Asymptotic formulas for curve operators in TQFT

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Abstract

Topological quantum field theories with gauge group SU_2 associate to each surface with marked points Σ and each integer $r > 0$ a vector space $V_r(\Sigma)$ and to each simple closed curve γ in Σ an Hermitian operator T_r^γ acting on that space. We show that the matrix elements of the operators T_r^γ have an asymptotic expansion in orders of $\frac{1}{r}$, and give a formula to compute the first two terms in terms of trace functions, generalizing results of Marché and Paul, see [MP].

This property, proved in [MP] for the punctured torus and the 4-holed sphere, was used to show that curve operators on these surfaces are Toeplitz operators and to derive the semiclassical behavior of some quantum invariants. Our result would be a first step in order to generalize this approach to arbitrary surfaces.

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Introduction

After the discovery of the Jones polynomial, Witten defined in 1989 in [W], by a method using Feynman path integrals, a family of new invariants of 3-manifolds, together with

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a structure of topological quantum field theory. Reshitikhin and Turaev formalised the ideas of Witten to construct their well-known family of 3-manifolds invariants [RT]: the invariants $Z_{2r}(M)$, $r \in \mathbb{N}^*$, together with a TQFT-structure for these invariants. An other method, more combinatorial, to define 3-manifold invariants and a TQFT from the ideas of Witten was later developed by [BHMV], this is the construction we will use in this paper, and that we will sketch in Section 1.

The TQFT defined in [BHMV] associates to each marked surface Σ vector spaces $V_r(\Sigma)$ and to each cobordism containing a tangle $(M, T, \Sigma_0, \Sigma_1)$ a morphism $V_r(\Sigma_0) \to V_r(\Sigma_1)$. In particular, the construction associates to γ a simple closed curve on Σ a curve operator $T_r^\gamma \in \mathrm{End}(V_r(\Sigma))$ (actually morphisms are associated to extended cobordisms of extended surfaces, but we can drop the extended structure when we consider only curve operators). The vector spaces $V_r(\Sigma)$ come with a natural Hermitian form. Moreover, for each banded graph Γ, such that the boundary of a regular neighborhood of Γ is Σ, [BHMV] described a natural Hermitian basis $(\varphi_c)_{c \in U_r}$, labelled by a certain set U_r of admissible colorings $c : E \to \mathbb{Z}$ of the set E of edges of Γ.

Then the matrix coefficients of the curve operators admit a presentation as below:

$$T_r^\gamma \varphi_c = \overline{c}(\gamma) \sum_{k:E \to \mathbb{Z}} F_k^\gamma(\frac{c}{r}, \frac{1}{r})\varphi_{c+k}$$

Here, $\overline{c}$ is an element of $H^1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$ associated to an admissible color c, and $\overline{c}(\gamma)$ is a sign factor, always trivial if Γ is planar. Furthermore, we have an open set $U \subset \mathbb{R}^E$, such that for $c \in U_r$, we have $\frac{c}{r} \in \overline{U}$, and F_k^γ are smooth functions on a neighborhood of $U \times \{0\}$ in $U \times [0, \infty)$.

This result extends a result of [MP], which studied the case of curve operators on the punctured torus and the four-holed sphere. In that paper, an asymptotic formula was given to characterize the first two asymptotic terms of the F_k^γ in orders of $\frac{1}{r}$ as Fourier coefficients of trace functions f_γ on the representation space $\mathcal{M}(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}_2)/\mathrm{SU}_2$, such that $f_\gamma : \rho \mapsto -\mathrm{tr}(\rho(\gamma))$. We generalize the asymptotic formula in [MP]. Indeed, for each relative spin-structure on $(\Gamma, \partial\Gamma)$ denoted by q, we write $\chi(\gamma) = (-1)^{q(\gamma)}$ and we introduce the so-called ψ-symbol of γ by the following formula:

$$\sigma_\chi^\gamma(\tau, \theta, \hbar) = \sum_{e \in E} F_k^\gamma(\tau, \hbar) e^{ik\theta} \chi(\gamma)$$

Then we have the asymptotic expansion:

$$\sigma_\chi^\gamma(\tau, \theta, \hbar)) = f_\gamma(R_\chi(\tau, \theta)) + \frac{\hbar}{2i} \sum_{e \in E} \frac{\partial^2}{\partial \tau_e \partial \theta_e} f_\gamma(R_\chi(\tau, \theta)) + o(\hbar)$$

where f_γ is the trace function on $\mathcal{M}(\Sigma)$ introduced above, and R_χ are some specific action-angle parametrizations of $\mathcal{M}(\Sigma)$. More precisely, the graph Γ induces a pants decomposition of Σ by curves $(C_e)_{e \in E}$ and a momentum mapping $h : \rho \to \left(\frac{1}{\pi} \mathrm{Acos}(-\frac{1}{2} f_{C_e})\right)_{e \in E}$, for which τ_e corresponds to action coordinates, and θ_e to angle coordinates.

The proof of [MP] in the case where Σ is the punctured torus and the four-holed sphere relied on explicit computations for some simple set of curves that generated the Kauffman algebra of Σ, then extending the result to general curves. This approach failed in higher genus as no simple set of generators is known. Instead, we developed a more conceptual and systematic method, which relies on the study of algebraic properties of the ψ-symbol and the Kauffman algebra of Σ.

The authors of [MP] used the asymptotic estimation to construct a framework for curve operators on the punctured torus and the four-holed sphere as Toeplitz operators on the sphere. This allowed to implement the WKB-approximation for eigenvectors, and deduce asymptotic expansions of quantum invariants (such as a new proof of the asymptotic expansion of $6j$-symbols, and an expression for the punctured S-matrix). Therefore, we hope to use our asymptotic expansions for general marked surface as a first step to give a framework of curve operators as Toeplitz operators on toric varieties, or at least apply the tools of microlocal analysis. Such a Toeplitz framework for curve operators would be a useful tool to study combinatorial TQFT. Indeed, in a different approach, Andersen introduced some geometrical curve operators in [A06,A10] that are Toeplitz operators to prove the asymptotic faithfullness of the quantum representations of the mapping class group. We think that viewing the standard curve operators as Toeplitz operators could provide other interesting applications and hopefully an original approach to the Witten conjecture for the expansion of Reshetikhin-Turaev invariants.

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1 Matrix coefficients of curve operators

1.1 A quick overview of TQFT and curve operators

To each surface Σ with marked points p_i colored by elements $\hat{c}_i$ of $\mathcal{C}_r = \{1, \cdots, r-1\}$, neglecting the so-called framing anomaly, the construction of [BHMV] associates a vector space $V_r(\Sigma, \hat{c})$ and to any cobordism (M, Σ_0, Σ_1) containing a link L a morphism

$$V_r(M, L) \ : \ V_r(\Sigma_0) \to V_r(\Sigma_1)$$

such that for every closed orientable 3-manifold M, we have $V_r(M) = Z_{2r}(M)$.

To give a more explicit picture of the TQFT and the vector spaces $V_r(\Sigma)$, we first need to introduce the notion of Kauffman bracket skein modules of 3-manifolds and Kauffman algebras of marked surfaces.

For M a 3-manifold (which can have a boundary), we define $K(M, A)$ as the quotient of free $\mathbb{C}[A^{\pm 1}]$-module generated by links modulo isotopy and the Kauffman relations (see Figure 1).

For $t \in \mathbb{C}^*$, we can define a Kauffman module evaluated in t: we write $K(M,t) = K(M,A) \underset{A=t}{\otimes} \mathbb{C}$.
Now if Σ is a surface with marked points $p_1,\ldots,p_n$, we denote by $K(\Sigma,A)$ the Kauffman module $K((\Sigma \setminus \{p_1,\ldots,p_n\}) \times [0,1], A)$.
We call a *multicurve* on Σ a disjoint union of simple curves on Σ, which is disjoint with the marked points of Σ. It is easy to see that $K(\Sigma,A)$ is spanned by multicurves on Σ, and actually multicurves give a basis of this vector space.
The module $K(\Sigma,A)$ has an algebra structure: the product $\gamma\cdot\delta$ of two elements of $K(\Sigma,A)$ is obtained by isotopying γ and δ so they are included in $\Sigma \times (\frac{1}{2},1)$ and $\Sigma \times (0,\frac{1}{2})$ respectively, then gluing the two parts into $\Sigma \times [0,1]$.
For $t \in \mathbb{C}^*$, we define $K(\Sigma,t) = K(\Sigma,A) \underset{A=t}{\otimes} \mathbb{C}$, which is also an algebra, and admits the set of multicurves as a basis. Using this basis, we can identify $K(\Sigma,t)$ with $K(\Sigma,-1)$ and we embed $K(\Sigma,-e^{\frac{i\pi\hbar}{2}}) = K(\Sigma,A) \underset{A=-e^{\frac{i\pi\hbar}{2}}}{\otimes} \mathbb{C}[[\hbar]]$ into $K(\Sigma,-1)[[\hbar]]$.

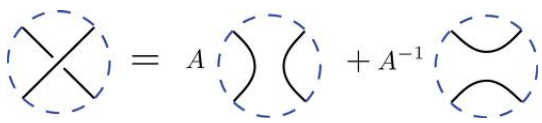


Figure 1: The first Kauffman relation. The other relation states that any trivial component is identified with $-A^2 - A^{-2}$

The vector spaces $V_r(\Sigma,\hat{c})$ have a definition as quotients of Kauffman modules at roots of unity, as below:

Definition and theorem: Let H be a handlebody with $\partial H = \Sigma$, a surface with marked points $p_1,\ldots,p_n$. Given a coloring $\hat{c}$ of the marked points, we chose $c_i - 1$ points in a small neighborhood of p_i for each i, and write P for the set of all such points. We define a relative Kauffman module $K(H,\hat{c},\zeta_r)$ as the $\mathbb{C}[A^{\pm 1}]$-module generated by banded tangles in H whose intersection with Σ is the set P. For $r \in \mathbb{N}^*$, we write $\zeta_r = -e^{\frac{i\pi}{2r}}$. For any embedding j of H in $\mathbf{S}^3$, we define the following sub-module of $K(H,\hat{c},\zeta_r)$:

$$N_r^j = \{x \in K(H,\hat{c},\zeta_r) \ / \ \forall y \in K(\mathbf{S}^3 \setminus \mathrm{Im}(j),\hat{c},\zeta_r), \quad \langle x| \overset{r}{\underset{i=1}{\otimes}} f_{c_i-1}|y\rangle = 0\}$$

where we write f_k for the k-th Jones-Wenzl idempotent, and $\langle x| \overset{r}{\underset{i=1}{\otimes}} f_{c_i-1}|y\rangle$ stands for the element of $K(\mathbf{S}^3,\zeta_r)$ obtained from x and y by pasting H with $\mathbf{S}^3 \setminus \mathrm{Im}(j)$, inserting Jones-Wenzl idempotent at each marked points. Then N_r^j is in fact independent of j, is of finite codimension, and we may define:

$$V_r(\Sigma,\hat{c}) = K(H,\hat{c},\zeta_r)/N_r^j.$$

With this setting, there is a simple description of the curve operator T_r^γ associated to a multicurve γ on Σ disjoint from the marked points $p_1,\ldots,p_n$, or more generally to an

element of $K(\Sigma,\zeta_r)$.
Indeed, we can take a element z of $K(H,\hat{c},\zeta_r)$ and stack a multicurve γ over it to obtain another element $\gamma\cdot z$ of $K(H,\hat{c},\zeta_r)$. The induced map factors through N_r^j, as for $n\in N_r^j$ and for any $z\in K(\mathbf{S}^3\setminus \mathrm{Im}(j),\hat{c},\zeta_r)$, we have $\langle \gamma\cdot n| \overset{r}{\underset{i=1}{\otimes}} f_{c_i-1}|z\rangle = \langle n|\overset{r}{\underset{i=1}{\otimes}} f_{c_i-1}|\gamma\cdot z\rangle$. Thus we have defined an endomorphism T_r^γ of $V_r(\Sigma,\hat{c})$ associated to $\gamma\in K(\Sigma,\zeta_r)$. Furthermore, the following map is a morphism of algebras:

$$\begin{array}{ccc} K(\Sigma,\zeta_r) & \longrightarrow & \mathrm{End}(V_r(\Sigma,\hat{c})) \\ \gamma & \longmapsto & T_r^\gamma \end{array}$$

In [BHMV] a Hermitian structure on $V_r(\Sigma,\hat{c})$ was constructed, coming from the bracket $\langle\cdot,\cdot\rangle$ that we introduced above, was provided and a Hermitian basis of $V_r(\Sigma,\hat{c})$.

Let $\mathcal{C}_r=\{1,\ldots,r-1\}$ be the set of colors (we shifted all colors by 1 comparing to the conventions of [BHMV]). We can construct a basis of the space $V_r(\Sigma,\hat{c})$ by the following procedure:

We start by considering a pants decomposition of Σ by a family of curves $\mathcal{C}=\{C_e\}_{e\in E}$ containing the components of $\partial\Sigma$. We also choose a banded graph Γ drawn on the surface Σ with a trivalent vertex v_P lying in each pants P of the decomposition, for every $e\in E$ an edge (that we will also call e) joining two trivalent vertices and intersecting once the curve C_e and disjoint from the other curves C_f, and finally n univalent vertices labeled by $p_1,\ldots,p_n$ corresponding to the marked points of Σ. A such graph is said *compatible* to the pants decomposition $\mathcal{C}$.

We call admissible coloring of Γ a map $c:E\to C_r$ such that the following conditions hold:

- for each edge e connected to a univalent vertex p_i one has $c_e=\hat{c}_i$.
- for any triple of edges e,f,g adjacent to the same vertex one has

(i) $c_e+c_f+c_g$ is odd

(ii) $c_e+c_f<c_g$

(iii) $c_e+c_f+c_g<2r$.

We will denote by U_r the set of such admissible colorings. The construction of [BHMV] provides for each admissible coloring c a vector $\varphi_c\in V_r(\Sigma,\hat{c})$ obtained by cabling the graph Γ by a specific combination of multicurves (we will detail this construction in Section 2). Moreover, the family (φ_c) when c runs over all admissible colorings is a Hermitian basis of $V_r(\Sigma,\hat{c})$.

For a multicurve γ, the operators T_r^γ are Hermitian operators for the Hermitian structure on $V_r(\Sigma,\hat{c})$ given by [BHMV]. The spectrum and the eigenvectors of T_r^γ are known: First, as all components of γ are disjoint, there exists a pants decomposition of Σ by a family of curve $\mathcal{C}=\{C_e\}_{e\in E}$ such that Γ is isotopic to the union of n_e parallel copies of C_e, for some integers $n_e \in \mathbb{N}$. Then the Hermitian basis (φ_c) coming from the pants decomposition $\mathcal{C}$ is an eigenbasis of T_r^γ, and we have

$$T_r^\gamma \varphi_c = \left(\prod_{e\in E} \big(-2\cos(\frac{\pi c_e}{r})\big)^{n_e} \right) \varphi_c$$

We should take note that the spectral radius $||T_r^\gamma||$ is thus always less than $2^{n(\gamma)}$, where we write $n(\gamma)$ for the number of components of the multicurve γ.

In this paper, we will make a fundamental use of the following theorem that describes the Kauffman algebra $K(\Sigma,-1)$:

Theorem [Bullock/Brumfiel-Hilden]: Consider the following affine algebraic variety: $\mathcal{M}'(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2(\mathbb{C})$. It is the space of characters of the fundamental group of $\Sigma \setminus \{p_1,\ldots,p_n\}$ in $\mathrm{SL}_2(\mathbb{C})$. Let also $\mathrm{Reg}(\mathcal{M}'(\Sigma))$ denote the algebra of regular functions from $\mathcal{M}'(\Sigma)$ to $\mathbb{C}$. The map

$$\begin{array}{rccl} \sigma: & K(\Sigma,-1) & \longrightarrow & \mathrm{Reg}(\mathcal{M}'(\Sigma)) \\ & \gamma & \longrightarrow & f_\gamma \text{ such that } \; f_\gamma(\rho) = -\mathrm{Tr}(\rho(\gamma)) \end{array}$$

is a injective morphism of algebras.

Bullock [Bul97] and Brumfiel-Hilden [BH] first independently proved this gives an isomorphism from $K(\Sigma,-1)$ modulo its nilradical to $\mathcal{M}'(\Sigma)$. Their work was later completed by Sikora [Si09] and independently by Charles and Marché [CM09] to give the previous theorem.

Finally, we end this preliminary section with a formula due to Goldman for products of elements of the Kauffman algebra at first order. We recall that $\mathcal{M}'(\Sigma)$ is a Poisson manifold for the Poisson structure given in [G86]. Using the previous theorem, the product of elements of $K(\Sigma,-1)$ corresponds to the product of trace functions on $\mathcal{M}'(\Sigma)$. By the works of Goldman [G86] and Turaev [TU91], the first order term in the product of elements in $K(\Sigma,-e^{\frac{i\pi\hbar}{2}})$ corresponds to the Poisson bracket of trace functions.

Theorem 1 : Let γ and δ be two multicurves, which can be viewed as elements of $K(\Sigma,-e^{\frac{i\pi\hbar}{2}}) = K(\Sigma,A) \underset{A=-e^{\frac{i\pi\hbar}{2}}}{\otimes} \mathbb{C}[[\hbar]]$. This space is isomorphic to a subspace of $\mathrm{Reg}(\mathcal{M}'(\Sigma))[[\hbar]]$ via the map σ of the previous theorem. Then we have:

$$\sigma(\gamma\cdot\delta) = f_\gamma f_\delta + \hbar\{f_\gamma, f_\delta\} + o(\hbar)$$

1.2 An asymptotic expression for matrix coefficients of curve operators

In this section, we fix a surface Σ with marked points $p_1,\ldots,p_n$. We pick $t_1,\ldots,t_n \in \mathbb{Q}\cap[0,1]$. and we define colorings of the marked points as $(\hat{c}_r)_i = rt_i$. Thus we will need to suppose that r is a multiple of D, the common denominator of the t_i, so that $(\hat{c}_r)_i$ are integers. For any multicurve γ, and r a mutiple of D, we recall that T_r^γ is the endomorphism of $V_r(\Sigma,\hat{c}_r)$ associated to γ by the TQFT. We fix a pants decomposition $\{C_e\}_{e\in E}$ of Σ and

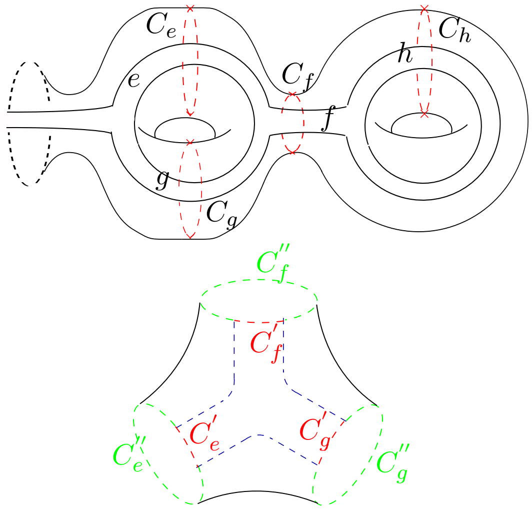


Figure 2: A banded graph compatible to a pants decomposition of Σ by the curve C_e and the associated cell decomposition of a pants into hexagons

a compatible trivalent banded graph Γ drawn on Σ, and thus a Hermitian basis $(\varphi_c)_{c\in U_r}$ of $V_r(\Sigma,\hat{c}_r)$ as explained above.

Note that the data of Γ give us a cell decomposition of Σ into a bunch of hexagons. The 1-skeleton corresponds to the boundary components of Γ and the curves C_e. For each $e\in E$, we name $C_e^{'}$ (respectively $C_e^{''}$) the segment $\Gamma\cap C_e$ (resp. $C_e\setminus \mathrm{Int}(C_e\cap\Gamma)$), see Figure 2.
We will associate to each admissible color c an element $\overline{c}$ of $H^1(\Sigma,\mathbb{Z}/2)$ by writing, for $\gamma\in H_1(\Sigma,\mathbb{Z}/2)$:

$$\overline{c}(\gamma)=\prod_{e\in E}(-1)^{(c_e-1)(C_e^{'*}(\gamma)+C_e^{''*}(\gamma))}$$

In this formula, $C_e^{'*}$ (resp. $C_e^{''*}$) is the cellular cochain dual to $C_e^{'}$ (resp. $C_e^{''}$). We can check that this formula defines indeed a cocycle as its value on the boundary of each hexagon is of the form $(-1)^{c_e+c_f+c_g-1}$ for e,f,g three adjacent edges, which equals to 1 as c is an admissible color.

We introduce a set U of "real admissible colorings", that we view as a set of limits of admissible colorings in U_r. It consists of maps $E\to(0,1)$ such that for any triple e,f, and g of edges of the graph associated to the pants decomposition that are adjacent to the same trivalent vertex, we have:

- $\tau_e+\tau_f<\tau_g$
- $\tau_e+\tau_f+\tau_g<2$
- for $e\in E$ adjacent to a marked point p_i we have $\tau_e=t_i$.

We can now give an expression of $T_r^{\gamma}\varphi_c$ as $\dfrac{c}{r}$ converges to an element $\tau\in U$ when $r\to+\infty$.

Theorem 1: Let γ be a multicurve in $\Sigma\setminus\{p_1,\dots,p_n\}$.

1. There is an open set $V_\gamma\subset U\times[0,1]$ containing $U\times\{0\}$ and functions $(F_k^\gamma)_{k:E\to\mathbb{Z}}$ that are smooth on V_γ such that we have for any $c\in U_r$,

$$T_r^\gamma\varphi_c=\overline{c}(\gamma)\sum_{k:E\to\mathbb{Z}}F_k^\gamma(\frac{c}{r},\frac{1}{r})\varphi_{c+k}$$

2. Let $I_e=\sharp(\gamma\cap C_e)$. If there exists $e\in E$ such that $k_e>I_e$ or such that $k_e\neq I_e(\mathrm{mod}\,2)$, then $F_k^\gamma=0$.

We remind that here, $\overline{c}$ is an element of $H^1(\Sigma,\mathbb{Z}/2)$, so that $\overline{c}(\gamma)$ is just a sign. So up to this oscillating sign which varies with the level r, the matrix coefficients of curves operators are converging when $r\to+\infty$. One might remark that this sign factor did not appear in [MP], but it can be shown that it is trivial when the banded trivalent graph Γ is planar (which was the case for the punctured torus and the four-holed sphere).
The two points of the theorem are proved by doing local computations, working in each pants of the decomposition of Σ. The proof is rather technical but not difficult and relies on fusion rules in TQFT. It will be detailed in Section 2.

The open sets V_γ are actually explicit: we will show that we can take

$$V_\gamma=\{(\tau,\hbar)\ /\ (\tau_e+\varepsilon_e\hbar I_e)_{e\in E}\in U\ ,\ \forall\varepsilon\in\{\pm1\}^E\}.$$

The coefficients F_k can be computed recursively for any surface Σ (together with pants decomposition $\mathcal{C}$ and trivalent banded graph Γ), but are uneasy to make explicit. However, we will provide a formula to get the first two term F_k^0 and F_k^1 of their asymptotic expansion in orders of $\hbar=\dfrac{1}{r}$:

$$F_k(\tau,\hbar)=F_k^0(\tau)+\hbar F_k^1(\tau)+o(\hbar)$$

To state our asymptotic formula, we have yet another definition to give.

For Γ a trivalent banded graph on Σ compatible with the pants decomposition $\mathcal{C}$, we define the *intersection algebra* A_Γ as

$$A_\Gamma=\bigoplus_{\alpha\in H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2)}\mathbb{C}\cdot[\alpha]$$

with the product structure given by $[\gamma]\cdot[\delta]=(-1)^{\langle\gamma,\delta\rangle}[\gamma+\delta]$. Here $\langle\cdot,\cdot\rangle$ shall be understood as the intersection form in $H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2)$. The associativity of the product in A_Γ follows easily from the bilinearity of the intersection form.
Note that A_Γ is a commutative (the intersection form in $H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2)$ is symmetric) $\mathbb{C}$-algebra of dimension 2^g, where g is the genus of Σ. Define $\hat{A}_\Gamma$ the set of characters $A_\Gamma\to\mathbb{C}$, then A_Γ is isomorphic to $\mathbb{C}^{\hat{A}_\Gamma}$, by the map $a\in A_\Gamma\mapsto(\chi(a))_{\chi\in\hat{A}_\Gamma}$.
Remember that Γ and $\mathcal{C}$ give a cell decomposition of Σ into hexagons, with two hexagons

associated to each trivalent vertex of Γ. In fact, we have a 2:1 covering $p:\Sigma\to\Gamma$ branched along $\partial\Gamma$ obtained by identifying the hexagons associated to the same vertex, and we write p_* for the map $H_1(\Sigma,\mathbb{Z}/2)\to H_1(\Gamma,\partial\Gamma,\mathbb{Z}/2)$ induced by p.
We can now turn to the statement of our asymptotic formula for the matrix coefficients F_k of curve operators:

Theorem 2 : Let γ be a multicurve on Σ. For all $\theta\in(\mathbb{R}/2\pi\mathbb{Z})^E$ and $(\tau,\hbar)\in V_\gamma$, we define the ψ-symbol of T_r^γ as the following element of A_Γ:

$$\sigma^\gamma(\tau,\theta,\hbar)=\sum_{k:E\to\mathbb{Z}}F_k(\tau,\hbar)e^{ik\cdot\theta}[p_*(\gamma)]$$

Then, $\sigma^\gamma(\tau,\theta,\hbar)$ has the following asymptotic expansion:

$$\sigma^\gamma(\tau,\theta,\hbar)=\sigma^\gamma(\tau,\theta,0)+\frac{\hbar}{2i}\sum_{e\in E}\frac{\partial^2}{\partial\tau_e\partial\theta_e}\sigma^\gamma(\tau,\theta,0)+o(\hbar)$$

and writing $\sigma^\gamma_\chi(\tau,\theta)=\chi(\sigma^\gamma(\tau,\theta,0))$, we have $\sigma^\gamma_\chi(\tau,\theta)=f_\gamma(R_\chi(\tau,\theta))=-\mathrm{Tr}(R_\chi(\tau,\theta)(\gamma))$, where the R_χ are action-angle parametrizations on the representation space $\mathcal{M}(\Sigma)=\mathrm{Hom}(\pi_1(\Sigma\setminus\{p_1,\dots,p_n\}),\mathrm{SU}_2)/\mathrm{SU}_2$, differing only by the origin of angles.
More precisely, there is a moment mapping on $\mathcal{M}(\Sigma)$ given by

$$\begin{array}{rccl} h: & \mathcal{M}(\Sigma) & \mapsto & U \\ & \rho & \mapsto & (h_{C_e}(\rho)=\frac{1}{\pi}\mathrm{Acos}(\frac{1}{2}\mathrm{Tr}(\rho(C_e))))_{e\in E}\end{array}$$

The variables h_{C_e} are independent Poisson commuting functions on $\mathcal{M}(\Sigma)$. By a result of Goldman (see [G86]), their Hamiltonian flow give an action of a torus on $\mathcal{M}(\Sigma)$, and thus induce *angle coordinates* θ_e on each level set of the h_{C_e}, so that (τ_e,θ_e) (where we write $\tau_e=h_{C_e}(\rho)$) are canonical coordinates on $\mathcal{M}_{\mathrm{irr}}(\Sigma)$, the subset of irreducible representations.

Let us add a few remarks on the definition of the ψ-symbol:

1. To begin with, the sum over $k:E\to\mathbb{Z}$ is actually a finite sum, as only a finite number of coefficients F_k^γ do not vanish by the second point of Theorem 1. Furthermore, we wrote $k\cdot\theta$ for $\sum_{e\in E}k_e\theta_e$. Also, we will often omit the p_* and just write $[\gamma]$ for the element $[p_*(\gamma)]$, when γ is a multicurve.
2. We can recover the matrix coefficents F_k^γ from the ψ-symbol by taking Fourier coefficients of $\sigma^\gamma(\tau,\cdot,\hbar)$.
3. We will often refer to the zero order in $\hbar$ of the ψ-symbol, that is $\sigma^\gamma(\tau,\theta,0)$, as the principal symbol of T_r^γ.
4. For a fixed $(\tau,\theta,\hbar)$, this definition only introduce $\gamma\mapsto\sigma^\gamma(\tau,\theta,\hbar)$ as a map from multicurves to A_Γ. We extend it by linearity to obtain $\sigma(\tau,\theta,\hbar)\ :\ K(\Sigma,-e^{\frac{i\pi\hbar}{2}})\to A_\Gamma[[\hbar]]$, as $K(\Sigma,-e^{\frac{i\pi\hbar}{2}})$ is spanned by multicurves.

The proof of Theorem 2, giving an asymptotic formula for the ψ-symbol, will be the goal of Sections 3 and 4. It will rely heavily on the following property of the ψ-symbol, that explains its compatibility with the product in $K(\Sigma, -e^{\frac{i\pi\hbar}{2}})$:

Proposition 1.1 : Let γ and δ be two multicurves on Σ. Then we have the following asymptotic expression:

$$\sigma^{\gamma\cdot\delta}(\tau,\theta,\hbar) = \left(\sigma^{\gamma}(\tau,\theta,\hbar)\sigma^{\delta}(\tau,\theta,\hbar) + \frac{\hbar}{i}\sum_{e}\partial_{\tau_e}\sigma^{\gamma}(\tau,\theta,\hbar)\partial_{\theta_e}\sigma^{\delta}(\tau,\theta,\hbar)\right) + o(\hbar)$$

According to this proposition, the principal symbol $\sigma(\tau,\theta,0) : K(\Sigma,-1) \to A_\Gamma$ is a morphism of algebras. Recall that there is an isomorphism between the algebras A_Γ and $\mathbb{C}^{\hat{A}_\Gamma}$, the components of which are characters that we denoted by $\chi : A_\Gamma \to \mathbb{C}$. Then the maps $\sigma_\chi = \chi \circ \sigma(\tau,\theta,0) : K(\Sigma,-1) \to \mathbb{C}$ constitutes a collection of algebra morphisms.
We will then use the theorem of Bullock to show that these morphisms have the form $f \in \mathrm{Reg}(\mathcal{M}'(\Sigma)) \to f(R_\chi)$, for some representations R_χ of $\pi_1(\Sigma \setminus \{p_1,\dots p_n\})$ into $\mathrm{SL}_2(\mathbb{C})$.

Finally, we identify precisely the representations R_χ and how they depend on (τ,θ) by checking the special values of the ψ-symbol on the curves C_e.

As for the computation of the first order term, we proceed in a similar fashion: first we will show, using only Proposition 1.1 that this term is related to derivations of algebras $K(\Sigma,-1) \to A$, then by studying the values of the ψ-symbol on the curves C_e and on another family of curves D_e, we show the first order term is indeed given by the formula in Theorem 2.

2 Computations of curve operators using fusion rules

This section is devoted to the skein theory computations that will be needed in order to prove Theorem 1. We describe the general form of matrix coefficents of curve operators, and give examples of explicit computations of the coefficients F_k^γ and the ψ-symbol σ^γ for some curves γ.

2.1 Fusion rules in a pants decomposition

In this paragraph, we will work with a fixed surface Σ, along with a pants decomposition by a family of curves $\mathcal{C} = \{C_e\}_{e\in E}$. We can consider $n_e \geq 1$ parallel copies $(C_e^k)_{1\leq k\leq n_e}$ of the curves C_e so that the curves C_e^k cut the surface Σ into a collection of pants $\{P_s\}_{s\in S}$ and annuli $\{A_e^k \ , e \in E \ , \ \ 1 \leq k \leq n_e - 1\}$.
We recall that to this pants decomposition is associated a Hermitian basis φ_c of $V_r(\Sigma)$, of which we remind the construction:

Let Γ be a banded trivalent graph compatible to the pants decomposition $\mathcal{C}$ of Σ as in Subsection 1.1. We recall that Γ is viewed as drawn on Σ. Given an admissible coloring $c \ : \ E \to C_r$, we define $\psi_c \in K(\Sigma;\hat{c};\zeta_r)$ as follows:

- Replace each edge e of Γ by $c_e - 1$ parallel copies of e lying on Σ.

- Insert in the middle of each edge the idempotent f_{c_e-1} where we write f_k for the k-th Jones-Wenzl idempotent (see [BHMV] for details).

- In the neighborhood of each trivalent vertex, join the three bunches of lines in Σ in the unique possible way avoiding crossings.

This family of vectors is actually an orthogonal basis of $V_r(\Sigma, \hat{c})$ for a natural Hermitian structure defined in [BHMV], that we do not recall here. We refer to Theorem 4.11 in [BHMV] for the proof and the following formula:

$$||\psi_c||^2 = (\frac{2}{r})^{\frac{\chi(\Gamma)}{2}} \frac{\prod_P \langle c_P^1, c_P^2, c_P^3 \rangle}{\prod_e \langle c_e \rangle} \tag{1}$$

Here the first product is over all vertex P corresponding to pants of the pants decomposition, the second over the edges e of the graph Γ. We write $\langle n \rangle$ for $\sin(\frac{\pi n}{r})$ and $\langle n \rangle!$ for $\prod_{i=1}^{n} \langle i \rangle$. For c_P^1, c_P^2, and c_P^3 the colors of the 3 edges adjacent to P, and we also set

$$\langle a, b, c \rangle = \frac{\langle \frac{a+b+c-1}{2} \rangle ! \langle \frac{a+b-c-1}{2} \rangle ! \langle \frac{a-b+c-1}{2} \rangle ! \langle \frac{b+c-a-1}{2} \rangle !}{\langle a-1 \rangle ! \langle b-1 \rangle ! \langle c-1 \rangle !}.$$

As we will work with TQFT vectors locally, inside a pants of the pants decomposition for example, we will need to give a local version of this norm. Notice that if we forget the global factor $(\frac{2}{r})^{\frac{\chi(\Gamma)}{2}}$ in the norm, we will not change the matrix coefficients of curve operators T_r^γ. Then, we will decide that the square of the norm of a trivalent graph is

$$\frac{\prod_P \langle c_P^1, c_P^2, c_P^3 \rangle}{\prod_{e \in E_2} \langle c_e \rangle \prod_{e \in E_1} \langle c_e \rangle^{\frac{1}{2}}}$$

where the products in the denominator are over E_2, the set of edges adjacent to 2 internal vertices (we include the marked points here), and E_1 the set of edges adjacent to 1 internal vertex and 1 external vertex (the other edges bear no contribution to the norm). With this definition, if we paste pieces of colored graph to get the graph Γ, we obtain the previous norm.

With this setting, we give a normalized version of fusion rules in TQFT. The fusion rules derived in [MV], give a way to compute the image of the vector φ_c by curves operators. We list the fusion rules that we will need below; our version differs from the rules in [MV], as we express them with the normalized vectors φ_c instead of the vectors ψ_c.

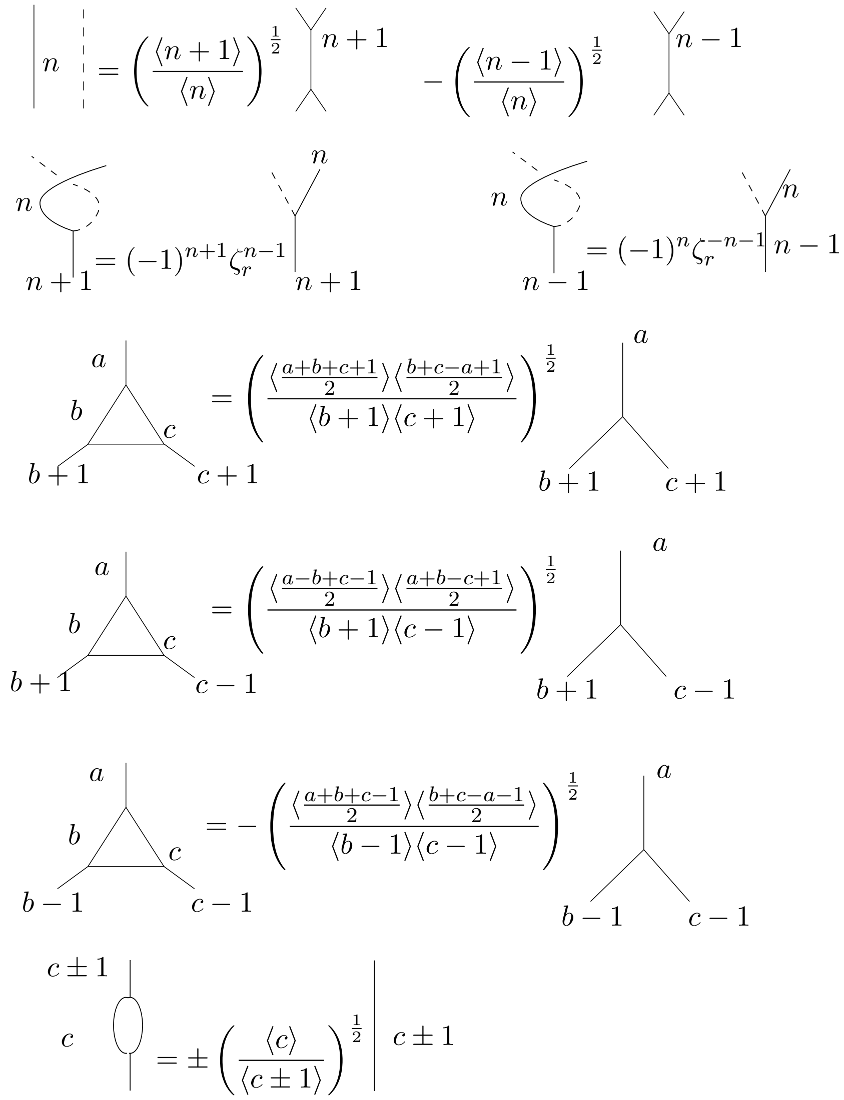


Figure 3: Fusion rules

These "normalized" fusion rules allow to simplify the union of a colored banded graph and a curve colored by 2. All edges for which we did not write a color are colored by 2. The first rule allows to merge an edge colored by 2 with an other one. The second line consists of the "half-twist formulas" of [MV]. When all curves have been merged with the graph, the 3rd, 4th and 5th lines can be used to remove trigons, and the last rule to remove bigons.

We will carry out the computations by using fusion rules only locally, that is only inside of a pair of pants of the pants decomposition , or inside an annulus in the neighborhood of one of the curves C_e.

Indeed, for γ a multicurve, by a classification provided by Dehn, we can suppose up to isotopy that the intersection of γ with each pants P_s of the decomposition looks like the 4th picture of Figure 4, and the intersection with each of the annulus A_e^k looks like one of the first three pictures of Figure 4.

Furthermore, in this isotopy class, the intersection of γ with each C_e is the smallest in the

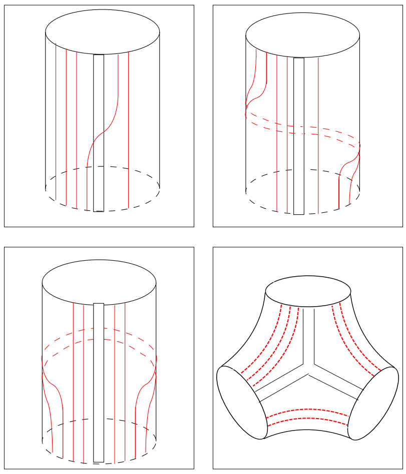
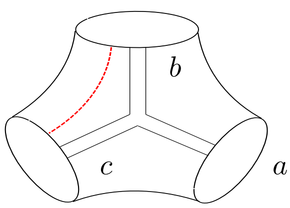


Figure 4: Dehn presentation of multicurves

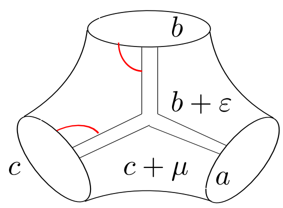
isotopy class of γ, see [FLP].

Now, we do the computations in two steps:
First, we use fusion rules to reduce each type of piece to elements corresponding to the trace of the graph Γ in a pants or annulus with a certain coloring, glued with "candlesticks".
We call a *candlestick* an element of the TQFT of an annulus that is the normalized vector associated to a banded trivalent graph in an annulus, consisting of a central edge joining the boundary components (with no twist), colored by $n \in \mathcal{C}_r$ on the bottom component, a collection of legs colored by 2, joining the central edge and the bottom component, as in Figure 5. The data that define a candlestick with k legs $C(n, \varepsilon, \Theta)$ is the color $n \in \mathcal{C}_r$ of the central edge at the bottom, the order Θ in which the legs join the central edge, and the shifts of the color of the central edge $(\varepsilon_i)_{i=1\ldots k}$ when we pass each vertex corresponding to a leg.

Reduction of the different pieces : Simple computations using fusion rules give us the following formulas when the pants or the annulus contain only one curve:



$$= \sum_{\varepsilon,\mu} F_{\varepsilon,\mu}(a,b,c,r)$$



where we set $F_{+,+}(a,b,c,r) = \left(\dfrac{\langle \frac{a+b+c+1}{2} \rangle \langle \frac{b+c-a+1}{2} \rangle}{\langle b \rangle \langle c \rangle} \right)^{\frac{1}{2}}$,

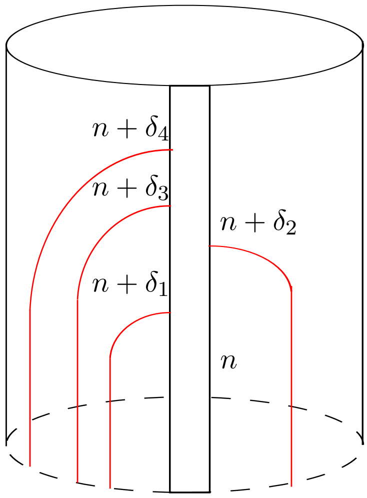
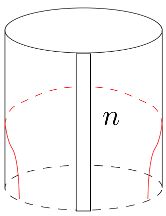


Figure 5: A candlestick $C(n,\varepsilon,\theta)$ with 4 legs

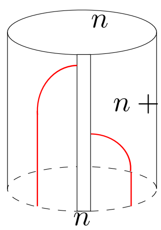
We wrote $\delta_i = \sum_{j=1}^{i} \varepsilon_j$ the partial sums of the shifts. In the bottom the central edge is colored by n, and the color is shifted by ε_i when Γ meets the i-th leg. Notice that the legs can go alternatively to the left or to the right of the central edge

$$F_{+,-}(a,b,c,r) = F_{-,+}(a,c,b,r) = -\left(\frac{\langle\frac{a-b+c-1}{2}\rangle\langle\frac{a+b-c-1}{2}\rangle}{\langle b\rangle\langle c\rangle}\right)^{\frac{1}{2}}$$

and $F_{-,-}(a,b,c,r) = -\left(\frac{\langle\frac{a+b+c-1}{2}\rangle\langle\frac{b+c-a-1}{2}\rangle}{\langle b\rangle\langle c\rangle}\right)^{\frac{1}{2}}$

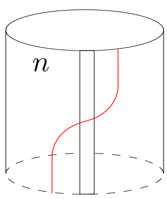


$$= \sum_{\varepsilon} G_\varepsilon(n,r)$$

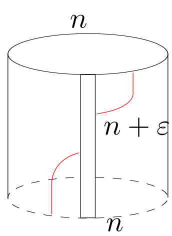


where $G_+(n,r) = (-1)^{n+1}\zeta_r^{-n+1}\left(\frac{\langle n+1\rangle}{\langle n\rangle}\right)^{\frac{1}{2}}$

and $G_-(n,r) = (-1)^{n+1}\zeta_r^{n+1}\left(\frac{\langle n-1\rangle}{\langle n\rangle}\right)^{\frac{1}{2}}$

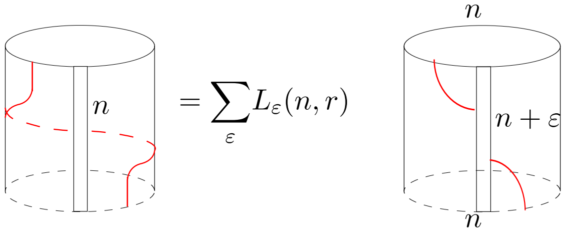


$$= \sum_{\varepsilon} H_\varepsilon(n,r)$$



where $H_+(n,r) = (-1)^{n+1}\zeta_r^{n-1}\left(\frac{\langle n+1\rangle}{\langle n\rangle}\right)^{\frac{1}{2}}$

and $H_-(n,r) = (-1)^{n+1}\zeta_r^{-n-1}\left(\frac{\langle n-1\rangle}{\langle n\rangle}\right)^{\frac{1}{2}}$



where $L_+(n,r) = (-1)^{n+1}\zeta_r^{n+2}\left(\frac{\langle n+1\rangle}{\langle n\rangle}\right)^{\frac{1}{2}}$

and $L_-(n,r) = (-1)^{n+1}\zeta_r^{-n+2}\left(\frac{\langle n-1\rangle}{\langle n\rangle}\right)^{\frac{1}{2}}$

All these coefficients are of the required form $\overline{c}(\gamma)F(\frac{c}{r},\frac{1}{r})$
where F is a smooth function defined on $V_\gamma = \{(\tau,\hbar)\ /\ \tau_e \pm \hbar I_e \in U\}$, and $\overline{c}(\gamma)$ is a sign factor, corresponding to taking the holonomy $\mathrm{Hol}(\varepsilon,\gamma)$ of the cocyle $\overline{c}$ with values in $\{\pm 1\}$ along γ already defined in Section 1 and displayed in Figure 6.

If we have many curves in a pants or annulus, we only need to choose an order to make

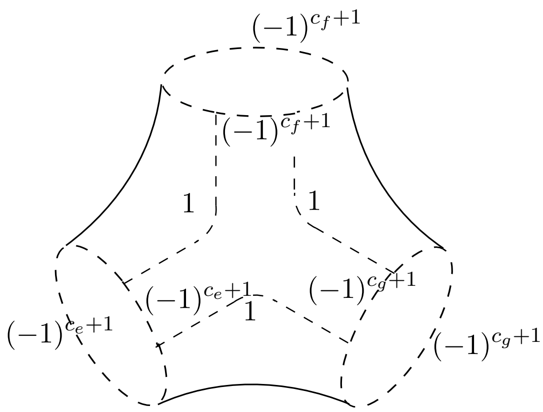
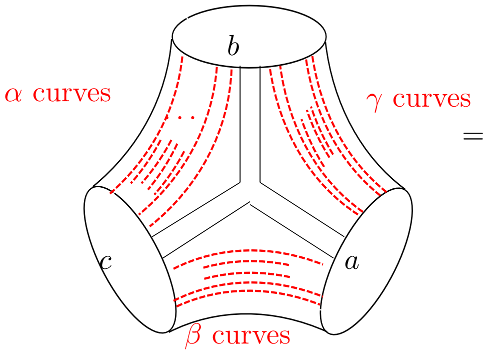
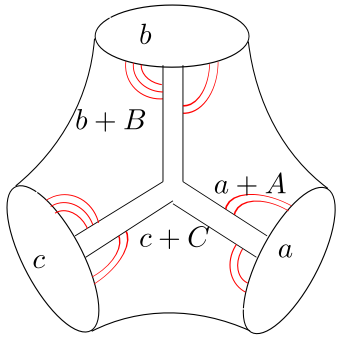


Figure 6: The cocycle $\overline{c}$ on the pants bounded by the curves C_e, C_f and C_g

the fusions, and apply the latter formulas. For example, in the case of the pants, we obtain:



$$= \sum_{\varepsilon,\mu,\nu} P_{\varepsilon,\mu,\nu}(a,b,c,r)$$



where we wrote $A = \sum_{i=1}^{\beta+\gamma} \varepsilon_i$, $B = \sum_{j=1}^{\alpha+\gamma} \mu_j$, and $C = \sum_{k=1}^{\alpha+\beta} \nu_k$

Here we have first used fusion on the α curves that go from C_b to C_c, then the β curves that run from C_a to C_c, and finally the γ curves from C_a to C_c. With this order for the fusions, the coefficients $P_{\varepsilon,\mu,\nu}(a,b,c,r)$ are products of three factors corresponding to each batch of fusions:

$$F_{\mu_1,\nu_1}(a,b,c,r)F_{\mu_2,\nu_2}(a,b+\mu_1,c+\nu_1,r)\dots F_{\mu_\alpha,\nu_\alpha}(a,b+\sum_{i=1}^{\alpha-1}\mu_i,c+\sum_{i=1}^{\alpha-1}\nu_i,r)$$

$$F_{\nu_{\alpha+1},\varepsilon_1}(b+\sum_{i=1}^{\alpha}\mu_i,a,c+\sum_{i=1}^{\alpha}\nu_i,r)\dots F_{\nu_{\alpha+\beta},\varepsilon_\beta}(b+\sum_{i=1}^{\alpha}\mu_i,a+\sum_{i=1}^{\beta-1}\varepsilon_i,c+\sum_{i=1}^{\alpha+\beta-1}\nu_i,r)$$

$$F_{\mu_{\alpha+1},\varepsilon_{\beta+1}}(c+\sum\nu,b+\sum_{i=1}^{\alpha}\mu_i,a+\sum_{i=1}^{\beta}\varepsilon_i,r)\dots F_{\mu_{\alpha+\gamma},\varepsilon_{\beta+\gamma}}(c+\sum\nu,b+\sum_{i=1}^{\alpha+\gamma-1}\mu_i,a+\sum_{i=1}^{\beta+\gamma-1}\varepsilon_i,r)$$

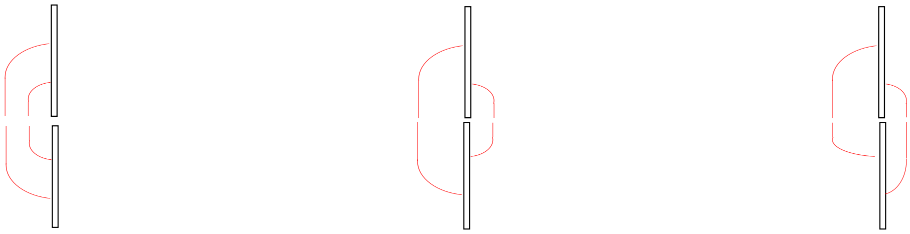
Notice that at every step of the fusion, the shifts in the color c_e are sums of ± 1 terms, one term for each arc intersecting C_e that has been merged with Γ. Hence the coefficients $P_{\varepsilon,\mu,\nu}$ is defined and smooth on the required domain $V_\gamma = \{(\tau,\hbar)\ /\ \tau_e \pm I_e^\gamma \hbar \in U\}$. Furthermore, at the end the shift in c_e is not greater than the number of curves that intersect C_e and of same parity as this number.

We now only need to explain what happens when we glue together two candlesticks.
First, remark that we can only paste candlesticks with the same number of legs, and the same bottom color n. Moreover, if we paste two candlesticks $C(n,\varepsilon,\Theta)$ and $C(n,\mu,\Theta')$ with $\sum_j \mu_j \neq \sum_i \varepsilon_i$, then we obtain always 0 (as the vector space $V_r(\Sigma)$ of a sphere Σ with two points marked by different colors is 0).

Proposition 2.1: The gluing of two candlesticks with k legs $C(n,\varepsilon,\Theta)$ et $C(n,\mu,\Theta')$ with $\sum_{i=1}^{k} \varepsilon_i = \sum_{j=1}^{k} \mu_j$ is proportionnal to a band colored by $n + \sum \varepsilon_i$ joining the two boundary components of the annulus with no twist, the proportionality constant being $G(\frac{n}{r},\frac{1}{r})$, where G is a smooth function on $\{(\tau,\hbar)\ /\ \tau \pm k\hbar \in (0,1)\}$.

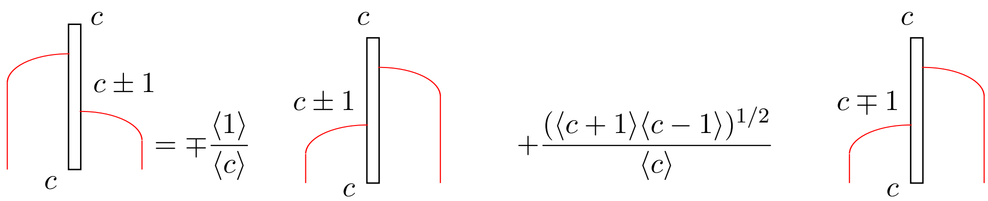
We should point out that in this proposition, the function G depends on Θ, Θ', ε and μ.

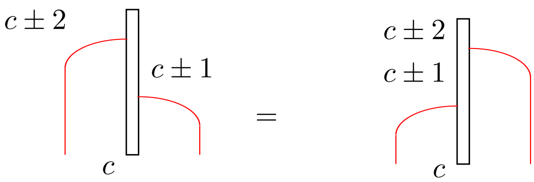
Proof : We prove this proposition by induction on the number of legs of the candlestick. If we paste two candlesticks with only one leg,we get directly the result from the fusion rule that eliminate bigons (see Figure 3), adding only a factor $(\frac{\langle c \pm 1\rangle}{\langle c\rangle})^{\frac{1}{2}}$. Now, if $n = 2$, the only delicate case is when the legs of the two part are disposed as in the third case of the figure below :



Case 1 Case 2 Case 3

Indeed, in case 1 and 2, we could simply eliminate two bigons. For the case 3, we use the following switching legs formulas:





This shows proposition (2.1) for $k \leq 2$.

Now, suppose we glue two candlesticks with $k + 1$ legs. We have two cases as in Figure 7:

In case 1, the upper leg of the upper candlestick and the bottom leg of the bottom candlestick both go to the right (or both to the left), the gluing is obtained by gluing two candlesticks with k legs, then suppressing a bigon. The factor we get is of the form $G(\frac{n}{r}, \frac{1}{r})\left(\frac{\langle n + \sum_{i=1}^{k+1}\varepsilon_i\rangle}{\langle n + \sum_{i=1}^{k}\varepsilon_i\rangle}\right)^{\frac{1}{2}}$, which is indeed a function of $(\frac{n}{r}, \frac{1}{r})$ that is smooth on the domain we claimed.

On the contrary, in case 2, the upper leg of the upper part and the bottom leg of the bottom part go to different sides, but by applying the formulas to switch legs, we can reduce

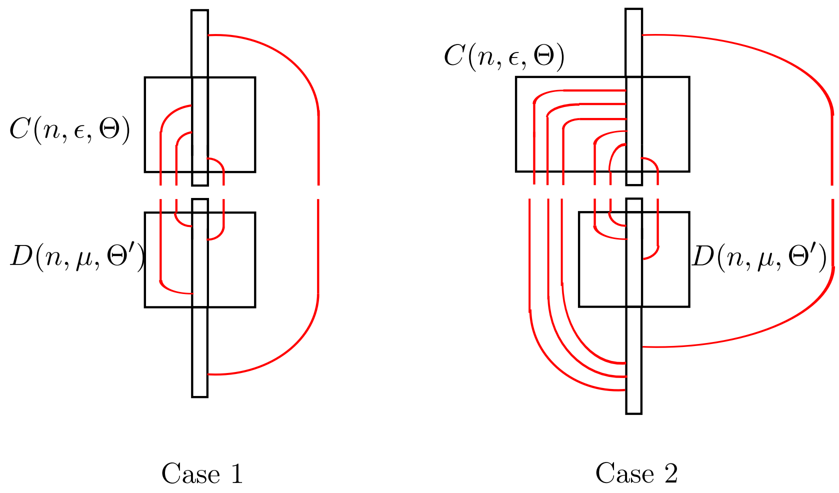


Figure 7: The two cases of pasting candlesticks with k legs

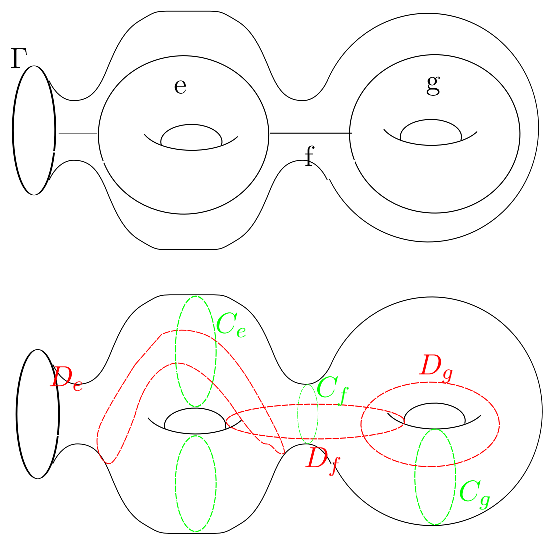


Figure 8: The family of curves D_e on Σ associated to the trivalent banded graph Γ

this to the former case.

2.2 Examples of ψ-symbol

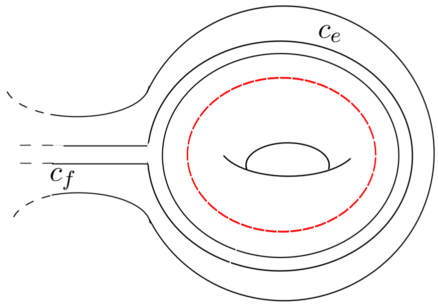
We derive expressions of ψ-symbols for two families of curves on Σ: the first family consists of the curves C_e of the pants decompostion itself, and the other of curves $(D_e)_{e\in E}$ that are in some sense dual to the curves C_e. The D_e are defined this way: if e is an internal edge that joints a vertex to itself, then D_e is a loop parallel to e. If e joints two different vertices, then D_e consists of two arcs parallel to e that we close into a loop as in Figure 8.

Note that C_e and D_f intersect each other if and only if $e = f$, and in this case they intersect once or twice, and that $p(C_e)$ and $p(D_e)$ are null homologous in $H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$.

Proposition 2.2: We have for any $e \in E$ and $c \in U_r$:

1. $T_r^{C_e}\varphi_c = -2\cos(\pi\frac{c_e}{r})\varphi_c$ et $\sigma^{C_e}(\frac{c}{r},\theta,\frac{1}{r}) = -2\cos(\pi\frac{c_e}{r})[0]$

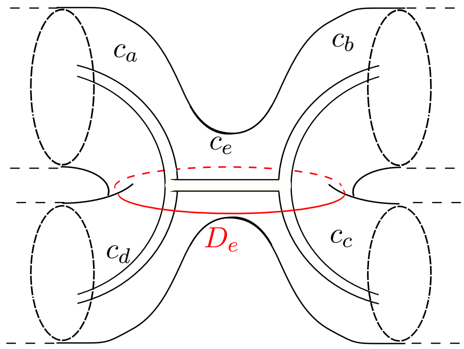
2. In the case where e is an edge joining an internal vertex to itself as in the picture:



we have $\sigma^{D_e}(\frac{c}{r},\theta,\frac{1}{r}) = \left(W(\pi\frac{c_e}{r},\pi\frac{c_f}{r},\frac{\pi}{r})e^{i\theta_e} + W(\pi\frac{c_e}{r},\pi\frac{c_f}{r},-\frac{\pi}{r})e^{-i\theta_e}\right)[0]$

where $W(\tau,\alpha,\hbar) = \left(\frac{\sin(\tau+\alpha/2+\hbar/2)\sin(\tau-\alpha/2+\hbar/2)}{\sin(\tau)\sin(\tau+\hbar)}\right)^{\frac{1}{2}}$

3. In the case where e is an edge between two distinct internal vertices as in the picture:



we have $\sigma^{D_e}(\frac{c}{r},\theta,\frac{1}{r}) = -\left(I(\pi\tau,\pi\hbar) + J(\pi\tau,\pi\hbar)e^{2i\theta_e} + J(\pi(\tau-2\hbar\delta_e),\pi\hbar)e^{-2i\theta_e}\right)[0]$

Here, we have set $\tau = \frac{c}{r}$, $\hbar = \frac{1}{r}$, δ_e for the Kronecker symbol,

$$I(\tau,\hbar) = 2\cos(\tau_c+\tau_d-\hbar)$$

$$+4\frac{\sin(\frac{\tau_a+\tau_d-\tau_e-\hbar}{2})\sin(\frac{\tau_a-\tau_d+\tau_e+\hbar}{2})\sin(\frac{\tau_b+\tau_c-\tau_e-\hbar}{2})\sin(\frac{\tau_b-\tau_c+\tau_e+\hbar}{2})}{\sin(\tau_e)\sin(\tau_e+\hbar)}$$

$$+4\frac{\sin(\frac{\tau_a+\tau_d+\tau_e-\hbar}{2})\sin(\frac{-\tau_a+\tau_d+\tau_e-\hbar}{2})\sin(\frac{\tau_b+\tau_c+\tau_e-\hbar}{2})\sin(\frac{-\tau_b+\tau_c+\tau_e-\hbar}{2})}{\sin(\tau_e)\sin(\tau_e-\hbar)}$$

and

$$J(\tau,\hbar) = 4(\frac{\sin(\frac{\tau_a+\tau_d-\tau_e-\hbar}{2})\sin(\frac{\tau_a-\tau_d+\tau_e+\hbar}{2})\sin(\frac{\tau_b+\tau_c-\tau_e-\hbar}{2})\sin(\frac{\tau_b-\tau_c+\tau_e+\hbar}{2})}{\sin(\tau_e)\sin(\tau_e+\hbar)}$$

$$\frac{\sin(\frac{\tau_a+\tau_d+\tau_e+\hbar}{2})\sin(\frac{-\tau_a+\tau_d+\tau_e+\hbar}{2})\sin(\frac{\tau_b+\tau_c+\tau_e+\hbar}{2})\sin(\frac{-\tau_b+\tau_c+\tau_e+\hbar}{2})}{\sin(\tau_e+\hbar)\sin(\tau_e+2\hbar)})$$

The expressions of $T_r^{C_e}$ and $T_r^{D_e}$ can be derived by using fusion rules. The computations are rather long in the last case, but straighforward.
These expressions, as well as the expressions of the ψ-symbol of the curves C_e and D_e were already given in [MP]. They also checked by hand that the formulas of Theorem 2 were satisfied by these curves. We will only derive from the formulas that the zero-th and first order term for these curves are related as in Theorem 2, as this is rather quick, and will come of use later:

Proposition 2.3: Let γ be any of the curves C_e or D_e.
Then $\sigma^\gamma(\tau,\theta,\hbar) = \sigma^\gamma(\tau,\theta,0) + \frac{\hbar}{2i}\sum_{e\in E}\frac{\partial^2}{\partial\tau_e\partial\theta_e}\sigma^\gamma(\tau,\theta,0) + o(\hbar)$

Proof : For C_e, there is not much to prove: as σ^{C_e} does not depend on $\hbar$, the first order term vanishes, and as σ^{C_e} does not depend on θ_e, $\frac{\partial^2}{\partial\tau_e\partial\theta_e}\sigma^\gamma(\tau,\theta,0)$ also vanishes.
For the curves D_e, we need to separate the case 1 where e joints a vertex to itself, and the case 2 where it joints two distinct vertices.
In case 1 depicted by the figure above, we have
$\sigma^{D_e}(\tau,\theta,\hbar) = (W(\pi\tau_e,\pi\tau_f,\pi\hbar)e^{i\theta_e} + W(\pi\tau_e,\pi\tau_f,-\pi\hbar)e^{-i\theta_e})[0]$. Notice that from the formula of W given above, we get $W(\pi\tau_e,\pi\tau_f,\pi\hbar) = W(\pi(\tau_e+\frac{\hbar}{2}),\pi\tau_f,0)+o(\hbar)$. Thus

$$\sigma^{D_e}(\tau,\theta,\hbar) = \sigma^{D_e}(\tau,\theta,0) + \frac{\hbar}{2}(W(\pi\tau_e,\pi\tau_f,0)e^{i\theta_e} - W(\pi\tau_e,\pi\tau_f,0)e^{-i\theta_e})[0] + o(\hbar)$$

$$= \sigma^{D_e}(\tau,\theta,0) + \frac{\hbar}{2i}\sum_{e\in E}\frac{\partial^2}{\partial\tau_e\partial\theta_e}\sigma^{D_e}(\tau,\theta,0) + o(\hbar)$$

as expected.
Finally, in the case 2 above, we have
$\sigma^{D_e}(\tau,\theta,\hbar) = -(I(\pi\tau,\pi\hbar) + J(\pi\tau,\pi\hbar)e^{2i\theta_e} + J(\pi(\tau-2\hbar\delta_e),\pi\hbar)e^{-2i\theta_e})[0]$. It is easily seen that $J(\tau,\hbar) = J(\tau+\hbar\delta_e,0)$ for δ_e Kronecker symbol. Thus the only thing we need is to prove that $I(\tau,\hbar) = I(\tau,0)+o(\hbar)$. This is a bit more tricky:
First, notice that we can write

$$I(\tau,\hbar) = 2\cos(\tau_c+\tau_d-\hbar) + \frac{1}{\sin(\tau_e)}(F(\tau_e+\hbar) - F(-\tau_e+\hbar)) + o(\hbar)$$

where

$$F(\tau_e) = 4\frac{\sin(\frac{\tau_a+\tau_d-\tau_e}{2})\sin(\frac{\tau_a-\tau_d+\tau_e}{2})\sin(\frac{\tau_b+\tau_c-\tau_e}{2})\sin(\frac{\tau_b-\tau_c+\tau_e}{2})}{\sin(\tau_e)}$$

$$= \frac{(\cos(\tau_d-\tau_e)-\cos(\tau_a))(\cos(\tau_c-\tau_e)-\cos(\tau_b))}{\sin(\tau_e)}$$

Therefore, the first order term of $I(\tau,\hbar)$ is $\hbar\left(2\sin(\tau_c+\tau_d) + \frac{2}{\sin(\tau_e)}\frac{d}{d\tau_e}\mathcal{P}(F)(\tau_e)\right)$, where

$\mathcal{P}(F)$ is the even part of the function F. From the formula above, we have

$$\mathcal{P}(F)(\tau_e) = \sin(\tau_c + \tau_d)\cos(\tau_e) - \cos(\tau_a)\sin(\tau_c) - \cos(\tau_b)\sin(\tau_d)$$

So that $\frac{1}{\sin(\tau_e)}\frac{d}{d\tau_e}\mathcal{P}(F)(\tau_e) = -\sin(\tau_c + \tau_d)$, and the first order of $I(\tau, \hbar)$ vanishes.

The computations of σ^{C_e} and σ^{D_e} were previously used by [MP] to prove a version of Theorem 2 for the punctured torus and the 4-holed sphere. Their approach was to derive from the formulas that the asymptotic estimate of Theorem 2 was valid for the curves C_e, D_e and $\tau_{C_e}(D_e)$ where τ_{C_e} denotes the Dehn twist along C_e. Then they used the compatibility of the ψ-symbol with the product in $K(\Sigma, -e^{\frac{i\pi\hbar}{2}})$ to prove that if Theorem 2 is verified by γ and δ two multicurves, then it is also true for their product $\gamma \cdot \delta$. This yielded Theorem 2 for all multicurves in the punctured torus and the 4-holed sphere, as the curves C_e, D_e, and $\tau_{C_e}(D_e)$ were sufficient to generate the Kauffman algebra.

However, this approach fails in higher genus, as this set of curves no longer generate the Kauffman algebra. Therefore, we develop an other approach to tackle the higher genus cases, which is also more conceptual and requires less computations. Our fundamental idea is to use the multiplicativity of the ψ-symbol together with the theorem of Bullock (reminded in Section 1) to view the zero-th and first order term of the ψ-symbol in terms of algebra morphism or derivation of algebras on $\mathrm{Reg}(\mathcal{M}'(\Sigma))$. We then only need to compare this general shape with the values of the ψ-symbol on a few curves to get the formula of Theorem 2. (In fact, we will only need the values on the C_e for the zero order term, while the first order term also required the values on D_e)

3 Principal symbol and representation spaces

This section will focus on the study of the principal symbol $\sigma^\gamma(\tau, \theta, 0)$, that is the zero-order of the ψ-symbol $\sigma^\gamma(\tau, \theta, \hbar)$. The goal of this paragraph is to establish the formula for the principal symbol, which is stated in the main theorem: $\sigma^\gamma_\chi(\tau, \theta, 0) = f_\gamma(R_\chi(\tau, \theta))$, where f_γ is the function on $\mathcal{M}(\Sigma)$ such that $f_\gamma(\rho) = -\mathrm{Tr}(\rho(\gamma))$, and R_χ is an action-angle parametrization of $\mathcal{M}(\Sigma)$.

3.1 The intersection algebra A_Γ and multiplicativity of the ψ-symbol

The aim of this section is to prove the property of compatibility of the ψ-symbol with the multiplication in $K(\Sigma, -1)$, given by Proposition 1.1:

Proposition 1.1 : Let γ and δ be two multicurves on Σ. Then we have the following asymptotic expression:

$$\sigma^{\gamma\cdot\delta}(\tau, \theta, \hbar) = \left(\sigma^\gamma(\tau, \theta, \hbar)\sigma^\delta(\tau, \theta, \hbar) + \frac{\hbar}{i}\sum_e \partial_{\tau_e}\sigma^\gamma(\tau, \theta, \hbar)\partial_{\theta_e}\sigma^\delta(\tau, \theta, \hbar)\right) + o(\hbar)$$

A version of this proposition appeared in [MP], but they worked with an other definition

of the ψ-symbol, which took values in $\mathbb{C}$, whereas in our definition, the ψ-symbol takes values in A_Γ.
Thus, before we turn to the proof the proposition, we would like to start this section by pointing out a few facts about the algebras A_Γ.

We know that A_Γ is a commutative $\mathbb{C}$-algebra of dimension 2^g, and that implies that it is isomorphic to the algebra $\mathbb{C}^{2^g}$. We would like to describe this isomorphism more explicitly.
First, we point out that A_Γ can be viewed as a quotient of a group algebra. Indeed, take the set $G = H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2) \times \{\pm 1\}$, and define the product on G by:

$$(\alpha, \varepsilon) \cdot (\beta, \mu) = (\alpha + \beta, \varepsilon\mu(-1)^{\langle \alpha, \beta\rangle}).$$

Here, $\langle \cdot, \cdot \rangle$ is the intersection form in $H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$. Then A_Γ is clearly the quotient $\mathbb{C}[G]/(\alpha, -1) \sim -(\alpha, 1)$. The algebra $\mathbb{C}[G]$ is isomorphic to the algebra $\mathbb{C}^{\hat{G}}$, where $\hat{G} = \mathrm{Hom}(G, \{\pm 1\})$ and the components of the isomorphism are of the form

$$\sum_{g\in G} \alpha_g g \to \sum_{g\in G} \alpha_g \rho(g)$$

where ρ goes over $\hat{G}$. Then the components of the isomorphism $A_\Gamma \to \mathbb{C}^{2^g}$ are given by

$$\sum_{\gamma \in H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)} \alpha_\gamma [\gamma] \to \sum_{\gamma \in H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)} \alpha_\gamma \rho(\gamma, 1)$$

where ρ goes over the set $\hat{A}_\Gamma$ of representations $G \to \{\pm 1\}$ such that $\rho(0, -1) = -1$. We point out that the quotients of two such representations corresponds to representations $H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2) \to \{\pm 1\}$, and $\hat{A}_\Gamma$ has a structure of affine space over $H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$.

Another way of thinking of this affine space structure is to view $\hat{A}_\Gamma$ as the set of "relative spin-structures" over $(\Gamma, \partial\Gamma)$. Indeed, define a linear map $\chi : A_\Gamma \mapsto \mathbb{C}$ by $\chi([\gamma]) = (-1)^{q(\gamma)}$. Then we have $\chi \in \hat{A}_\Gamma$ if and only if for any $\gamma, \delta \in H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$,we have $q(\gamma + \delta) = q(\gamma) + q(\delta) + \langle \gamma, \delta \rangle$, where $\langle \cdot, \cdot \rangle$ is the intersection form in $H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$.

Now the principal symbol $\sigma^\gamma(\tau, \theta, 0)$ is a morphism $K(\Sigma, -1) \to A_\Gamma$, we also label his components $\sigma^\gamma_\chi(\tau, \theta) = \chi(\sigma^\gamma(\tau, \theta, 0))$ for every $\chi \in \hat{A}_\Gamma$.

Proof of Proposition 1.1: We fix $r > 0$ and we take γ and δ two multicurves on Σ. According to Theorem 1, the matrix coefficients of the operator T^γ_r can be written as:

$$T^\gamma_r \varphi_c = \overline{c}(\gamma) \sum_{k: E \to \mathbb{Z}} F^\gamma_k(\tau, \hbar) \varphi_{c+k}$$

with the F^γ_k being smooth functions on V_γ such that $F^\gamma_k = 0$ as soon as there is some $e \in E$

such that $|k_e| > I_e^\gamma$ or $k_e \not\equiv I_e^\gamma (\mathrm{mod}\ 2)$.
As $\gamma \in K(\Sigma, -e^{\frac{i\pi}{2r}}) \to T_r^\gamma \in \mathrm{End}(V_r(\Sigma))$ is an morphism of algebras, we have:

$$T_r^{\gamma\cdot\delta}\varphi_c = T_r^\gamma(T_r^\delta \varphi_c)$$

writing $\tau = \dfrac{c}{r}$ and $\hbar = \dfrac{1}{r}$, and substituing the above expression of the matrix coefficients, we get

$$T_r^{\gamma\cdot\delta}\varphi_c = \sum_{m:E\to\mathbb{Z}} \left(\sum_{k+l=m} F_l^\gamma(\tau + k\hbar, \hbar) F_k^\delta(\tau,\hbar)\overline{c}(\gamma)\overline{c+k}(\gamma)\right)\varphi_{c+m}$$

$$= \overline{c}(\gamma)\overline{c}(\delta) i(\gamma,\delta) \sum_{m:E\to\mathbb{Z}} \left(\sum_{k+l=m} F_l^\gamma(\tau + k\hbar, \hbar) F_k^\delta(\tau,\hbar)\right)\varphi_{c+m}$$

Here, we could factor $\overline{c+k}(\gamma)$ out of the sum, as if $F_k^\delta \neq 0$ then $k_e \equiv I_e^\delta[2]$ and $\overline{k}(\gamma) = i(\gamma,\delta) = \prod\limits_{e\in E}(-1)^{I_e^\delta(C_e^{'*}(\gamma) + C_e^{''*}(\gamma))}$. We later show that $i(\gamma,\delta)$ correspond to the intersection in $H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$ of the projections of γ and δ.
Now as $K(\Sigma, -e^{\frac{i\pi\hbar}{2}})$ is generated by multicurves, we can write $\gamma\cdot\delta = \sum\limits_\lambda f_\lambda(\hbar)\lambda$, and in this sum, $f_\lambda \neq 0$ only when $[\lambda] = [\gamma]+[\delta] \in H_1(\Sigma, \mathbb{Z}/2)$, according to the Kauffman relations. When it is the case, we have $\overline{c}(\lambda) = \overline{c}(\gamma)\overline{c}(\delta)$. We can write another formula for the curve operator of the product:

$$T_r^{\gamma\cdot\delta}\varphi_c = \sum_m \left(\sum_\lambda \overline{c}(\lambda) f_\lambda(\hbar) F_m^\lambda(\tau,\hbar)\right)\varphi_{c+m}$$

So, identifying coefficients in the two formulas, we get:

$$\sum_\lambda f_\lambda(\hbar) F_m^\lambda(\tau,\hbar) = \left(\sum_{k+l=m} F_l^\gamma(\tau + k\hbar, \hbar) F_k^\delta(\tau,\hbar)\right) i(\gamma,\delta)$$

Now, remember that we defined the ψ-symbol of an arbitrary element of $K(\Sigma, -e^{\frac{i\pi\hbar}{2}})$ by extending lineary the formula for multicurves. Thus, we have:

$$\sigma^{\gamma\cdot\delta}(\tau,\theta,\hbar) = \sum_m\sum_\lambda f_\lambda(\hbar) F_m^\lambda(\tau,\hbar) e^{im\theta}[\lambda]$$

recalling that $[\lambda] = [\gamma]+[\delta]$ and using the previous identity of coefficients:

$$= i(\gamma,\delta)\sum_m \left(\sum_{k+l=m} F_l^\gamma(\tau + k\hbar, \hbar) F_k^\delta(\tau,\hbar)\right) e^{im\theta}[\gamma+\delta]$$

$$= i(\gamma,\delta)\langle p_*(\gamma), p_*(\delta)\rangle \left(\sigma^\gamma(\tau,\theta,\hbar)\sigma^\delta(\tau,\theta,\hbar) + \frac{\hbar}{i}\sum_{e\in E} \partial_{\tau_e}\sigma^\gamma(\tau,\theta,\hbar)\partial_{\theta_e}\sigma^\delta(\tau,\theta,\hbar)\right) + o(\hbar)$$

All that is left to prove is to show that the formula given for $i(\gamma,\delta)$ actually computes the

intersection in $H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$.
Let γ and δ be two curves in Γ. Up to isotopy, we can suppose that δ lies in the interior of Γ and γ follows the edges of the cell decomposition of Γ. Then the intersection points lie only in the curves $p(C_e) = L_e$. The number of intersection points of γ and δ in L_e is congruent modulo 2 to $\sharp(\delta \cap L_e)L_e^*(\gamma)$ where L_e^* is the dual to the cell L_e. If $\tilde{\gamma}$ and $\tilde{\delta}$ are lifts of γ and δ to Σ, then $L_e^*(\gamma) = C_e^{'*}(\tilde{\gamma}) + C_e^{''*}(\tilde{\gamma})$ and $\sharp(\delta \cap L_e) = \sharp(\tilde{\delta} \cap C_e)$ modulo 2, hence the formula for $i(\tilde{\gamma}, \tilde{\delta})$ computes the number of intersection points of γ and δ modulo 2.

3.2 Principal symbol and the SL_2-character variety

This section aims to relate the components of the principal symbol σ_χ and functions on the space of representations $\pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$.
We will start our study of the principal symbol by the following proposition, which describes which values $\sigma_\chi^\gamma(\tau, \theta)$ can take:

Proposition 3.1 :

1. For any multicurve γ and $\chi \in \hat{A}_\Gamma$, we have $\sigma_\chi^\gamma(\tau, \theta) \in \mathbb{R}$
2. For any multicurve γ and $\chi \in \hat{A}_\Gamma$, we have $|\sigma_\chi^\gamma(\tau, \theta)| \leq 2^{n(\gamma)}$ where $n(\gamma)$ is the number of components of γ.

Proof : 1. We recall that the components of the ψ-symbol σ_χ^γ are complex-valued. The stated property comes from the fact that curve operators are Hermitian: for any multicurve γ, and every r, the operator T_r^γ is a Hermitian endomorphism of $V_r(\Sigma)$. By definition, we have $T_r^\gamma \varphi_c = \sum_k F_k^\gamma(\frac{c}{r}, \frac{1}{r})\varphi_{c+k}$. As the basis $(\varphi_c)_{c \in U_r}$ is a Hermitian basis, we get $F_{-k}^\gamma(\frac{c+k}{r}, \frac{1}{r}) = \overline{F_k^\gamma(\frac{c}{r}, \frac{1}{r})}$ for all $c \in U_r$. Then for $r \to +\infty$ we have $F_{-k}^\gamma(\tau, 0) = \overline{F_k^\gamma(\tau, 0)}$. Hence $\sigma_\chi^\gamma(\tau, \theta) = \chi(\gamma)\sum_k F_k^\gamma(\tau, 0)e^{ik\cdot\theta} \in \mathbb{R}$, $\forall (\tau, \theta) \in U \times (\mathbb{R}/2\pi\mathbb{Z})^E$

2. We want to find a majoration of $|\sigma_\chi^\gamma(\tau, \theta)|$, where γ is a multicurve. By definition, we have $\sigma_\chi^\gamma(\tau, \theta) = \chi(\gamma)\sum_k F_k^\gamma(\tau, 0)e^{ik\cdot\theta}$. From one hand, we know that the coefficients F_k^γ are zero as soon as there is e such that $|k_e| > I_e^\gamma = \sharp(\gamma \cap C_e)$. The number of non-zero coefficients is then lower than $M_\gamma = \prod_{e \in E}(2I_e^\gamma + 1)$. On the other hand, for any $r \geq 2$ and $c \in U_r$:

$$F_k^\gamma(\frac{c}{r}, \frac{1}{r}) = \langle T_r^\gamma \varphi_c, \varphi_{c+k} \rangle \leq ||T_r^\gamma||$$

We reminded in the preliminary section the spectral radius of T_r^γ is always smaller than $2^{n(\gamma)}$. Thus we have $|F_k^\gamma(\frac{c}{r}, \frac{1}{r})| \leq 2^{n(\gamma)}$ for every r > 0 and every $c \in U_r$. Taking the limit, we get $|F_k^\gamma(\tau, 0)| \leq 2^{n(\gamma)}$.
These two estimations only allow us to write $|\sigma_\chi^\gamma(\tau, \theta)| \leq M_\gamma 2^{n(\gamma)}$. To obtained to the promised inequality, we use the multiplicativity of $\sigma_\chi(\tau, \theta)$:

We have, for any integer p: $|\sigma_\chi^{\gamma^p}(\tau,\theta)| = |\sigma_\chi^\gamma(\tau,\theta)|^p$. But γ^p is also a multicurve, obtained by taking p parallel copies of each component of γ.
We deduce that $|\sigma_\chi^{\gamma^p}(\tau,\theta)| \leq M_{\gamma^p} 2^{n(\gamma^p)} \leq p^{3g-3} M_\gamma 2^{pn(\gamma)}$ and so we get by taking $p \to +\infty$ that $|\sigma_\chi^\gamma(\tau,\theta)| \leq 2^{n(\gamma)}$ for all $(\tau,\theta) \in U \times (\mathbb{R}/2\pi\mathbb{Z})^E$.

Now, recall that the components of the ψ-symbol

$$\sigma_\chi(\tau,\theta) \ : \ K(\Sigma,-1) \to \mathbb{C}$$

are morphisms of algebras. There is a simple description of all such morphism of algebras: indeed, by Bullock's theorem that we recalled in the first section, we have an isomorphism

$$K(\Sigma,-1) \simeq \mathrm{Reg}(\mathcal{M}'(\Sigma))$$

where $\mathcal{M}'(\Sigma)$ stands for $\mathrm{Hom}(\pi_1\Sigma, SL_2(\mathbb{C}))//SL_2(\mathbb{C})$. A morphism of algebras ϕ from $\mathrm{Reg}(\mathcal{M}'(\Sigma))$ to $\mathbb{C}$ is always of the form

$$\phi \ : \ f \in \mathrm{Reg}(\mathcal{M}'(\Sigma)) \ \to \ f(\rho)$$

for some $\rho \in \mathcal{M}'(\Sigma)$. We deduce the existence of maps

$$R_\chi \ : \ U \times (\mathbb{R}/2\pi\mathbb{Z})^E \to \mathcal{M}'(\Sigma)$$

such that $\sigma_\chi^\gamma(\tau,\theta) = f_\gamma(R_\chi(\tau,\theta))$.

3.3 A system of actions-angles coordinates on the SU_2-character variety

This paragraph will be devoted to study the map R_χ more closely, the aim is to prove that it actually gives action-angle coordinates on the character variety $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}_2)/\mathrm{SU}_2$ that we will denote by $\mathcal{M}(\Sigma)$.
In $\mathcal{M}(\Sigma)$ there is an open dense subset $\mathcal{M}_{\mathrm{irr}}(\Sigma)$ consisting of all conjugacy of irreducible representations. It is a well-known fact that $\mathcal{M}_{\mathrm{irr}}(\Sigma)$ has a structure of (smooth) symplectic variety therefore we call it the smooth part of $\mathcal{M}(\Sigma)$.
At first sight, the maps R_χ take values in $\mathcal{M}'(\Sigma)$. Again, we have a subset $\mathcal{M}'_{\mathrm{irr}}(\Sigma) \subset \mathcal{M}'(\Sigma)$ consisting of conjugacy classes of irreducible representations, and there is a structure of (smooth) complex symplectic variety on it (that restricts to that of $\mathcal{M}_{\mathrm{irr}}(\Sigma)$).

We remark the following two points:

First, we point out that R_χ is always an irreducible representation. Indeed, it is easy to see that for reducible representation, we have for e,f,g three adjacent edges

$$h_{C_e}(\rho) + h_{C_f}(\rho) = h_{C_g}(\rho)$$

for one of the three ordering of e,f,g. This can not happen for $R_\chi(\tau,\theta)$ as $(h_{C_e})_{e\in E}$ maps it to $\tau \in U$, and we have strict inequalities $\tau_g < \tau_e + \tau_f$.

Our second point is that the map R_χ is smooth. This has a sense as by our first remark its image is in the smooth part of $\mathcal{M}'(\Sigma)$. As stated in Definition 2.1, the map $(\tau, \theta) \to \sigma^\gamma(\tau, \theta, 0)$ is a smooth function on $U \times (\mathbb{R}/2\pi\mathbb{Z})^E$, for all $\gamma \in K(\Sigma, -1)$. So the map $(\tau, \theta) \to \mathrm{Tr}(R_\chi(\tau, \theta)(\gamma))$ is smooth for every $\gamma \in \pi_1(\Sigma)$. As the space $\mathcal{M}'(\Sigma)$ can be parametrized by a finite collection of coordinates $\rho \to \mathrm{Tr}(\rho(\gamma_j))$, where $\gamma_j \in \pi_1(\Sigma)$, the map $R_\chi : U \times (\mathbb{R}/2\pi\mathbb{Z})^E \to \mathcal{M}'(\Sigma)$ is smooth.

Proposition 3.2: The maps R_χ take in fact values in $\mathcal{M}(\Sigma) = \mathrm{Hom}(\pi_1\Sigma, \mathrm{SU}_2)/\mathrm{SU}_2$

Proof : Indeed, we have seen with Proposition 3.1 that $\sigma^\gamma_\chi(\tau, \theta)$ is real-valued. We can use a well-known lemma:

Lemma : Any irreductible subgroup $G \subset \mathrm{SL}_2(\mathbb{C})$ such that the trace of all elements of G are real is conjugated to either a subgroup of $\mathrm{SL}_2(\mathbb{R})$ or a subgroup of SU_2.

The proof of this lemma is based only on elementary algebra, manipulating trace of products of elements of G. A detailed proof can be found for example in [HK, p.3040-3041].
As we have $\sigma^\gamma_\chi(\tau, \theta, 0) = -\mathrm{Tr}(R_\chi(\tau, \theta)(\gamma)) \in \mathbb{R}$, we get that $R_\chi(\tau, \theta)$ is conjugated to either a representation in $\mathrm{SL}_2(\mathbb{R})$ or a representation in SU_2.

To prove Proposition 3.2, we still need to dismiss the case where the image of $R_\chi(\tau, \theta)$ would be conjugated to a subgroup of $\mathrm{SL}_2(\mathbb{R})$. To this end, we use the point 2. of Proposition 3.1, who states that $|\mathrm{Tr}(R_\chi(\tau, \theta)\gamma)| \leq 2$ for every $\gamma \in \pi_1(\Sigma)$ representing a simple closed curve on Σ. We have to show the following lemma:

Lemma 2.1 : Let $\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{R})$ be a representation such that $|\mathrm{Tr}(\rho(\gamma))| \leq 2$ for any simple closed curve γ on Σ. Then ρ is an abelian representation.

Proof of Lemma 2.1 : We point out that this lemma is sufficient to get rid of the case $R_\chi(\pi_1(\Sigma)) \subset \mathrm{SL}_2(\mathbb{R})$, as $R_\chi(\tau, \theta)$ is always an irreducible representation.
We carry out the proof of Lemma 2.1 in two steps:
First, we assume that there exists a simple non separating closed curve γ such that $\rho(\gamma)$ is elliptic, that is the absolute value of its trace is < 2. Up to a composition by an element of the mapping class group, we can assume that $\gamma = a_1$ where $a_1, b_1, \ldots, b_g$ is a standard system of generators of $\pi_1(\Sigma)$. The commutator $[a_1, b_1]$ is in the conjugacy class of a simple closed curve. Besides, for A, B two elements of $\mathrm{SL}_2(\mathbb{R})$, with A elliptic, we have the following implication:

$$\mathrm{Tr}([A, B]) \leq 2 \implies \text{A and B are elliptic and commute.}$$

If A is elliptic, we can indeed, in a well-chosen basis, write $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. Let

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, we compute that

$\mathrm{Tr}([A,B]) = 2(ad-bc)\cos^2(\theta)+(a^2+b^2+c^2+d^2)\sin^2(\theta) = 2+((a-d)^2+(b+c)^2)\sin^2(\theta)$. So the commutator is hyperbolic except if $a = d$ and $b = -c$, and then B commutes with A. Moreover, as $\det(B) = 1$ we also get that B is elliptic.

This property allows us to conclude that $\rho(a_1)$ and $\rho(b_1)$, that we supposed elliptic must commute. Then a_1 and $b_1 a_k$ (or $b_1 b_k$) are also in the conjugacy classes of two simple closed curved that intersect each other only once. Then the whole image of the representation must be commutative, and is composed of elliptic elements, thus is conjugated to a subgroup of $\mathrm{SO}_2(\mathbb{R})$.

Now if all simple non separating closed curves are sent to parabolic elements, we can proceed in a similar fashion, thanks to the following property:

If $A, B \in \mathrm{SL}_2(\mathbb{R})$ are such that A is parabolic and $[A,B]$ also, then A and B are parabolic and commuting.

As before, in some basis, we can write $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, we compute that $\mathrm{Tr}([A,B]) = 2 + c^2$. Thus $\mathrm{Tr}([A,B]) = 2 \implies c = 0$ and A and B are parabolic and commute. This ends the proof of the lemma.

Proposition 3.3: For any $\chi \in \hat{A}_\Gamma$, the map $R_\chi \; : \; (\tau,\theta) \in U \times (\mathbb{R}/2\pi\mathbb{Z})^E \to R_\chi(\tau,\theta) \in \mathcal{M}(\Sigma)$ gives action-angle coordinates on the symplectic variety $\mathcal{M}_{\mathrm{irr}}(\Sigma)$.

We remind that when a pants decomposition $\mathcal{C} = \{C_e\}_{e\in E}$ of Σ is given, the family of functions $h_{C_e} = \frac{1}{\pi}\mathrm{Acos}(-\frac{f_{C_e}}{2})$ constitutes a moment mapping $h \; : \; \mathcal{M}(\Sigma) \to \overline{U}$. The variables τ_e are the action coordinates associated to this moment mapping:

$$h_{C_e}(R_\chi(\tau,\theta)) = \frac{1}{\pi}\mathrm{Acos}(-\frac{f_{C_e}(R_\chi(\tau,\theta))}{2}) = \frac{1}{\pi}\mathrm{Acos}(-\frac{\sigma_\chi^{C_e}(\tau,\theta)}{2}) = \tau_e$$

where the third equality comes from the computation of the operator $T_r^{C_e}$ given in Section 2: for any coloring c of E, we have $T_r^{C_e}\varphi_c = -2\cos(\frac{\pi c}{r})\varphi_c$, so that $\sigma_\chi^{C_e}(\tau,\theta,\hbar) = F_0^{C_e}(\tau,\hbar)\chi([0]) = -2\cos(\pi\tau_e)$.

The fact that (τ,θ) is a system of action-angle coordinates on $\mathcal{M}(\Sigma)$ can be described in the following way:

$$\omega = \frac{1}{2\pi}\sum_{e\in E} d\tau_e \wedge d\theta_e$$

where ω refers to the symplectic form on the variety $\mathcal{M}(\Sigma)$.

It also amouts to the fact that the vector fields $\frac{1}{2\pi}\partial_{\theta_e}$ and $X_{h_{C_e}}$ (the symplectic gradient associated to the function h_{C_e}) on $\mathcal{M}(\Sigma)$ are equals. This equality of vector fields can be rewritten in terms of Poisson brackets:

$$\forall f \in C^{\infty}(\mathcal{M}(\Sigma),\mathbb{C}),\ \forall \tau,\theta \text{ we have } \{h_{C_e}, f\} = \frac{1}{2\pi}\frac{\partial}{\partial \theta_e} f(R_{\chi}(\tau,\theta))$$

We only need to verify this equality when f is one of the function f_γ, where $\gamma \in K(\Sigma,-1)$ as the Poisson bracket is a first order differential operator, and any function f on $\mathcal{M}(\Sigma)$ can be approximated at order 1 by a trace function f_γ. By linearity, we can show it only for γ a multicurve.
To compute such Poisson brackets, we can apply the formula of Goldman [G86] that we recalled in Section 1:
We note ε the linear map

$$\varepsilon \ : \ K(\Sigma,-e^{\frac{i\pi\hbar}{2}}) \to K(\Sigma,-1) \simeq \mathrm{Reg}(\mathcal{M}'(\Sigma))$$

$$\sum_{\gamma \text{ multicurve}} c_\gamma(\hbar)\gamma \to \sum_{\gamma \text{ multicurve}} c_\gamma(0)\gamma$$

For γ and $\delta \in K(\Sigma,-e^{\frac{i\pi\hbar}{2}})$ we have:

$$\{f_{\varepsilon(\gamma)}, f_{\varepsilon(\delta)}\} = f_{\varepsilon(\frac{i}{2\pi\hbar}[\gamma,\delta])}$$

with $[\gamma,\delta] = \gamma\cdot\delta - \delta\cdot\gamma \in K(\Sigma,-e^{\frac{i\pi\hbar}{2}})$.
We apply the above formula to compute $\{h_{C_e}, f_\gamma\}$ for any $\gamma \in K(\Sigma,-e^{\frac{i\pi\hbar}{2}})$: We recall that $h_{C_e} = \frac{1}{\pi}\mathrm{Acos}(-\frac{f_{C_e}}{2})$. Our strategy to compute the Poisson bracket is to approximate h_{C_e} with polynomials in f_{C_e}.
On a neighboorhood V of $R_\chi(\tau,\theta)$, f_γ has values in an open set $(-2+\eta, 2-\eta) \subset [-2,2]$. We choose a sequence of polynomials P_j such that P_j converge to the map $x \mapsto \frac{1}{\pi}\mathrm{Acos}(-\frac{x}{2})$ on $(-2+\eta,2-\eta)$ in the C^1-topology. The Poisson bracket being a differential operator of order one, we have that $\{P_j(f_{C_e}), f_\gamma\}$ converges uniformly on V to $\{h_{C_e}, f_\gamma\}$ when $j \to +\infty$.

Now, the maps

$$\{\cdot, f_\gamma\} \ : \ C^{\infty}(\mathcal{M}(\Sigma)) \to C^{\infty}(\mathcal{M}(\Sigma)) \text{ and}$$

$$\frac{i}{2\pi\hbar}[\cdot,\gamma] \ : \ K(\Sigma,-1) \to K(\Sigma,-1)$$

being derivations of algebras, we have by Goldman's formula:

$$\{P_j(f_{C_e}), f_\gamma\}(R_\chi(\tau,\theta)) = f_{\varepsilon(\frac{i}{2\pi\hbar}[P_j(C_e),\gamma])}(R_\chi(\tau,\theta)) = \sigma_\chi^{\varepsilon(\frac{i}{2\pi\hbar}[P_j(C_e),\gamma])}(\tau,\theta,0)$$

We compute this last quantity: we recall that we wrote $T_r^\gamma \varphi_c = \sum_k F_k^\gamma(\tau,\hbar)\varphi_{c+k}$ and we gave in Section 2.2 the expression $T_r^{C_e}\varphi_c = -2\cos(\pi\tau_e)\varphi_c$. We deduce that $T_r^{P_j(C_e)}\varphi_c =$

$P_j(-2\cos(\pi\tau_e))\varphi_c$. For $c \in U_r$, we have:

$$T_r^{[P_j(C_e),\gamma]}\varphi_c = \sum_k P_j(-2\cos(\pi(\tau_e + k_e\hbar)))F_k^\gamma(\tau,\hbar)\varphi_{c+k} - \sum_k P_j(-2\cos(\pi\tau_e))F_k^\gamma(\tau,\hbar)\varphi_{c+k}$$

so that as $[C_e^k] = [0]$ in A_Γ we obtain:

$$\sigma_\chi^{\varepsilon(\frac{i}{2\pi\hbar}[P_j(C_e),\gamma])}(\tau,\theta,0) = \frac{i}{2\pi}\sum_k \left.\frac{P_j(-2\cos(\pi(\tau_e + k_e\hbar))) - P_j(-2\cos(\pi\tau_e))}{\hbar}\right|_{\hbar=0} F_k^\gamma(\tau,0)e^{ik\cdot\theta}\chi(\gamma)$$

When j tends to $+\infty$, as P_j approach the function $x \mapsto \frac{1}{\pi}\mathrm{Acos}(-\frac{x}{2})$, this quantity goes to

$$\frac{1}{2\pi}\sum_k ik_e F_k^\gamma(\tau,\hbar)e^{ik\cdot\theta}\chi(\gamma) = \frac{1}{2\pi}\frac{\partial}{\partial_{\theta_e}}\sigma_\chi^\gamma(\tau,\theta,0) = \frac{1}{2\pi}\frac{\partial}{\partial_{\theta_e}}f_\gamma(R_\chi(\tau,\theta))$$

The last equality ends the proof: we have indeed $\{h_{C_e}, f_\gamma\}(R_\chi(\tau,\theta)) = \frac{1}{2\pi}\frac{\partial}{\partial\theta_e}f_\gamma(R_\chi(\tau,\theta))$ for every multicurve γ, and R_χ give a action-angle parametrization of $\mathcal{M}_{\mathrm{irr}}(\Sigma)$.

Origin of angle coordinates

Finally, we want to investigate how exactly R_χ varies with $\chi \in \hat{A}_\Gamma$. We recall that according to Section 3.1, the values of two different morphisms χ and χ' on $[\gamma]$ differ by a representation $\rho : H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2) \to \{\pm 1\}$.

Let us be more precise about angle coordinates. We recall that we have an Hamiltonian $h : \mathcal{M}_{\mathrm{irr}}(\Sigma) \to U$, given by $(h(\rho))_e = \frac{1}{\pi}\mathrm{Acos}(-\frac{\mathrm{Tr}(\rho(C_e))}{2})$. The Hamiltonian flow gives an action of $\mathbb{R}^E$ on $\mathcal{M}_{\mathrm{irr}}(\Sigma)$. This action has a kernel

$$\Lambda = \mathrm{Vect}_{\mathbb{Z}}\{(2\pi u_e)_{e\in E},\quad \pi(u_e + u_f + u_g)_{(e,f,g)\in S}\}$$

where $(u_e)_{e\in E}$ is the canonical basis of $\mathbb{R}^E$, E is the set of edges of Γ, and S is the set of triple of edges adjacent to the same vertex in Γ. We also define $\Lambda' = \mathrm{Vect}_{\mathbb{Z}}(\pi u_e) \supset \Lambda$. The quotient Λ'/Λ then acts on $\mathcal{M}^{\mathrm{irr}}(\Sigma)$ by $\pi u_e \cdot \rho(\gamma) = (-1)^{(C_e,\gamma)}\rho(\gamma)$, where $(\cdot,\cdot)$ is the intersection form in Σ.

Now that we know that the maps R_χ give action-angle coordinates on $\mathcal{M}_{\mathrm{irr}}(\Sigma)$, the only ambiguity is the choice of the origin of the angle part. That is we must have for any $\chi, \chi' \in \hat{A}_\Gamma$ that $R_{\chi'}(\tau,\theta) = R_\chi(\tau, \theta + v_{\chi,\chi'})$ for a fixed vector $v_{\chi,\chi'} \in \mathbb{R}/\Lambda$.

We use the values of R_χ on the curves D_e to get the origin of angle coordinates. We have $\mathrm{Tr}(R_\chi(\tau,\theta)(D_e)) = -\sigma_\chi^{D_e}(\tau,\theta,0) = -2W(\pi\tau,0)\cos(\theta_e)$ if e joins a vertex to itself, otherwise the same quantity equals $I(\pi\tau,0) + 2J(\pi\tau,0)\cos(2\theta_e)$. We see that in the first case, $\theta_e = 0$ is the unique minimum of $\mathrm{Tr}(R_\chi(\tau,\theta)(D_e))$, so that the origin of this coordinate is the same for all $\chi \in \hat{A}_\Gamma$. In the second case, $\theta_e \mapsto \mathrm{Tr}(R_\chi(\tau,\theta)(D_e))$ has exactly two

maxima, one for $\theta_e = 0$, one for $\theta_e = \pi$. So θ is fixed modulo πu_e. Thus for $\chi, \chi' \in \hat{A}_\Gamma$, we have $v_{\chi,\chi'} \in \Lambda'/\Lambda$.
Taking two elements χ, χ' in $\hat{A}_\Gamma$ we know that they differ by some $\rho : H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2) \to \{\pm 1\}$. It is possible to recover the vector $v_{\chi,\chi'} \in \Lambda'/\Lambda$ from the representation ρ: indeed, by Poincare duality, one can write $\rho(p_*(\gamma)) = (-1)^{\langle C,\gamma\rangle}$ where $C \in H_1(\Sigma, \mathbb{Z}/2)$, p_* is the projection $H_1(\Sigma, \mathbb{Z}/2) \to H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$ and $\langle \cdot, \cdot \rangle$ is the intersection form in $H_1(\Sigma, \mathbb{Z}/2)$. Remember that p_* maps each C_e to zero, so that the intersection of C with each C_e must vanish. As the C_e generate a Lagrangian of $H_1(\Sigma, \mathbb{Z}/2)$, C is a linear combination of the C_e and this yields a vector $v_\rho \in \Lambda'/\Lambda$ such that $R_{\rho\chi}(\tau, \theta) = R_\chi(\tau, \theta + v_\rho)$.

We note that when Γ is a planar graph we can drop these complicated consideration of angle origins and we could have taken the ψ-symbol to be just $\mathbb{C}$-valued. Indeed, in this case the intersection form in $H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$ is trivial, and the image of $H_1(\Sigma, \mathbb{Z}/2) \to H_1(\Gamma, \partial\Gamma, \mathbb{Z}/2)$ is $\{0\}$, so that all components of the ψ-symbol are the same.

4 First order of the ψ-symbol

In this section, we investigate the first order term in $\hbar$ of the asymptotic expansion of the ψ-symbol. We identify this term by relating it to the principal symbol, of which we already know a formula.
We remind that for a multicurve γ, the map $(\tau, \hbar, \theta) \to \sigma^\gamma(\tau, \theta, \hbar)$ is defined as a finite sum of smooth functions on V_γ, and V_γ is a neighborhood of $U \times \{0\}$ in $U \times [0,1]$. We may write, for any multicurve γ:

$$\sigma^\gamma(\tau, \theta, \hbar) = \sigma^\gamma(\tau, \theta, 0) + \hbar(\Delta_\gamma(\tau, \theta) + D_\gamma(\tau, \theta)) + o(\hbar)$$

Here, $\Delta_\gamma(\tau, \theta)$ refers to the expected first order as in Theorem 2 that is:

$$\Delta_\gamma(\tau, \theta) = \frac{1}{2i} \sum_e \frac{\partial^2}{\partial \tau_e \partial \theta_e} \sigma^\gamma(\tau, \theta, 0).$$

Hence what we want to prove in this section is that the default $D_\gamma(\tau, \theta)$ is zero for all γ and $(\tau, \theta) \in U \times (\mathbb{R}/2\pi\mathbb{Z})^E$.
We remark that the previous expressions define $\Delta(\tau, \theta)$ and $D(\tau, \theta)$ as maps from the set of multicurves to A_Γ, what we can extend by linearity to linear maps $K(\Sigma, -e^{\frac{i\pi\hbar}{2}}) \to A_\Gamma[[\hbar]]$. Furthermore, Δ_γ and D_γ are some linear combinations of partial derivatives of the smooth functions F_k on V_γ, they are both smooth on $U \times (\mathbb{Z}/2\pi\mathbb{Z})^E$.

Proposition 4.1 : For any multicurve γ and for all (τ, θ), the default $D_\gamma(\tau, \theta)$ vanishes, so that the first order term of $\sigma^\gamma(\tau, \theta, \hbar)$ is $\Delta_\gamma(\tau, \theta)$.

The proof relies on the two following lemmas:

Lemma 4.1 : Let (τ, θ) be in $U \times (\mathbb{R}/2\pi\mathbb{Z})^E$. We will provide $\mathbb{C}$ with a structure of $K(\Sigma, -1)$-module (or equivalently of $\mathrm{Reg}(\mathcal{M}'(\Sigma))$-module): for $x \in \mathbb{C}$ and $f \in \mathrm{Reg}(\mathcal{M}'(\Sigma))$, we define $f \cdot x = f(R_\chi(\tau, \theta))x$. Then the corresponding component of the default $\gamma \mapsto \chi(D_\gamma(\tau, \theta))$ is a derivation of $K(\Sigma, -1)$-modules from $K(\Sigma, -1)$ to $\mathbb{C}$.

Lemma 4.2 : For the same structure of $\mathrm{Reg}(\mathcal{M}'(\Sigma))$-module on $\mathbb{C}$ as above, we have an isomorphism $\mathrm{Der}(\mathrm{Reg}(\mathcal{M}'(\Sigma)), \mathbb{C}) \simeq T_{R_\chi(\tau,\theta)}\mathcal{M}(\Sigma)$ sending a vector $X \in T_{R_\chi(\tau,\theta)}\mathcal{M}(\Sigma)$ to the derivation $f \to \mathcal{L}_X f(R_\chi(\tau, \theta))$, and the vector fields $(\partial\tau_e, \partial\theta_e)$ give a basis of the tangent spaces $T_{R_\chi(\tau,\theta)}\mathcal{M}(\Sigma)$.

Proof of lemma 4.1 : We use Proposition 1.1 to determine how the default $D(\tau, \theta)$ behaves with the product of elements in $K(\Sigma, -e^{\frac{i\pi\hbar}{2}})$. We work with one component σ_χ of the ψ-symbol at a time. For $\gamma \in K(\Sigma, -1)$, we will note $E_\gamma = \chi(\Delta_\gamma + D_\gamma)$, so that we can write $\sigma_\chi^\gamma(\tau, \theta, \hbar) = \sigma^\gamma(\tau, \theta, 0) + \hbar E_\gamma(\tau, \theta) + o(\hbar)$.
Then, applying $\chi \in \hat{A}_\Gamma$ to Proposition 1.1 we have:

$$\sigma_\chi^{\gamma\cdot\delta}(\tau, \theta, \hbar) = \sigma_\chi^\gamma(\tau, \theta, \hbar)\sigma_\chi^\delta(\tau, \theta, \hbar) + \frac{\hbar}{i}\sum_e \partial_{\tau_e}\sigma_\chi^\gamma(\tau, \theta, \hbar)\partial_{\theta_e}\sigma_\chi^\delta(\tau, \theta, \hbar) + o(\hbar)$$

We have $\sigma_\chi^\gamma(\tau, \theta, 0) = f_\gamma(R_\chi(\tau, \theta)$. Recall that by the formula of Goldman given in Section 1.1, $f_{\gamma\cdot\delta} = f_\gamma f_\delta + \hbar\frac{\pi}{i}\{f_\gamma, f_\delta\} + o(\hbar)$. So, isolating terms of order 1 in $\hbar$, we get:

$$\begin{aligned}
&\frac{\pi}{i}\{f_\gamma, f_\delta\}(R_\chi(\tau, \theta)) + E_{\gamma\cdot\delta}(\tau, \theta) \\
&\quad = E_\gamma(\tau, \theta)f_\delta(R_\chi(\tau, \theta)) + E_\delta(\tau, \theta)f_\gamma(R_\chi(\tau, \theta)) + \frac{1}{i}\sum_e \partial_{\tau_e} f_\gamma(R_\chi(\tau, \theta))\partial_{\theta_e} f_\delta(R_\chi(\tau, \theta))
\end{aligned}$$

but $\{f_\gamma, f_\delta\} = \frac{1}{2\pi}\sum_e (\partial_{\tau_e} f_\gamma \partial_{\theta_e} f_\delta - \partial_{\tau_e} f_\delta \partial_{\theta_e} f_\gamma)$. We deduce that

$$E_{\gamma\cdot\delta} = E_\gamma \sigma_\chi^\delta + E_\delta \sigma_\chi^\gamma + \frac{1}{2i}\sum_e \left(\partial_{\tau_e}\sigma_\chi^\gamma \partial_{\theta_e}\sigma_\chi^\delta + \partial_{\theta_e}\sigma_\chi^\gamma \partial_{\tau_e}\sigma_\chi^\delta\right)$$

However, as for $\gamma, \delta \in K(\Sigma, -1)$ we have $f_{\gamma\cdot\delta} = f_\gamma f_\delta$, and $\chi(\Delta_\gamma) = \frac{1}{2i}\sum_e \frac{\partial^2 f_\gamma}{\partial\tau_e \partial\theta_e} \circ R_\chi$, the Leibniz rules implies that $\chi(\Delta_\gamma)$ satisfies the same law of composition:

$$\chi(\Delta_{\gamma\cdot\delta}) = \chi(\Delta_\gamma)f_\delta + \chi(\Delta_\delta)f_\gamma + \frac{1}{2i}\sum_e (\partial_{\tau_e} f_\gamma \partial_{\theta_e} f_\delta + \partial_{\theta_e} f_\gamma \partial_{\tau_e} f_\delta)$$

This concludes the proof of Lemma 4.1: $\chi \circ D$ is a derivation.

Proof of Lemma 4.2 : It is well-known that $\mathcal{M}'(\Sigma)$ is an affine algebraic variety whose

smooth points form the open dense subset $\mathcal{M}'_{\mathrm{irr}}(\Sigma)$ (see [Si09] for instance). The point $R_\chi(\tau,\theta)$ is thus a smooth point of $\mathcal{M}'(\Sigma)$ for any $(\tau,\theta)\in U\times(\mathbb{R}/2\pi\mathbb{Z}^E$.
Then the proof comes from elementary considerations of algebraic geometry: when V is an affine algebraic variety, and x a point of V, we put a structure of $\mathrm{Reg}(V)$-module on $\mathbb{C}$ by defining $f\cdot\lambda=f(x)\lambda$. Then $\mathrm{Der}_x(V,\mathbb{C})$ identifies with $T_xV=m_x/(m_x)^2$ the algebraic tangent space to V at x (where $m_x=\{f\ /\ f(x)=0\}$), and the algebraic tangent space at a smooth point is the same of as the tangent space of V at x in the sense of differential manifolds. As the affine variety $\mathcal{M}'(\Sigma)$ is smooth on the image of R_χ, by this general property, derivations of $\mathrm{Reg}(\mathcal{M}(\Sigma))$ can be viewed as vectors of the tangent space. As $(\tau,\theta)\mapsto R_\chi(\tau,\theta)$ is a parametrization of $\mathcal{M}(\Sigma)$, the vector fields $(\partial\tau_e,\partial\theta_e)$ give a basis of the tangent space $T_{R_\chi(\theta,\tau)}\mathcal{M}(\Sigma)$ for each (τ,θ).

Proof of Proposition 4.1 from the Lemmas 4.1 et 4.2. Combining these two lemmas allows us to assert that $\chi(D(\tau,\theta))$, viewed as a map $\mathrm{Reg}(\mathcal{M}'(\Sigma))\to\mathbb{C}$, is of the form $f\to\mathcal{L}_Xf(R_\chi(\tau,\theta))$ for some $X\in T_{R_\chi(\tau,\theta)}\mathcal{M}'(\Sigma)$ and we may write
$X=\sum_e a_e\dfrac{\partial}{\partial_{\tau_e}}+b_e\dfrac{\partial}{\partial_{\theta_e}}$ for some coefficients $a_e,b_e:\mathcal{M}(\Sigma)\to\mathbb{C}$. As the default is smooth, so are the coefficients a_e and b_e.
We want to prove that these coefficients all vanish. To this end, we remind that we proved in Section 2.2 that the default vanishes for the curves C_e and D_e. Besides, we have the formula of Section 2:
We have $\sigma^{C_e}(\tau,\theta,\hbar)=-2\cos(\pi\tau_e)[0]$, so that $\chi(D_{C_e})(\tau,\theta)=2a_e\pi\sin(\pi\tau_e)$. As the default vanishes on C_e, we must have $a_e=0$.
To show the vanishing of the b_e, we use the formulas for D_e:
In the first kind of curve D_e, described in Section 2.2, we have $f_{D_e}(R_\chi(\tau,\theta))=\sigma_\chi^{D_e}(\tau,\theta,0)=2W(\pi\tau,0)\cos(\theta_e)$ where W does not vanish for $\tau\in U$.
We know that default D_{D_e} vanishes, so we have
$\chi(D_{D_e}(\tau,\theta))=b_e\dfrac{\partial}{\partial\theta_e}f_{D_e}(R_\chi(\tau,\theta))=-2b_e\pi\sin(\theta_e)W(\pi\tau,0)=0$. This yields $b_e=0$.

In the second case, we have $f_{D_e}(R_\chi(\tau,\theta))=\sigma_\chi^{D_e}(\tau,\theta,0)=-2J(\pi\tau,0)\cos(2\theta_e)-I(\pi\tau,0)$ for the functions I and J defined in Section 2.2, that are non-vanishing for $\tau\in U$.
Again as
$\chi(D_{D_e}(\tau,\theta))=b_e\dfrac{\partial}{\partial\theta_e}f_{D_e}(R_\chi(\tau,\theta))=4\pi b_e\sin(2\theta_e)J(\pi\tau,0)$ vanishes, we must have $b_e=0$.
It follows that the default $\gamma\mapsto D_\gamma$ is the zero derivation on $K(\Sigma,-1)\mapsto A_\Gamma$, which is the last ingredient we needed to complete the proof of the Proposition 4.1.

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