HODGE COHOMOLOGY OF ITERATED FIBRED CUSP METRICS ON WITT SPACES

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Abstract. On a Witt space, we identify the L^2 cohomology of iterated fibred cusp metrics with the middle perversity intersection cohomology of the corresponding stratified space.

The Hodge theorem for smooth compact manifolds establishes an important link between two analytic invariants of a manifold, the vector space of (L^2) harmonic forms over the manifold and the (L^2) cohomology, and a topological invariant of the manifold, the cohomology with real coefficients, calculated using cellular, simplicial or smooth deRham theory. In the 1980's Cheeger, together with Goresky and MacPherson [3], [4], discovered that this important link extends to a subset of pseudomanifolds called Witt spaces, where the relationship is between L^2 cohomology with respect to an (incomplete) iterated conical metric and the (unique) middle perversity intersection cohomology, again with real coefficients. Several subsequent papers by various authors have further explored the links between intersection cohomology, and especially the middle perversity intersection cohomology when it is unique, and both L^2 harmonic forms and L^2 cohomology of noncompact or incomplete manifolds.

We will not attempt to list all such papers, but rather focus on a few results which share a geometric similarity. The first such result is the proof of the Zucker conjecture by Saper and Stern, which says that the L^2 cohomology of a Hermitian locally symmetric space is isomorphic to the middle perversity intersection cohomology of its Borel-Bailey compactification [8]. The second result, by Timmerscheidt, is that the L^2 cohomology of the complement of a normal crossings divisor in a Kähler manifold, endowed with a particular complete metric, is isomorphic to the cohomology of its (smooth) compactification [6]. In [7], the first author and her collaborators proved that the space of L^2 harmonic forms for a manifold with fibration boundary, endowed with a particular type of complete metric, is isomorphic to the image of lower middle perversity intersection cohomology in upper middle perversity intersection cohomology of its fibrewise compactification. All of these results have in common that the metric in question is complete, and near each compactifying stratum has a hyperbolic cusp type behaviour on the link of that stratum.

The purpose of this paper is to prove a general Hodge theorem along these geometric lines. That is, for any Witt space, X, it identifies the middle perversity intersection cohomology of X with the L^2 cohomology of the regular set $X \setminus X_{\text{sing}}$, endowed with a complete "iterated fibred cusp" metric, which near the singular strata has hyperbolic cusp type behaviour on the link. The Kodaira decomposition theorem tells us that when the L^2 cohomology with respect to some metric is finite dimensional, it is isomorphic to the space of L^2 harmonic forms, so as a consequence, we also get an isomorphism to the space of L^2 harmonic forms for this metric. By reference to the theory of Hilbert complexes, see [2], we additionally obtain the corollary that the Hodge Laplacian associated to such a metric is Fredholm as an unbounded operator on the space of L^2 differential forms.

First we must give the definition of the metrics we will consider. Let X be a smooth stratified space in the sense of [1], see also [10] and [5]. In this case, there is an associated manifold with fibred corners, $\widetilde{M}$. To each boundary hypersurface of $\widetilde{M}$, we can make a choice of boundary defining function so that for each $p \in X_{\text{sing}}$, there exists a neighborhood $\overline{\mathcal{U}}$ of p of the form

$$\overline{\mathcal{U}} = C_1(\overline{L}) \times V,$$

where V is an open ball in the Euclidean space, $\overline{L}$ is a smooth stratified space and

$$C_1(\overline{L}) = [0,1)_x \times \overline{L}/\{0\} \times \overline{L}$$

is the cone over $\overline{L}$. Under this identification, the parameter x can be chosen to corresponds to the boundary defining functions of the associated boundary hypersurface of the manifold with fibred corners $\widetilde{M}$. In this case, we say $\overline{\mathcal{U}}$ is a **regular neighborhood** of p in X. Such an associated set of boundary defining functions and neighborhoods is called a set of "control data". We can now define the metrics we will consider on $M := X \setminus X_{\text{sing}} = \widetilde{M} \setminus \partial \widetilde{M}$.

Definition 1. *A **quasi iterated fibred cusp metric** is defined inductively on the depth of X to be a complete Riemannian metric g on the regular set M of X such that for any $p \in X_{\text{sing}}$ and any regular neighborhood $\overline{\mathcal{U}} = C_1(\overline{L}) \times V$ of p, the restriction of g to $\mathcal{U} = \overline{\mathcal{U}} \setminus (\overline{\mathcal{U}} \cap X_{\text{sing}})$ is quasi isometric to a metric of the form*

$$g_{\mathcal{U}} = \frac{dx^2}{x^2} + g_V + x^2 g_L, \tag{1}$$

where g_V is a Riemannian metric on V and g_L is a quasi iterated fibred cusp metric on the interior L of $\overline{L}$.

We make two notes about these metrics. First of all, if we rearrange and change coordinates in (1), letting $r = -\log(x) \in (\epsilon, \infty)$, we find that the metric takes on the more familiar generalised hyperbolic cusp form on fibres:

$$g_{\mathcal{U}} = dr^2 + e^{-2r} g_L + g_V.$$

Second, we note that the iterated fibred cusp metrics of [5] constitute a special case of quasi iterated fibred cusp metrics. In fact, quasi iterated fibred cusp metrics could alternatively be defined as complete Riemannian metrics on the regular set M of X that are quasi-isometric to iterated fibred cusp metrics. Since the the L^2 cohomology on a Riemannian manifold depends only on the quasi-isometry class of the metric, considering this larger class of metrics leads to no extra difficulties. This also means that, when calculating the L^2 cohomology on a regular neighborhood U, we are free to consider a model metric as in Equation (1).

We may now state our result, which is for smooth stratified spaces satisfying the Witt condition [9], namely, smooth stratified spaces with no codimension 1 stratum, and such that for any regular neighborhood $\overline{\mathcal{U}} = C_1(\overline{L}) \times V$, the link $\overline{L}$ is either odd dimensional or has vanishing lower middle perversity intersection cohomology in degree $(\dim L)/2$.

Theorem 2. *Let g be a metric on the interior M of a smooth stratified Witt space, X, which is quasi-isometric to an iterated fibred cusp metric. Then the L^2 cohomology of (M, g) and the space of L^2 harmonic forms are naturally isomorphic to the middle perversity intersection cohomology of X,*

$$\mathcal{H}^*_{L^2}(M, g) \cong L^2H^*(M, g) \cong \mathrm{IH}^*_{\underline{m}}(X).$$

Before we prove this result, we recall that the L^2 cohomology of a complete manifold may be computed from its complex of smooth L^2 forms:

$$\cdots \xrightarrow{d} L^2_d\Omega^{k-1}(M, g) \xrightarrow{d} L^2_d\Omega^k(M, g) \xrightarrow{d} L^2_d\Omega^{k+1}(M, g) \xrightarrow{d} \cdots,$$

where

$$L^2_d\Omega^{k-1}(M, g) = \{\omega \in L^2\Omega^k(M, g)\ ;\ d\omega \in L^2\Omega^{k+1}(M, g)\}.$$

That is, the L^2-cohomology is given by

$$L^2H^k(M, g) := \{\omega \in L^2\Omega^k(M, g)\ ;\ d\omega = 0\}/\{d\eta\ ;\ \eta \in L^2_d\Omega^k(M, g)\}.$$

We can turn this complex into a complex of sheaves over X as follows. Over each regular neighborhood $\overline{\mathcal{U}}$ in X, we define the complex of L^2-forms associated to the iterated fibred cusp metric g by:

$$\cdots \xrightarrow{d} L^2_d\Omega^{k-1}(\overline{\mathcal{U}}, g) \xrightarrow{d} L^2_d\Omega^{k}(\overline{\mathcal{U}}, g) \xrightarrow{d} L^2_d\Omega^{k+1}(\overline{\mathcal{U}}, g) \xrightarrow{d} \cdots ,$$

where

$$L^2_d\Omega^{k-1}(\overline{\mathcal{U}}, g) = \{\omega \in L^2\Omega^k(\mathcal{U}, g) \ ; \ d\omega \in L^2\Omega^{k+1}(\mathcal{U}, g)\}.$$

This gives a complex of presheaves on X,

$$\cdots \xrightarrow{d} L^2_d\Omega^{k-1} \xrightarrow{d} L^2_d\Omega^{k} \xrightarrow{d} L^2_d\Omega^{k+1} \xrightarrow{d} \cdots .$$

These pre-sheaves may be sheafified to form the complex of sheaves that we also denote $L^2_d\Omega^k$ on X. Before proving our main theorem, the following result will be useful.

Lemma 3. *For all $k \in \mathbb{N}_0$, the sheaf $L^2_d\Omega^k$ on X is fine.*

Proof. We need to show that the sheaf $\mathrm{Hom}_{\mathbb{Z}}(L^2_d\Omega^k, L^2_d\Omega^k)$ admits partitions of unity. Let $\{\overline{\mathcal{U}}_i\}$ be an open cover of X by open sets in M and regular neighborhoods in X. Since X is compact, we can assume this cover contains finitely many open sets. By slightly shrinking each of the $\overline{\mathcal{U}}_i$, we can get another cover $\{\overline{W}_i\}$ of the same form, where $\overline{W}_i \subset \overline{\mathcal{U}}_i$ for each i. For each $\overline{\mathcal{U}}_i$ contained in M, we can then consider a smooth nonnegative function $\rho_i \in \mathcal{C}^\infty_c(\overline{\mathcal{U}}_i)$ such that ρ_i is nowhere zero on $\overline{W}_i \subset \overline{\mathcal{U}}_i$. When $\overline{\mathcal{U}}_i$ is a regular neighborhood of a point $p_i \in \partial\overline{M}$ of the form

$$\overline{\mathcal{U}}_i = C_1(\overline{L}_i) \times V_i,$$

we can take a nonnegative function $\rho_i \in \mathcal{C}^0_c(\overline{\mathcal{U}}_i)$ with ρ_i nowhere on $\overline{W}_i$ to be of the form $\rho_i = \pi_i^*\phi_i$ for some $\phi_i \in \mathcal{C}^\infty_c([0,1) \times V_i)$, where $\pi_i : C_1(\overline{L}) \times V_i \to [0,1)_x \times V_i$ is the obvious projection. In this way, we insure $d\rho_i \in L^\infty\Omega^1(M, g)$. This means the functions $\psi_i = \frac{\rho_i}{\sum_j \rho_j}$ form a partition of unity for the sheaf $\mathrm{Hom}_{\mathbb{Z}}(L^2_d\Omega^k, L^2_d\Omega^k)$.

□

We can now prove Theorem 2.

Proof. We proceed by induction on the depth of X. If X has depth zero, the result follows from the standard Hodge theorem on compact manifolds. Suppose now that X has depth d and that the theorem holds for all Witt spaces of depth less than d. Given a point p in $X \setminus M$, we can find a regular neighborhood $\overline{\mathcal{U}}$ of X of the form

$$\overline{\mathcal{U}} = C_1(\overline{L}) \times V,$$

where V is an open ball in the Euclidean space and $\overline{L}$ is a smooth Witt stratified space. If g_L is an iterated fibred cusp metric on the interior L of $\overline{L}$ and g_V is a choice of Riemannian metric on V, then the metric

$$g_{\mathcal{U}} = \frac{dx^2}{x^2} + g_V + x^2 g_L$$

is an iterated fibered cusp metric on $\mathcal{U} = \overline{\mathcal{U}} \setminus (\mathcal{U} \cap X_{\text{sing}})$. In particular, $g_{\mathcal{U}}$ is quasi-isometric to the restriction of g on $\mathcal{U}$, so we can use $g_{\mathcal{U}}$ instead of of g to compute the L^2-cohomology of $\mathcal{U}$. But for the metric $g_{\mathcal{U}}$, we can apply the Kunnëth formula of Zucker [11], which gives,

$$L^2H^k(\mathcal{U}, g) = \bigoplus_{i=0}^{1} \mathrm{WH}^i((0,1), \frac{dx^2}{x^2}, k - i - \frac{\ell}{2}) \otimes L^2H^k(L, g_L),$$

where $\ell = \dim L$. Now, we have that $WH^1((0,1), \frac{dx^2}{x^2}, a) = \{0\}$ for $a \neq 0$ and is infinite dimensional when $a = 0$, while

$$\mathrm{WH}^0((0,1), \frac{dx^2}{x^2}, a) = \begin{cases} \{0\}, & a \geq 0, \\ \mathbb{R}, & a < 0. \end{cases}$$

On the other hand, by our induction hypothesis, $L^2H^*(L, g_L) \cong \mathrm{IH}^*_{\underline{m}}(\overline{L})$. In particular, the Witt condition then implies that $L^2H^{\frac{\ell}{2}}(L, g_L) = \{0\}$ if ℓ is even. Putting these observations together, we obtain that

$$L^2H^k(\mathcal{U}) = \begin{cases} \mathrm{IH}^k_{\underline{m}}(L), & k < \frac{\ell}{2}, \\ \{0\}, & k \geq \frac{\ell}{2}. \end{cases}$$

In terms of the lower middle perversity $\underline{m}(\ell) = \lfloor \frac{\ell-1}{2} \rfloor$, this can be rewritten as

$$L^2H^k(\mathcal{U}) = \begin{cases} \mathrm{IH}^k_{\underline{m}}(L), & k \leq \ell - 2 - \underline{m}(\ell), \\ \{0\}, & k > \ell - 2 - \underline{m}(\ell). \end{cases}$$

Since $p \in X \setminus M$ was arbitrary, we can conclude by [7, Proposition 1] that there is a natural isomorphism

$$L^2H^*(M, g_{\text{ifc}}) \cong \mathrm{IH}^*_{\underline{m}}(X).$$

Since $\mathrm{IH}^*_{\underline{m}}(X)$ is finite dimensional, this means that, the ranges of d and its formal adjoint $\delta_{g_{\text{ifc}}}$ are closed. From the Kodaira decomposition theorem, we conclude that the space of L^2 harmonic forms $\mathcal{H}^*_{L^2}(M, g_{\text{ifc}})$ is naturally identified with $L^2H^*(M, g_{\text{ifc}})$. □

This easily leads to the following consequences for the signature operator.

Corollary 4. *Let g be an iterated fibred cusp metric on the interior M of a smooth Witt stratified space X. Then the associated signature operator $D_g = d + \delta_g$ is Fredholm with index given by*

$$\operatorname{ind}(D) = \operatorname{sgn}(\mathrm{IH}^*_{\underline{m}}(X)).$$

Proof. This follows from the fact that the sheaf $L^2_d\Omega^k$ is a locally self-dual sheaf, and the axiomatic characterisation of intersection cohomology. □

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