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# GEOMETRY OF OPTIMAL CONTROL FOR CONTROL-AFFINE SYSTEMS

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Abstract. Motivated by the ubiquity of control-affine systems in optimal control theory, we investigate the geometry of point-affine control systems with metric structures in dimensions two and three. We compute local isometric invariants for point-affine distributions of constant type with metric structures for systems with 2 states and 1 control and systems with 3 states and 1 control, and use Pontryagin's Maximum Principle to find geodesic trajectories for homogeneous examples. Even in these low dimensions, the behavior of these systems is surprisingly rich and varied.

## 1. Introduction

In [1], we investigated the local structure of *point-affine distributions.* A rank-$s$ point-affine distribution on an $n$-dimensional manifold $M$ is a subbundle $\mathcal{F}$ of the tangent bundle $TM$ such that, for each $x \in M$, the fiber $\mathcal{F}_x = T_xM \cap \mathcal{F}$ is an $s$-dimensional affine subspace of $T_xM$ that contains a distinguished point. In local coordinates, the points of $\mathcal{F}$ are parametrized by $s+1$ pointwise independent smooth vector fields $v_0(x), v_1(x), \ldots, v_s(x)$ for which $\mathcal{F}_x = v_0(x) + \text{span}\,(v_1(x), \ldots, v_s(x))$ and $v_0(x)$ is the distinguished point in $\mathcal{F}_x$.

Our interest in point-affine distributions is motivated by a family of ordinary differential equations that occurs in control theory: the control-affine systems. A control system is a system of underdetermined ODEs

$$\dot{x} = f(x, u),$$

where $x \in M$ and $u$ takes values in an $s$-dimensional manifold $\mathcal{U}$. The system is *control-affine* if the right-hand side is affine linear in the control variables $u$; i.e., if the system locally has the form

$$\dot{x}(t) = v_0(x) + \sum_{i=1}^{s} v_i(x)u^i(t), \tag{1.1}$$

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where the controls $u^1, \ldots, u^s$ appear linearly in the right hand side and $v_0, \ldots, v_s$ are $s+1$ independent vector fields. Replacing $v_0$, which is called the *drift* vector field, with a linear combination of $v_1, \ldots, v_s$ added to $v_0$ would yield an equivalent system of differential equations. In many instances, however, there is a distinguished null value for the controls (imagine turning off all motors), and this null value determines a distinguished drift vector field. In these instances, we always choose $v_0$ to be the distinguished drift vector field. Consequently, the null value for the controls will be

$$u^1 = \cdots = u^s = 0.$$

While the control-affine systems (1.1) may appear to be rather special, these systems are ubiquitous. In fact, any control system whatsoever becomes control-affine after a single prolongation, so these systems actually encompass all control systems, at the cost of increasing the number of state variables.

In [1] we studied local diffeomorphism invariants for these point-affine structures. A local equivalence for two point affine structures is a local diffeomorphism of $M$ whose derivative maps one distinguished drift vector field to the other, and maps one affine sub-bundle to the other (see [1] for precise definitions). With this notion of local equivalence, we were able to determine local normal forms for rank-1 point-affine structures when the manifold $M$ had dimension 2 or 3. In some cases the normal forms are parametrized by arbitrary functions.

The current paper seeks to refine the previous results by adding a metric structure to the point-affine structure. We do so by introducing a positive definite quadratic cost functional $Q : \mathcal{F} \to \mathbb{R}$. In local coordinates, where

$$w = v_0(x) + \sum_{i=1}^{s} v_i(x) u^i \in \mathcal{F}_x,$$

we will define

$$Q_x(w) = \sum g_{ij}(x) u^i u^j,$$

where the matrix $(g_{ij}(x))$ is positive definite and the components are smooth functions of $x$. This is a natural extension of the well-studied notion of a sub-Riemannian metric on a linear distribution, which represents a quadratic cost functional for a driftless system. (See, e.g., [3], [4], [5].)

With the added metric structure, we refine our notion of local point-affine equivalence to that of a local *point-affine isometry*. A local point-affine isometry is a local point-affine equivalence that additionally preserves the quadratic cost functional.

Let $\gamma(t) = x(t)$ be a trajectory for (1.1). The added metric structure allows us to assign the following energy cost functional to $\gamma(t)$:

$$E(\gamma) = \frac{1}{2} \int_\gamma Q_{x(t)}\big(\dot{x}(t)\big) dt. \tag{1.2}$$

Naturally associated to (1.2) is the *optimal control problem* of finding trajectories of (1.1) that minimize (1.2). We will use Pontryagin's maximum principle to find an ODE system on $T^*M$ with the property that any minimal cost trajectory for (1.1) must be the projection of some solution for the ODE system on $T^*M$.

We shall use the normal forms from [1] as starting points. In each case we will add a metric structure to the point-affine structure. Even in these low-dimensional cases, the analysis can be quite involved. To simplify matters, we will narrow our focus to homogeneous examples; i.e., examples that admit a symmetry group which acts transitively on $M$. Despite the low dimensions ($n = 2$ or 3 and $s = 1$) and the simplifying assumption of homogeneity, we will see that these structures exhibit surprisingly rich and varied behavior.

## 2. 2 States, 1 Control

We first examine optimal control for point-affine systems with 2 states and 1 control. In [1], we found two local normal forms under point-affine equivalence:

(1) **Case 1.1:** $\mathcal{F} = \frac{\partial}{\partial x^1} + \operatorname{span}\left(\frac{\partial}{\partial x^2}\right)$. The dual coframing

$$\eta^1 = dx^1, \qquad \eta^2 = \lambda dx^2$$

to the framing

$$v_1 = \frac{\partial}{\partial x^1}, \qquad v_2 = \frac{1}{\lambda}\frac{\partial}{\partial x^2}$$

(well-defined up to scaling in $v_2$) had structure equations

$$\begin{aligned} d\eta^1 &= 0 \\ d\eta^2 &\equiv 0 \qquad \text{mod } \eta^2. \end{aligned}$$

(2) **Case 1.2:** $\mathcal{F} = x^2\left(\frac{\partial}{\partial x^1} + J\frac{\partial}{\partial x^2}\right) + \operatorname{span}\left(\frac{\partial}{\partial x^2}\right)$ We found a canonical coframing $(\eta^1, \eta^2)$ with structure equations

$$\begin{aligned} d\eta^1 &= \eta^1 \wedge \eta^2 \\ d\eta^2 &= T^2_{12}\eta^1 \wedge \eta^2. \end{aligned}$$

We chose local coordinates so that $\eta^1 = \frac{1}{x^2}dx^1$. The first structure equation then implies that

$$\eta^2 = \frac{1}{x^2}\left(dx^2 - J dx^1\right)$$

for some function $J$, and

$$T^2_{12} = x^2\frac{\partial J}{\partial x^2} - J.$$

The dual framing is

$$v_1 = x^2\left(\frac{\partial}{\partial x^1} + J\frac{\partial}{\partial x^2}\right), \qquad v_2 = x^2\frac{\partial}{\partial x^2}.$$

Now we add a cost functional in each case and compute homogeneous examples. The assumption of homogeneity is equivalent to the condition that all structure functions $T^i_{jk}$ appearing in the structure equations for a canonical coframing are constants. (See [2] for details.)

2.1. **Case 1.1.** $\mathcal{F} = \frac{\partial}{\partial x^1} + \text{span}\left(\frac{\partial}{\partial x^2}\right)$.

The corresponding control system is

$$\begin{aligned}\dot{x}^1 &= 1\\ \dot{x}^2 &= u.\end{aligned} \tag{2.1}$$

A cost functional (1.2) for this system may be written as

$$Q(\dot{x}) = \frac{1}{2}G(x)u^2, \quad G(x) > 0.$$

An adapted framing $(v_1, v_2)$ on $\mathbb{R}^2$ may be defined by choosing $v_1$ to be the drift vector field $v_1 = \frac{\partial}{\partial x^1}$ and $v_2$ to be the unit vector

$$v_2 = \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^2}$$

in the linear distribution $L_{\mathcal{F}}$ obtained by translating $\mathcal{F}$ to the zero section. This framing is canonically defined up to the sign of $v_2$. The dual coframing $(\eta^1, \eta^2)$ to this framing is given by

$$\eta^1 = dx^1, \qquad \eta^2 = \sqrt{G}dx^2, \tag{2.2}$$

with structure equations

$$\begin{aligned}d\eta^1 &= 0\\ d\eta^2 &= \frac{G_{x^1}}{2G}\eta^1 \wedge \eta^2.\end{aligned} \tag{2.3}$$

This coframing is canonically defined by the affine distribution and the cost functional, *independent* of the choice of local coordinates $(x^1, x^2)$ on $\mathbb{R}^2$.

Local coordinates for which this coframing is described by (2.2) are determined only up to transformations of the form

$$x^1 = \tilde{x}^1 + a, \qquad x^2 = \phi(\tilde{x}^2). \tag{2.4}$$

From (2.3), the structure is homogeneous if and only if $\frac{G_{x^1}}{2G}$ is equal to a constant $c_1$. In this case, we have

$$G(x^1, x^2) = G_0(x^2)e^{2c_1x^1}.$$

Since this implies that

$$\eta^2 = e^{c_1x^1}\sqrt{G_0(x^2)}dx^2,$$

we can make a change of variables

$$\tilde{x}^1 = x^1, \qquad \tilde{x}^2 = \int \sqrt{G_0(x^2)}dx^2$$

(note that this transformation is of the form (2.4)) to arrange that

$$\eta^2 = e^{c_1 \tilde{x}^1} d\tilde{x}^2,$$

which implies that $\tilde{G}(\tilde{x}^1, \tilde{x}^2) = e^{2c_1 \tilde{x}^1}$.

Now consider the problem of computing optimal trajectories for (2.1). The Hamiltonian for the energy functional (1.2) is

$$\begin{aligned} \mathcal{H} &= p_1 \dot{x}^1 + p_2 \dot{x}^2 - Q(\dot{x}) \\ &= p_1 + p_2 u - \frac{1}{2} G(x) u^2 \\ &= p_1 + p_2 u - \frac{1}{2} e^{2c_1 x^1} u^2. \end{aligned}$$

By Pontryagin's maximum principle, a necessary condition for optimal trajectories is that the control function $u(t)$ is chosen so as to maximize $\mathcal{H}$. Since $u$ is unrestricted and $\frac{1}{2} e^{2c_1 x^1} > 0$, $\max_u \mathcal{H}$ occurs when

$$0 = \frac{\partial \mathcal{H}}{\partial u} = p_2 - e^{2c_1 x^1} u,$$

that is, when

$$u = p_2 e^{-2c_1 x^1}. \tag{2.5}$$

So along an optimal trajectory, we have

$$\begin{aligned} \mathcal{H} &= p_1 + (p_2)^2 e^{-2c_1 x^1} - \frac{1}{2} (p_2)^2 e^{-2c_1 x^1} \\ &= p_1 + \frac{1}{2} (p_2)^2 e^{-2c_1 x^1}. \end{aligned}$$

Moreover, $\mathcal{H}$ is constant along trajectories, and so we have

$$p_1 + \frac{1}{2} (p_2)^2 e^{-2c_1 x^1} = k. \tag{2.6}$$

Hamilton's equations

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x}$$

take the form

$$\begin{aligned} \dot{x}^1 &= 1 \\ \dot{x}^2 &= u \\ \dot{p}_1 &= c_1 e^{-2c_1 x^1} (p_2)^2 \\ \dot{p}_2 &= 0. \end{aligned} \tag{2.7}$$

Now, using (2.5), the second equation in (2.7) becomes

$$\dot{x}^2 = p_2 e^{-2c_1 x^1},$$

so optimal trajectories are solutions of the system

$$\begin{aligned} \dot{x}^1 &= 1 \\ \dot{x}^2 &= p_2 e^{-2c_1 x^1} \\ \dot{p}_1 &= c_1 (p_2)^2 e^{-2c_1 x^1} \\ \dot{p}_2 &= 0. \end{aligned} \tag{2.8}$$

The fourth equation in (2.8) implies that $p_2$ is constant; say, $p_2 = c_2$. Then optimal trajectories are solutions of the system

$$\begin{aligned} \dot{x}^1 &= 1 \\ \dot{x}^2 &= c_2 e^{-2c_1 x^1}. \end{aligned} \tag{2.9}$$

This system can be integrated explicitly:

- If $c_1 = 0$, then the solutions are:

$$\begin{aligned} x^1 &= t \\ x^2 &= c_2 t + c_3. \end{aligned} \tag{2.10}$$

These solutions correspond to the family of curves

$$x^2 = c_2 x^1 + c_3$$

in the $(x^1, x^2)$-plane. Thus, the set of critical curves consists of all non-vertical straight lines in the $(x^1, x^2)$ plane, oriented in the direction of increasing $x^1$. Sample optimal trajectories are shown in Figure 1.

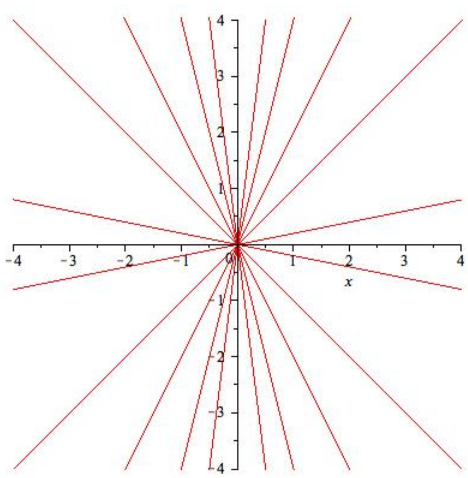


FIGURE 1.

- If $c_1 \neq 0$, then the solutions are:

$$x^1 = t$$
$$x^2 = -\frac{1}{2c_1} c_2 e^{-2c_1 t}. \tag{2.11}$$

These solutions correspond to the family of curves

$$x^2 = -\frac{1}{2c_1} c_2 e^{-2c_1 x^1}$$

in the $(x^1, x^2)$-plane. Thus, the set of critical curves consists of a family of exponential curves in the $(x^1, x^2)$ plane, oriented in the direction of increasing $x^1$. Sample optimal trajectories are shown in Figure 2.

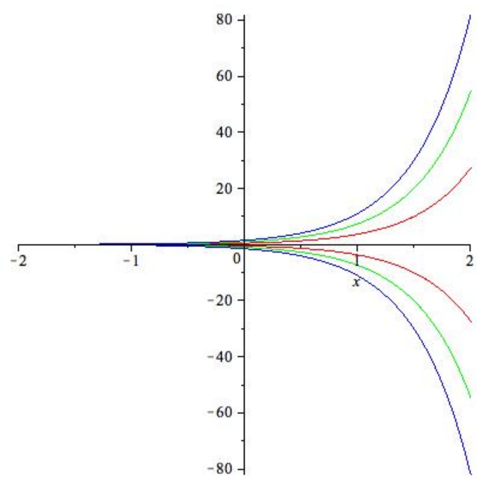


FIGURE 2.

2.2. **Case 1.2.** $\mathcal{F} = x^2\left(\frac{\partial}{\partial x^1} + J\frac{\partial}{\partial x^2}\right) + \text{span}\left(\frac{\partial}{\partial x^2}\right)$. Since the canonical framing for point-affine equivalence in this case had $v_2 = x^2\frac{\partial}{\partial x^2}$, we will write the corresponding control system as

$$\dot{x}^1 = x^2$$
$$\dot{x}^2 = x^2 J + x^2 u. \tag{2.12}$$

Note that our assumption that (2.12) is strictly affine requires that we restrict to the open set where $x^2 \neq 0$.

A cost functional (1.2) for this system may be written as

$$Q(\dot{x}) = \frac{1}{2}G(x)u^2, \quad G(x) > 0.$$

An adapted framing $(v_1, v_2)$ on $\mathbb{R}^2$ may be defined by choosing $v_1$ to be the drift vector field

$$v_1 = x^2\left(\frac{\partial}{\partial x^2} + J\frac{\partial}{\partial x^2}\right)$$

and $v_2$ to be the unit vector

$$v_2 = \frac{x^2}{\sqrt{G}}\frac{\partial}{\partial x^2}$$

in $L_{\mathcal{F}}$. This framing is canonically defined up to the sign of $v_2$. The dual coframing $(\eta^1, \eta^2)$ to this framing is given by

$$\eta^1 = \frac{1}{x^2}dx^1, \qquad \eta^2 = \frac{\sqrt{G}}{x^2}\left(dx^2 - J dx^1\right),$$

with structure equations

$$\begin{aligned} d\eta^1 &= \frac{1}{\sqrt{G}}\eta^1 \wedge \eta^2 \\ d\eta^2 &= \left[x^2\left(\sqrt{G}\right)_{x^1} + x^2 J\left(\sqrt{G}\right)_{x^2} + \sqrt{G}\left(x^2 J_{x^2} - J\right)\right]\eta^1 \wedge \eta^2. \end{aligned} \tag{2.13}$$

Again, this coframing is canonically defined, independent of the choice of local coordinates on $\mathbb{R}^2$.

Local coordinates for which $\eta^2 = \frac{1}{x^2}dx^1$ are determined up to transformations of the form

$$x^1 = \phi(\tilde{x}^1), \qquad x^2 = \tilde{x}^2\phi'(\tilde{x}^1). \tag{2.14}$$

From (2.13), the structure is homogeneous if and only if $G$ is equal to a constant $g_0 > 0$, and the function

$$x^2 J_{x^2} - J$$

is constant, which is the case if and only if

$$J = x^2 J_1(x^1) + j_0$$

where $j_0$ is constant.

Under a transformation of the form (2.14), we have

$$\begin{aligned} \eta^2 &= \frac{\sqrt{g_0}}{x^2}\left(dx^2 - J(x^1, x^2)dx^1\right) \\ &= \frac{\sqrt{g_0}}{\tilde{x}^2\phi'(\tilde{x}^1)}\left(\left(\phi'(\tilde{x}^1)d\tilde{x}^2 + \tilde{x}^2\phi''(\tilde{x}^2)d\tilde{x}^1\right) - J(\phi(\tilde{x}^1), \tilde{x}^2\phi'(\tilde{x}^1))\phi'(\tilde{x}^1)d\tilde{x}^1\right) \\ &= \frac{\sqrt{g_0}}{\tilde{x}^2}\left(d\tilde{x}^2 - \left(J\left(\phi(\tilde{x}^1), \tilde{x}^2\phi'(\tilde{x}^1)\right) - \tilde{x}^2\frac{\phi''(\tilde{x}^1)}{\phi'(\tilde{x}^1)}\right)d\tilde{x}^1\right). \end{aligned}$$

Therefore, the function $\tilde{J}(\tilde{x}^1, \tilde{x}^2)$ corresponding to the new coordinates $(\tilde{x}^1, \tilde{x}^2)$ is defined by the condition that

$$\tilde{J}(\tilde{x}^1, \tilde{x}^2) = J\big(\phi(\tilde{x}^1), \tilde{x}^2\phi'(\tilde{x}^1)\big) - \tilde{x}^2 \frac{\phi''(\tilde{x}^1)}{\phi'(\tilde{x}^1)}.$$

So in the homogeneous case, where

$$J = x^2 J_1(x^1) + j_0,$$

we can make such a change of variables to arrange that $\tilde{J}_1(\tilde{x}^1) = 0$, and therefore $\tilde{J} = j_0$. Moreover, these coordinates are unique up to an affine transformation

$$x^1 = a\tilde{x}^1 + b, \qquad x^2 = a\tilde{x}^2.$$

Now consider the problem of computing optimal trajectories for (2.12). The Hamiltonian for the energy functional (1.2) is

$$\begin{aligned}\mathcal{H} &= p_1\dot{x}^1 + p_2\dot{x}^2 - Q(\dot{x}) \\ &= p_1x^2 + p_2x^2\big(J(x) + u\big) - \frac{1}{2}G(x)u^2 \\ &= p_1x^2 + p_2x^2\big(j_0 + u\big) - \frac{1}{2}g_0u^2.\end{aligned}$$

Setting $\frac{\partial \mathcal{H}}{\partial u} = 0$, as required by Pontryagin's maximum principle, provides the necessary condition

$$u = \frac{p_2x^2}{g_0} \tag{2.15}$$

for an optimal trajectory. So along an optimal trajectory, we have

$$\mathcal{H} = p_1x^2 + p_2x^2j_0 + \frac{1}{2g_0}\big(p_2x^2\big)^2.$$

Hamilton's equations take the form

$$\begin{aligned}\dot{x}^1 &= x^2 \\ \dot{x}^2 &= x^2j_0 + \frac{p_2\big(x^2\big)^2}{g_0} \\ \dot{p}_1 &= 0 \\ \dot{p}_2 &= -p_1 - p_2j_0 - \frac{(p_2)^2x^2}{g_0}.\end{aligned} \tag{2.16}$$

It is straightforward to show that the three functions

- $\mathcal{H} = p_1x^2 + p_2x^2j_0 + \frac{1}{2g_0}(p_2x^2)^2$
- $p_1$
- $p_1x^1 + p_2x^2$.

are first integrals for this system. This observation alone would in principle allow us to construct unparametrized solution curves for the system. But in fact, we can solve this system fully, as follows.

The third equation in (2.16) implies that $p_1$ is constant; say, $p_1 = c_1$. Now observe that

$$\begin{aligned}\frac{d}{dt}(p_2x^2) &= p_2\dot{x}^2 + x^2\dot{p}_2 \\ &= p_2x^2\left(j_0 + \frac{p_2x^2}{g_0}\right) + x^2\left(-c_1 - p_2j_0 - \frac{(p_2)^2x^2}{g_0}\right) \qquad (2.17)\\ &= -c_1x^2.\end{aligned}$$

If $c_1 = 0$, then (2.17) implies that $p_2x^2$ is equal to a constant $k_2$, and so

$$\begin{aligned}\dot{x}^2 &= x^2\left(j_0 + \frac{k_2}{g_0}\right) \\ &= c_2x^2.\end{aligned}$$

There are two subcases, depending on the value of $c_2$.

- If $c_2 = 0$, then
$$x^2 = c_3,$$
and since $\dot{x}^1 = x^2$, we have
$$x^1 = c_3t + c_4.$$
These solutions correspond to the family of curves
$$x^2 = c_3$$
in the $(x^1, x^2)$-plane. These curves are all horizontal lines, oriented in the direction of increasing $x^1$ when $x^2 > 0$ and decreasing $x^1$ when $x^2 < 0$. Sample optimal trajectories are shown in Figure 3.

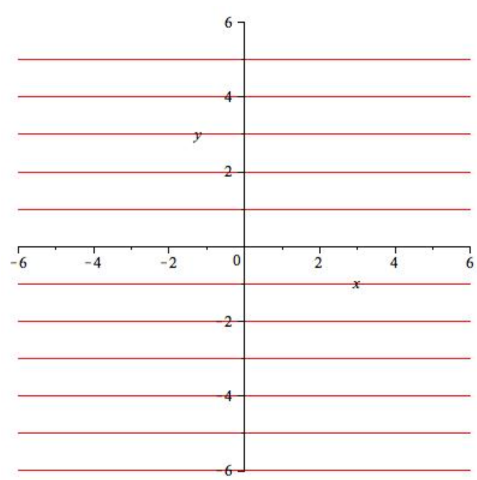


FIGURE 3.

- If $c_2 \neq 0$, then

$$x^2 = c_3 e^{c_2 t}, \tag{2.18}$$

and since $\dot{x}^1 = x^2$, we have

$$x^1 = \frac{c_3}{c_2} e^{c_2 t} + c_4. \tag{2.19}$$

These solutions correspond to the family of curves

$$x^2 = c_2(x^1 - c_4)$$

in the $(x^1, x^2)$-plane. These curves are all non-vertical, non-horizontal lines, oriented in the direction of increasing $x^1$ when $x^2 > 0$ and decreasing $x^1$ when $x^2 < 0$. Sample optimal trajectories are shown in Figure 4.

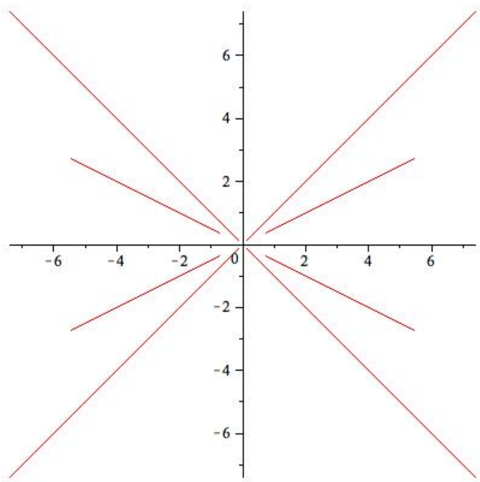


FIGURE 4.

On the other hand, if $c_1 \neq 0$, then

$$\begin{aligned} \frac{d^2}{dt^2}(p_2 x^2) &= -c_1 \dot{x}^2 \\ &= -c_1 x^2 \left( j_0 + \frac{p_2 x^2}{g_0} \right) \\ &= \frac{d}{dt}(p_2 x^2) \left( j_0 + \frac{p_2 x^2}{g_0} \right). \end{aligned} \tag{2.20}$$

Integrating this equation once gives

$$\frac{d}{dt}(p_2 x^2) = j_0(p_2 x^2) + \frac{(p_2 x^2)^2}{2 g_0} + c_2. \tag{2.21}$$

There are three subcases, depending on the value of $k = g_0(j_0^2 g_0 - 2c_2)$.

- If $k = 0$, then the solution to (2.21) is

$$p_2 x^2 = -\frac{g_0(2 + j_0(t + c_3))}{t + c_3},$$

and from equation (2.17),

$$x^2 = -\frac{1}{c_1}\frac{d}{dt}(p_2 x^2) = -\frac{2g_0}{c_1(t + c_3)^2}.$$

Then since $\dot{x}^1 = x^2 = -\frac{1}{c_1}\frac{d}{dt}(p_2 x^2)$, we have

$$\begin{aligned} x^1 &= -\frac{1}{c_1}(p_2 x^2) + c_4 \\ &= \frac{g_0(2 + j_0(t + c_3))}{c_1(t + c_3)} + c_4. \end{aligned}$$

These solutions correspond to the family of curves

$$x^2 = -\frac{1}{2c_1 g_0}\left(c_1 x^1 - (j_0 g_0 + c_1 c_4)\right)^2$$

in the $(x^1, x^2)$-plane. These curves are all parabolas with vertex lying on the $x^1$-axis. Since we must have $x^2 \neq 0$, the set of critical curves consists of all branches of parabolas with vertex on the $x^2$-axis, oriented in the direction of increasing $x^1$ when $x^2 > 0$ and decreasing $x^1$ when $x^2 < 0$. Sample optimal trajectories are shown in Figure 5.

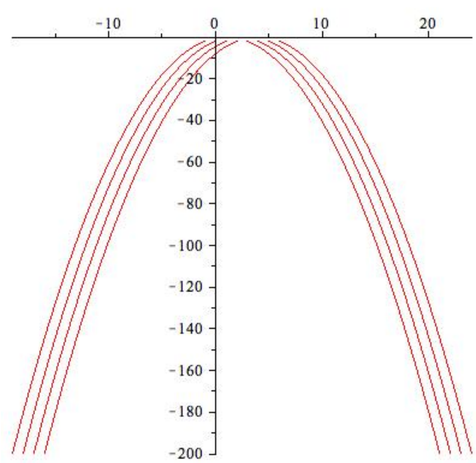


Figure 5.

- If $k > 0$, then the solution to (2.21) is

$$p_2 x^2 = -\sqrt{k}\tanh\left(\frac{\sqrt{k}}{2g_0}(t + c_3)\right) - j_0 g_0,$$

and from equation (2.17),

$$x^2 = -\frac{1}{c_1}\frac{d}{dt}(p_2x^2) = \frac{k}{2c_1g_0}\operatorname{sech}^2\left(\frac{\sqrt{k}}{2g_0}(t+c_3)\right).$$

Then since $\dot{x}^1 = x^2 = -\frac{1}{c_1}\frac{d}{dt}(p_2x^2)$, we have

$$\begin{aligned} x^1 &= -\frac{1}{c_1}(p_2x^2) + c_4 \\ &= \frac{1}{c_1}\left(\sqrt{k}\tanh\left(\frac{\sqrt{k}}{2g_0}(t+c_3)\right) + j_0g_0\right) + c_4. \end{aligned}$$

These solutions correspond to the family of curves

$$x^2 = -\frac{1}{2c_1g_0}\left[\left(c_1x^1 - (j_0g_0 + c_1c_4)\right)^2 - k\right]$$

in the $(x^1, x^2)$-plane. These curves are all parabolas opening towards the $x^1$-axis. Thus the set of critical curves consists of parabolic arcs opening towards the $x^1$-axis, approaching the axis as $t \to \pm\infty$, and oriented in the direction of increasing $x^1$ when $x^2 > 0$ and decreasing $x^1$ when $x^2 < 0$. Sample optimal trajectories are shown in Figure 6.

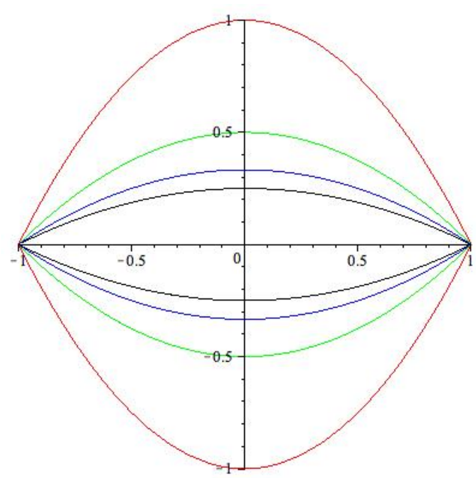


FIGURE 6.

- If $k < 0$, then the solution to (2.21) is

$$p_2x^2 = \sqrt{-k}\tan\left(\frac{\sqrt{-k}}{2g_0}(t+c_3)\right) - j_0g_0,$$

and from equation (2.17),

$$x^2 = -\frac{1}{c_1}\frac{d}{dt}(p_2x^2) = \frac{k}{2c_1g_0}\sec^2\left(\frac{\sqrt{-k}}{2g_0}(t+c_3)\right).$$

Then since $\dot{x}^1 = x^2 = -\frac{1}{c_1}\frac{d}{dt}(p_2 x^2)$, we have

$$\begin{aligned} x^1 &= -\frac{1}{c_1}(p_2 x^2) + c_4 \\ &= -\frac{1}{c_1}\left(\sqrt{-k}\tan\left(\frac{\sqrt{-k}}{2g_0}(t+c_3)\right) - j_0 g_0\right) + c_4. \end{aligned}$$

These solutions correspond to the family of curves

$$x^2 = -\frac{1}{2c_1 g_0}\left[\left(c_1 x^1 - (j_0 g_0 + c_1 c_4)\right)^2 - k\right]$$

in the $(x^1, x^2)$-plane. These curves are all parabolas opening away from the $x^1$-axis. Thus the set of critical curves consists of parabolic arcs opening away from the $x^1$-axis, becoming unbounded in finite time, and oriented in the direction of increasing $x^1$ when $x^2 > 0$ and decreasing $x^1$ when $x^2 < 0$. Sample optimal trajectories are shown in Figure 7.

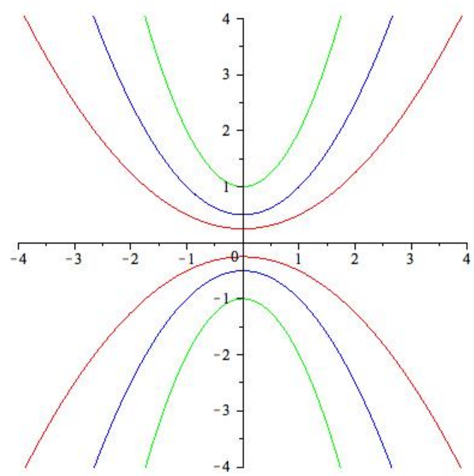


Figure 7.

## 3. 3 States, 1 Control

In [1], we found three local normal forms under point-affine equivalence:

(1) **Case 2.1:** $\mathcal{F} = \left(\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + J\frac{\partial}{\partial x^3}\right) + \text{span}\left(\frac{\partial}{\partial x^3}\right)$. The dual coframing

$$\begin{aligned} \eta^1 &= dx^1, \\ \eta^2 &= dx^3 - J\,dx^1 - \frac{1}{2}J_{x^3}(dx^2 - x^3\,dx^1), \\ \eta^3 &= dx^2 - x^3\,dx^1 \end{aligned}$$

to the framing

$$\begin{aligned} v_1 &= \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + J \frac{\partial}{\partial x^3}, \\ v_2 &= \frac{\partial}{\partial x^3}, \\ v_3 &= -[v_1, v_2] = \frac{\partial}{\partial x^2} + \frac{1}{2} J_{x^3} \frac{\partial}{\partial x^3} \end{aligned}$$

(well-defined up to dilation in the $(v_2, v_3)$-plane) had structure equations

$$\begin{aligned} d\eta^1 &= 0 \\ d\eta^2 &\equiv T^2_{13} \eta^1 \wedge \eta^3 && \mod \eta^2 \\ d\eta^3 &= \eta^1 \wedge \eta^2 && \mod \eta^3. \end{aligned}$$

(2) **Case 2.2:** $\mathcal{F} = \left(x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + J\left(x^2 \frac{\partial}{\partial x^3}\right)\right) + \text{span}\left(\frac{\partial}{\partial x^3}\right)$. We found a canonical coframing $\eta^1, \eta^2, \eta^3$ with structure equations

$$\begin{aligned} d\eta^1 &= \eta^1 \wedge \eta^3 \\ d\eta^2 &= T^2_{12} \eta^1 \wedge \eta^2 + T^2_{13} \eta^1 \wedge \eta^3 + T^2_{23} \eta^2 \wedge \eta^3 \\ d\eta^3 &= \eta^1 \wedge \eta^2 + T^2_{12} \eta^1 \wedge \eta^3. \end{aligned}$$

Based on the first structure equation, we chose partial local coordinates $(x^1, x^2)$ so that $\eta^1 = \frac{1}{x^2} dx^1$. The first structure equation then implies that there exists a third independent local coordinate function $x^3$ such that

$$\eta^3 = \frac{1}{x^2} dx^2 - \frac{x^3}{(x^2)^2} dx^1.$$

The third structure equation then implies that

$$\eta^2 = \frac{1}{x^2} dx^3 - \frac{1}{x^2} J\, dx^1 - \frac{1}{2}\left(J_{x^3} + \left(\frac{x^3}{x^2}\right)^2\right)\left(dx^2 - \frac{x^3}{x^2} dx^1\right)$$

for some function $J$. The dual framing to the coframing $(\eta^1, \eta^2, \eta^3)$ is

$$\begin{aligned} v_1 &= x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + J\left(x^2 \frac{\partial}{\partial x^3}\right), \\ v_2 &= x^2 \frac{\partial}{\partial x^3}, \\ v_3 &= -[v_1, v_2] = x^2 \frac{\partial}{\partial x^2} + \frac{1}{2}\left((x^2)^2 J_{x^3} + x^3\right) \frac{\partial}{\partial x^3}. \end{aligned}$$

(3) **Case 2.3:**

$\mathcal{F} = \left(\frac{\partial}{\partial x^1} + J\left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H \frac{\partial}{\partial x^3}\right)\right) + \text{span}\left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H \frac{\partial}{\partial x^3}\right),$

where $\frac{\partial H}{\partial x^1} \neq 0$. We found a canonical coframing $\eta^1, \eta^2, \eta^3$ with structure equations

$$\begin{aligned} d\eta^1 &= T^1_{13}\eta^1 \wedge \eta^3 - \epsilon\eta^2 \wedge \eta^3 \\ d\eta^2 &= T^2_{12}\eta^1 \wedge \eta^2 + T^2_{13}\eta^1 \wedge \eta^3 + T^2_{23}\eta^2 \wedge \eta^3 \\ d\eta^3 &= \eta^1 \wedge \eta^2 + T^2_{12}\eta^1 \wedge \eta^3 + T^3_{23}\eta^2 \wedge \eta^3, \end{aligned}$$

where $\epsilon = \pm 1 = \mathrm{sgn}(H_{x^1})$. Based on the structure equations, we chose local coordinates so that

$$\begin{aligned} \eta^1 &= dx^1 - x^3\, dx^2 \\ \eta^2 &\equiv \epsilon\sqrt{\epsilon H_{x^1}}\left(dx^2 - J(dx^1 - x^3\, dx^2)\right) \mod \eta^3 \\ \eta^3 &= \frac{1}{\sqrt{\epsilon H_{x^1}}}\left(H\, dx^2 - dx^3\right). \end{aligned}$$

The dual framing is

$$\begin{aligned} v_1 &= \frac{\partial}{\partial x^1} + J\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right) \\ v_2 &= \frac{\epsilon}{\sqrt{\epsilon H_{x^1}}}\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right), \\ v_3 &= -[v_1, v_2] \equiv -\sqrt{\epsilon H_{x^1}}\frac{\partial}{\partial x^3} \mod v_2. \end{aligned}$$

Now we add a cost functional in each case and compute homogeneous examples. The assumption of homogeneity is equivalent to the condition that all structure functions $T^i_{jk}$ appearing in the structure equations for a canonical coframing are constants. (See [2] for details.)

3.1. **Case 2.1.** $\mathcal{F} = \left(\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + J\frac{\partial}{\partial x^3}\right) + \mathrm{span}\left(\frac{\partial}{\partial x^3}\right)$.

The corresponding control system is

$$\begin{aligned} \dot{x}^1 &= 1 \\ \dot{x}^2 &= x^3 \\ \dot{x}^3 &= J + u. \end{aligned} \tag{3.1}$$

A cost functional (1.2) for this system may be written as

$$Q(\dot{x}) = \frac{1}{2}G(x)u^2, \quad G(x) > 0.$$

An adapted framing $(v_1, v_2, v_3)$ on $\mathbb{R}^3$ may be defined by choosing $v_1$ to be the drift vector field

$$v_1 = \frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + J\frac{\partial}{\partial x^3},$$

$v_2$ to be the unit vector

$$v_2 = \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^3}$$

in $L_{\mathcal{F}}$, and

$$v_3 = -[v_1, v_2].$$

This framing is canonically defined up to the signs of $v_2$ and $v_3$. The dual coframing $(\eta^1, \eta^2, \eta^3)$ to this framing is given by

$$\begin{aligned} \eta^1 &= dx^1, \\ \eta^2 &\equiv \sqrt{G}(dx^3 - J\, dx^1) \mod \eta^3, \\ \eta^3 &= \sqrt{G}(dx^2 - x^3\, dx^1). \end{aligned} \tag{3.2}$$

This coframing is canonically defined, independent of the choice of local coordinates on $\mathbb{R}^3$.

Local coordinates for which the coframing takes the form (3.2) are determined up to transformations of the form

$$\begin{aligned} x^1 &= \tilde{x}^1 + a \\ x^2 &= \phi(\tilde{x}^1, \tilde{x}^2) \\ x^3 &= \phi_{\tilde{x}^1}(\tilde{x}^1, \tilde{x}^2) + \tilde{x}^3 \phi_{\tilde{x}^2}(\tilde{x}^1, \tilde{x}^2) \end{aligned} \tag{3.3}$$

with $\phi_{\tilde{x}^2} \neq 0$. Under such a transformation we have

$$\sqrt{\tilde{G}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)} = \sqrt{G(x^1, x^2, x^3)}\, \phi_{\tilde{x}^2} \tag{3.4}$$

$$\tilde{J}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{1}{\phi_{\tilde{x}^2}} \left( J(x^1, x^2, x^3) - \phi_{\tilde{x}^2\tilde{x}^2}(\tilde{x}^3)^2 - 2\phi_{\tilde{x}^1\tilde{x}^2}\tilde{x}^3 - \phi_{\tilde{x}^1\tilde{x}^1} \right), \tag{3.5}$$

with $x^1, x^2, x^3$ as in (3.3).

Now suppose that the structure is homogeneous. First consider the structure equation for $d\eta^3$. A computation shows that

$$d\eta^3 \equiv \frac{G_{x^3}}{2G^{3/2}} \eta^2 \wedge \eta^3 \mod \eta^1.$$

Therefore, $\frac{G_{x^3}}{2G^{3/2}}$ must be equal to a constant $-c_1$. (The minus sign is included for convenience in what follows.) The remaining analysis varies considerably depending on whether $c_1$ is zero or nonzero.

3.1.1. *Case 2.1.1: $c_1 = 0$.* If $c_1 = 0$, then $G_{x^3} = 0$, and so

$$G(x^1, x^2, x^3) = G_0(x^1, x^2).$$

According to (3.4), by a local change of coordinates of the form (3.3) with $\phi$ a solution of the PDE

$$\phi_{\tilde{x}^2}(\tilde{x}^1, \tilde{x}^2) = \frac{1}{G_0(\tilde{x}^1, \phi(\tilde{x}^1, \tilde{x}^2))},$$

we can arrange that $\tilde{G}_0(\tilde{x}^1, \tilde{x}^2) = 1$. This condition is preserved by transformations of the form (3.3) with

$$\phi(\tilde{x}^1, \tilde{x}^2) = \tilde{x}^2 + \phi_0(\tilde{x}^1). \tag{3.6}$$

With the assumption that $G(x^1, x^2, x^3) = 1$, the equation for $d\eta^3$ reduces to

$$d\eta^3 = \eta^1 \wedge \eta^2 + J_{x^3} \eta^1 \wedge \eta^3.$$

Therefore, $J_{x^3}$ must be equal to a constant $c_3$, and so

$$J(x^1, x^2, x^3) = c_3 x^3 + J_0(x^1, x^2).$$

Now the equation for $d\eta^2$ becomes

$$d\eta^2 = (J_0)_{x^2}\, \eta^1 \wedge \eta^3.$$

Therefore, $(J_0)_{x^2}$ must be equal to a constant $c_2$, and so

$$J_0(x^1, x^2) = c_2 x^2 + J_1(x^1).$$

With $\phi$ as in (3.6) and

$$J(x^1, x^2, x^3) = c_2 x^2 + c_3 x^3 + J_1(x^1),$$

equation (3.5) reduces to

$$\tilde{J}_1(\tilde{x}^1) = J_1(\tilde{x}^1 + a) - \left(\phi_0''(\tilde{x}^1) - c_3 \phi_0'(\tilde{x}^1) - c_2 \phi_0(\tilde{x}^1)\right).$$

Therefore, we can choose local coordinates to arrange that $\tilde{J}_1(\tilde{x}^1) = 0$.

To summarize, we have constructed local coordinates for which

$$G(x^1, x^2, x^3) = 1, \qquad J(x^1, x^2, x^3) = c_2 x^2 + c_3 x^3.$$

These coordinates are determined up to transformations of the form

$$x^1 = \tilde{x}^1 + a, \qquad x^2 = \tilde{x}^2 + \phi_0(\tilde{x}^1), \qquad x^3 = \tilde{x}^3 + \phi_0'(\tilde{x}^1),$$

where $\phi_0(\tilde{x}^1)$ is a solution of the ODE

$$\phi_0''(\tilde{x}^1) - c_3 \phi_0'(\tilde{x}^1) - c_2 \phi_0(\tilde{x}^1) = 0.$$

Now consider the problem of computing optimal trajectories for (3.1). The Hamiltonian for the energy functional (1.2) is

$$\begin{aligned} \mathcal{H} &= p_1 \dot{x}^1 + p_2 \dot{x}^2 + p_3 \dot{x}^3 - Q(\dot{x}) \\ &= p_1 + p_2 x^3 + p_3 (J + u) - \frac{1}{2} G(x) u^2 \\ &= p_1 + p_2 x^3 + p_3 (c_2 x^2 + c_3 x^3 + u) - \frac{1}{2} u^2. \end{aligned}$$

Setting $\frac{\partial \mathcal{H}}{\partial u} = 0$, as required by Pontryagin's maximum principle, provides the necessary condition

$$u = p_3 \tag{3.7}$$

for an optimal trajectory. So along an optimal trajectory, we have

$$\mathcal{H} = p_1 + p_2 x^3 + p_3 (c_2 x^2 + c_3 x^3) + \frac{1}{2} (p_3)^2.$$

Hamilton's equations take the form

$$\begin{aligned}\dot{x}^1 &= 1\\ \dot{x}^2 &= x^3\\ \dot{x}^3 &= c_2x^2 + c_3x^3 + p_3\\ \dot{p}_1 &= 0\\ \dot{p}_2 &= -c_2p_3\\ \dot{p}_3 &= -p_2 - c_3p_3.\end{aligned} \tag{3.8}$$

The last two equations in (3.8) can be written as

$$\ddot{p}_2 + c_3\dot{p}_2 - c_2p_2 = 0,$$

and the function $p_3 = -\frac{1}{c_2}\dot{p}_2$ satisfies this same ODE. Then the equations for $\dot{x}^2, \dot{x}^3$ can be written as

$$\ddot{x}^2 - c_3\dot{x}^2 - c_2x^2 = p_3(t),$$

where $p_3(t)$ is an arbitrary solution of the ODE

$$\ddot{p}_3 + c_3\dot{p}_3 - c_2p_3 = 0.$$

Therefore, $x^2(t)$ is an arbitrary solution of the 4th-order ODE

$$\left(\frac{d^2}{dt^2} + c_3\frac{d}{dt} - c_2\right)\left(\frac{d^2}{dt^2} - c_3\frac{d}{dt} - c_2\right)x^2(t) = 0,$$

and for any such $x^2(t)$, we have

$$x^1(t) = t + t_0, \qquad x^3(t) = \dot{x}^2(t).$$

A sample optimal trajectory is shown in Figure 8.

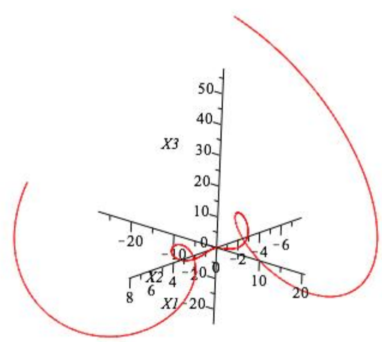


FIGURE 8.

3.1.2. *Case 2.1.2:* $c_1 \neq 0$. If $c_1 \neq 0$, then

$$G(x^1, x^2, x^3) = \frac{1}{(c_1 x^3 + G_0(x^1, x^2))^2}.$$

According to (3.4), by a local change of coordinates of the form (3.3) with $\phi$ a solution of the PDE

$$\phi_{x^1}(\tilde{x}^1, \tilde{x}^2) = \frac{1}{c_1} G_0\left(\tilde{x}^1, \phi(\tilde{x}^1, \tilde{x}^2)\right),$$

we can arrange that $\tilde{G}_0(\tilde{x}^1, \tilde{x}^2) = 0$. This condition is preserved by transformations of the form (3.3) with

$$\phi(\tilde{x}^1, \tilde{x}^2) = \phi_0(\tilde{x}^2). \tag{3.9}$$

With the assumption that $G(x^1, x^2, x^3) = \frac{1}{(c_1 x^3)^2}$, the equation for $d\eta^3$ reduces to

$$d\eta^3 = \eta^1 \wedge \eta^2 - \frac{(2J - x^3 J_{x^3})}{x^3} \eta^1 \wedge \eta^3 - c_1 \eta^2 \wedge \eta^3.$$

Therefore, $\frac{(2J - x^3 J_{x^3})}{x^3}$ must be equal to a constant $c_3$, and so

$$J(x^1, x^2, x^3) = c_3 x^3 + J_0(x^1, x^2)(x^3)^2.$$

Now the equation for $d\eta^2$ becomes

$$d\eta^2 = -x^3 (J_0)_{x^1}\, \eta^1 \wedge \eta^3.$$

The quantity $-x^3 (J_0)_{x^1}$ can only be constant if $(J_0)_{x^1} = 0$; therefore, we must have

$$J_0(x^1, x^2) = J_1(x^2).$$

With $\phi$ as in (3.9) and

$$J(x^1, x^2, x^3) = c_3 x^3 + J_1(x^2)(x^3)^2,$$

equation (3.5) reduces to

$$\tilde{J}_1(\tilde{x}^2) = J_1(\phi_0(\tilde{x}^2)) \phi_0'(\tilde{x}^2) - \frac{\phi_0''(\tilde{x}^2)}{\phi_0'(\tilde{x}^2)}.$$

Therefore, we can choose local coordinates to arrange that $\tilde{J}_1(\tilde{x}^2) = 0$.

To summarize, we have constructed local coordinates for which

$$G(x^1, x^2, x^3) = \frac{1}{(c_1 x^3)^2}, \qquad J(x^1, x^2, x^3) = c_3 x^3.$$

These coordinates are determined up to transformations of the form

$$x^1 = \tilde{x}^1 + a, \qquad x^2 = b\tilde{x}^2 + c, \qquad x^3 = b\tilde{x}^3 + c.$$

Now consider the problem of computing optimal trajectories for (3.1). The Hamiltonian for the energy functional (1.2) is

$$\begin{aligned}\mathcal{H} &= p_1\dot{x}^1 + p_2\dot{x}^2 + p_3\dot{x}^3 - Q(\dot{x})\\ &= p_1 + p_2x^3 + p_3(J+u) - \frac{1}{2}G(x)u^2\\ &= p_1 + p_2x^3 + p_3(c_3x^3+u) - \frac{1}{2(c_1x^3)^2}u^2.\end{aligned}$$

Setting $\frac{\partial\mathcal{H}}{\partial u} = 0$, as required by Pontryagin's maximum principle, provides the necessary condition

$$u = (c_1x^3)^2p_3 \tag{3.10}$$

for an optimal trajectory. So along an optimal trajectory, we have

$$\mathcal{H} = p_1 + p_2x^3 + c_3p_3x^3 + \frac{1}{2}(c_1x^3p_3)^2.$$

Hamilton's equations take the form

$$\begin{aligned}\dot{x}^1 &= 1\\ \dot{x}^2 &= x^3\\ \dot{x}^3 &= c_3x^3 + (c_1x^3)^2p_3\\ \dot{p}_1 &= 0\\ \dot{p}_2 &= 0\\ \dot{p}_3 &= -p_2 - c_3p_3 - (c_1p_3)^2x^3.\end{aligned} \tag{3.11}$$

The equation for $\dot{p}_2$ in (3.11) implies that $p_2(t)$ is equal to a constant $c_2$. Then (3.11) implies that

$$\begin{aligned}(\dot{p_3x^3}) &= -c_2x^3\\ \dot{x^3} &= c_3x^3 + c_1x^3(p_3x^3).\end{aligned} \tag{3.12}$$

These equations can be solved as follows:

- If $c_2 = 0$, then the function $p_3x^3$ is constant, and so the equation for $\dot{x}^3$ becomes

  $$\dot{x}^3 = \tilde{c}x^3$$

  for some constant $\tilde{c}$. If $\tilde{c} = 0$, then the solution trajectories are given by

  $$x^1(t) = t + t_0, \qquad x^2(t) = at + b, \qquad x^3(t) = a$$

  for some constants $a, b$. Sample optimal trajectories are shown in Figure 9.

  If $\tilde{c} \neq 0$, then the solution trajectories are given by

  $$x^1(t) = t + t_0, \qquad x^2(t) = \frac{a}{\tilde{c}}e^{\tilde{c}t} + b, \qquad x^3(t) = ae^{\tilde{c}t}$$

  for some constants $a, b$. Sample optimal trajectories are shown in Figure 10.

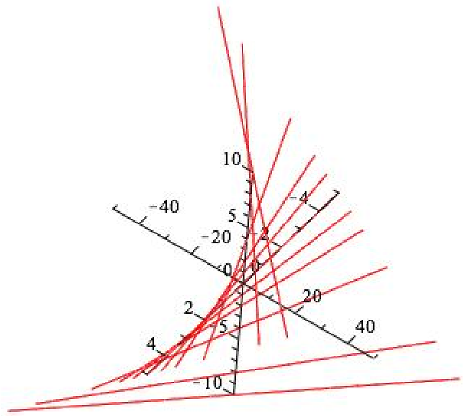


FIGURE 9.

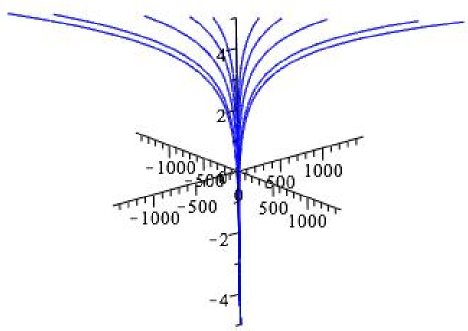


FIGURE 10.

- If $c_2 \neq 0$, then (3.12) can be written as the 2nd-order ODE for the function $z(t) = p_3(t)x^3(t)$:

$$\ddot{z} = (c_3 + c_1^2 z)\dot{z}.$$

Integrating once yields

$$\dot{z} = \frac{1}{2}(c_1 z)^2 + c_3 z + c_4$$

for some constant $c_4$. Depending on the values of the constants, the solution $z(t)$ has one of the following forms:

$$(1) \quad z(t) = a\tan(bt+c)+d, \qquad c_3^2 - 2c_1c_4 < 0;$$

$$(2) \quad z(t) = a\tanh(bt+c)+d, \qquad c_3^2 - 2c_1c_4 > 0;$$

$$(3) \quad z(t) = \frac{1}{at+b} + c, \qquad c_3^2 - 2c_1c_4 = 0.$$

Then we have

$$x^3 = -\frac{1}{c_2}\dot{z} = \dot{x}^2,$$

and so the corresponding solution trajectories are given (with slightly modified constants) by:

$$(1) \quad \begin{cases} x^1(t) = t + t_0 \\ x^2(t) = a\tan(bt+c)+d, \\ x^3(t) = ab\sec^2(bt+c); \end{cases}$$

$$(2) \quad \begin{cases} x^1(t) = t + t_0 \\ x^2(t) = a\tanh(bt+c)+d, \\ x^3(t) = ab\,\mathrm{sech}^2(bt+c); \end{cases}$$

$$(3) \quad \begin{cases} x^1(t) = t + t_0 \\ x^2(t) = \dfrac{1}{at+b} + c, \\ x^3(t) = -\dfrac{a}{(at+b)^2}. \end{cases}$$

Sample optimal trajectories for the first two cases are shown in Figure 11.

3.2. **Case 2.2.** $\mathcal{F} = \left(x^2\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} + J\left(x^2\frac{\partial}{\partial x^3}\right)\right) + \mathrm{span}\left(\frac{\partial}{\partial x^3}\right)$.

The corresponding control system is

$$\begin{aligned} \dot{x}^1 &= x^2 \\ \dot{x}^2 &= x^3 \\ \dot{x}^3 &= x^2(J+u). \end{aligned} \tag{3.13}$$

A cost functional (1.2) for this system may be written as

$$Q(\dot{x}) = \frac{1}{2}G(x)u^2, \quad G(x) > 0.$$

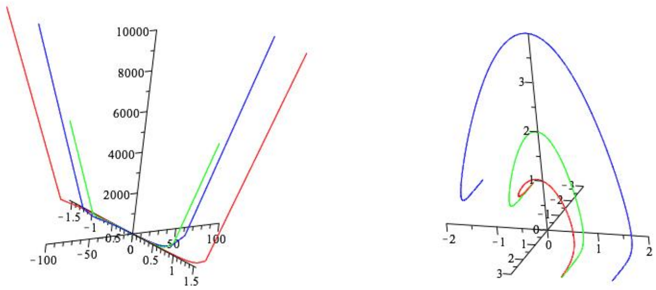


FIGURE 11.

An adapted framing $(v_1, v_2, v_3)$ on $\mathbb{R}^3$ may be defined by choosing $v_1$ to be the drift vector field

$$v_1 = x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + J \left( x^2 \frac{\partial}{\partial x^3} \right),$$

$v_2$ to be the unit vector

$$v_2 = \frac{1}{\sqrt{G}} \left( x^2 \frac{\partial}{\partial x^3} \right)$$

in $L_{\mathcal{F}}$, and

$$v_3 = -[v_1, v_2].$$

This framing is canonically defined up to the signs of $v_2$ and $v_3$. The dual coframing $(\eta^1, \eta^2, \eta^3)$ to this framing is given by

$$\begin{aligned} \eta^1 &= \frac{1}{x^2}\, dx^1, \\ \eta^2 &\equiv \frac{\sqrt{G}}{x^2}\, (dx^3 - J\, dx^1) \mod \eta^3, \\ \eta^3 &= \sqrt{G} \left( \frac{1}{x^2}\, dx^2 - \frac{x^3}{(x^2)^2}\, dx^1 \right). \end{aligned} \tag{3.14}$$

This coframing is canonically defined, independent of the choice of local coordinates on $\mathbb{R}^3$.

Local coordinates for which the coframing takes the form (3.14) are determined up to transformations of the form

$$\begin{aligned} x^1 &= \phi(\tilde{x}^1) \\ x^2 &= \phi'(\tilde{x}^1)\tilde{x}^2 \\ x^3 &= \phi'(\tilde{x}^1)\tilde{x}^3 + \phi''(\tilde{x}^1)(\tilde{x}^2)^2 \end{aligned} \tag{3.15}$$

with $\phi'(\tilde{x}^1) \neq 0$. Under such a transformation we have

$$\tilde{G}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = G(x^1, x^2, x^3) \tag{3.16}$$

$$\tilde{J}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = J(x^1, x^2, x^3) - \frac{1}{\phi'(\tilde{x}^1)} \left(\phi'''(\tilde{x}^1)(\tilde{x}^2)^2 + 3\phi''(\tilde{x}^1)\tilde{x}^3\right), \tag{3.17}$$

with $x^1, x^2, x^3$ as in (3.15).

Now suppose that the structure is homogeneous. The structure equation for $d\eta^1$ is

$$d\eta^1 = \frac{1}{\sqrt{G}}\, \eta^1 \wedge \eta^3.$$

Therefore, $G(x^1, x^2, x^3)$ must be equal to a constant $g$. Now the equation for $d\eta^3$ becomes

$$d\eta^3 = \eta^1 \wedge \eta^2 + \left(x^2 J_{x^3} - 3\frac{x^3}{x^2}\right) \eta^1 \wedge \eta^3.$$

Therefore, the quantity $\left(x^2 J_{x^3} - 3\frac{x^3}{x^2}\right)$ must be equal to a constant $a$, and so

$$J(x^1, x^2, x^3) = \frac{3}{2}\left(\frac{x^3}{x^2}\right)^2 + a\frac{x^3}{x^2} + J_0(x^1, x^2).$$

Now the equation for $d\eta^2$ reduces to

$$d\eta^2 = \left(x^2 (J_0)_{x^2} - 2J_0 - a\frac{x^3}{x^2}\right) \eta^1 \wedge \eta^3 - \frac{1}{\sqrt{g}}\, \eta^2 \wedge \eta^3.$$

The quantity $\left(x^2 (J_0)_{x^2} - 2J_0 - a\frac{x^3}{x^2}\right)$ can only be constant if $a = 0$; therefore, we must have $a = 0$ and

$$x^2 (J_0)_{x^2} - 2J_0 = -2c_1$$

for some constant $c_1$. Therefore,

$$J_0(x^1, x^2) = c_1 + J_1(x^1)(x^2)^2,$$

and

$$J(x^1, x^2, x^3) = \frac{3}{2}\left(\frac{x^3}{x^2}\right)^2 + c_1 + J_1(x^1)(x^2)^2.$$

With $\phi$ as in (3.15) and $J$ as above, equation (3.17) reduces to

$$\tilde{J}_1(\tilde{x}^1) = (\phi'(\tilde{x}^1)^2 J_1(\phi(\tilde{x}^1)) - \frac{\phi'''(\tilde{x}^1)}{\phi'(\tilde{x}^1)} + \frac{3}{2}\frac{\phi''(\tilde{x}^1)}{(\phi'(\tilde{x}^1))^2}.$$

Therefore, we can choose local coordinates to arrange that $\tilde{J}_1(\tilde{x}^1) = 0$. This condition is preserved by transformations of the form (3.15) with

$$\frac{\phi'''(\tilde{x}^1)}{\phi'(\tilde{x}^1)} - \frac{3}{2}\frac{\phi''(\tilde{x}^1)}{(\phi'(\tilde{x}^1))^2} = 0.$$

This implies that $\phi$ is a linear fractional transformation; i.e.,

$$\phi(\tilde{x}^1) = \frac{a\tilde{x}^1 + b}{c\tilde{x}^1 + d}.$$

To summarize, we have constructed local coordinates for which

$$G(x^1, x^2, x^3) = g, \qquad J(x^1, x^2, x^3) = \frac{3}{2}\left(\frac{x^3}{x^2}\right)^2 + c_1.$$

These coordinates are determined up to transformations of the form

$$x^1 = \frac{a\tilde{x}^1 + b}{c\tilde{x}^1 + d}, \qquad x^2 = \frac{ad - bc}{(c\tilde{x}^1 + d)^2}\tilde{x}^2, \qquad x^3 = \frac{ad - bc}{(c\tilde{x}^1 + d)^2}\tilde{x}^3 - \frac{2c(ad - bc)}{(c\tilde{x}^1 + d)^3}\tilde{x}^2.$$

Now consider the problem of computing optimal trajectories for (3.13). The Hamiltonian for the energy functional (1.2) is

$$\begin{aligned}
\mathcal{H} &= p_1\dot{x}^1 + p_2\dot{x}^2 + p_3\dot{x}^3 - Q(\dot{x}) \\
&= p_1x^2 + p_2x^3 + p_3x^2(J + u) - \frac{1}{2}G(x)u^2 \\
&= p_1x^2 + p_2x^3 + p_3x^2\left(\frac{3}{2}\left(\frac{x^3}{x^2}\right)^2 + c_1 + u\right) - \frac{1}{2}gu^2.
\end{aligned}$$

Setting $\frac{\partial \mathcal{H}}{\partial u} = 0$, as required by Pontryagin's maximum principle, provides the necessary condition

$$u = \frac{p_3x^2}{g} \tag{3.18}$$

for an optimal trajectory. So along an optimal trajectory, we have

$$\mathcal{H} = p_1x^2 + p_2x^3 + p_3x^2\left(\frac{3}{2}\left(\frac{x^3}{x^2}\right)^2 + c_1\right) + \frac{1}{2g}(p_3x^2)^2.$$

Hamilton's equations take the form

$$\begin{aligned}
\dot{x}^1 &= x^2 \\
\dot{x}^2 &= x^3 \\
\dot{x}^3 &= \frac{3}{2}\frac{(x^3)^2}{x^2} + c_1x^2 + \frac{1}{g}p_3(x^2)^2 \\
\dot{p}_1 &= 0 \\
\dot{p}_2 &= -p_1 + \frac{3}{2}\frac{p_3(x^3)^2}{(x^2)^2} - c_1p_3 - \frac{1}{g}(p_3)^2x^2 \\
\dot{p}_3 &= -p_2 - 3\frac{p_3x^3}{x^2}.
\end{aligned} \tag{3.19}$$

The system (3.19) has three independent first integrals in addition to the Hamiltonian $\mathcal{H}$ (which is automatically a first integral): it is straightforward to show, using (3.19), that the functions

(1) $I_1 = p_1$;

(2) $I_2 = p_1 x^1 + p_2 x^2 + p_3 x^3$;
(3) $I_3 = p_1 (x^1)^2 + 2p_2 x^1 x^2 + 2p_3 x^1 x^3 + 2p_3 (x^2)^2$.

are each constant on any solution curve. We can use these conserved quantities to reduce the system (3.19), as follows: on any solution curve of (3.19), we have

$$I_1 = k_1, \qquad I_2 = k_2, \qquad I_3 = k_3$$

for some constants $k_1, k_2, k_3$. These equations can be solved for $p_1, p_2, p_3$ to obtain:

$$\begin{aligned} p_1 &= k_1 \\ p_2 &= k_1 \left( -\frac{x^1}{x^2} - \frac{(x^1)^2 x^3}{2(x^2)^3} \right) + k_2 \left( \frac{1}{x^2} + \frac{x^1 x^3}{(x^2)^3} \right) + k_3 \left( -\frac{x^3}{2(x^2)^3} \right) \\ p_3 &= k_1 \left( \frac{(x^1)^2}{2(x^2)^2} \right) + k_2 \left( -\frac{x^1}{(x^2)^2} \right) + k_3 \left( \frac{1}{2(x^2)^2} \right). \end{aligned}$$

These equations can be substituted into (3.19) to obtain a closed, first-order ODE system for the functions $x^1, x^2, x^3$, depending on the parameters $k_1, k_2, k_3$; moreover, making the same substitution in the Hamiltonian $\mathcal{H}$ yields a conserved quantity for this system. (The precise expressions for the system and the conserved quantity are complicated and unenlightening, so we will not write them out explicitly here.) The resulting ODE systems cannot be solved analytically, but numerical integration yields sample trajectories as shown in Figure 12.

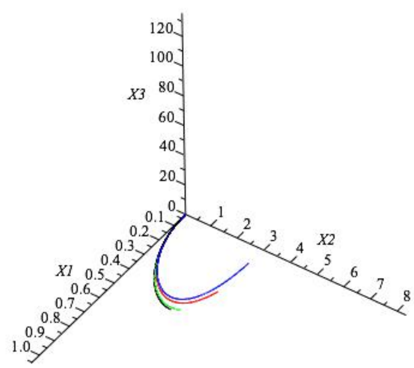


FIGURE 12.

### 3.3. **Case 2.3.** $\mathcal{F} = \left(\frac{\partial}{\partial x^1} + J\left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H \frac{\partial}{\partial x^3}\right)\right) + \text{span}\left(x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H \frac{\partial}{\partial x^3}\right)$.

The corresponding control system is

$$\begin{aligned} \dot{x}^1 &= 1 + x^3(J+u) \\ \dot{x}^2 &= J+u \\ \dot{x}^3 &= H(J+u). \end{aligned} \tag{3.20}$$

A cost functional (1.2) for this system may be written as

$$Q(\dot{x}) = \frac{1}{2}G(x)u^2, \quad G(x) > 0.$$

An adapted framing $(v_1, v_2, v_3)$ on $\mathbb{R}^3$ may be defined by choosing $v_1$ to be the drift vector field

$$v_1 = \frac{\partial}{\partial x^1} + J\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right),$$

$v_2$ to be the unit vector

$$v_2 = \frac{1}{\sqrt{G}}\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right)$$

in $L_{\mathcal{F}}$, and

$$v_3 = -[v_1, v_2].$$

This framing is canonically defined up to the signs of $v_2$ and $v_3$. The dual coframing $(\eta^1, \eta^2, \eta^3)$ to this framing is given by

$$\begin{aligned} \eta^1 &= dx^1 - x^3\, dx^2, \\ \eta^2 &\equiv \sqrt{G}\left(dx^2 - J(dx^1 - x^3\, dx^2)\right) \mod \eta^3, \\ \eta^3 &= \frac{\sqrt{G}}{H_{x^1}}\left(H\, dx^2 - dx^3\right). \end{aligned} \tag{3.21}$$

This coframing is canonically defined, independent of the choice of local coordinates on $\mathbb{R}^3$.

Finding local coordinate transformations which preserve the expressions (3.21) is considerably more complicated than in the previous cases and will require some care. Let $(x^1, x^2, x^3)$ and $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ be two local coordinates systems with respect to which the coframing $(\eta^1, \eta^2, \eta^3)$ takes the form (3.21). Then we must have

$$\eta^1 = dx^1 - x^3\, dx^2 = d\tilde{x}^1 - \tilde{x}^3\, d\tilde{x}^2. \tag{3.22}$$

Taking the exterior derivative of (3.22) yields

$$d\eta^1 = dx^2 \wedge dx^3 = d\tilde{x}^2 \wedge d\tilde{x}^3. \tag{3.23}$$

In particular,

$$\text{span}(dx^2, dx^3) = \text{span}(d\tilde{x}^2, d\tilde{x}^3).$$

Therefore we must have

$$x^2 = \bar{\phi}(\tilde{x}^2, \tilde{x}^3), \qquad x^3 = \bar{\psi}(\tilde{x}^2, \tilde{x}^3) \tag{3.24}$$

for some functions $\bar{\phi}(\tilde{x}^2, \tilde{x}^3)$, $\bar{\psi}(\tilde{x}^2, \tilde{x}^3)$. Equation (3.23) then implies that the functions $\bar{\phi}, \bar{\psi}$ satisfy the PDE

$$\bar{\phi}_{\tilde{x}^2}\bar{\psi}_{\tilde{x}^3} - \bar{\phi}_{\tilde{x}^3}\bar{\psi}_{\tilde{x}^2} = 1. \tag{3.25}$$

Unfortunately, equation (3.25) cannot be solved explicitly in terms of arbitrary functions of $\tilde{x}^2, \tilde{x}^3$. However, it *can* be solved implicitly with a slightly different setup. Instead of (3.24), suppose that we define our coordinate transformation by

$$\tilde{x}^2 = \phi(x^2, \tilde{x}^3), \qquad x^3 = \psi(x^2, \tilde{x}^3) \tag{3.26}$$

Then equation (3.23) is equivalent to the condition

$$\phi_{x^2} = \psi_{\tilde{x}^3}.$$

(In addition, both terms in this equation must be nonzero.) This is equivalent to the condition that there exists a function $\Phi(x^2, \tilde{x}^3)$ such that

$$\phi(x^2, \tilde{x}^3) = \Phi_{\tilde{x}^3}, \qquad \psi(x^2, \tilde{x}^3) = \Phi_{x^2}.$$

Then equation (3.22) implies that

$$x^1 = \tilde{x}^1 + \Phi(x^2, \tilde{x}^3) - \tilde{x}^3\Phi_{\tilde{x}^3}(x^2, \tilde{x}^3).$$

Thus, the local coordinate transformations which preserve the expression for $\eta^1$ in (3.21) are defined implicitly by

$$\begin{aligned} x^1 &= \tilde{x}^1 + \Phi(x^2, \tilde{x}^3) - \tilde{x}^3\Phi_{\tilde{x}^3}(x^2, \tilde{x}^3) \\ \tilde{x}^2 &= \Phi_{\tilde{x}^3}(x^2, \tilde{x}^3) \\ x^3 &= \Phi_{x^2}(x^2, \tilde{x}^3), \end{aligned} \tag{3.27}$$

where $\Phi(x^2, \tilde{x}^3)$ is an arbitrary smooth function of two variables with $\Phi_{x^2\tilde{x}^3} \neq 0$.

Next we will compute how the function $H(x^1, x^2, x^3)$ transforms under a coordinate transformation of the form (3.27). (When we consider the implications of homogeneity, it will turn out that $G$ and $J$ can be expressed in terms of $H$ and its derivatives; thus there is no need to explicitly compute the effects of the transformation (3.27) on $G$ and $J$.) Consider the expression for $\eta^3$ in (3.21). We must have

$$\eta^3 = \frac{\sqrt{G(x)}}{H_{x^1}(x)}\left(H(x)\,dx^2 - dx^3\right) = \frac{\sqrt{\tilde{G}(\tilde{x})}}{\tilde{H}_{\tilde{x}^1}(\tilde{x})}\left(\tilde{H}(\tilde{x})\,d\tilde{x}^2 - d\tilde{x}^3\right). \tag{3.28}$$

From (3.27), we have

$$\begin{aligned} d\tilde{x}^2 &= \Phi_{x^2\tilde{x}^3}\,dx^2 + \Phi_{\tilde{x}^3\tilde{x}^3}\,d\tilde{x}^3 \\ dx^3 &= \Phi_{x^2x^2}\,dx^2 + \Phi_{x^2\tilde{x}^3}\,d\tilde{x}^3. \end{aligned}$$

Substituting these expressions into (3.28) yields

$$\frac{\sqrt{G(x)}}{H_{x^1}(x)}\left((H(x)-\Phi_{x^2x^2})\,dx^2-\Phi_{x^2\tilde{x}^3}\,d\tilde{x}^3\right)=\frac{\sqrt{\tilde{G}(\tilde{x})}}{\tilde{H}_{\tilde{x}^1}(\tilde{x})}\left(\tilde{H}(\tilde{x})\Phi_{x^2\tilde{x}^3}\,dx^2+(\tilde{H}(\tilde{x})\Phi_{\tilde{x}^3\tilde{x}^3}-1)\,d\tilde{x}^3\right). \tag{3.29}$$

Equating the ratios of the coefficients of $dx^2$ and $d\tilde{x}^3$ on both sides of (3.29) yields

$$\frac{(H(x)-\Phi_{x^2x^2})}{-\Phi_{x^2\tilde{x}^3}}=\frac{\tilde{H}(\tilde{x})\Phi_{x^2\tilde{x}^3}}{(\tilde{H}(\tilde{x})\Phi_{\tilde{x}^3\tilde{x}^3}-1)},$$

which implies that

$$H(x^1,x^2,x^3)=\frac{((\Phi_{x^2\tilde{x}^3})^2-\Phi_{x^2x^2}\Phi_{\tilde{x}^3\tilde{x}^3})\tilde{H}(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3)+\Phi_{x^2x^2}}{1-\Phi_{\tilde{x}^3\tilde{x}^3}\tilde{H}(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3)}. \tag{3.30}$$

Now suppose that the structure is homogeneous. Unlike in the previous cases, the assumption of homogeneity will imply some relations among the constants appearing in the structure equations. Our frame adaptation

$$v_3=-[v_1,v_2]$$

implies that the structure equations have the form

$$\begin{aligned} d\eta^1 &= T^1_{13}\eta^1\wedge\eta^3+T^1_{23}\eta^2\wedge\eta^3\\ d\eta^2 &= T^2_{13}\eta^1\wedge\eta^3+T^2_{23}\eta^2\wedge\eta^3\\ d\eta^3 &= \eta^1\wedge\eta^2+T^3_{13}\eta^1\wedge\eta^3+T^3_{23}\eta^2\wedge\eta^3. \end{aligned} \tag{3.31}$$

In the homogeneous case, the functions $T^i_{jk}$ are all constant, and differentiating equations (3.31) implies that

$$\begin{aligned} 0=d(d\eta^1) &= (T^1_{23}T^3_{13}-T^1_{13}T^3_{23})\,\eta^1\wedge\eta^2\wedge\eta^3\\ 0=d(d\eta^2) &= (T^2_{23}T^3_{13}-T^2_{13}T^3_{23})\,\eta^1\wedge\eta^2\wedge\eta^3\\ 0=d(d\eta^3) &= -(T^1_{13}+T^2_{23})\,\eta^1\wedge\eta^2\wedge\eta^3. \end{aligned}$$

Therefore, the vectors

$$[T^1_{13}\quad T^1_{23}],\qquad [T^2_{13}\quad T^2_{23}],\qquad [T^3_{13}\quad T^3_{23}] \tag{3.32}$$

are all scalar multiples of each other (unless $T^3_{13}=T^3_{23}=0$), and moreover,

$$T^2_{23}=-T^1_{13}.$$

In most of the computations that follow, these relations will be self-evident; however, at one point they will have implications for the function $H$.

The structure equation for $d\eta^1$ is

$$d\eta^1=-\frac{JH_{x^1}}{\sqrt{G}}\,\eta^1\wedge\eta^3-\frac{H_{x^1}}{G}\,\eta^2\wedge\eta^3.$$

Therefore, we must have

$$J = \frac{c_1}{\sqrt{c_2 H_{x^1}}}, \qquad G = \frac{1}{c_2} H_{x^1}$$

for some constants $c_1, c_2$ with $c_2 \neq 0$, and then the equation for $d\eta^1$ becomes

$$d\eta^1 = -c_1\, \eta^1 \wedge \eta^3 - c_2\, \eta^2 \wedge \eta^3.$$

**Remark 3.1.** Since $G > 0$, $c_2$ must have the same sign as $H_{x^1}$. For simplicity, we will assume that both are positive; the analysis when both are negative would be similar.

Now the equation for $d\eta^3$ reduces to

$$d\eta^3 = \eta^1 \wedge \eta^2$$
$$-\frac{1}{2\sqrt{c_2}} \left( \frac{(H_{x^1x^2} + x^3 H_{x^1x^1} + H H_{x^1x^3} - 2H_{x^1} H_{x^3})}{(H_{x^1})^{3/2}} \right) \left( c_1\, \eta^1 \wedge \eta^3 + c_2\, \eta^2 \wedge \eta^3 \right).$$

Therefore,

$$\frac{(H_{x^1x^2} + x^3 H_{x^1x^1} + H H_{x^1x^3} - 2H_{x^1} H_{x^3})}{(H_{x^1})^{3/2}} = -2c_3 \tag{3.33}$$

for some constant $c_3$. Substituting the derivative of (3.33) with respect to $x^1$ into the equation for $d\eta^2$ yields

$$d\eta^2 = \left( \frac{3}{4} \left( \frac{H_{x^1x^1}}{H_{x^1}} \right)^2 - \frac{1}{2} \frac{H_{x^1x^1x^1}}{H_{x^1}} + \frac{c_1^2}{c_2} \right) \eta^1 \wedge \eta^3 + c_1\, \eta^2 \wedge \eta^3.$$

Observe that:

- The coefficient of $\eta^2 \wedge \eta^3$ in $d\eta^2$ is equal to minus the coefficient of $\eta^1 \wedge \eta^3$ in $d\eta^1$, as we previously observed that it must be.
- If $c_3 \neq 0$, then the ratio of the $\eta^1 \wedge \eta^3$ and $\eta^2 \wedge \eta^3$ coefficients in $d\eta^2$ must be equal to $\frac{c_1}{c_2}$ (which is the ratio of these coefficients in $d\eta^1$), and hence the $\eta^1 \wedge \eta^3$ coefficient in $d\eta^2$ must be equal to $\frac{c_1^2}{c_2}$.

Therefore, if $c_3 \neq 0$, then $H$ satisfies the PDE

$$\frac{3}{4} \left( \frac{H_{x^1x^1}}{H_{x^1}} \right)^2 - \frac{1}{2} \frac{H_{x^1x^1x^1}}{H_{x^1}} = 0. \tag{3.34}$$

The solutions of (3.34) are precisely the linear fractional transformations in the $x^1$ variable, and so we must have

$$H(x^1, x^2, x^3) = \frac{F_1(x^2, x^3) x^1 + F_0(x^2, x^3)}{G_1(x^2, x^3) x^1 + G_0(x^2, x^3)} \tag{3.35}$$

for some functions $F_0(x^2, x^3)$, $F_1(x^2, x^3)$, $G_0(x^2, x^3)$, $G_1(x^2, x^3)$.

**Remark 3.2.** If $c_3 = 0$, then the vectors $[T^1_{13} \;\; T^1_{23}]$, $[T^2_{13} \;\; T^2_{23}]$ are no longer required to be linearly independent, and so the Schwarzian derivative of $H$ with respect to $x^1$ appearing in equation (3.34) is only required to be constant, but not necessarily equal to zero. This assumption leads to a significantly more complicated process for normalizing the function $H$ via an appropriate choice of local coordinates, and we will not pursue the analysis in this case here.

Now we compute how the function (3.35) transforms under a local coordinate transformation of the form (3.27).

**Lemma 3.3.** *There exists a local coordinate transformation of the form* (3.27) *such that* $\tilde{H}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ *is linear in* $\tilde{x}^1$*; i.e.,*

$$\tilde{H}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \tilde{F}_1(\tilde{x}^2, \tilde{x}^3)\tilde{x}^1 + \tilde{F}_0(\tilde{x}^2, \tilde{x}^3), \tag{3.36}$$

*with* $\tilde{F}_1 \neq 0$.

*Proof.* Equation (3.30) can be written as

$$\tilde{H}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{H(x^1, x^2, x^3) - \Phi_{x^2x^2}}{\Phi_{\tilde{x}^3\tilde{x}^3} H(x^1, x^2, x^3) + ((\Phi_{x^2\tilde{x}^3})^2 - \Phi_{x^2x^2}\Phi_{\tilde{x}^3\tilde{x}^3})}. \tag{3.37}$$

Substituting (3.35) into this equation yields

$$\begin{aligned}
\frac{\tilde{F}_1\tilde{x}^1 + \tilde{F}_0}{\tilde{G}_1\tilde{x}^1 + \tilde{G}_0} &= \frac{\left(\frac{F_1x^1+F_0}{G_1x^1+G_0}\right) - \Phi_{x^2x^2}}{\Phi_{\tilde{x}^3\tilde{x}^3}\left(\frac{F_1x^1+F_0}{G_1x^1+G_0}\right) + ((\Phi_{x^2\tilde{x}^3})^2 - \Phi_{x^2x^2}\Phi_{\tilde{x}^3\tilde{x}^3})} \\
&= \frac{\left(F_1x^1 + F_0\right) - \Phi_{x^2x^2}\left(G_1x^1 + G_0\right)}{\Phi_{\tilde{x}^3\tilde{x}^3}\left(F_1x^1 + F_0\right) + ((\Phi_{x^2\tilde{x}^3})^2 - \Phi_{x^2x^2}\Phi_{\tilde{x}^3\tilde{x}^3})\left(G_1x^1 + G_0\right)} \\
&= \frac{\left[F_1 - \Phi_{x^2x^2}G_1\right]x^1 + \left[F_0 - \Phi_{x^2x^2}G_0\right]}{\left[\Phi_{\tilde{x}^3\tilde{x}^3}F_1 + ((\Phi_{x^2\tilde{x}^3})^2 - \Phi_{x^2x^2}\Phi_{\tilde{x}^3\tilde{x}^3})\,G_1\right]x^1 + \left[\Phi_{\tilde{x}^3\tilde{x}^3}F_0 + ((\Phi_{x^2\tilde{x}^3})^2 - \Phi_{x^2x^2}\Phi_{\tilde{x}^3\tilde{x}^3})\,G_0\right]}
\end{aligned}$$

The coefficients of $\tilde{x}^1$ on the left-hand side of this equation are the same as the coefficients of $x^1$ on the right-hand side, so the condition that $\tilde{G}_1 = 0$ is equivalent to

$$\begin{aligned}
0 &= \Phi_{\tilde{x}^3\tilde{x}^3}(x^2, \tilde{x}^3)F_1(x^2, x^3) \\
&\qquad + \left((\Phi_{x^2\tilde{x}^3}(x^2, \tilde{x}^3))^2 - \Phi_{x^2x^2}(x^2, \tilde{x}^3)\Phi_{\tilde{x}^3\tilde{x}^3}(x^2, \tilde{x}^3)\right)G_1(x^2, x^3) \\
&= \Phi_{\tilde{x}^3\tilde{x}^3}(x^2, \tilde{x}^3)F_1(x^2, \Phi_{x^2}(x^2, \tilde{x}^3)) \\
&\qquad + \left((\Phi_{x^2\tilde{x}^3}(x^2, \tilde{x}^3))^2 - \Phi_{x^2x^2}(x^2, \tilde{x}^3)\Phi_{\tilde{x}^3\tilde{x}^3}(x^2, \tilde{x}^3)\right)G_1(x^2, \Phi_{x^2}(x^2, \tilde{x}^3)).
\end{aligned}$$

Any solution $\Phi(x^2, \tilde{x}^3)$ of this equation will induce a local coordinate transformation for which $\tilde{H}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ has the form (3.36), as desired. The condition $\tilde{F}_1 \neq 0$ follows from the requirement that $H_{x^1} \neq 0$. (In fact, our assumption that $H_{x^1} > 0$ implies that $\tilde{F}_1 > 0$.)

□

Local coordinates for which $H$ has the form (3.36) are determined up to transformations of the form (3.27) with

$$\Phi_{\tilde{x}^3\tilde{x}^3} = 0;$$

i.e.,

$$\Phi(x^2, \tilde{x}^3) = \Phi_1(x^2)\tilde{x}^3 + \Phi_0(x^2). \tag{3.38}$$

With $\Phi$ as above, the local coordinate transformation (3.27) reduces to

$$\begin{aligned} x^1 &= \tilde{x}^1 + \Phi_0(x^2) \\ \tilde{x}^2 &= \Phi_1(x^2) \\ x^3 &= \Phi_0'(x^2) + \Phi_1'(x^2)\tilde{x}^3. \end{aligned} \tag{3.39}$$

With the assumption that $H$ has the form

$$H(x^1, x^2, x^3) = F_1(x^2, x^3)x^1 + F_0(x^2, x^3),$$

differentiating equation (3.33) with respect to $x^1$ yields

$$-\frac{(F_1)_{x^3}}{\sqrt{F_1}} = 0.$$

Therefore,

$$F_1(x^2, x^3) = F_1(x^2).$$

Now equation (3.30) reduces to

$$F_1(x^2)x^1 + F_0(x^2, x^3) = \left(\Phi_1'(x^2)\right)^2 \left(\tilde{F}_1(\tilde{x}^2)\tilde{x}^1 + \tilde{F}_0(\tilde{x}^2, \tilde{x}^3)\right) + \Phi_0''(x^2) + \Phi_1''(x^2)\tilde{x}^3,$$

which, taking (3.39) into account, becomes

$$\begin{aligned} F_1(x^2)\tilde{x}^1 + \left(F_1(x^2)\Phi_0(x^2) + F_0(x^2, \Phi_0'(x^2) + \Phi_1'(x^2)\tilde{x}^3)\right) = \\ \left(\Phi_1'(x^2)\right)^2 \tilde{F}_1(\Phi_1(x^2))\tilde{x}^1 \\ + \left(\left(\Phi_1'(x^2)\right)^2 \tilde{F}_0(\Phi_1(x^2), \tilde{x}^3) + \Phi_0''(x^2) + \Phi_1''(x^2)\tilde{x}^3\right). \end{aligned} \tag{3.40}$$

Equating the coefficients of $\tilde{x}^1$ on both sides yields

$$F_1(x^2) = \left(\Phi_1'(x^2)\right)^2 \tilde{F}_1(\Phi_1(x^2)).$$

Thus any solution $\Phi_1(x^2)$ of the equation

$$\Phi_1'(x^2) = \sqrt{F_1(x^2)}$$

will induce a local coordinate transformation for which

$$\tilde{F}(\tilde{x}^2) = 1.$$

Local coordinates for which $F_1(x^2) = 1$ are determined up to transformations of the form (3.39) with

$$\Phi_1'(x^2) = 1;$$

i.e.,

$$\Phi(x^2, \tilde{x}^3) = x^2\tilde{x}^3 + a\tilde{x}^3 + \Phi_0(x^2)$$

for some constant $a$. With $\Phi$ as above, the local coordinate transformation (3.27) reduces to

$$\begin{aligned} x^1 &= \tilde{x}^1 + \Phi_0(x^2) \\ \tilde{x}^2 &= x^2 + a \\ x^3 &= \tilde{x}^3 + \Phi_0'(x^2). \end{aligned} \tag{3.41}$$

Now equation (3.33) takes the form

$$(F_0)_{x^3} = c_3.$$

Therefore,

$$F_0(x^2, x^3) = c_3 x^3 + F_2(x^2).$$

Now equation (3.40) reduces to

$$\Phi_0(x^2) + c_3\Phi_0'(x^2) + F_2(x^2) = \tilde{F}_2(x^2 + a) + \Phi_0''(x^2).$$

Thus any solution $\Phi_0(x^2)$ of the equation

$$\Phi_0(x^2) + c_3\Phi_0'(x^2) - \Phi_0''(x^2) = -F_2(x^2)$$

will induce a local coordinate transformation for which

$$\tilde{F}_2(\tilde{x}^2) = 0.$$

Local coordinates for which $F_2(x^2) = 0$ are determined up to transformations of the form (3.41) with

$$\Phi_0(x^2) + c_3\Phi_0'(x^2) - \Phi_0''(x^2) = 0;$$

i.e.,

$$\Phi_0(x^2) = b_1 e^{r_1 x^2} + b_2 e^{r_2 x^2},$$

where $b_1, b_2$ are constants and

$$r_1 = \frac{c_3 + \sqrt{c_3^2 + 4}}{2}, \qquad r_2 = \frac{c_3 - \sqrt{c_3^2 + 4}}{2}. \tag{3.42}$$

To summarize, we have constructed local coordinates for which

$$G(x^1, x^2, x^3) = \frac{1}{c_2}, \qquad J(x^1, x^2, x^3) = \frac{c_1}{\sqrt{c_2}}, \qquad H(x^1, x^2, x^3) = x^1 + c_3 x^3.$$

These coordinates are determined up to transformations of the form

$$\begin{aligned} x^1 &= \tilde{x}^1 + b_1 e^{r_1 x^2} + b_2 e^{r_2 x^2} \\ \tilde{x}^2 &= x^2 + a \\ x^3 &= \tilde{x}^3 + b_1 r_1 e^{r_1 x^2} + b_2 r_2 e^{r_2 x^2}. \end{aligned}$$

Now consider the problem of computing optimal trajectories for (3.20). The Hamiltonian for the energy functional (1.2) is

$$
\begin{aligned}
\mathcal{H} &= p_1\dot{x}^1 + p_2\dot{x}^2 + p_3\dot{x}^3 - Q(\dot{x}) \\
&= p_1\left(1 + x^3(J+u)\right) + p_2(J+u) + p_3 H(J+u) - \frac{1}{2}G(x)u^2 \\
&= p_1\left(1 + x^3\left(\frac{c_1}{\sqrt{c_2}} + u\right)\right) + p_2\left(\frac{c_1}{\sqrt{c_2}} + u\right) + p_3\left(x^1 + c_3x^3\right)\left(\frac{c_1}{\sqrt{c_2}} + u\right) - \frac{1}{2c_2}u^2.
\end{aligned}
$$

Setting $\frac{\partial \mathcal{H}}{\partial u} = 0$, as required by Pontryagin's maximum principle, provides the necessary condition

$$
u = c_2(p_1x^3 + p_2 + p_3x^1 + c_3p_3x^3) \tag{3.43}
$$

for an optimal trajectory. So along an optimal trajectory, we have

$$
\begin{aligned}
\mathcal{H} = p_1 + p_3x^1 + \frac{c_1}{\sqrt{c_2}}\left(p_1x^3 + p_2 + c_3p_3x^3\right)& \\
+ \left(p_1x^3 + p_2 + c_3p_3x^3\right)\left(c_2(p_1x^3 + p_2 + p_3x^1 + c_3p_3x^3)\right)&.
\end{aligned}
$$

Hamilton's equations take the form

$$
\begin{aligned}
\dot{x}^1 &= 1 + x^3\left(\frac{c_1}{\sqrt{c_2}} + c_2(p_1x^3 + p_2 + p_3x^1 + c_3p_3x^3)\right) \\
\dot{x}^2 &= \frac{c_1}{\sqrt{c_2}} + c_2(p_1x^3 + p_2 + p_3x^1 + c_3p_3x^3) \\
\dot{x}^3 &= (x^1 + c_3x^3)\left(\frac{c_1}{\sqrt{c_2}} + c_2(p_1x^3 + p_2 + p_3x^1 + c_3p_3x^3)\right) \\
\dot{p}_1 &= -p_3\left(\frac{c_1}{\sqrt{c_2}} + c_2(p_1x^3 + p_2 + p_3x^1 + c_3p_3x^3)\right) \\
\dot{p}_2 &= 0 \\
\dot{p}_3 &= -\Bigg((p_1 + c_3p_3)\left(\frac{c_1}{\sqrt{c_2}} + c_2(p_1x^3 + p_2 + p_3x^1 + c_3p_3x^3)\right) \\
&\qquad + \left((p_1 + c_3p_3)x^3 + p_2\right)\left(c_2p_1 + c_2c_3p_3\right)\Bigg).
\end{aligned}
\tag{3.44}
$$

The system (3.44) has three independent first integrals in addition to the Hamiltonian $\mathcal{H}$ (which is automatically a first integral): it is straightforward to show, using (3.44), that the functions

(1) $I_1 = (p_1 + r_1p_3)e^{r_1x^2}$;
(2) $I_2 = (p_1 + r_2p_3)e^{r_2x^2}$;
(3) $I_3 = p_2$,

where $r_1, r_2$ are as in (3.42), are each constant on any solution curve. We can use these conserved quantities to reduce the system (3.44), as follows:

on any solution curve of (3.44), we have

$$I_1 = k_1, \qquad I_2 = k_2, \qquad I_3 = k_3$$

for some constants $k_1, k_2, k_3$. These equations can be solved for $p_1, p_2, p_3$ to obtain:

$$p_1 = \frac{1}{\sqrt{c_3^2 + 4}} \left( k_1 r_2 e^{-r_1 x^2} - k_2 r_1 e^{-r_2 x^2} \right)$$

$$p_2 = k_3$$

$$p_3 = \frac{1}{\sqrt{c_3^2 + 4}} \left( -k_1 e^{-r_1 x^2} + k_2 e^{-r_2 x^2} \right).$$

These equations can be substituted into (3.44) to obtain a closed, first-order ODE system for the functions $x^1, x^2, x^3$, depending on the parameters $k_1, k_2, k_3$; moreover, making the same substitution in the Hamiltonian $\mathcal{H}$ yields a conserved quantity for this system. (The precise expressions for the system and the conserved quantity are complicated and unenlightening, so we will not write them out explicitly here.) The resulting ODE systems cannot be solved analytically, but numerical integration yields sample trajectories (with $c_3 > 0$ and $c_3 < 0$) as shown in Figure 13.

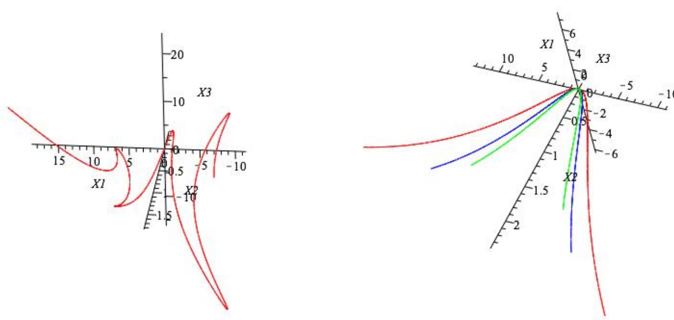

FIGURE 13.

## 4. Conclusion

What is perhaps most interesting about these results is how the behavior of control-affine systems in low dimensions varies from that of control-linear (i.e., driftless) systems. As we observed in [1], functional invariants appear in much lower dimension for affine distributions (beginning with $n = 2, s = 1$) than for linear distributions, where there are no functional invariants in dimensions below $n = 5,\ s = 2$.

With the addition of a quadratic cost functional, we see a similar phenomenon: for linear distributions with a quadratic cost functional, there are no functional invariants for any $n$ when $s = 1$, since local coordinates can always be chosen so that a unit vector field for the cost functional is represented by the vector field $\frac{\partial}{\partial x^1}$. But for affine distributions with $s = 1$, there are numerous functional invariants, and even the homogeneous examples exhibit a wide variety of behavior for the optimal trajectories.

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