then*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ \leq \; & (b-a)\left\{ |f'(a)|\,[A] + |f'(b)|\,[B] \right\} \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} A &= \int_0^{\frac{1}{6}} h\left(t\left[\frac{1}{6}-t\right]\right)dt + \int_{\frac{1}{6}}^{\frac{1}{2}} h\left(t\left[t-\frac{1}{6}\right]\right)dt + \int_{\frac{1}{2}}^{\frac{5}{6}} h\left(t\left[\frac{5}{6}-t\right]\right)dt + \int_{\frac{5}{6}}^{1} h\left(t\left[t-\frac{5}{6}\right]\right)dt \\ B &= \int_0^{\frac{1}{6}} h\left[(1-t)\left(\frac{1}{6}-t\right)\right]dt + \int_{\frac{1}{6}}^{\frac{1}{2}} h\left[(1-t)\left(t-\frac{1}{6}\right)\right]dt \\ &\quad + \int_{\frac{1}{2}}^{\frac{5}{6}} h\left[(1-t)\left(\frac{5}{6}-t\right)\right]dt + \int_{\frac{5}{6}}^{1} h\left[(1-t)\left(t-\frac{5}{6}\right)\right]dt. \end{aligned}$$

Proof. From Lemma 1, $h-$convexity of $|f'|$ and properties of absolute value, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] \right| \\ = \; & (b-a)\left| \int_0^1 k(t)\, f'(ta+(1-t)b)\,dt \right| \\ \leq \; & (b-a)\left\{ \int_0^{\frac{1}{2}} \left|t-\frac{1}{6}\right| |f'(ta+(1-t)b)|\,dt + \int_{\frac{1}{2}}^{1} \left|t-\frac{5}{6}\right| |f'(ta+(1-t)b)|\,dt \right\} \\ \leq \; & (b-a)\left\{ \int_0^{\frac{1}{2}} \left|t-\frac{1}{6}\right| \left(h(t)|f'(a)| + h(1-t)|f'(b)|\right)dt \right. \\ & \left. + \int_{\frac{1}{2}}^{1} \left|t-\frac{5}{6}\right| \left(h(t)|f'(a)| + h(1-t)|f'(b)|\right)dt \right\} \\ = \; & (b-a)\left\{ \int_0^{\frac{1}{6}} \left(\frac{1}{6}-t\right)\left(h(t)|f'(a)| + h(1-t)|f'(b)|\right)dt \right. \\ & + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(t-\frac{1}{6}\right)\left(h(t)|f'(a)| + h(1-t)|f'(b)|\right)dt \\ & + \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6}-t\right)\left(h(t)|f'(a)| + h(1-t)|f'(b)|\right)dt \\ & \left. + \int_{\frac{5}{6}}^{1} \left(t-\frac{5}{6}\right)\left(h(t)|f'(a)| + h(1-t)|f'(b)|\right)dt \right\}. \end{aligned}$$

By properties of function h, we can write

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\ & (b-a)\left\{\int_0^{\frac{1}{6}}\left[h\left(t\left(\frac{1}{6}-t\right)\right)|f'(a)|+h\left((1-t)\left(\frac{1}{6}-t\right)\right)|f'(b)|\right]dt\right. \\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left[h\left(t\left(t-\frac{1}{6}\right)\right)|f'(a)|+h\left((1-t)\left(t-\frac{1}{6}\right)\right)|f'(b)|\right]dt \\
&+\int_{\frac{1}{2}}^{\frac{5}{6}}\left[h\left(t\left(\frac{5}{6}-t\right)\right)|f'(a)|+h\left((1-t)\left(\frac{5}{6}-t\right)\right)|f'(b)|\right]dt \\
&\left.+\int_{\frac{5}{6}}^{1}\left[h\left(t\left(t-\frac{5}{6}\right)\right)|f'(a)|+h\left((1-t)\left(t-\frac{5}{6}\right)\right)|f'(b)|\right]dt\right\}
\end{aligned}$$

$$\begin{aligned}
=\ & (b-a)|f'(a)|\left\{\int_0^{\frac{1}{6}}h\left(t\left[\frac{1}{6}-t\right]\right)dt+\int_{\frac{1}{6}}^{\frac{1}{2}}h\left(t\left[t-\frac{1}{6}\right]\right)dt\right. \\
&\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}h\left(t\left[\frac{5}{6}-t\right]\right)dt+\int_{\frac{5}{6}}^{1}h\left(t\left[t-\frac{5}{6}\right]\right)dt\right\} \\
&+(b-a)|f'(b)|\left\{\int_0^{\frac{1}{6}}h\left((1-t)\left(\frac{1}{6}-t\right)\right)dt+\int_{\frac{1}{6}}^{\frac{1}{2}}h\left((1-t)\left(t-\frac{1}{6}\right)\right)dt\right. \\
&\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}h\left[(1-t)\left(\frac{5}{6}-t\right)\right]dt+\int_{\frac{5}{6}}^{1}h\left[(1-t)\left(t-\frac{5}{6}\right)\right]dt\right\}
\end{aligned}$$

which completes the proof. □

Corollary 4. *In Theorem 4, if* $f(a)=f\left(\frac{a+b}{2}\right)=f(b)$ *and* $h(t)=1$, *then we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & (b-a)\left\{|f'(a)|+|f'(b)|\right\}.
\end{aligned}$$

Remark 2. *In Theorem 4, if we choose* $h(t)=t$, *then we obtain the inequality (1.7).*

Theorem 5. *Let* $h:J\subset\mathbb{R}\to\mathbb{R}$ *be non-negative functions,* $f:I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$ *where* $a,b\in I^{\circ}$ *with* $a<b$. *If* $|f'|$ *is* $h-$*concave on* I, *then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\ & \frac{b-a}{12}\left(\frac{2+2^{p+2}}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{h\left(\frac{1}{2}\right)}\right]^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right|.
\end{aligned}$$

Proof. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;& (b-a)\left|\int_0^1 k(t)\,f'(ta+(1-t)b)\,dt\right|\\ \leq\;& (b-a)\left(\int_0^1 |k(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\end{aligned}$$

Since $|f'|$ is h-concave on I, by inequalities (1.8) we have

$$\int_0^1 |f'(ta+(1-t)b)|^q\,dt\leq\frac{1}{2h\left(\frac{1}{2}\right)}\left|f'\left(\frac{a+b}{2}\right)\right|^q.$$

Therefore, we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;& (b-a)\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right\}^{\frac{1}{p}}\left[\frac{1}{2h\left(\frac{1}{2}\right)}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}.\end{aligned}$$

Since

$$\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt=\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^p dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^p dt=\frac{1}{p+1}\left(\frac{1}{6^{p+1}}+\frac{1}{3^{p+1}}\right)$$

and

$$\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt=\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^p dt+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)^p dt=\frac{1}{p+1}\left(\frac{1}{6^{p+1}}+\frac{1}{3^{p+1}}\right),$$

we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;& \frac{b-a}{12}\left(\frac{2+2^{p+2}}{p+1}\right)^{\frac{1}{p}}\left[\frac{1}{h\left(\frac{1}{2}\right)}\right]^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right|.\end{aligned}$$

This completes the proof. □

Corollary 5. *In Theorem 5, if we choose $h(t)=t$, then we obtain*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\\ \leq\;& \frac{b-a}{6}\left(\frac{1+2^{p+1}}{p+1}\right)^{\frac{1}{p}}\left|f'\left(\frac{a+b}{2}\right)\right|.\end{aligned}$$

3. APPLICATIONS TO SPECIAL MEANS

We now consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$. We take

(1) *Arithmetic mean* :

$$A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}^{+}.$$

(2) *Logarithmic mean*:

$$L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|, \ \alpha, \beta \neq 0, \ \alpha, \beta \in \mathbb{R}^{+}.$$

(3) *Generalized log − mean*:

$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z}\backslash\{-1, 0\}, \ \alpha, \beta \in \mathbb{R}^{+}.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

Proposition 1. *Let $a, b \in \mathbb{R}^{+}$, $0 < a < b$ and $n \in \mathbb{N}, n > 1$. Then, we have*

$$\begin{aligned} & \left| L_n^n(a,b) - \frac{1}{3}\left[A(a^n, b^n) - A^n(a,b)\right] \right| \\ \leq \ & n\frac{(b-a)}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left[\frac{a^{n-1}}{2} + b^{n-1}\left(2 - \left(\frac{1}{2}\right)^{q}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Proof. The assertion follows from Corollary 3 applied for $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. □

Proposition 2. *Let $a, b \in \mathbb{R}^{+}$, $a < b$. Then, we have*

$$\left|L_n^n(a,b) - A^n(a,b)\right| \leq (b-a)\left[\frac{1}{a^2} + \frac{1}{b^2}\right].$$

Proof. The assertion follows from Corollary 4 applied for $f(x) = \frac{1}{x}$, $x \in [a, b]$. □

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