UNIFORM FAMILIES OF ERGODIC OPERATOR NETS

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ABSTRACT. We study mean ergodicity in amenable operator semigroups and establish the connection to the convergence of strong and weak ergodic nets. We then use these results in order to show the convergence of uniform families of ergodic nets as they appear in topological Wiener-Wintner theorems.

The classical mean ergodic theorem (see [10, Chapter 2.1]) is concerned with the convergence of the Cesàro means $\frac{1}{N}\sum_{n=0}^{N-1} S^n$ for some power bounded operator S on a Banach space X. The natural extension of the Cesàro means for representations $\mathbf{S}$ of general semigroups is the notion of an *ergodic net* as introduced by Eberlein [7] and Sato [18]. In the first part of this paper we discuss and slightly modify this concept in order to adopt it better for the study of operator semigroups. Sato showed in [18] that in amenable semigroups there always exist weak ergodic nets. We extend this result and show that even strong ergodic nets exist. Using this fact we then state a mean ergodic theorem connecting the convergence of strong and weak ergodic nets and the existence of a zero element in the closed convex hull of $\mathbf{S}$.

In the second part we develop the right framework for investigating uniform convergence in topological Wiener-Wintner theorems. Assani [2] and Robinson [16] studied those theorems and asked when averages of the form $\frac{1}{N}\sum_{n=0}^{N-1}\lambda^n S^n$ converge uniformly in $\lambda \in \mathbb{T}$ for some operator S on the space of continuous functions. Subsequently, their results have been generalised in different ways by Walters [19], Santos and Walkden [17] and Lenz [11],[12]. We propose and study *uniform families of ergodic nets* as an appropriate concept for treating and unifying these and other results.

1. Amenable and mean ergodic operator semigroups

We start from a semitopological semigroup G and refer to Berglund et al. [3, Chapter 1.3] for an introduction to this theory. Let X be a Banach space and denote by $\mathscr{L}(X)$ the set of bounded linear operators on X. We further assume that $\mathbf{S} = \{S_g : g \in G\}$ is a bounded representation of G on X, i.e.,

(i) $S_g \in \mathscr{L}(X)$ for all $g \in G$ and $\sup_{g\in G}\|S_g\| < \infty$,

(ii) $S_g S_h = S_{hg}$ for all $g, h \in G$,

(iii) $g \mapsto S_g x$ is continuous for all $x \in X$.

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If $\mathbf{S}$ is such a bounded representation, then we denote by $\operatorname{co}\mathbf{S}$ its convex hull and by $\overline{\operatorname{co}}\mathbf{S}$ the closure with respect to the strong operator topology. Notice that $\mathbf{S}$ as well as $\operatorname{co}\mathbf{S}$ and $\overline{\operatorname{co}}\mathbf{S}$ are topological semigroups with respect to the strong and semitopological semigroups with respect to the weak operator topology.

An *invariant mean* on the space $C_b(G)$ of bounded continuous functions on G is a linear functional $m \in C_b(G)'$ satisfying

(1) $\langle m, \mathbb{1}\rangle = 1$ and

(2) $\langle m, R_g f\rangle = \langle m, L_g f\rangle = \langle m, f\rangle \quad \forall g \in G, f \in C_b(G)$, where

$R_g f(h) = f(hg)$ and $L_g f(h) = f(gh)$ for $h \in G$.

The semigroup G is *amenable* if there exists an invariant mean on $C_b(G)$ (c.f. Berglund et al. [3, Chapter 2.3] or the survey article of Day [5]).

Notice that if $\mathbf{S} := \{S_g : g \in G\}$ is a bounded representation of an amenable semigroup G on X, then $\mathbf{S}$ endowed with the strong as well as the weak operator topology is also amenable. Indeed, if $\tilde{m} \in C_b(G)'$ is an invariant mean on $C_b(G)$, then $m \in C_b(\mathbf{S})'$ given by

$$\langle m, f\rangle := \langle \tilde{m}, \tilde{f}\rangle \quad (f \in C_b(\mathbf{S}))$$

defines an invariant mean on $C_b(\mathbf{S})$, where $\tilde{f}(g) = f(S_g)$.

In the following, the space $\mathscr{L}(X)$ will be endowed with the strong operator topology unless stated otherwise.

Definition 1.1. A net $(A_\alpha^{\mathbf{S}})_{\alpha\in\mathcal{A}}$ of operators in $\mathscr{L}(X)$ is called a *strong* $\mathbf{S}$*-ergodic net* if the following conditions hold.

(1) $A_\alpha^{\mathbf{S}} \in \overline{\operatorname{co}}\mathbf{S}$ for all $\alpha \in \mathcal{A}$.

(2) $(A_\alpha^{\mathbf{S}})$ is *strongly asymptotically invariant*, i.e.,

$\lim_\alpha A_\alpha^{\mathbf{S}}x - A_\alpha^{\mathbf{S}}S_g x = \lim_\alpha A_\alpha^{\mathbf{S}}x - S_g A_\alpha^{\mathbf{S}}x = 0$ for all $x \in X$ and $g \in G$.

The net $(A_\alpha^{\mathbf{S}})$ is called a *weak* $\mathbf{S}$*-ergodic net* if $(A_\alpha^{\mathbf{S}})$ is *weakly asymptotically invariant*, i.e., if the limit in (2) is taken with respect to the weak topology $\sigma(X, X')$ on X.

We note that our definition of an ergodic net is somewhat different to that of Eberlein [7], Sato [18] and Krengel [10, Chapter 2.2]. Instead of condition (1) they require only the following weaker condition.

(1') $A_\alpha^{\mathbf{S}}x \in \overline{\operatorname{co}}\mathbf{S}x$ for all $\alpha \in \mathcal{A}$ and $x \in X$.

However, the existence of (even strong) ergodic nets in the sense of Definition 1.1 is ensured by Corollary 1.5. Moreover, both definitions lead to the same convergence results (see Theorem 1.7 below). The reason is that if the limit of $A_\alpha^{\mathbf{S}}x$ exists for all $x \in X$ and is denoted by Px, then the operator P satisfies $P \in \overline{\operatorname{co}}\mathbf{S}$ rather than only $Px \in \overline{\operatorname{co}}\mathbf{S}x$ for all $x \in X$ (see Nagel [14, Theorem 1.2]).

Here are some typical examples of ergodic nets.

Examples 1.2. (a) Let $S \in \mathscr{L}(X)$ with $\|S\| \leq 1$ and consider the representation $\mathbf{S} = \{S^n : n \in \mathbb{N}\}$ of the semigroup $(\mathbb{N}, +)$ on X. Then the *Cesàro means* $(A_N^{\mathbf{S}})_{N\in\mathbb{N}}$ given by

$$A_N^{\mathbf{S}} := \frac{1}{N}\sum_{n=0}^{N-1} S^n$$

form a strong $\mathbf{S}$-ergodic net.

(b) In the situation of (a), the *Abel means* $(A_r^{\mathbf{S}})_{0<r<1}$ given by

$$A_r^{\mathbf{S}} := (1-r)\sum_{n=0}^{\infty} r^n S^n$$

form a strong $\mathbf{S}$-ergodic net.

(c) Consider the semigroup $(\mathbb{R}_+, +)$ being represented on X by a bounded C_0-semigroup $\mathbf{S} = \{S(t) : t \in \mathbb{R}_+\}$. Then $(A_s^{\mathbf{S}})_{s\in\mathbb{R}_+}$ given by

$$A_s^{\mathbf{S}}x := \frac{1}{s}\int_0^s S(t)x\,dt \quad (x \in X)$$

is a strong $\mathbf{S}$-ergodic net.

(d) Let $\mathbf{S} = \{S_g : g \in G\}$ be a bounded representation on X of an abelian semigroup G. Order the elements of $\operatorname{co}\mathbf{S}$ by setting $U \leq V$ if there exists $W \in \operatorname{co}\mathbf{S}$ such that $V = WU$. Then $(A_U^S)_{U\in\operatorname{co}\mathbf{S}}$ given by

$$A_U^{\mathbf{S}} := U$$

is a strong $\mathbf{S}$-ergodic net.

(e) Let H be a locally compact group with Haar measure $|\cdot|$ and let $G \subset H$ be a subsemigroup. Suppose that there exists a *Følner net* $(F_\alpha)_{\alpha\in\mathcal{A}}$ in G (also called *summing net*, see [15, Chapter 4]), i.e., a net of compact sets such that $|F_\alpha| > 0$ for all $\alpha \in \mathcal{A}$ and

$$\lim_\alpha \frac{|F_\alpha g \Delta F_\alpha|}{|F_\alpha|} = \lim_\alpha \frac{|gF_\alpha \Delta F_\alpha|}{|F_\alpha|} = 0 \quad \forall g \in G,$$

where $A\Delta B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of two sets A and B. Suppose that $\mathbf{S} := \{S_g : g \in G\}$ is a bounded representation of G on X. Then $(A_\alpha^{\mathbf{S}})_{\alpha\in\mathcal{A}}$ given by

$$A_\alpha^{\mathbf{S}}x := \frac{1}{|F_\alpha|}\int_{F_\alpha} S_g x\,dg \;\; (x \in X)$$

is a strong $\mathbf{S}$-ergodic net.

If G is an amenable group in the situation of Example 1.2 (e), then there always exists a Følner net $(F_\alpha)_{\alpha\in\mathcal{A}}$ in G (see [15, Theorem 4.16]). Hence, in amenable groups there always exist strong $\mathbf{S}$-ergodic nets for each representation $\mathbf{S}$. In [18, Proposition 1] Sato showed the existence of weak ergodic nets in amenable operator semigroups. We give a proof adapted to our situation.

Proposition 1.3. *Let G be represented on X by a bounded amenable semigroup $\mathbf{S} = \{S_g : g \in G\}$. Then there exists a weak $\mathbf{S}$-ergodic net in $\mathscr{L}(X)$.*

Proof. Denote by B the closed unit ball of $C_b(\mathbf{S})'$ and by $\operatorname{ex} B$ the set of extremal points of B. If $m \in C_b(\mathbf{S})'$ is an invariant mean, then $m \in B = \overline{\operatorname{co}} \operatorname{ex} B$ by the Krein-Milman theorem, where the closure is taken with respect to the weak*-topology σ^*. Since $\operatorname{ex} B = \{\delta_{S_g} : g \in G\}$, this implies that there exists a net $(\sum_{i=1}^{N_\alpha} \lambda_{i,\alpha}\delta_{S_{g_i}})_{\alpha\in\mathcal{A}} \subset \operatorname{co}\{\delta_{S_g} : g \in G\}$ with $\sigma^*\text{-}\lim_\alpha \sum_{i=1}^{N_\alpha} \lambda_{i,\alpha}\delta_{S_{g_i}} = m$. Since m is invariant, we obtain

$$\lim_\alpha \sum_{i=1}^{N_\alpha} \lambda_{i,\alpha}\delta_{S_{g_i}}(f - R_{S_g}f) = \lim_\alpha \sum_{i=1}^{N_\alpha} \lambda_{i,\alpha}\delta_{S_{g_i}}(f - L_{S_g}f) = 0 \quad \forall g \in G, f \in C_b(\mathbf{S}).$$

Define the net $(A_\alpha^{\mathbf{S}})_{\alpha\in\mathcal{A}}$ by $A_\alpha^{\mathbf{S}} := \sum_{i=1}^{N_\alpha} \lambda_{i,\alpha}S_{g_i} \in \operatorname{co}\mathbf{S}$ for $\alpha \in \mathcal{A}$. To see that $(A_\alpha^{\mathbf{S}})_{\alpha\in\mathcal{A}}$ is weakly asymptotically invariant, let $x \in X$ and $x' \in X'$ and define $f_{x,x'} \in C_b(\mathbf{S})$ by $f_{x,x'}(S_g) := \langle S_g x, x'\rangle$ for $g \in G$. Then for all $g \in G$ we have

$$\begin{aligned}\left\langle A_\alpha^{\mathbf{S}}x - A_\alpha^{\mathbf{S}}S_g x, x'\right\rangle &= \sum_{i=1}^{N_\alpha} \lambda_{i,\alpha}\left(\left\langle S_{g_i}x, x'\right\rangle - \left\langle S_{g_i}S_g x, x'\right\rangle\right)\\ &= \sum_{i=1}^{N_\alpha} \lambda_{i,\alpha}(f_{x,x'}(S_{g_i}) - R_{S_g}f_{x,x'}(S_{g_i}))\\ &= \sum_{i=1}^{N_\alpha} \lambda_{i,\alpha}\delta_{S_{g_i}}(f_{x,x'} - R_{S_g}f_{x,x'}) \longrightarrow 0\end{aligned}$$

and

$$\left\langle A_\alpha^{\mathbf{S}}x - S_g A_\alpha^{\mathbf{S}}x, x'\right\rangle \longrightarrow 0$$

analogously. Hence $(A_\alpha^{\mathbf{S}})_{\alpha\in\mathcal{A}}$ is a weak $\mathbf{S}$-ergodic net. □

It seems to be unknown that the existence of weak ergodic nets actually implies the existence of strong ergodic nets.

Theorem 1.4. *Let G be represented on X by a bounded semigroup $\mathbf{S} = \{S_g : g \in G\}$. Then the following assertions are equivalent.*

(1) There exists a weak $\mathbf{S}$-ergodic net.

(2) There exists a strong $\mathbf{S}$-ergodic net.

Proof. (1)⇒(2): Consider the locally convex space $E := \prod_{(g,x)\in G\times X} X \times X$ endowed with the product topology, where $X \times X$ carries the product (norm-)topology. Define the linear map

$$\Phi : \mathscr{L}(X) \to E, \quad \Phi(T) = (TS_g x - Tx, S_g Tx - Tx)_{(g,x)\in G\times X}.$$

By 17.13(iii) in [9] the weak topology $\sigma(E, E')$ on the product E coincides with the product of the weak topologies of the coordinate spaces. Hence, if $(A_\alpha^{\mathbf{S}})_{\alpha\in\mathcal{A}}$ is a weak $\mathbf{S}$-ergodic net on X, then $\Phi(A_\alpha^{\mathbf{S}}) \to 0$ with respect to the weak topology on E and thus $0 \in \overline{\Phi(\overline{\operatorname{co}}\mathbf{S})}^{\sigma(E,E')}$. Since the weak and strong closure coincide on the convex set $\Phi(\overline{\operatorname{co}}\mathbf{S})$, there exists a net $(B_\beta^{\mathbf{S}})_{\beta\in\mathcal{B}} \subset \overline{\operatorname{co}}\mathbf{S}$ with $\Phi(B_\beta^{\mathbf{S}}) \to 0$ in the topology of E. By the definition of this topology this means $\|B_\beta^{\mathbf{S}}S_g x - B_\beta^{\mathbf{S}}x\| \to 0$ and $\|S_g B_\beta^{\mathbf{S}}x - B_\beta^{\mathbf{S}}x\| \to 0$ for all $(g, x) \in G \times X$ and hence $(B_\beta^{\mathbf{S}})_{\beta\in\mathcal{B}}$ is a strong $\mathbf{S}$-ergodic net.

(2)⇒(1) is clear. □

The following corollary is a direct consequence of Proposition 1.3 and Theorem 1.4.

Corollary 1.5. *Let G be represented on X by a bounded amenable semigroup $\mathbf{S} = \{S_g : g \in G\}$. Then there exists a strong $\mathbf{S}$-ergodic net.*

The question of convergence of ergodic nets leads to the concept of *mean ergodicity*. We use the following abstract notion.

Definition 1.6. The semigroup $\mathbf{S}$ is called *mean ergodic* if $\overline{\mathrm{co}}\mathbf{S}$ contains a zero element P (c.f. [3, Chapter 1.1]), called the *mean ergodic projection of* $\mathbf{S}$.

Notice that for P being a zero element of $\overline{\mathrm{co}}\mathbf{S}$ it suffices that $PS_g = S_gP = P$ for all $g \in G$.

Nagel [14] and Sato [18] studied those semigroups and their results are summarized in Krengel [10, Chapter 2].

In the next theorem we collect a series of properties equivalent to mean ergodicity. Most of them can be found in Krengel [10, Chapter 2, Theorem 1.9], but we give a proof for completeness.

Denote the fixed spaces of $\mathbf{S}$ and $\mathbf{S}'$ by $\operatorname{Fix}\mathbf{S} = \{x \in X : S_gx = x \ \forall g \in G\}$ and $\operatorname{Fix}\mathbf{S}' = \{x' \in X' : S'_gx' = x' \ \forall g \in G\}$ respectively and the linear span of the set $\operatorname{rg}(I - \mathbf{S}) = \{x - S_gx : x \in X, g \in G\}$ by $\operatorname{lin}\operatorname{rg}(I - \mathbf{S})$.

Theorem 1.7. *Let G be represented on X by a bounded amenable semigroup $\mathbf{S} = \{S_g : g \in G\}$. Then the following assertions are equivalent.*

(1) $\mathbf{S}$ *is mean ergodic with mean ergodic projection* P.

(2) $\overline{\mathrm{co}}\mathbf{S}x \cap \operatorname{Fix}\mathbf{S} \neq \emptyset$ *for all* $x \in X$.

(3) $\operatorname{Fix}\mathbf{S}$ *separates* $\operatorname{Fix}\mathbf{S}'$.

(4) $X = \operatorname{Fix}\mathbf{S} \oplus \overline{\operatorname{lin}}\operatorname{rg}(I - \mathbf{S})$.

(5) $A_\alpha^{\mathbf{S}}x$ *has a weak cluster point for some/every weak* $\mathbf{S}$*-ergodic net* $(A_\alpha^{\mathbf{S}})$ *and all* $x \in X$.

(6) $A_\alpha^{\mathbf{S}}x$ *converges weakly for some/every weak* $\mathbf{S}$*-ergodic net* $(A_\alpha^{\mathbf{S}})$ *and all* $x \in X$.

(7) $A_\alpha^{\mathbf{S}}x$ *converges weakly for some/every strong* $\mathbf{S}$*-ergodic net* $(A_\alpha^{\mathbf{S}})$ *and all* $x \in X$.

(8) $A_\alpha^{\mathbf{S}}x$ *converges strongly for some/every strong* $\mathbf{S}$*-ergodic net* $(A_\alpha^{\mathbf{S}})$ *and all* $x \in X$.

The limit P of the nets $(A_\alpha^{\mathbf{S}})$ in the weak (resp. strong) operator topology is the mean ergodic projection of $\mathbf{S}$ *and equals the projection onto* $\operatorname{Fix}\mathbf{S}$ *along* $\overline{\operatorname{lin}}\operatorname{rg}(I - \mathbf{S})$.

Proof. (1)⇒(2): Since $S_gP = P$ for all $g \in G$, we have $Px \in \overline{\mathrm{co}}\mathbf{S}x \cap \operatorname{Fix}\mathbf{S}$ for all $x \in X$.

(2)⇒(3): Let $0 \neq x' \in \operatorname{Fix}\mathbf{S}'$. Take $x \in X$ such that $\langle x', x\rangle \neq 0$. If $y \in \overline{\mathrm{co}}\mathbf{S}x \cap \operatorname{Fix}\mathbf{S}$ then we have $\langle x', y\rangle = \langle x', x\rangle \neq 0$. Hence $\operatorname{Fix}\mathbf{S}$ separates $\operatorname{Fix}\mathbf{S}'$.

(3)⇒(4): Let $x' \in X'$ vanish on $\operatorname{Fix}\mathbf{S} \oplus \overline{\operatorname{lin}}\operatorname{rg}(I - \mathbf{S})$. Then in particular $\langle x', y\rangle = \langle x', S_gy\rangle = \left\langle S'_gx', y\right\rangle$ for all $y \in X$ and $g \in G$. Hence $x' \in \operatorname{Fix}\mathbf{S}'$. Since $\operatorname{Fix}\mathbf{S}$ separates $\operatorname{Fix}\mathbf{S}'$ and x' vanishes on $\operatorname{Fix}\mathbf{S}$, this implies $x' = 0$. Hence $\operatorname{Fix}\mathbf{S} \oplus \overline{\operatorname{lin}}\operatorname{rg}(I - \mathbf{S})$ is dense in X by the Hahn-Banach theorem and it remains to show that $\operatorname{Fix}\mathbf{S} \oplus \overline{\operatorname{lin}}\operatorname{rg}(I - \mathbf{S})$ is closed. Let $(A_\alpha^{\mathbf{S}})$ be a weak

$\mathbf{S}$-ergodic net and define $D := \{x \in X : \sigma\text{-}\lim_\alpha A_\alpha^{\mathbf{S}} x \text{ exists}\}$. Then $D = \operatorname{Fix}\mathbf{S} \oplus \overline{\operatorname{lin}}\operatorname{rg}(I - \mathbf{S})$ and D is closed since $(A_\alpha^{\mathbf{S}})$ is uniformly bounded.

(4)⇒(6): Let $(A_\alpha^{\mathbf{S}})$ be any weak $\mathbf{S}$-ergodic net. Then $A_\alpha^{\mathbf{S}} x$ converges weakly for all $x \in X$. Indeed, the convergence on $\operatorname{Fix}\mathbf{S}$ is clear and the weak convergence to 0 on $\operatorname{lin}\operatorname{rg}(I - \mathbf{S})$ follows from weak asymptotic invariance and linearity of $(A_\alpha^{\mathbf{S}})$. Since $\{x \in X : \sigma\text{-}\lim_\alpha A_\alpha^{\mathbf{S}} x \text{ exists}\}$ is closed we obtain weak convergence on all of $\operatorname{Fix}\mathbf{S} \oplus \overline{\operatorname{lin}}\operatorname{rg}(I - \mathbf{S})$.

(4)⇒(8): An analogous reasoning as in (4)⇒(6) yields the strong convergence of $A_\alpha^{\mathbf{S}} x$ for every strong $\mathbf{S}$-ergodic net $(A_\alpha^{\mathbf{S}})$ and every $x \in X$.

(5)⇒(1): Let $(A_\alpha^{\mathbf{S}})$ be a weak $\mathbf{S}$-ergodic net and define Px as the weak limit of a convergent subnet $(A_{\beta_x}^{\mathbf{S}} x)$ of $(A_\alpha^{\mathbf{S}} x)$ for each $x \in X$. Then $Px \in \overline{\operatorname{co}}\mathbf{S}x$ for all $x \in X$ and thus $P \in \overline{\operatorname{co}}\mathbf{S}$ by [14, Theorem 1.2]. Furthermore for all $x \in X$ and $g \in G$ we have $Px - PS_g x = \sigma\text{-}\lim_{\beta_x} A_{\beta_x}^{\mathbf{S}} x - A_{\beta_x}^{\mathbf{S}} S_g x = 0$ and $Px - S_g Px = \sigma\text{-}\lim_{\beta_x} A_{\beta_x}^{\mathbf{S}} x - S_g A_{\beta_x}^{\mathbf{S}} x = 0$ by weak asymptotic invariance. Hence $S_g P = PS_g = P$ for all $g \in G$ and thus $\mathbf{S}$ is mean ergodic.

The remaining implications are trivial. □

The next result can be found in Nagel [14, Satz 1.8] (see also Ghaffari [8, Theorem 1]). We give a different proof.

Corollary 1.8. *Let G be represented on X by a bounded amenable semigroup $\mathbf{S} = \{S_g : g \in G\}$. If $\mathbf{S}$ is relatively compact with respect to the weak operator topology, then $\mathbf{S}$ is mean ergodic.*

Proof. Since $\mathbf{S}x$ is relatively weakly compact, we obtain that $\overline{\operatorname{co}}\mathbf{S}x$ is weakly compact for all $x \in X$ by the Krein-Šmulian Theorem. Hence, if $(A_\alpha^{\mathbf{S}})$ is a weak $\mathbf{S}$-ergodic net, then $A_\alpha^{\mathbf{S}} x$ has a weak cluster point for each $x \in X$. The mean ergodicity of $\mathbf{S}$ then follows from Theorem 1.7. □

If the Banach space satisfies additional geometric properties, contractivity of the semigroup implies amenability and mean ergodicity. For uniformly convex spaces with strictly convex dual unit balls this has been shown by Alaoglu and Birkhoff [1, Theorem 6] using the so-called *minimal method*. In [4, Theorem 6'] Day observed that the same method still works if uniform convexity is replaced by strict convexity.

Corollary 1.9. *Let X be a reflexive Banach space such that the unit balls of X and X' are strictly convex. If the semigroup G is represented on X by a semigroup of contractions $\mathbf{S} = \{S_g : g \in G\}$, then $\mathbf{S}$ is mean ergodic.*

Proof. If $\mathbf{S}$ is a contractive semigroup in $\mathscr{L}(X)$ and the unit balls of X and of X' are strictly convex, then $\mathbf{S}$ is amenable by [6, Corollary 4.14]. Since $\mathbf{S}$ is bounded on the reflexive space X, it follows that $\mathbf{S}$ is relatively compact with respect to the weak operator topology. Hence Corollary 1.8 implies that $\mathbf{S}$ is mean ergodic. □

In some situations (see e.g. Assani [2, Theorem 2.10], Walters [19, Theorem 4], Lenz [12, Theorem 1]) one is interested in convergence of an ergodic net only on some given $x \in X$. The following result is a direct consequence of Theorem 1.7 by considering the restriction of $\mathbf{S}$ to the closed invariant subspace $Y_x := \overline{\mathrm{lin}}\mathbf{S}x$.

Proposition 1.10. *Let G be represented on X by a bounded amenable semigroup $\mathbf{S} = \{S_g : g \in G\}$ and let $x \in X$. Then the following assertions are equivalent.*

1. *$\mathbf{S}$ is mean ergodic on Y_x with mean ergodic projection P_x.*
2. *$\overline{\mathrm{co}}\mathbf{S}x \cap \mathrm{Fix}\,\mathbf{S} \neq \emptyset$.*
3. *$\mathrm{Fix}\,\mathbf{S}|_{Y_x}$ separates $\mathrm{Fix}\,\mathbf{S}|_{Y_x}'$.*
4. *$x \in \mathrm{Fix}\,\mathbf{S} \oplus \overline{\mathrm{lin}}\,\mathrm{rg}(I - \mathbf{S})$.*
5. *$A_\alpha^{\mathbf{S}}x$ has a weak cluster point for some/every weak $\mathbf{S}$-ergodic net $(A_\alpha^{\mathbf{S}})$.*
6. *$A_\alpha^{\mathbf{S}}x$ converges weakly for some/every weak $\mathbf{S}$-ergodic net $(A_\alpha^{\mathbf{S}})$.*
7. *$A_\alpha^{\mathbf{S}}x$ converges weakly for some/every strong $\mathbf{S}$-ergodic net $(A_\alpha^{\mathbf{S}})$.*
8. *$A_\alpha^{\mathbf{S}}x$ converges strongly for some/every strong $\mathbf{S}$-ergodic net $(A_\alpha^{\mathbf{S}})$.*

2. Uniform families of ergodic operator nets

We now use the above results on mean ergodic semigroups in order to study the convergence of uniform families of ergodic nets.

Let I be an index set and suppose that the semigroup G is represented on X by bounded amenable semigroups $\mathbf{S}_i = \{S_{i,g} : g \in G\}$ for each $i \in I$. Moreover, we assume that the $\mathbf{S}_i$ are *uniformly bounded*, i.e., $\sup_{i\in I}\sup_{g\in G}\|S_{i,g}\| < \infty$.

Definition 2.1. Let $\mathcal{A}$ be a directed set and let $(A_\alpha^{\mathbf{S}_i})_{\alpha\in\mathcal{A}} \subset \mathscr{L}(X)$ be a net of operators for each $i \in I$. Then $\{(A_\alpha^{\mathbf{S}_i})_{\alpha\in\mathcal{A}} : i \in I\}$ is a *uniform family of ergodic nets* if

(1) $\forall \alpha \in \mathcal{A}, \forall \varepsilon > 0, \forall x_1, \dots, x_m \in X, \exists N \in \mathbb{N}$ such that for each $i \in I$ there exists a convex combination $\sum_{j=1}^N c_{i,j}S_{i,g_j} \in \mathrm{co}\,\mathbf{S}_i$ satisfying

$$\sup_{i\in I}\|A_\alpha^{\mathbf{S}_i}x_k - \textstyle\sum_{j=1}^N c_{i,j}S_{i,g_j}x_k\| < \varepsilon \quad \forall k \in \{1, \dots, m\};$$

(2) $\lim_\alpha \sup_{i\in I}\|A_\alpha^{\mathbf{S}_i}x - A_\alpha^{\mathbf{S}_i}S_{i,g}x\| = \lim_\alpha \sup_{i\in I}\|A_\alpha^{\mathbf{S}_i}x - S_{i,g}A_\alpha^{\mathbf{S}_i}x\| = 0 \quad \forall g \in G, x \in X.$

Notice that if $\{(A_\alpha^{\mathbf{S}_i})_{\alpha\in\mathcal{A}} : i \in I\}$ is a uniform family of ergodic nets, then each $(A_\alpha^{\mathbf{S}_i})_{\alpha\in\mathcal{A}}$ is a strong $\mathbf{S}_i$-ergodic net.

Here are some examples of uniform families of ergodic nets.

Proposition 2.2. *(a) Let $S \in \mathscr{L}(X)$ with $\|S\| \leq 1$. Consider the semigroup $(\mathbb{N}, +)$ being represented on X by the families $\mathbf{S}_\lambda = \{(\lambda S)^n : n \in \mathbb{N}\}$ for $\lambda \in \mathbb{T}$. Then*

$$\left\{\left(\frac{1}{N}\sum_{n=0}^{N-1}(\lambda S)^n\right)_{N\in\mathbb{N}} : \lambda \in \mathbb{T}\right\}$$

is a uniform family of ergodic nets.

(b) In the situation of (a),

$$\left\{\left((1-r)\sum_{n=0}^\infty r^n\lambda^n S^n\right)_{0<r<1} : \lambda\in\mathbb{T}\right\}$$

is a uniform family of ergodic nets.

(c) Let K be a compact space and $\varphi : K \to K$ a continuous transformation. Let H be a Hilbert space and $S : f \mapsto f\circ\varphi$ the Koopman operator corresponding to φ on $C(K,H)$, the space of continuous H-valued functions on K. Denote by $U(H)$ the set of unitary operators on H and by Λ the set of continuous maps $\gamma : K \to U(H)$. Consider the semigroup $(\mathbb{N},+)$ and its representations on $C(K,H)$ given by the families $\mathbf{S}_\gamma = \{(\gamma S)^n : n\in\mathbb{N}\}$ for $\gamma\in\Lambda$, where $(\gamma S)f(x) = \gamma(x)Sf(x)$ for $x\in K$ and $f\in C(K,H)$. Then

$$\left\{\left(\frac{1}{N}\sum_{n=0}^{N-1}(\gamma S)^n\right)_{N\in\mathbb{N}} : \gamma\in\Lambda\right\}$$

is a uniform family of ergodic nets.

(d) Let $(S(t))_{t\in\mathbb{R}_+}$ be a bounded C_0-semigroup on X. Consider the semigroup $(\mathbb{R}_+,+)$ being represented on X by the families $\mathbf{S}_r = \{e^{2\pi irt}S(t) : t\in\mathbb{R}_+\}$ for $r\in B$, where $B\subset\mathbb{R}$ is bounded. Then

$$\left\{\left(\frac{1}{s}\int_0^s e^{2\pi irt}S(t)\,dt\right)_{s\in\mathbb{R}_+} : r\in B\right\}$$

is a uniform family of ergodic nets.

(e) Let $\mathbf{S} = \{S_g : g\in G\}$ be a bounded representation on X of an abelian semigroup G. Order the elements of $\operatorname{co}\mathbf{S}$ by setting $U\le V$ if there exists $W\in\operatorname{co}\mathbf{S}$ such that $V = WU$. Denote by $\widehat{G}$ the character semigroup of G, i.e., the set of continuous multiplicative maps $G\to\mathbb{T}$, and consider the representations $\mathbf{S}_\chi = \{\chi(g)S_g : g\in G\}$ of G on X for $\chi\in\widehat{G}$. Then

$$\left\{\left(\sum_{i=1}^N c_i\chi(g_i)S_{g_i}\right)_{\sum_{i=1}^N c_iS_{g_i}\in\operatorname{co}\mathbf{S}} : \chi\in\widehat{G}\right\}$$

is a uniform family of ergodic nets.

(f) Let H be a locally compact group with Haar measure $|\cdot|$ and let $G\subset H$ be a subsemigroup. Suppose that there exists a Følner net *$(F_\alpha)_{\alpha\in\mathcal{A}}$ in G. Suppose that $\mathbf{S} := \{S_g : g\in G\}$ is a bounded representation of G on X. Consider the representations $\mathbf{S}_\chi = \{\chi(g)S_g : g\in G\}$ of G on X for $\chi\in\Lambda$, where $\Lambda\subset\widehat{H}$ is uniformly equicontinuous on compact sets. Then*

$$\left\{\left(\frac{1}{|F_\alpha|}\int_{F_\alpha}\chi(g)S_g\,dg\right)_{\alpha\in\mathcal{A}} : \chi\in\Lambda\right\}$$

is a uniform family of ergodic nets.

Proof. (a) Property (1) of Definition 2.1 is clear. To see (2), let $x\in X$ and $k\in\mathbb{N}$. Then

$$\begin{aligned}\sup_{\lambda\in\mathbb{T}}\left\|\frac{1}{N}\sum_{n=0}^{N-1}(\lambda S)^n x - (\lambda S)^{n+k}x\right\| &\le \sup_{\lambda\in\mathbb{T}}\frac{1}{N}\sum_{n=0}^{k-1}\|(\lambda S)^n x\| + \frac{1}{N}\sum_{n=N}^{N-1+k}\|(\lambda S)^n x\| \\ &\le \frac{2k}{N}\|x\| \xrightarrow[N\to\infty]{} 0.\end{aligned}$$

(b) (1): Let $0 < r < 1$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $r^N < \frac{\varepsilon}{2}$. Then for all $\lambda \in \mathbb{T}$ we have

$$\begin{aligned}
&\|(1-r)\sum_{n=0}^{\infty} r^n\lambda^n S^n - \frac{(1-r)}{(1-r^N)}\sum_{n=0}^{N-1} r^n\lambda^n S^n\| \\
&\leq \left\|(1-r)\sum_{n=0}^{\infty} r^n\lambda^n S^n - (1-r)\sum_{n=0}^{N-1} r^n\lambda^n S^n\right\| + \\
&\qquad \left\|(1-r)\sum_{n=0}^{N-1} r^n\lambda^n S^n - \frac{(1-r)}{(1-r^N)}\sum_{n=0}^{N-1} r^n\lambda^n S^n\right\| \\
&\leq (1-r)\sum_{n=N}^{\infty} r^n + (1-r)\left|1 - \frac{1}{(1-r^N)}\right|\sum_{n=0}^{N-1} r^n \\
&\leq r^N + r^N < \varepsilon.
\end{aligned}$$

(2): Let $x \in X$ and $k \in \mathbb{N}$. Define for each $\lambda \in \mathbb{T}$ the sequence $x^{(\lambda)}$ by $x_n^{(\lambda)} := (\lambda S)^n x - (\lambda S)^{n+k}x$ for $n \in \mathbb{N}$. Then it follows from (a) that $\sup_\lambda \|\frac{1}{N}\sum_{n=0}^{N-1} x_n^{(\lambda)}\| \longrightarrow 0$. It is well known that Cesàro convergence implies the convergence of the Abel means to the same limit (see [13, Proposition 2.3]). One checks that if the Cesàro convergence is uniform in $\lambda \in \mathbb{T}$, then the convergence of the Abel means is also uniform. Hence we obtain $\lim_{r\uparrow 1}\sup_{\lambda\in\mathbb{T}} \|(1-r)\sum_{n=0}^{\infty} r^n x_n^{(\lambda)}\| = 0$.

(c) (1) is clear. To see (2) let $f \in C(K, H)$ and $k \in \mathbb{N}$. Then

$$\|(\gamma S)f\| = \sup_{x\in K}\|\gamma(x)f(\varphi(x))\|_H = \sup_{x\in K}\|f(\varphi(x))\|_H \leq \|f\|,$$

since $\gamma(x)$ is unitary for all $x \in K$. Hence

$$\begin{aligned}
\sup_{\gamma\in\Lambda}\left\|\frac{1}{N}\sum_{n=0}^{N-1}(\gamma S)^n f - (\gamma S)^{n+k}f\right\| &\leq \sup_{\gamma\in\Lambda}\frac{1}{N}\sum_{n=0}^{k-1}\|(\gamma S)^n f\| + \frac{1}{N}\sum_{n=N}^{N-1+k}\|(\gamma S)^n f\| \\
&\leq \frac{2k}{N}\|f\| \xrightarrow[N\to\infty]{} 0.
\end{aligned}$$

(d) This is a special case of (f) for the Følner net $([0,s])_{s>0}$ in $\mathbb{R}_+$ and the set $\Lambda := \{\chi_r : \mathbb{R}_+ \to \mathbb{T} : r \in B\}$, where $\chi_r(t) = e^{2\pi irt}$ for $t \in \mathbb{R}_+$. Notice that $\Lambda \subset \widehat{\mathbb{R}}$ is uniformly equicontinuous on compact sets since B is bounded.

(e) (1) is clear. To see (2) let $x \in X$, $g \in G$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{2M^2\|x\|}{N} < \varepsilon$, where $M = \sup_{g\in G}\|S_g\|$. Then for all $V = W\frac{1}{N}\sum_{n=0}^{N-1} S_g^n \geq \frac{1}{N}\sum_{n=0}^{N-1} S_g^n$, where $W = \sum_{i=0}^{k-1} c_i S_{g_i} \in \operatorname{co}\mathbf{S}$, we have

$$\begin{aligned}\sup_{\chi\in\widehat{G}} &\left\| \sum_{i=0}^{k-1} \sum_{n=0}^{N-1} \frac{1}{N} c_i \chi(g_i g^n) S_{g_i g^n} x - \sum_{i=0}^{k-1} \sum_{n=0}^{N-1} \frac{1}{N} c_i \chi(g_i g^n) S_{g_i g^n} \chi(g) S_g x \right\| \\ &\leq \sup_{\chi\in\widehat{G}} \left\| \sum_{i=0}^{k-1} c_i \chi(g_i) S_{g_i} \right\| \cdot \left\| \frac{1}{N} \sum_{n=0}^{N-1} (\chi(g)S_g)^n (x - \chi(g) S_g x) \right\| \\ &\leq \sup_{\chi\in\widehat{G}} M \frac{1}{N} \|x - (\chi(g)S_g)^N x\| \leq M \frac{1}{N} 2M \|x\| < \varepsilon.\end{aligned}$$

(f) (1): Let $\alpha \in \mathcal{A}$, $\varepsilon > 0$ and $x_1, \ldots, x_m \in X$. Since F_α is compact and Λ is uniformly equicontinuous on F_α the family $\{g \mapsto \chi(g)S_g x_k : \chi \in \Lambda\}$ is also uniformly equicontinuous on F_α for each $k \in \{1, \ldots, m\}$. Hence for each $k \in \{1, \ldots, m\}$ we can choose an open neighbourhood U_k of the unity of H satisfying

$$g, h \in G,\ \ h^{-1} g \in U_k \ \ \Rightarrow \ \sup_{\chi\in\Lambda} \|\chi(g) S_g x_k - \chi(h) S_g x_k\| < \varepsilon.$$

Then $U := \bigcap_{k=1}^m U_k$ is still an open neighbourhood of unity. Since F_α is compact there exists $g_1, \ldots, g_N \in F_\alpha$ such that $F_\alpha \subset \bigcup_{n=1}^N g_n U$. Defining $V_1 := g_1 U \cap F_\alpha$ and $V_n := (g_n U \cap F_\alpha) \setminus V_{n-1}$ for $n = 2, \ldots, N$ we obtain a disjoint union $F_\alpha = \bigcup_{n=1}^N V_n$. Hence for all $\chi \in \Lambda$ and $k \in \{1, \ldots, m\}$ we have

$$\begin{aligned}\left\| \frac{1}{|F_\alpha|} \int_{F_\alpha} \chi(g) S_g x_k dg - \sum_{n=1}^N \frac{|V_n|}{|F_\alpha|} \chi(g_n) S_{g_n} x_k \right\| &\leq \frac{1}{|F_\alpha|} \sum_{n=1}^N \int_{V_n} \underbrace{\|\chi(g) S_g x_k - \chi(g_n) S_{g_n} x_k\|}_{<\varepsilon} dg \\ &< \frac{1}{|F_\alpha|} \sum_{n=1}^N |V_n| \varepsilon = \varepsilon.\end{aligned}$$

(2): If $x \in X$ and $h \in G$, then we have

$$\begin{aligned}\sup_{\chi\in\Lambda} \left\| \frac{1}{|F_\alpha|} \int_{F_\alpha} \chi(g) S_g x - \chi(gh) S_{gh} x \, dg \right\| &\leq \sup_{\chi\in\Lambda} \frac{1}{|F_\alpha|} \int_{F_\alpha \triangle F_\alpha h} \|\chi(g) S_g x\| \\ &\leq \frac{|F_\alpha \triangle F_\alpha h|}{|F_\alpha|} \sup_{g\in G} \|S_g\| \|x\| \longrightarrow 0\end{aligned}$$

and

$$\sup_{\chi\in\Lambda} \left\| \frac{1}{|F_\alpha|} \int_{F_\alpha} \chi(g) S_g x - \chi(hg) S_{hg} x \, dg \right\| \longrightarrow 0$$

analogously.

□

Now, let $\{(A_\alpha^{\mathbf{S}_i})_{\alpha\in\mathcal{A}} : i \in I\}$ be a uniform family of ergodic nets. If $x \in X$ and $\mathbf{S}_i$ is mean ergodic on $\overline{\operatorname{lin}}\mathbf{S}_i x$ for each $i \in I$, then it follows from Proposition 1.10 that $\|A_\alpha^{\mathbf{S}_i} x - P_i x\| \to 0$

for all $i \in I$, where P_i denotes the mean ergodic projection of $\mathbf{S}_i|_{\overline{\text{lin}}\mathbf{S}_i x}$. The question arises, when this convergence is uniform in $i \in I$. The following elementary example shows that in general we cannot expect uniform convergence.

Example 2.3. Let $X = \mathbb{C}$ and let $S = I_{\mathbb{C}} \in L(\mathbb{C})$ be the identity operator on $\mathbb{C}$. Consider the semigroup $(\mathbb{N}, +)$ being represented on $\mathbb{C}$ by the families $\mathbf{S}_\lambda = \{\lambda^n I_{\mathbb{C}} : n \in \mathbb{N}\}$ for $\lambda \in \mathbb{T}$. Then for each $\lambda \in \mathbb{T}$ the Cesàro means $\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n$ converge, but the convergence is not uniform in $\lambda \in \mathbb{T}$.

However, the following theorem gives a sufficient condition for the convergence to be uniform.

Lemma 2.4. *If $\{(A_\alpha^{\mathbf{S}_i})_{\alpha \in \mathcal{A}} : i \in I\}$ is a uniform family of ergodic nets, then*

$$\lim_\alpha \sup_{i \in I} \|A_\alpha^{\mathbf{S}_i} x - A_\alpha^{\mathbf{S}_i} A_\beta^{\mathbf{S}_i} x\| = 0 \quad \forall \beta \in \mathcal{A}, x \in X.$$

Proof. Let $\beta \in \mathcal{A}$, $\varepsilon > 0$ and $x \in X$. Since $\{(A_\alpha^{\mathbf{S}_i})_{\alpha \in \mathcal{A}} : i \in I\}$ is a uniform family of ergodic nets, there exists $N \in \mathbb{N}$ and for each $i \in I$ a convex combination $\sum_{j=1}^N c_{i,j} S_{i,g_j} \in \operatorname{co} \mathbf{S}_i$ such that $\sup_{i \in I} \|A_\beta^{\mathbf{S}_i} x - \sum_{j=1}^N c_{i,j} S_{i,g_j} x\| < \varepsilon / M$, where $M = \sup_{i \in I} \sup_{g \in G} \|S_{i,g}\|$. Now, choose $\alpha_0 \in \mathcal{A}$ such that for all $\alpha > \alpha_0$

$$\sup_{i \in I} \|A_\alpha^{\mathbf{S}_i} x - A_\alpha^{\mathbf{S}_i} S_{i,g_j} x\| < \varepsilon \quad \forall j = 1, \dots, N.$$

Then for all $\alpha > \alpha_0$ and $i \in I$ we obtain

$$\begin{aligned} \|A_\alpha^{\mathbf{S}_i} x - A_\alpha^{\mathbf{S}_i} A_\beta^{\mathbf{S}_i} x\| &\le \|A_\alpha^{\mathbf{S}_i} x - A_\alpha^{\mathbf{S}_i} \textstyle\sum_{j=1}^N c_{i,j} S_{i,g_j} x\| + \|A_\alpha^{\mathbf{S}_i} \textstyle\sum_{j=1}^N c_{i,j} S_{i,g_j} x - A_\alpha^{\mathbf{S}_i} A_\beta^{\mathbf{S}_i} x\| \\ &\le \textstyle\sum_{j=1}^N c_{i,j} \|A_\alpha^{\mathbf{S}_i} x - A_\alpha^{\mathbf{S}_i} S_{i,g_j} x\| + M \|\textstyle\sum_{j=1}^N c_{i,j} S_{i,g_j} x - A_\beta^{\mathbf{S}_i} x\| \\ &< 2\varepsilon. \end{aligned}$$

□

Theorem 2.5. *Let I be a compact set and let G be represented on X by the uniformly bounded families $\mathbf{S}_i = \{S_{i,g} : g \in G\}$ for all $i \in I$. Let $\{(A_\alpha^{\mathbf{S}_i})_{\alpha \in \mathcal{A}} : i \in I\}$ be a uniform family of ergodic nets. Take $x \in X$ and assume that*

(a) $\mathbf{S}_i$ is mean ergodic on $\overline{\text{lin}}\mathbf{S}_i x$ with mean ergodic projection P_i for all $i \in I$,

(b) $I \to \mathbb{R}_+$, $i \mapsto \|A_\alpha^{\mathbf{S}_i} x - P_i x\|$ is continuous for all $\alpha \in \mathcal{A}$.

Then

$$\lim_\alpha \sup_{i \in I} \|A_\alpha^{\mathbf{S}_i} x - P_i x\| = 0.$$

Proof. It follows from Proposition 1.10 and the hypotheses that the function $f_\alpha : I \to \mathbb{R}_+$ defined by $f_\alpha(i) = \|A_\alpha^{\mathbf{S}_i} x - P_i x\|$ is continuous for each $\alpha \in \mathcal{A}$ and $\lim_\alpha f_\alpha(i) = 0$ for all $i \in I$. By compactness and continuity we obtain a net $(i_\alpha)_{\alpha \in \mathcal{A}} \subset I$ with $\sup_{i \in I} f_\alpha(i) = f_\alpha(i_\alpha)$ for all $\alpha \in \mathcal{A}$. To show that $\lim_\alpha \sup_{i \in I} f_\alpha(i) = 0$ it thus suffices to show that every subnet of $(f_\alpha(i_\alpha))$ has a subnet converging to 0. Let $(f_{\alpha_k}(i_{\alpha_k}))$ be a subnet of $(f_\alpha(i_\alpha))$ and let $\varepsilon > 0$. Since I is compact, we can choose a subnet of (i_{α_k}), also denoted by (i_{α_k}), such that $i_{\alpha_k} \to i_0$ for some $i_0 \in I$. Since f_α converges pointwise to 0, we can take $\beta \in \mathcal{A}$ such that $f_\beta(i_0) < \varepsilon / M$, where $M = \sup_{i\in I}\sup_{g\in G}\|S_{i,g}\|$. By continuity of f_β there exists k_1 such that for all $k > k_1$ we have $f_\beta(i_{\alpha_k}) - f_\beta(i_0) < \varepsilon/M$. By Lemma 2.4 there exists $k_2 > k_1$ such that for all $k > k_2$

$$\begin{aligned} f_{\alpha_k}(i_{\alpha_k}) &\le \|A_{\alpha_k}^{\mathbf{S}_{i_{\alpha_k}}} x - A_{\alpha_k}^{\mathbf{S}_{i_{\alpha_k}}} A_\beta^{\mathbf{S}_{i_{\alpha_k}}} x\| + \|A_{\alpha_k}^{\mathbf{S}_{i_{\alpha_k}}} A_\beta^{\mathbf{S}_{i_{\alpha_k}}} x - A_{\alpha_k}^{\mathbf{S}_{i_{\alpha_k}}} P_{i_{\alpha_k}} x\| \\ &\le \varepsilon + M\|A_\beta^{\mathbf{S}_{i_{\alpha_k}}} x - P_{i_{\alpha_k}} x\| \\ &\le \varepsilon + M(f_\beta(i_{\alpha_k}) - f_\beta(i_0)) + M f_\beta(i_0) \\ &\le 3\varepsilon. \end{aligned}$$

Hence $\lim_\alpha \sup_{i\in I} f_\alpha(i) = 0$. □

We now apply the above theory to operator semigroups on the space $C(K)$ of complex valued continuous functions on a compact metric space K.

Corollary 2.6. *Let $\varphi : K \to K$ be a continuous map, $S : f \mapsto f \circ \varphi$ the corresponding Koopman operator on $C(K)$, and assume that there exists a unique φ-invariant Borel probability measure μ on K. If $f \in C(K)$ satisfies $P_\lambda f = 0$ for all $\lambda \in \mathbb{T}$, where P_λ denotes the mean ergodic projection of $\{(\lambda S)^n : n \in \mathbb{N}\}$ on $L^2(K,\mu)$, then*

(1) $\lim\limits_{N\to\infty} \sup\limits_{\lambda\in\mathbb{T}} \left\| \frac{1}{|F_N|} \sum_{n\in F_N} \lambda^n S^n f \right\|_\infty = 0$ *for each Følner sequence $(F_N)_{N\in\mathbb{N}}$ in $\mathbb{N}$,*

(2) $\lim\limits_{r\uparrow 1} \sup\limits_{\lambda\in\mathbb{T}} \|(1-r)\sum_{n=0}^\infty r^n \lambda^n S^n f\|_\infty = 0.$

Proof. Under the hypotheses it follows from Robinson [16, Theorem 1.1] and Proposition 1.10 that the semigroups $\mathbf{S}_\lambda = \{(\lambda S)^n : n \in \mathbb{N}\}$ are mean ergodic on $\overline{\operatorname{lin}}\mathbf{S}_\lambda f$ for each $\lambda \in \mathbb{T}$. Let now $(F_N)_{N\in\mathbb{N}}$ be a Følner sequence in $\mathbb{N}$ and consider the uniform family of ergodic nets

$$\left\{ \left(\frac{1}{|F_N|} \sum_{n\in F_N} \lambda^n S^n \right)_{N\in\mathbb{N}} : \lambda \in \mathbb{T} \right\}$$

(c.f. Proposition 2.2 (f)). If $P_\lambda f = 0$ for all $\lambda \in \mathbb{T}$, then also condition (b) in Theorem 2.5 is satisfied since the map $\lambda \mapsto \frac{1}{|F_N|}\sum_{n\in F_N} \lambda^n S^n f$ is continuous for each $N \in \mathbb{N}$. Hence $\lim_{N\to\infty}\sup_{\lambda\in\mathbb{T}} \|\frac{1}{|F_N|}\sum_{n\in F_N}\lambda^n S^n f\|_\infty = 0$ by Theorem 2.5. The same reasoning applied to the uniform family of ergodic nets

$$\left\{ ((1-r)\sum_{n=0}^\infty r^n \lambda^n S^n)_{0<r<1} : \lambda \in \mathbb{T} \right\}$$

yields the second assertion. □

Remark 2.7. In [2, Theorem 2.10] Assani proved the first assertion of Corollary 2.6 for the Følner sequence $F_N = \{0, \dots, N-1\}$ in $\mathbb{N}$. Generalisations of this result can be found in Walters [19, Theorem 5], Santos and Walkden [17, Prop. 4.3] and Lenz [12, Theorem 2]. We will systematically study and unify these cases in a subsequent paper.

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References

1. L. Alaoglu and G. Birkhoff, *General ergodic theorems*, Ann. of Math. (2), 41 (1940), pp. 293–309.
2. I. Assani, *Wiener Wintner Ergodic Theorems*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.
3. J. F. Berglund, H. D. Junghenn, and P. Milnes, *Analysis on Semigroups*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons Inc., New York, 1989.
4. M. M. Day, *Reflexive Banach spaces not isomorphic to uniformly convex spaces*, Bull. Amer. Math. Soc., 47 (1941), pp. 313–317.
5. M. M. Day, *Semigroups and amenability*, in Semigroups (Proc. Sympos., Wayne State Univ., Detroit, Mich., 1968), Academic Press, New York, 1969, pp. 5–53.
6. K. de Leeuw and I. Glicksberg, *Applications of almost periodic compactifications*, Acta Math. 105 (1961), pp. 63–97.
7. W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. 67 (1949), pp. 217–240.
8. A. Ghaffari, *Ergodic theory of amenable semigroup actions*, Proc. Indian Acad. Sci. Math. Sci. 117 (2007), pp. 177–183.
9. J. L. Kelley and I. Namioka, *Linear Topological Spaces*, D. Van Nostrand Co., Inc., Princeton, N.J., 1963.
10. U. Krengel, *Ergodic Theorems*, vol. 6 of de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, 1985.
11. D. Lenz, *Aperiodic order via dynamical systems: diffraction for sets of finite local complexity*, in Ergodic Theory, vol. 485 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2009, pp. 91–112.
12. ———, *Continuity of eigenfunctions of uniquely ergodic dynamical systems and intensity of Bragg peaks*, Comm. Math. Phys. 287 (2009), pp. 225–258.
13. Y.-C. Li, R. Sato, and S.-Y. Shaw, *Convergence theorems and Tauberian theorems for functions and sequences in Banach spaces and Banach lattices*, Israel J. Math. 162 (2007), pp. 109–149.
14. R. J. Nagel, *Mittelergodische Halbgruppen linearer Operatoren*, Ann. Inst. Fourier (Grenoble) 23 (1973), pp. 75–87.
15. A. L. T. Paterson, *Amenability*, vol. 29 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1988.
16. E. A. Robinson, Jr., *On uniform convergence in the Wiener-Wintner theorem*, J. London Math. Soc. 49 (1994), pp. 493–501.
17. S. I. Santos and C. Walkden, *Topological Wiener-Wintner ergodic theorems via non-abelian Lie group extensions*, Ergodic Theory Dynam. Systems 27 (2007), pp. 1633–1650.
18. R. Satō, *On abstract mean ergodic theorems*, Tôhoku Math. J. 30 (1978), pp. 575–581.
19. P. Walters, *Topological Wiener-Wintner ergodic theorems and a random L^2 ergodic theorem*, Ergodic Theory Dynam. Systems 16 (1996), pp. 179–206.

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