

# Estimates for approximation numbers of some classes of composition operators on the Hardy space

Daniel Li, Hervé Queffélec, Luis Rodríguez-Piazza\*

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**Abstract.** We give estimates for the approximation numbers of composition operators on  $H^2$ , in terms of some modulus of continuity. For symbols whose image is contained in a polygon, we get that these approximation numbers are dominated by  $e^{-c\sqrt{n}}$ . When the symbol is continuous on the closed unit disk and has a domain touching the boundary non-tangentially at a finite number of points, with a good behavior at the boundary around those points, we can improve this upper estimate. A lower estimate is given when this symbol has a good radial behavior at some point. As an application we get that, for the cusp map, the approximation numbers are equivalent, up to constants, to  $e^{-cn/\log n}$ , very near to the minimal value  $e^{-cn}$ . We also see the limitations of our methods. To finish, we improve a result of O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi, in showing that for every compact set  $K$  of the unit circle  $\mathbb{T}$  with Lebesgue measure 0, there exists a compact composition operator  $C_\varphi: H^2 \rightarrow H^2$ , which is in all Schatten classes, and such that  $\varphi = 1$  on  $K$  and  $|\varphi| < 1$  outside  $K$ .

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## 1 Introduction and notation

If the approximation numbers of some classes of operators on Hilbert spaces are well understood (for example, those of Hankel operators: see [16]), it is not the case of those of composition operators. Though their behavior remains mysterious, some recent results are obtained in [14] and [12] for approximation numbers of composition operators on the Hardy space  $H^2$ . In [14], it is proved that one always has  $a_n(C_\varphi) \gtrsim e^{-cn}$  for some  $c > 0$  ([14], Theorem 3.1) and

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that this speed of decay can only be got when the symbol  $\varphi$  maps the unit disk  $\mathbb{D}$  into a disk centered at 0 of radius strictly less than 1, i.e.  $\|\varphi\|_\infty < 1$  ([14], Theorem 3.4).

In this paper, we give estimates which are somewhat general, in terms of some modulus of continuity. In Section 2, we obtain an upper estimate when the symbol  $\varphi$  is continuous on the closed unit disk and has an image touching non-tangentially the unit circle at a finite number of points, with a good behavior on the boundary around this point. As an application, we show that for symbols  $\varphi$  whose image is contained in a polygon  $a_n(C_\varphi) \leq a e^{-b\sqrt{n}}$ , for some constants  $a, b > 0$ ; this has to be compared with [12], Proposition 2.7, where it is shown that if  $\varphi$  is a univalent symbol such that  $\varphi(\mathbb{D})$  contains an angular sector centered on the unit circle and with opening  $\theta\pi$ ,  $0 < \theta < 1$ , then  $a_n(C_\varphi) \geq a e^{-b\sqrt{n}}$ , for some (other) positive constants  $a$  and  $b$ , depending only on  $\theta$ . In Section 3, we obtain a lower bound when  $\varphi$  has a good radial behavior at the contact point. Both proofs use Blaschke products. This allows to recover the estimation  $a_n(C_{\lambda_\theta}) \approx e^{-c\sqrt{n}}$  obtained in [14], Proposition 6.3, and [12], Theorem 2.1 for the lens map  $\lambda_\theta$ . In Section 4.1, we give another example, the cusp map, for which  $a_n(C_\varphi) \approx e^{-cn/\log n}$ , very near the minimum value  $e^{-cn}$ . We end that section by considering a one-parameter class of symbols, first studied by J. Shapiro and P. D. Taylor [22] and seeing the limitations of our methods. In Section 5, we improve a result of E.A. Gallardo-Gutiérrez and M.J. González (previously generalized by O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi [5], Theorem 3.1). It is known that for every compact composition operator  $C_\varphi: H^2 \rightarrow H^2$ , the set  $E_\varphi = \{e^{i\theta}; |\varphi^*(e^{i\theta})| = 1\}$  has Lebesgue measure 0. These authors showed ([6]), with a rather difficult construction, that there exists a compact composition operator  $C_\varphi: H^2 \rightarrow H^2$  such that the Hausdorff dimension of  $E_\varphi$  is equal to 1 (and in [5], it is shown that for any negligible compact set  $K$ , there is a Hilbert-Schmidt operator  $C_\varphi$  such that  $E_\varphi = K$ ). We improve this result in showing that for every compact set  $K$  of the unit circle  $\mathbb{T}$  with Lebesgue measure 0, there exists a compact composition operator  $C_\varphi: H^2 \rightarrow H^2$ , which is even in all Schatten classes, and such that  $E_\varphi = K$ .

*Notation.* We denote by  $\mathbb{D}$  the open unit disk and by  $\mathbb{T} = \partial\mathbb{D}$  the unit circle;  $m$  is the normalized Lebesgue measure on  $\mathbb{T}$ :  $dm(t) = dt/2\pi$ . The disk algebra  $A(\mathbb{D})$  is the space of functions which are continuous on the closed unit disk  $\overline{\mathbb{D}}$  and analytic in the open unit disk. If  $H^2$  is the usual Hardy space on  $\mathbb{D}$ , every analytic self-map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  (also called *Schur function*) defines, by Littlewood's subordination principle, a bounded operator  $C_\varphi: H^2 \rightarrow H^2$  by  $C_\varphi(f) = f \circ \varphi$ , called the *composition operator of symbol  $\varphi$* .

Recall that if  $T: E \rightarrow F$  is a bounded operator between two Banach spaces, the *approximation numbers*  $a_n(T)$  of  $T$  are defined by:

$$a_n(T) = \inf\{\|T - R\|; \text{rank}(R) < n\}, \quad n = 1, 2, \dots$$

The sequence  $(a_n(T))_n$  is non-increasing and, when  $F$  has the Approximation Property,  $T$  is compact if and only if  $a_n(T)$  tends to 0.

**Definition 1.1** A modulus of continuity  $\omega$  is a continuous function

$$\omega: [0, A] \rightarrow \mathbb{R}^+,$$

which is increasing, sub-additive, and vanishes at zero.

Some examples are:

$$\omega(h) = h^\alpha, \quad 0 < \alpha \leq 1; \quad \omega(h) = h \log \frac{1}{h}; \quad \omega(h) = \frac{1}{\log \frac{1}{h}}.$$

For any modulus of continuity  $\omega$ , there is a concave modulus of continuity  $\omega'$  such that  $\omega \leq \omega' \leq 2\omega$  (see [17] for example); therefore we may and shall assume that  $\omega$  is concave on  $[0, A]$ . In that case,  $\omega^{-1}$  is convex, and

$$(1.1) \quad r_\omega(x) := \frac{\omega^{-1}(x)}{x}$$

is non-decreasing.

The notation  $u(t) \lesssim v(t)$  means that  $u(t) \leq Av(t)$  for some constant  $A > 0$  and  $u(t) \approx v(t)$  means that both  $u(t) \lesssim v(t)$  and  $v(t) \lesssim u(t)$ .

## 2 Upper bound and boundary behavior

**Definition 2.1** Let  $\omega$  be a modulus of continuity and  $\varphi$  a symbol in the disk algebra  $A(\mathbb{D})$ . Let  $\xi_0 \in \partial\mathbb{D} \cap \varphi(\overline{\mathbb{D}})$ . We say that the symbol  $\varphi$  has an  $\omega$ -regular behavior at  $\xi_0$  if, setting:

$$(2.1) \quad \gamma(t) = \varphi(e^{it}),$$

and  $E_{\xi_0} = \{t; \gamma(t) = \xi_0\}$ , there exists  $r_0 > 0$  such that:

1) for some positive constant  $C > 0$ , one has, for every  $t_0 \in E_{\xi_0}$  and  $|t - t_0| \leq r_0$ :

$$(2.2) \quad |\gamma(t) - \gamma(t_0)| \leq C(1 - |\gamma(t)|).$$

2) for some positive constant  $c > 0$ , one has, for every  $t_0 \in E_{\xi_0}$  and  $|t - t_0| \leq r_0$ :

$$(2.3) \quad c\omega(|t - t_0|) \leq |\gamma(t) - \gamma(t_0)|.$$

The first condition implies that the image of  $\varphi$  touches  $\partial\mathbb{D}$  at the point  $\xi_0$ , and non-tangentially. The second one implies that  $\varphi$  does not stay long near  $\xi_0 = \gamma(t_0)$ .

Note that, due to (2.3), the intervals  $[t - r_0/2, t + r_0/2]$ , for  $t \in E_{\xi_0}$  are pairwise disjoint and therefore the set  $E_{\xi_0}$  must be finite.

We shall make the following assumption (to avoid the Lipschitz class):

$$(2.4) \quad \lim_{h \rightarrow 0^+} \frac{\omega(h)}{h} = \infty; \quad \text{equivalently} \quad \lim_{h \rightarrow 0^+} \frac{\omega^{-1}(h)}{h} = 0.$$

Indeed, assume that  $\gamma$  is  $K$ -Lipschitz at some point  $t_0 \in [0, 2\pi]$ , namely  $|\varphi(e^{it}) - \varphi(e^{it_0})| \leq K|t - t_0|$ , with  $|\varphi(e^{it_0})| = 1$ ; then

$$\begin{aligned} m(\{t \in [0, 2\pi]; |\varphi(e^{it}) - \varphi(e^{it_0})| \leq h\}) \\ \geq m(\{t \in [0, 2\pi]; |t - t_0| \leq h/K\}) = h/2\pi K; \end{aligned}$$

hence this measure is not  $o(h)$  and the composition operator  $C_\varphi$  is not compact ([15], or [3], Theorem 3.12).

In order to treat the case where the image of  $\varphi$  is a polygon, we need to generalize the above definition. We ask not only that  $\varphi$  is  $\omega$ -regular at the points  $\xi_1, \dots, \xi_p$  of contact of  $\varphi(\overline{\mathbb{D}})$  with  $\partial\mathbb{D}$ , but a little bit more.

**Definition 2.2** *Assume that  $\varphi(\overline{\mathbb{D}}) \cap \partial\mathbb{D} = \{\xi_1, \dots, \xi_p\}$ . We say that  $\varphi$  is globally-regular if there exists a modulus of continuity  $\omega$  such that, writing  $E_{\xi_j} = \{t; \gamma(t) = \xi_j\}$ , one has, for some  $r_1, \dots, r_p > 0$*

$$\mathbb{T} = \bigcup_{j=1}^p (E_{\xi_j} + [-r_j, r_j])$$

and for some positive constants  $C, c > 0$ ,

1') one has, for  $j = 1, \dots, p$ , every  $t_j \in E_{\xi_j}$  and  $|t - t_j| \leq r_j$ :

$$(2.5) \quad |\gamma(t) - \gamma(t_j)| \leq C(1 - |\gamma(t)|).$$

2') one has, for  $j = 1, \dots, p$ , every  $t_j \in E_{\xi_j}$  and  $|t - t_j| \leq r_j$ :

$$(2.6) \quad c\omega(|t - t_j|) \leq |\gamma(t) - \gamma(t_j)|.$$

Let us note that condition 1') is equivalent to say that  $\varphi(\overline{\mathbb{D}})$  is contained in a polygon inside  $\overline{\mathbb{D}}$  whose vertices contain  $\xi_1, \dots, \xi_p$ , and these are the only vertices in the boundary  $\partial\mathbb{D}$ . Of course, we may assume that (2.5) and (2.6) hold only when  $t$  is in a neighborhood of  $t_j$ , since they will then hold for  $|t - t_j| \leq r_j$ , provided we change the constants  $C, c$ .

Before stating our theorem, let us introduce a notation. If  $\varphi$  is as in Definition 2.2 and  $\sigma, \kappa > 0$  are some constants, we set:

$$(2.7) \quad d_N = \left\lceil \sigma \log \frac{\kappa 2^{-N}}{\omega^{-1}(\kappa 2^{-N})} \right\rceil + 1,$$

where  $\lceil \cdot \rceil$  stands for the integer part. For every integer  $q \geq 1$ , we denote by

$$(2.8) \quad N = N_q \quad \text{the largest integer such that } pNd_N < q$$

( $N_q = 1$  if no such  $N$  exists).

We then have the following result.

**Theorem 2.3** *Let  $\varphi$  be a symbol in  $A(\mathbb{D})$  whose image touches  $\partial\mathbb{D}$  at the points  $\xi_1, \dots, \xi_p$ , and nowhere else. Assume that  $\varphi$  is globally-regular. Then, there are constants  $\kappa, K, L > 0$ , depending only on  $\varphi$ , such that, using the notation (2.7) and (2.8), one has, for every  $q \geq 1$ :*

$$(2.9) \quad a_q(C_\varphi) \leq K \sqrt{\frac{\omega^{-1}(\kappa 2^{-N_q})}{\kappa 2^{-N_q}}}.$$

Before proving this theorem, let us indicate two applications. In these examples, we can give an upper estimate for all approximation numbers  $a_n(C_\varphi)$ ,  $n \geq 1$  because we can interpolate between the integers  $Nd_N$  and  $(N+1)d_{N+1}$ , which is not the case in general.

1)  $\omega(h) = h^\theta$ ,  $0 < \theta < 1$ , as this is the case for inscribed polygons (see the proof of the foregoing Theorem 2.4; here  $\theta = \max\{\theta_1, \dots, \theta_p\}$ , where  $\theta_1\pi, \dots, \theta_p\pi$  are the values of the angles of the polygon), as well as, with  $p = 2$ , for lens maps  $\lambda_\theta$  (see [21], page 27, for the definition; see also [12]). We have here  $\omega^{-1}(h) = h^{1/\theta}$ . Hence  $d_N \approx N$ ,  $N_q \approx \sqrt{q}$ , and we then get from (2.9) that  $a_q(C_\varphi) \leq \alpha 2^{-\delta N}$  for  $q \gtrsim N^2$ , with  $\delta > 0$ . Equivalently, for suitable constants  $\alpha, \beta > 0$ ,

$$(2.10) \quad a_n(C_\varphi) \leq \alpha e^{-\beta\sqrt{n}},$$

which is the result obtained in [12], Theorem 2.1.

2)  $\omega(h) = \frac{1}{(\log 1/h)^\alpha}$ ,  $0 < \alpha \leq 1$ , as this is the case, when  $\alpha = 1$ , for the cusp map, defined below in Section 4.1 (with  $p = 1$ ). Then, we have  $\omega^{-1}(h) = e^{-h^{-1/\alpha}}$  and  $d_N \approx 2^{N/\alpha}$ , so that  $N_q \approx \log q$  and  $2^{N_q/\alpha} \approx q/\log q$ . Now, a simple computation gives:

$$(2.11) \quad a_n(C_\varphi) \leq \alpha e^{-\beta n / \log n}.$$

Without assuming some regularity, one has the following general upper estimate.

**Theorem 2.4** *Let  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  be an analytic self-map whose image is contained in a polygon  $\mathbf{P}$  with vertices on the unit circle. Then, there exist constants  $\alpha, \beta > 0$ ,  $\beta$  depending only on  $\mathbf{P}$ , such that:*

$$(2.12) \quad a_n(C_\varphi) \leq \alpha e^{-\beta\sqrt{n}}.$$

In [12], Proposition 2.7, it is shown that if  $\varphi$  is a univalent symbol such that  $\varphi(\mathbb{D})$  contains an angular sector centered on the unit circle and with opening  $\theta\pi$ ,  $0 < \theta < 1$ , then  $a_n(C_\varphi) \geq \alpha e^{-\beta\sqrt{n}}$ , for some (other) positive constants  $\alpha$  and  $\beta$ , depending only on  $\theta$ . Note that the injectivity of the symbol is there necessary, since there exists (see the proof of Corollary 5.4 in [14]), for every sequence  $(\varepsilon_n)$  of positive numbers tending to 0, a symbol  $\varphi$  whose image is  $\mathbb{D} \setminus \{0\}$ , and hence

contains polygons), which is 2-valent, and for which  $a_n(C_\varphi) \lesssim e^{-\varepsilon n}$ . This bound may be much smaller than  $e^{-\beta\sqrt{n}}$ .

**Proof of Theorem 2.3.** It follows the lines of that of [12], Theorem 2.1.

Recall ([12], Lemma 2.4) that for every Blaschke product  $B$  with less than  $N$  zeros (each of them being counted with its multiplicity), one has:

$$(2.13) \quad [a_N(C_\varphi)]^2 \lesssim \sup_{0 < h < 1, |\xi|=1} \frac{1}{h} \int_{S(\xi, h)} |B(z)|^2 dm_\varphi(z),$$

where  $S(\xi, h) = \{z \in \overline{\mathbb{D}}; |z - \xi| \leq h\}$  and  $m_\varphi$  is the pull-back measure by  $\varphi$  of the normalized Lebesgue measure  $m$  on  $\mathbb{T}$ .

The proof will come from an adequate choice of a Blaschke product.

Fix a positive integer  $N$ .

Set, for  $j = 1, \dots, p$  and  $k = 1, 2, \dots$ :

$$(2.14) \quad p_{j,k} = (1 - 2^{-k})\xi_j$$

and consider the Blaschke product of length  $pNd$  ( $d$  being a positive integer, to be specified later) given by:

$$(2.15) \quad B(z) = \prod_{j=1}^p \prod_{k=1}^N \left[ \frac{z - p_{j,k}}{1 - \overline{p_{j,k}}z} \right]^d.$$

Recall that we have set

$$(2.16) \quad \gamma(t) = \varphi(e^{it}).$$

To use (2.13), note that if  $|\gamma(t) - \xi| \leq h$ , then, for some  $j = 1, \dots, p$  and some  $t_j \in E_{\xi_j}$ , one has  $|t - t_j| \leq r_j$  and, by (2.5),  $|\gamma(t) - \xi_j| \leq C(1 - |\gamma(t)|) \leq C|\gamma(t) - \xi| \leq Ch$ . Therefore, denoting by  $L_j$  the number of elements of  $E_{\xi_j}$  (which is finite by the remark following Definition 2.1):

$$[a_N(C_\varphi)]^2 \lesssim \sup_{0 < h < 1} \frac{1}{h} \sum_{j=1}^p L_j \int_{\{|\gamma(t) - \xi_j| \leq Ch\} \cap \{|t - t_j| \leq r_j\}} |B[\gamma(t)]|^2 \frac{dt}{2\pi},$$

and we only need to majorize the integrals:

$$I_j(h) = \int_{\{|\gamma(t) - \xi_j| \leq Ch\} \cap \{|t - t_j| \leq r_j\}} |B(\gamma(t))|^2 \frac{dt}{2\pi}.$$

Moreover, it suffices, by interpolation, to do that with  $h = h_n$ , where  $h_n = 2^{-n}$ .

By (2.6), for  $|t - t_j| \leq r_j$  and  $|\gamma(t) - \xi_j| \leq Ch_n$ , one has  $c\omega(|t - t_j|) \leq |\gamma(t) - \xi_j| \leq Ch_n = C2^{-n}$ , which implies that

$$(2.17) \quad |t - t_j| \leq \omega^{-1}(c^{-1}C2^{-n}).$$

Let

$$(2.18) \quad s_n = \omega^{-1}(c^{-1}C 2^{-n}).$$

One has:

$$I_j(h_n) \leq \int_{\{|t-t_j| \leq s_n\} \cap \{|t-t_j| \leq r_j\}} |B(\gamma(t))|^2 \frac{dt}{2\pi}.$$

For  $n \geq N$ , we simply majorize  $|B(\gamma(t))|$  by 1 and we get:

$$\begin{aligned} \frac{1}{h_n} I_j(h_n) &\leq \frac{1}{h_n} \frac{2s_n}{2\pi} = \frac{c^{-1}C}{\pi} \frac{1}{c^{-1}C 2^{-n}} \omega^{-1}(c^{-1}C 2^{-n}) \\ &\leq \frac{c^{-1}C}{\pi} \frac{\omega^{-1}(c^{-1}C 2^{-N})}{c^{-1}C 2^{-N}}, \end{aligned}$$

since the function  $\omega^{-1}(x)/x$  is non-decreasing.

When  $n \leq N - 1$ , we write:

$$\begin{aligned} I_j(h_n) &\leq \int_{\{|t-t_j| \leq s_N\} \cap \{|t-t_j| \leq r_j\}} |B(\gamma(t))|^2 \frac{dt}{2\pi} \\ &\quad + \int_{\{s_N < |t-t_j| \leq s_n\} \cap \{|t-t_j| \leq r_j\}} |B(\gamma(t))|^2 \frac{dt}{2\pi}. \end{aligned}$$

The first integral is estimated as above. For the second one, we claim that:

**Claim 2.5** *For some constant  $\chi < 1$ , one has, for  $j = 1, \dots, p$  and every  $t_j \in E_{\xi_j}$ :*

$$(2.19) \quad |B(\gamma(t))| \leq \chi^d \quad \text{when } |t - t_j| > s_N \text{ and } |t - t_j| \leq r_j.$$

To see that, we shall use [12], Lemma 2.3. Let us recall that this lemma asserts that for  $w, w_0 \in \mathbb{D}$  satisfying  $|w - w_0| \leq M \min(1 - |w|, 1 - |w_0|)$  for some positive constant  $M$ , one has:

$$(2.20) \quad \left| \frac{w - w_0}{1 - \overline{w_0}w} \right| \leq \frac{M}{\sqrt{M^2 + 1}}.$$

Let  $t$  such that  $|t - t_j| \leq r_j$  and  $|t - t_j| > s_N$ . We have, on the one hand,  $\omega(|t - t_j|) \geq \omega(s_N) = c^{-1}C 2^{-N}$ , and, on the other hand, since  $|\gamma(t_j)| = |\xi_j| = 1$

$$c\omega(|t - t_j|) \leq |\gamma(t) - \gamma(t_j)| \leq C(1 - |\gamma(t)|);$$

hence  $1 - |\gamma(t)| \geq 2^{-N}$ .

Let  $1 \leq k \leq N$  such that  $2^{-k} \leq 1 - |\gamma(t)| < 2^{-k+1}$ . Since  $|p_{j,k}| = 1 - 2^{-k}$ , we have:

$$|\gamma(t) - p_{j,k}| \leq |\gamma(t) - \xi_j| + |\xi_j - p_{j,k}| \leq C(1 - |\gamma(t)|) + 2^{-k} \leq (2C + 1)2^{-k}.$$

Hence

$$|\gamma(t) - p_{j,k}| \leq M \min(1 - |\gamma(t)|, 1 - |p_{j,k}|),$$

with  $M = 2C + 1$ . By (2.20), we get  $\left| \frac{\gamma(t) - p_{jk}}{1 - \overline{p_{jk}} \gamma(t)} \right| \leq \chi$ , where  $\chi = M/\sqrt{M^2 + 1}$  is  $< 1$ , and therefore  $|B[\gamma(t)]| \leq \chi^d$ .  $\square$

We can now end the proof of Theorem 2.3. We get:

$$\begin{aligned} & \frac{1}{h_n} \int_{\{s_N < |t-t_j| \leq s_n\} \cap \{|t-t_j| \leq r_j\}} |B(\gamma(t))|^2 \frac{dt}{2\pi} \\ & \leq \frac{1}{h_n} \frac{2s_n}{2\pi} \chi^{2d} = \frac{1}{h_n} \frac{\omega^{-1}(c^{-1}C 2^{-n})}{\pi} \chi^{2d} \\ & = \frac{c^{-1}C}{\pi} \frac{\omega^{-1}(c^{-1}C 2^{-n})}{c^{-1}C 2^{-n}} \chi^{2d} \\ & \leq \frac{1}{\pi} \omega^{-1}(c^{-1}C) \chi^{2d}, \end{aligned}$$

since  $\omega^{-1}(x)/x$  is non-decreasing.

We therefore get, setting  $\kappa = c^{-1}C$  and  $L = L_1 + \dots + L_p$ :

$$\begin{aligned} & \frac{1}{h_n} \sum_{j=1}^p L_j \int_{\{|\gamma(t) - \xi_j| \leq Ch_n\} \cap \{|t-t_j| \leq r_j\}} |B[\gamma(t)]|^2 \frac{dt}{2\pi} \\ & \leq \frac{\kappa L}{\pi} \frac{\omega^{-1}(\kappa 2^{-N})}{\kappa 2^{-N}} + \frac{L \omega^{-1}(\kappa)}{\pi} \chi^{2d}. \end{aligned}$$

Choose now  $d = d_N$ , where  $d_N$  is defined by (2.7), with  $\sigma = 1/\log(\chi^{-2})$ . Then  $\chi^{2d} \leq \omega^{-1}(\kappa 2^{-N})/(\kappa 2^{-N})$ , and, since the Blaschke product  $B$  has now  $pNd_N$  zeroes, we get, for some positive constant  $K$ :

$$a_{pNd_N+1}(C_\varphi) \leq K \sqrt{\frac{\omega^{-1}(\kappa 2^{-N})}{\kappa 2^{-N}}},$$

and that ends the proof of Theorem 2.3.  $\square$

**Proof of Theorem 2.4.** It suffices to consider the case when  $\varphi$  is a conformal map from  $\mathbb{D}$  onto  $\mathbf{P}$ . Indeed, let  $\psi$  be such a conformal map. In the general case, our assumption allows to write  $\varphi = \psi \circ u$ , where  $u = \psi^{-1} \circ \varphi: \mathbb{D} \rightarrow \mathbb{D}$  is analytic. It follows that  $C_\varphi = C_u \circ C_\psi$  and that  $a_n(C_\varphi) \leq \|C_u\| a_n(C_\psi)$ . Therefore, we may and shall assume that  $\varphi$  itself is this conformal map.

Let us denote by  $\xi_1, \dots, \xi_p$  the vertices of  $\mathbf{P}$ . Let  $0 < \pi\mu_j < \pi$  be the exterior angle of  $\mathbf{P}$  at  $\xi_j$ , namely the complement to  $\pi$  of the interior angle; so that:

$$\sum_{j=1}^p \mu_j = 2, \quad \text{and} \quad 0 < \mu_j < 1.$$

If one sets  $\theta_j = 1 - \mu_j$ , one has  $0 < \theta_j < 1$ .

We then use the explicit form of  $\varphi$  given by the Schwarz-Christoffel formula ([18], page 193):

$$(2.21) \quad \varphi(z) = A \int_0^z \frac{dw}{(a_1 - w)^{\mu_1} \dots (a_p - w)^{\mu_p}} + B,$$



for some constants  $A \neq 0$  and  $B \in \mathbb{C}$  and where  $a_1, \dots, a_p \in \partial\mathbb{D}$  are such that  $\xi_j = \varphi(a_j)$ ,  $j = 1, \dots, p$ . If, as before, we write  $\gamma(t) = \varphi(e^{it})$ , we have  $\xi_j = \gamma(t_j)$ , with  $a_j = e^{it_j}$  (note that here  $E_{\xi_j} = \{t_j\}$ ).

As we already said, condition (2.5) is trivially satisfied for a polygon.

To end the proof, we use Theorem 2.3 and its Example 1. For that it suffices to show that, for  $|t - t_j|$  small enough, we have:

$$(2.22) \quad |\gamma(t) - \xi_j| \approx |t - t_j|^{\theta_j}.$$

If  $z \in \mathbb{D}$  is close to  $a_j$ , it follows from (2.21) that we can write

$$\varphi(z) = A \int_0^z f_j(w) \frac{dw}{(a_j - w)^{\mu_j}} + B,$$

where  $f_j$  is holomorphic near  $a_j$  and  $f_j(a_j) \neq 0$  since

$$|f_j(a_j)| = \prod_{k \neq j, 1 \leq k \leq p} |a_j - a_k|^{-\mu_k}.$$

Write  $f_j(w) = f_j(a_j) + (a_j - w)g_j(w)$  where  $g_j$  is holomorphic near  $a_j$ . We get:

$$\begin{aligned} \varphi(z) &= Af_j(a_j) \int_0^z \frac{dw}{(a_j - w)^{\mu_j}} + B + \int_0^z g_j(w)(a_j - w)^{\theta_j} dw \\ &:= Af_j(a_j) \int_0^z \frac{dw}{(a_j - w)^{\mu_j}} + B + \psi_j(z), \end{aligned}$$

which can still be written (since  $\theta_j > 0$ ):

$$(2.23) \quad \varphi(z) = \lambda_j(a_j - z)^{\theta_j} + c_j + \psi_j(z),$$

where  $\lambda_j \neq 0$ ,  $c_j \in \mathbb{C}$ ,  $\psi_j$  is Lipschitz near  $a_j$  and  $\xi_j = \varphi(a_j) = c_j + \psi_j(a_j)$ . Now, we easily get (2.22). Indeed, for  $t$  near  $t_j$ , it follows from (2.23) that (recall that  $\gamma(t) = \varphi(e^{it})$  and  $\gamma(t_j) = \xi_j$ ):

$$|\gamma(t) - \gamma(t_j)| = |\lambda_j| |e^{it} - e^{it_j}|^{\theta_j} + O(|t - t_j|),$$

which the claimed estimate (2.22) since  $\lambda_j \neq 0$  and  $|t - t_j|$  is negligible compared to  $|t - t_j|^{\theta_j} \approx |e^{it} - e^{it_j}|^{\theta_j}$ .  $\square$

### 3 Lower bound and radial behavior

We shall consider symbols  $\varphi$  taking real values in the real axis (*i.e.* its Taylor series has real coefficients) and such that  $\lim_{r \rightarrow 1^-} \varphi(r) = 1$ , with a given speed.

**Definition 3.1** *We say that the analytic map  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  is real if it takes real values on  $] - 1, 1[$ , and that  $\varphi$  is an  $\omega$ -radial symbol if it is real and there is a modulus of continuity  $\omega: [0, 1] \rightarrow [0, 2]$  such that:*

$$(3.1) \quad 1 - \varphi(r) \leq \omega(1 - r), \quad 0 \leq r < 1.$$

With those definitions and notations, one has:

**Theorem 3.2** *Let  $\varphi$  be a real and  $\omega$ -radial symbol. Then, for the approximation numbers  $a_n(C_\varphi)$  of the composition operator  $C_\varphi$  of symbol  $\varphi$ , one has the following lower bound:*

$$(3.2) \quad a_n(C_\varphi) \geq c \sup_{0 < \sigma < 1} \sqrt{\frac{\omega^{-1}(a\sigma^n)}{a\sigma^n}} \exp\left[-\frac{20}{1-\sigma}\right],$$

where  $a = 1 - \varphi(0) > 0$  and  $c$  is another constant depending only on  $\varphi$ .

Observe that, for the lens map  $\lambda_\theta$  (see [12], Lemma 2.5), we have  $\omega^{-1}(h) \approx h^{1/\theta}$ , so that adjusting  $\sigma = 1 - 1/\sqrt{n}$ , we get

$$(3.3) \quad a_n(C_{\lambda_\theta}) \geq c \exp(-C\sqrt{n}),$$

which is the result of [14], Proposition 6.3.

For the cusp map  $\varphi$  (see Section 4.1), we have  $\omega^{-1}(h) \approx e^{-C'/h}$ , so that taking  $\sigma = \exp(-\log n/2n)$ , we get:

$$(3.4) \quad a_n(C_\varphi) \geq c \exp(-Cn/\log n).$$

We shall use the same methods as for lens maps (see [14], Proposition 6.3).

We need a lemma. Recall (see [8] pages 194–195, or [19] pages 302–303) that if  $(z_j)$  is a Blaschke sequence, its Carleson constant  $\delta$  is defined as  $\delta = \inf_{j \geq 1} (1 - |z_j|^2) |B'(z_j)|$ , where  $B$  is the Blaschke product whose zeros are the  $z_j$ 's. Now (see [7], Chapter VII, Theorem 1.1), every  $H^\infty$ -interpolation sequence  $(z_j)$  is a Blaschke sequence and its Carleson constant  $\delta$  is connected to its interpolation constant  $C$  by the inequalities

$$(3.5) \quad 1/\delta \leq C \leq \kappa/\delta^2$$

where  $\kappa$  is an absolute constant (actually  $C \leq \kappa_1(1/\delta)(1 + \log 1/\delta)$ ). Now, if  $(z_j)$  is a  $H^\infty$ -interpolation sequence with constant  $C$ , the sequence of the normalized reproducing kernels  $f_j = K_{z_j}/\|K_{z_j}\|$  satisfies

$$C^{-1} \left( \sum |\lambda_j|^2 \right)^{1/2} \leq \left\| \sum \lambda_j f_j \right\|_{H^2} \leq C \left( \sum |\lambda_j|^2 \right)^{1/2}$$

(see [14], Lemma 2.2).

**Lemma 3.3** *Let  $\varphi: \mathbb{D} \rightarrow \mathbb{D}$  be an analytic self-map. Let  $u = (u_1, \dots, u_n)$  be a finite sequence in  $\mathbb{D}$  and set  $v_j = \varphi(u_j)$ ,  $v = (v_1, \dots, v_n)$ . Denote by  $\delta_v$  the Carleson constant of the finite sequence  $v$  and set*

$$\mu_n^2 = \inf_{1 \leq j \leq n} \frac{1 - |u_j|^2}{1 - |\varphi(u_j)|^2}.$$

*Then, for some constant  $c' > 0$ , we have the lower bound:*

$$(3.6) \quad a_n(C_\varphi) \geq c' \delta_v^4 \mu_n.$$

**Proof.** Recall first that the Carleson constant  $\delta$  of a Blaschke sequence  $(z_j)$  is also equal to:

$$\delta = \inf_{k \geq 1} \prod_{j \neq k} \rho(z_k, z_j),$$

where  $\rho(z, \zeta) = \left| \frac{z - \zeta}{1 - \bar{z}\zeta} \right|$  is the pseudo-hyperbolic distance between  $z$  and  $\zeta$ . Now, the Schwarz-Pick Lemma (see [1], Theorem 3.2) asserts that every analytic self-map of  $\mathbb{D}$  contracts the pseudo-hyperbolic distance. Hence  $\rho(\varphi(u_j), \varphi(u_k)) \leq \rho(u_j, u_k)$  and so, if  $\delta_u$  and  $\delta_v$  denote the Carleson constants of  $u$  and  $v$ :

$$\delta_u \geq \delta_v.$$

Let now  $R$  be an operator of rank  $< n$ . There exists a function  $f = \sum_{j=1}^n \lambda_j K_{u_j} \in H^2 \cap \ker R$  with  $\|f\| = 1$ . We thus have:

$$\begin{aligned} \|C_\varphi^* - R\|^2 &\geq \|C_\varphi^*(f) - R(f)\|_2^2 = \|C_\varphi^*(f)\|_2^2 = \left\| \sum_{j=1}^n \lambda_j K_{v_j} \right\|_2^2 \\ &\geq C_v^{-2} \sum_{j=1}^n |\lambda_j|^2 \|K_{v_j}\|_2^2 = C_v^{-2} \sum_{j=1}^n \frac{|\lambda_j|^2}{1 - |v_j|^2} \\ &\geq C_v^{-2} \mu_n^2 \sum_{j=1}^n \frac{|\lambda_j|^2}{1 - |u_j|^2} \\ &\geq C_u^{-2} C_v^{-2} \mu_n^2 \|f\|_2^2 = C_u^{-2} C_v^{-2} \mu_n^2 \\ &\geq \kappa^{-4} \delta_u^4 \delta_v^4 \mu_n^2 \geq \kappa^{-4} \delta_v^8 \mu_n^2, \end{aligned}$$

and hence  $a_n(C_\varphi) \geq \kappa^{-2} \delta_v^4 \mu_n$ .  $\square$

**Remark.** This lemma allows to give, in the Hardy case, a simpler proof of Theorem 4.1 in [14], avoiding the use of Lemma 2.3 and Lemma 2.4 (concerning the backward shift) in that paper. Recall that this theorem says that for every non-increasing sequence  $(\varepsilon_n)_{n \geq 1}$  of positive real numbers tending to 0, there exists a univalent symbol  $\varphi$  such that  $\varphi(0) = 0$  and  $C_\varphi: H^2 \rightarrow H^2$  is compact, but  $a_n(C_\varphi) \gtrsim \varepsilon_n$  for every  $n \geq 1$ . Let us sketch briefly the argument. We use the notation of [14], Lemma 4.6. The symbol  $\varphi$  is defined as  $\varphi(z) = \sigma^{-1}(e^{-1}\sigma(z))$ , where  $\sigma$  is some conformal map  $\sigma: \mathbb{D} \rightarrow \Omega$ . We set  $A_j = (1/C_0) \log(1/\varepsilon_{j+1})$ ,  $r_j = \sigma^{-1}(e^j)$ . Then  $\varphi(r_{j+1}) = r_j$  and (see [14], pages 444–446):

$$\frac{1 - r_{j+1}}{1 - r_j} \geq \exp(-2C_0 A_j).$$

We shall apply the above Lemma 3.3 with  $u_j = r_j$ . Then  $v_j = \varphi(u_j) = r_{j-1}$ . Hence

$$\frac{1 - |u_j|^2}{1 - |v_j|^2} \geq \frac{1}{2} \frac{1 - |u_j|}{1 - |v_j|} = \frac{1}{2} \frac{1 - r_j}{1 - r_{j-1}} \geq \frac{1}{2} \exp(-2C_0 A_{j-1}) = \frac{1}{2} \varepsilon_j^2 \geq \frac{1}{2} \varepsilon_n^2.$$

It follows that  $\mu_n \geq \varepsilon_n/\sqrt{2}$ .

On the other hand,  $(r_j)_{j \geq 1}$  is an interpolating sequence (see [14], Lemma 4.6); hence there is a constant  $\delta > 0$  (which does not depend on  $n \geq 1$ ) such that  $\delta_v \geq \delta$ . Therefore Lemma 3.3 gives

$$a_n(C_\varphi) \geq c\delta^4\varepsilon_n,$$

which gives Theorem 4.1 of [14].  $\square$

**Proof of Theorem 3.2.** Fix  $0 < \sigma < 1$  and define inductively  $u_j \in [0, 1)$  by  $u_0 = 0$  and the relation

$$1 - \varphi(u_{j+1}) = \sigma[1 - \varphi(u_j)] \quad \text{with } 1 > u_{j+1} > u_j$$

(using the intermediate value theorem).

Setting  $v_j = \varphi(u_j)$ , we have  $-1 < v_j < 1$ ,

$$(3.7) \quad \frac{1 - v_{j+1}}{1 - v_j} = \sigma,$$

and

$$(3.8) \quad 1 - v_n = a\sigma^n, \quad \text{with } a = 1 - \varphi(0).$$

Now observe that, for  $1 \leq j \leq n$ , one has, due to the positivity of  $u_j$  and  $v_j$ , to (3.1), and the fact that  $r_\omega(x) = \omega^{-1}(x)/x$  is increasing:

$$\begin{aligned} \frac{1 - |u_j|^2}{1 - |v_j|^2} &\geq \frac{1 - u_j}{2(1 - v_j)} \geq \frac{1}{2} \frac{\omega^{-1}(1 - v_j)}{1 - v_j} = \frac{1}{2} r_\omega(1 - v_j) \\ &\geq \frac{1}{2} r_\omega(1 - v_n) = \frac{1}{2} r_\omega(a\sigma^n), \end{aligned}$$

which proves that  $\mu_n^2 \geq r_\omega(a\sigma^n)/2$ . Furthermore, the sequence  $(v_j)$  satisfies, by (3.7), a condition very similar to Newman's condition with parameter  $\sigma$ . In fact, for  $k > j$ , we have

$$\frac{|v_k - v_j|}{|1 - v_k v_j|} = \frac{(1 - v_j) - (1 - v_k)}{(1 - v_j) + v_j(1 - v_k)} \geq \frac{(1 - v_j) - (1 - v_k)}{(1 - v_j) + (1 - v_k)} = \frac{1 - \sigma^{k-j}}{1 + \sigma^{k-j}}.$$

Analogously, for  $j > k$ , we have  $\frac{|v_k - v_j|}{|1 - v_k v_j|} \geq \frac{1 - \sigma^{j-k}}{1 + \sigma^{j-k}}$ . Thus, as in the proof of [4], Theorem 9.2, we have, for every  $k$ ,

$$\prod_{j \neq k} \rho(v_j, v_k) = \prod_{j \neq k} \frac{|v_k - v_j|}{|1 - v_k v_j|} \geq \prod_{l=1}^{\infty} \left( \frac{1 - \sigma^l}{1 + \sigma^l} \right)^2.$$

Consequently,  $\delta_v \geq \prod_{l=1}^{\infty} \left( \frac{1 - \sigma^l}{1 + \sigma^l} \right)^2 \geq \exp\left(-\frac{5}{1 - \sigma}\right)$ , by [14], Lemma 6.4. Finally, use (3.6) to get:

$$a_n(C_\varphi) \geq c' \delta_v^4 \mu_n \geq c \exp\left(-\frac{20}{1 - \sigma}\right) \sqrt{r_\omega(a\sigma^n)}.$$

Taking the supremum over  $\sigma$ , that ends the proof of Theorem 3.2.  $\square$

**Remark.** The proof shows that

$$(3.9) \quad a_n(C_\varphi) \geq \sup_{u_1, \dots, u_n \in (0,1)} \inf_{\substack{f \in \langle K_{u_1}, \dots, K_{u_n} \rangle \\ \|f\|=1}} \|C_\varphi^* f\|,$$

where  $\langle K_{u_1}, \dots, K_{u_n} \rangle$  is the linear space generated by  $n$  distinct reproducing kernels  $K_{u_1}, \dots, K_{u_n}$ . But if  $B$  is the Blaschke product with zeros  $u_1, \dots, u_n$ , then  $\langle K_{u_1}, \dots, K_{u_n} \rangle = (BH^2)^\perp$ , the *model space* associated to  $B$ . Hence

$$(3.10) \quad a_n(C_\varphi) \geq \sup_B \inf_{\substack{f \in (BH^2)^\perp \\ \|f\|=1}} \|C_\varphi^* f\|,$$

where the supremum is taken over all Blaschke products with  $n$  zeros on the real axis  $(0, 1)$ . This has to be compared with the upper bound (which gives (2.13), see [12], proof of Lemma 2.4):

$$(3.11) \quad a_n(C_\varphi) \leq \inf_B \|C_\varphi|_{BH^2}\| = \inf_B \sup_{\substack{f \in BH^2 \\ \|f\|=1}} \|C_\varphi f\|,$$

where the infimum is over the Blaschke products with less than  $n$  zeros (in the Hilbert space  $H^2$ , the approximation number  $a_n(C_\varphi)$  is equal to the Gelfand number  $c_n(C_\varphi)$ , which is, by definition, less or equal to  $\|C_\varphi|_{BH^2}\|$ , since  $BH^2$  is of codimension  $< n$ ).

## 4 Examples

### 4.1 The cusp map

**Definition 4.1** *The cusp map is the conformal mapping  $\varphi$  sending the unit disk  $\mathbb{D}$  onto the domain represented on Figure 1.*

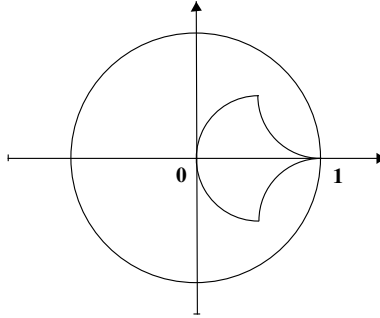


Figure 1: *Cusp map domain*

This map was first introduced in [11] (see also [13]). Explicitly,  $\varphi$  is defined as follows.

We first map  $\mathbb{D}$  onto the half-disk  $\mathbb{D}^+ = \{z \in \mathbb{D}; \Re z > 0\}$ . To do that, map  $\mathbb{D}$  onto itself by  $z \mapsto iz$ ; then map  $\mathbb{D}$  onto the upper half-plane  $\mathbb{H} = \{z \in \mathbb{C}; \Im z > 0\}$  by:

$$T(u) = i \frac{1+u}{1-u}.$$

Take the square root to map  $\mathbb{H}$  in the first quadrant  $Q_1 = \{z \in \mathbb{H}; \Re z > 0\}$ , and go back to the half-disk  $\{z \in \mathbb{D}; \Im z < 0\}$  by  $T^{-1}$ :  $T^{-1}(s) = \frac{1+is}{is-1}$ ; finally, make a rotation by  $i$  to go onto  $\mathbb{D}^+$ . We get:

$$(4.1) \quad \varphi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1}.$$

One has  $\varphi_0(1) = 0$ ,  $\varphi_0(-1) = 1$ ,  $\varphi_0(i) = -i$  and  $\varphi_0(-i) = i$ . The half-circle  $\{z \in \mathbb{T}; \Re z \geq 0\}$  is mapped onto the segment  $[-i, i]$  and the segment  $[-1, 1]$  onto the segment  $[0, 1]$ .

Set now, successively,

$$(4.2) \quad \varphi_1(z) = \log \varphi_0(z), \quad \varphi_2(z) = -\frac{2}{\pi} \varphi_1(z) + 1, \quad \varphi_3(z) = \frac{1}{\varphi_2(z)},$$

and finally:

$$(4.3) \quad \varphi(z) = 1 - \varphi_3(z).$$

Hence:

$$(4.4) \quad 1 - \varphi(z) = \frac{1}{1 + \frac{2}{\pi} \log(1/|\varphi_0(z)|) - i \frac{2}{\pi} \arg \varphi_0(z)}.$$

$\varphi_2$  maps  $\mathbb{D}$  onto the semiband  $\{z \in \mathbb{C}; \Re z > 1 \text{ and } |\Im z| < 1\}$ . One has  $\varphi(1) = 1$ ,  $\varphi(-1) = 0$ ,  $\varphi(i) = (1+i)/2$  and  $\varphi(-i) = (1-i)/2$ .

The domain  $\varphi(\mathbb{D})$  is edged by three circular arcs of radii  $1/2$  and of respective centers  $1/2$ ,  $1+i/2$  and  $1-i/2$ . The real interval  $] -1, 1[$  is mapped onto the real interval  $]0, 1[$  and the half-circle  $\{e^{i\theta}; |\theta| \leq \pi/2\}$  is sent onto the two circular arcs tangent at 1 to the real axis.

#### Lemma 4.2

1) For  $0 < r < 1$ , let  $\gamma = \frac{\pi}{4} - \arctan r = \arctan[(1-r)/(1+r)]$ ; then:

$$(4.5) \quad \varphi_0(r) = \tan(\gamma/2).$$

Hence, when  $r$  tends to  $1_-$ , one has:

$$(4.6) \quad 1 - \varphi(r) \sim \frac{\pi}{2} \frac{1}{\log(1/\gamma)} \sim \frac{\pi}{2} \frac{1}{\log(1/(1-r))}.$$

2) For  $|\theta| < \pi/2$ , one has:

$$(4.7) \quad \varphi_0(e^{i\theta}) = -i \frac{\tan(\theta/2)}{1 + \sqrt{1 - \tan^2(\theta/2)}}.$$

Hence, when  $\theta$  tends to 0, one has:

$$(4.8) \quad 1 - \varphi_0(e^{i\theta}) \sim \frac{\pi}{2} \frac{1}{\log(1/|\theta|)}.$$

**Proof.** 1) One has:

$$T(ir) = \frac{r-i}{ir-1} = -\frac{2r}{1+r^2} + i \frac{1-r^2}{1+r^2} = -\sin \alpha + i \cos \alpha,$$

with  $r = \tan(\alpha/2)$ ; hence  $T(ir) = \cos(\alpha + \pi/2) + i \sin(\alpha + \pi/2) = e^{i(\alpha + \pi/2)}$ . Set  $\beta = \frac{\alpha}{2} + \frac{\pi}{4}$ ; one gets:

$$\varphi_0(r) = \frac{e^{i\beta} - i}{-ie^{i\beta} + 1} = \frac{\cos \beta}{1 + \sin \beta} = \frac{\sin \gamma}{1 + \cos \gamma} = \tan(\gamma/2)$$

with  $\gamma = (\pi/2) - \beta = (\pi/4) - (\alpha/2) = (\pi/4) - \tan^{-1} r$ . Then (4.6) follows.

2) Let  $\tau = \frac{\pi}{2} - \theta$ ; one has:

$$T(ie^{i\theta}) = \frac{e^{i\theta} - i}{ie^{i\theta} - 1} = \frac{-\cos \theta}{1 + \sin \theta} = \frac{-\sin \tau}{1 + \cos \tau} = -\tan(\tau/2).$$

Note that  $0 < \tau/2 < \pi/2$  since  $|\theta| < \pi/2$ ; hence  $\tan(\tau/2) > 0$ . Therefore:

$$\varphi_0(e^{i\theta}) = \frac{i\sqrt{\tan(\tau/2)} - i}{-i.i\sqrt{\tan(\tau/2)} + 1} = i \frac{\sqrt{\tan(\tau/2)} - 1}{\sqrt{\tan(\tau/2)} + 1}.$$

But

$$\tan(\tau/2) = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1 - \tan(\theta/2)}{1 + \tan(\theta/2)};$$

it follows that:

$$\begin{aligned} \varphi_0(e^{i\theta}) &= i \frac{\sqrt{1 - \tan(\theta/2)} - \sqrt{1 + \tan(\theta/2)}}{\sqrt{1 - \tan(\theta/2)} + \sqrt{1 + \tan(\theta/2)}} \\ &= i \frac{(1 - \tan(\theta/2)) - (1 + \tan(\theta/2))}{(\sqrt{1 - \tan(\theta/2)} + \sqrt{1 + \tan(\theta/2)})^2} \\ &= -i \frac{\tan(\theta/2)}{1 + \sqrt{1 - \tan^2(\theta/2)}}. \end{aligned}$$

Now, since  $\varphi_0(e^{i\theta}) \sim -i\theta/4$  as  $\theta$  tends to 0, we get (4.8).  $\square$

It follows from this lemma and from Theorem 2.3 and Theorem 3.2 that one has the following estimate.

**Theorem 4.3** For the approximation numbers  $a_n(C_\varphi)$  of the composition operator  $C_\varphi: H^2 \rightarrow H^2$  of symbol the cusp map  $\varphi$ , we have:

$$(4.9) \quad e^{-c_1 n/\log n} \lesssim a_n(C_\varphi) \lesssim e^{-c_2 n/\log n}, \quad n = 2, 3, \dots,$$

for some constants  $c_1 > c_2 > 0$ .

**Proof.** 1) *Upper estimate.* Note first that, since the domain  $\varphi(\mathbb{D})$  is contained in the right half-plane and in the symmetric angular sector of vertex 1 and opening  $\pi/2$ , there is a constant  $C > 0$  such that  $|1 - \gamma(t)| \leq C(1 - |\gamma(t)|)$  and we have (2.2). Then (4.8) in Lemma 4.2 gives (2.3). The upper estimate is hence given in Theorem 2.3 and (2.11).

2) *Lower estimate.* By Lemma 4.2, (4.6), one has (3.1). Since  $\varphi$  is a real symbol, the upper estimate follows from Theorem 3.2, and (3.4).  $\square$

## 4.2 The Shapiro-Taylor map

This one-parameter map  $\varsigma_\theta$ ,  $\theta > 0$ , was introduced by J. Shapiro and P. Taylor in 1973 ([22]) and was further studied, with a slightly different definition, in [9], Section 5. J. Shapiro and P. Taylor proved that  $C_{\varsigma_\theta}: H^2 \rightarrow H^2$  is always compact, but is Hilbert-Schmidt if and only if  $\theta > 2$ . It is proved in [9], Theorem 5.1, that  $C_{\varsigma_\theta}$  is in the Schatten class  $S_p$  if and only if  $p > 4/\theta$ .

Here, we shall use these maps  $\varsigma_\theta$  to see the limitations of our previous methods.

We first recall their definition.

For  $\varepsilon > 0$ , we set  $V_\varepsilon = \{z \in \mathbb{C}; \Re z > 0 \text{ and } |z| < \varepsilon\}$ . For  $\varepsilon = \varepsilon_\theta > 0$  small enough, one can define

$$(4.10) \quad f_\theta(z) = z(-\log z)^\theta,$$

for  $z \in V_\varepsilon$ , where  $\log z$  will be the principal determination of the logarithm. Let now  $g_\theta$  be the conformal mapping from  $\mathbb{D}$  onto  $V_\varepsilon$ , which maps  $\mathbb{T} = \partial\mathbb{D}$  onto  $\partial V_\varepsilon$ , defined by  $g_\theta(z) = \varepsilon \varphi_0(z)$ , where  $\varphi_0$  is given in (4.1).

Then, we define:

$$(4.11) \quad \varsigma_\theta = \exp(-f_\theta \circ g_\theta).$$

One has  $\varsigma_\theta(1) = 1$  and  $g_\theta(e^{it}) \sim -it/4$  as  $t$  tends to 0, by Lemma 4.2; hence, when  $t$  is near of 0:

$$|1 - \varsigma_\theta(e^{it})| \approx |f_\theta[g_\theta(e^{it})]| \approx |t| [\log(1/|t|)]^\theta.$$

If we were allowed to apply Theorem 2.3, we would get that  $a_n(C_{\varsigma_\theta}) \lesssim 1/n^{\theta/4}$ , which would be in accordance with the fact that  $C_{\varsigma_\theta}$  is in the Schatten class  $S_p$  if and only if  $p > 4/\theta$ . However, condition (2.2) is not satisfied: by [9], equations (5.5) and (5.6), one has  $1 - |\varsigma_\theta(e^{it})| \approx |t|(\log 1/|t|)^{\theta-1}$ , whereas  $|1 - \varsigma_\theta(e^{it})| \approx |t|(\log 1/|t|)^\theta$ .



On the other hand, by the Lemma 4.2 again,  $g_\theta(r) \sim \varepsilon(1-r)/4$  as  $r$  tends to 1; hence, when  $r$  is near to 1:

$$1 - \varsigma_\theta(r) \approx (1-r)(\log 1/(1-r))^\theta,$$

so  $\varsigma_\theta$  is a real  $\omega$ -radial symbol with  $\omega(t) = t(\log 1/t)^\theta$ . Hence, we get from Theorem 3.2:

$$a_n(C_{\varsigma_\theta}) \gtrsim \frac{1}{n^{\theta/2}},$$

taking  $\sigma = 1/e$  in (3.2). However, this lower estimate is not the right one, since  $C_{\varsigma_\theta}$  is in  $S_p$  if and only if  $p > 4/\theta$ .

## 5 Contact points

It is well-known (and easy to prove) that for every compact composition operator  $C_\varphi: H^2 \rightarrow H^2$ , the set of contact points

$$E_\varphi = \{e^{i\theta}; |\varphi^*(e^{i\theta})| = 1\}$$

has Lebesgue measure 0. A natural question is: to what extent is this negligible set arbitrary? The following partial answer was given by E.A. Gallardo-Gutiérrez and M.J. González in [6].

**Theorem 5.1 (E.A. Gallardo-Gutiérrez and M.J. González)** *There is a compact composition operator  $C_\varphi$  on  $H^2$  such that the Hausdorff dimension of  $E_\varphi$  is one.*

This was generalized by O. El-Fallah, K. Kellay, M. Shabankhah, and H. Youssfi ([5], Theorem 3.1):

**Theorem 5.2 (O. El-Fallah, K. Kellay, M. Shabankhah, H. Youssfi)** *For every compact set  $K$  of measure 0 in  $\mathbb{T}$ , there exists a Schur function  $\varphi \in A(\mathbb{D})$ , the disk algebra, such that the associated composition operator  $C_\varphi$  is Hilbert-Schmidt on  $H^2$  and  $E_\varphi = K$ .*

As an application of our previous results, we shall extend these results, with a very simple proof. Our composition operator will not even be compact, or Hilbert-Schmidt, but in all Schatten classes  $S_p$ , and moreover its approximation numbers will be as small as possible.

**Theorem 5.3** *Let  $K$  be a Lebesgue-negligible compact set of the circle  $\mathbb{T}$ . Then, there exists a Schur function  $\psi \in A(\mathbb{D})$ , the disk algebra, such that  $E_\psi = K$ ,  $\psi(e^{i\theta}) = 1$  for all  $e^{i\theta} \in K$ , and:*

$$(5.1) \quad a_n(C_\psi) \leq a \exp(-bn/\log n).$$

*In particular,  $C_\psi \in \bigcap_{p>0} S_p$ .*

**Proof.** According to the Rudin-Carleson theorem ([2]), we can find  $\chi \in A(\mathbb{D})$  such that

$$\chi = 1 \text{ on } K \quad \text{and} \quad |\chi| < 1 \text{ on } \overline{\mathbb{D}} \setminus K.$$

Consider now the cusp map  $\varphi$ , defined in Section 4.1. One has  $\varphi \in A(\mathbb{D})$ ,  $\varphi(1) = 1$  and

$$a_n(C_\varphi) \leq a' \exp(-bn/\log n).$$

We now spread the point 1 by composing with the function  $\chi$ , which is equal to 1 on the whole of  $K$ . We check that the composed map  $\psi = \varphi \circ \chi$  has the required properties.

That  $\psi \in A(\mathbb{D})$  is clear. For  $z \in K$ , one has  $\psi(z) = \varphi(1) = 1$ , and for  $z \in \overline{\mathbb{D}} \setminus K$ , one has  $|\chi(z)| < 1$ ; hence  $|\psi(z)| < 1$ .

To finish, since  $C_\psi = C_\chi \circ C_\varphi$ , we have

$$a_n(C_\psi) \leq \|C_\chi\| a_n(C_\varphi) \leq a' \|C_\chi\| \exp(-bn/\log n) := \sigma_n,$$

proving the result (with  $a = a' \|C_\chi\|$ ), since clearly  $\sum_{n=1}^{\infty} \sigma_n^p < \infty$  for each  $p > 0$ .  $\square$

Actually, we can improve on the previous theorem by proving the following result. This result is optimal because if  $\|\psi\|_\infty = 1$ , we know (see [14], Theorem 3.4) that  $\liminf_{n \rightarrow \infty} [a_n(C_\psi)]^{1/n} = 1$ , so we cannot hope to get rid with the forthcoming vanishing sequence  $(\varepsilon_n)_n$ .

**Theorem 5.4** *Let  $K$  be a Lebesgue-negligible compact set of the circle  $\mathbb{T}$  and  $(\varepsilon_n)_n$  a sequence of positive real numbers with limit zero. Then, there exists a Schur function  $\varphi \in A(\mathbb{D})$  such that  $E_\varphi = K$ ,  $\varphi(e^{i\theta}) = 1$  for all  $e^{i\theta} \in K$ , and*

$$(5.2) \quad a_n(C_\varphi) \leq C \exp(-n\varepsilon_n),$$

where  $C$  is a positive constant.

This theorem is a straightforward consequence of the following lemma. Recall that the Carleson function of the Schur function  $\psi: \mathbb{D} \rightarrow \mathbb{D}$  is defined by:

$$\rho_\psi(h) = \sup_{|\xi|=1} m(\{t \in \mathbb{T}; |\psi(e^{it})| \geq 1-h \text{ and } |\arg(\psi(e^{it})\bar{\xi})| \leq \pi h\}).$$

**Lemma 5.5** *Let  $\delta$  be a nondecreasing positive function on  $(0, 1]$  tending to 0 as  $h \rightarrow 0$ . Then, there exists a Schur function  $\psi \in A(\mathbb{D})$  such that  $\psi(1) = 1$ ,  $|\psi(\xi)| < 1$  for  $\xi \in \mathbb{T} \setminus \{1\}$ , and such that  $\rho_\psi(h) \leq \delta(h)$ , for  $h > 0$  small enough.*

Once we have the lemma, in view of the upper bound in [14], Theorem 5.1, for approximation numbers, we can adjust the function  $\delta$  so as to have  $a_n(C_\psi) \leq Ke^{-n\varepsilon_n}$ . Then, we compose  $\psi$  with a peaking function  $\chi$  as in the previous section and the map  $\varphi = \psi \circ \chi$  fulfills the requirements of Theorem 5.4, with  $C = K\|\chi\|$ .  $\square$

**Proof of Lemma 5.5.** We use a slight modification of the map  $g$  constructed in [10], pages 66–67. Instead of taking a conformal map from  $\mathbb{D}$  to the domain used in [10], we modify this domain by limiting it to the right-hand side (by, say, a semicircle), as on the Figure 2. Let  $\Omega$  this domain. This domain is limited by the two hyperbolas  $y = 1/x$  and  $y = (1/x) + 4\pi$ . The limiting semicircle is chosen in order that  $\Im w \geq 1$  for  $w \in \Omega$ . The lower part of the “saw-teeth” have an imaginary part equal to  $4\pi n$ . If  $a \in \Omega$  is fixed and  $\Omega_n$  is the part of the domain  $\Omega$  such that  $\Im w < 4\pi n$ , the horizontal sizes of the “saw-teeth” are chosen in order that the harmonic measure  $\omega_\Omega(a, \partial\Omega \setminus \partial\Omega_n)$  is  $\leq \delta_n := \delta(1/16\pi(n+1))$ . Note that  $\partial\Omega \setminus \partial\Omega_n \supseteq \{w \in \partial\Omega; \Im w > 4\pi n\}$  (see [10], Lemma 4.2).

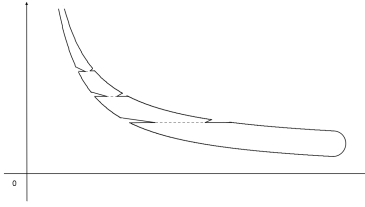


Figure 2: *Domain  $\Omega$*

By Carathéodory-Osgood’s Theorem (see [20], Theorem IX.4.9), there is a unique homeomorphism  $g$  from  $\overline{\mathbb{D}}$  onto  $\overline{\Omega} \cup \{\infty\}$  which maps conformally  $\mathbb{D}$  onto  $\Omega$  and such that  $g(0) = a$  and  $g(1) = \infty$  (we may choose these two values because if  $h: \overline{\mathbb{D}} \rightarrow \overline{\Omega} \cup \{\infty\}$  is such a map, and  $u$  is the automorphism of  $\overline{\mathbb{D}}$  such that  $u(0) = h^{-1}(a)$  and  $u(1) = h^{-1}(\infty)$ , then  $g = h \circ u$  suits – alternatively, having chosen  $h(0) = a$ , then, if  $h(e^{i\theta_0}) = \infty$ , we take  $g(z) = h(e^{i\theta_0}z)$ ).

We define  $\psi = (g - i)/(g + i)$ . Then  $\psi: \mathbb{D} \rightarrow \mathbb{D}$  is a Schur function and  $\psi \in A(\mathbb{D})$ . Moreover, since the domain  $\Omega$  is bounded horizontally, we have  $\psi(1) = 1$  and  $|\psi(e^{it})| < 1$  for  $0 < t < 2\pi$ .

Now,  $\rho_\psi(h) \leq m(\{z \in \mathbb{T}; |\psi(z)| > 1 - h\})$ . Writing  $g = u + iv$ , one has:

$$|\psi|^2 = \frac{u^2 + (v-1)^2}{u^2 + (v+1)^2} = 1 - \frac{4v}{u^2 + (v+1)^2}.$$

Since  $(1-h)^2 \geq 1-2h$ , the condition  $|\psi(z)| > 1-h$  implies that  $\frac{2v}{u^2+(v+1)^2} \leq h$ . But  $0 < u \leq 1+2\pi \leq 8$  and  $(v+1)^2 \leq 4v^2$  (since  $v \geq 1$ ); we get hence  $\frac{v}{32+2v^2} \leq h$ , or  $\frac{32}{v} + 2v \geq \frac{1}{h}$ . Using again the fact that  $v \geq 1$ , one obtains  $2v \geq \frac{1}{h} - 32$ , and hence  $2v \geq \frac{1}{2h}$  for  $0 < h \leq 1/64$ . Therefore, for  $0 < h \leq 1/64$ ,

$$\rho_\psi(h) \leq m(\{z \in \mathbb{T}; \Im \psi(z) \geq 1/4h\}).$$

Now, for  $n \geq 2$  and  $1/16\pi(n+1) \leq h < 1/16\pi n$ , one gets hence:

$$\begin{aligned} \rho_\psi(h) &\leq m(\{z \in \mathbb{T}; \Im \psi(z) > 4\pi n\}) \\ &= \omega_\Omega(a, \{w \in \partial\Omega; \Im w > 4\pi n\}) \leq \omega_\Omega(a, \partial\Omega \setminus \partial\Omega_n) \leq \delta_n \leq \delta(h), \end{aligned}$$

proving Lemma 5.5. □

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Daniel Li, Univ Lille Nord de France, U-Artois,  
 Laboratoire de Mathématiques de Lens EA 2462 & Fédération CNRS Nord-Pas-de-Calais FR 2956,  
 Faculté des Sciences Jean Perrin, Rue Jean Souvraz, S.P. 18,  
 F-62 300 LENS, FRANCE  
 daniel.li@euler.univ-artois.fr

Hervé Queffélec, Univ Lille Nord de France,  
 USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524,  
 Fédération CNRS Nord-Pas-de-Calais FR 2956,  
 F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE  
 Herve.Queffelec@univ-lille1.fr

Luis Rodríguez-Piazza, Universidad de Sevilla,  
 Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS,  
 Apartado de Correos 1160,  
 41 080 SEVILLA, SPAIN  
 piazza@us.es