DIFFERENCE BODIES IN COMPLEX VECTOR SPACES

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Abstract. A complete classification is obtained of continuous, translation invariant, Minkowski valuations on an m-dimensional complex vector space which are covariant under the complex special linear group.

1. Introduction

The classification of real- or body-valued valuations satisfying certain natural properties has attracted a lot of attention in the last years. The first fundamental classification result dates back to 1957, when Hadwiger classified the continuous, translation invariant real-valued valuations which are also invariant under the rotations of the Euclidean space. Since then many generalizations of this result have been obtained.

We denote by V a real vector space of dimension n and by $\mathcal{K}(V)$ the space of compact convex bodies in V. An operator $Z : \mathcal{K}(V) \to (A, +)$ with $(A, +)$ an abelian semi-group is called a *valuation* if it satisfies the following additivity property

$$Z(K \cup L) + Z(K \cap L) = Z(K) + Z(L),$$

for all $K, L \in \mathcal{K}(V)$ such that $K \cup L \in \mathcal{K}(V)$.

The classical case consists of taking $(A, +)$ as the real numbers with the usual sum. A particular class of real-valued valuations consists of those which are continuous – with respect to the Hausdorff topology – and translation invariant, i.e. $Z(K + x) = Z(K)$ for every $x \in V$. Some of the most important and recent results on the theory of continuous translation invariant valuations can be found in [3, 7, 23, 34, 36]. This theory has been extended to the more general framework of manifolds instead of a real vector space, see for instance [5, 11]. Apart from the continuity and the translation invariance of a real-valued valuation, we can impose invariance under some group acting transitively on the sphere (for instance, the unitary group). Then, we always get a finite dimensional real vector space (see [6]). Its dimension, a basis and the arising integral geometry have been studied intensively. For some references on this direction see [4, 10, 12, 13, 14].

Some other important particular cases of valuations are given, for instance, when considering the vector space of symmetric tensors (see [2, 9, 20,

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30, 37] for more information on tensor-valued valuations), or $(\mathcal{K}(V),+)$ with $+$ the Minkowski sum between two convex bodies (i.e. $K+L=\{x+y : x\in K,\, y\in L\}$). Convex body valued valuations with addition the Minkowski sum are called *Minkowski valuations*.

In this paper, we are interested in dealing with Minkwoski valuations. Some results on Minkowksi valuations not described in this paper can be found, for instance, in [18, 19, 21, 32, 33, 42, 43, 45]. Some papers dealing with convex geometry, but working in a complex vector space as ambient space – as we do – instead of a real vector space are [25, 26, 27].

Two fundamental properties of Minkowski valuations are the contravariance and the covariance with respect to the special linear group $\mathrm{SL}(V,\mathbb{R})$. A valuation $Z:\mathcal{K}(V)\to\mathcal{K}(V^*)$ is $\mathrm{SL}(V,\mathbb{R})$-*contravariant* if

$$Z(gK)=g^{-*}Z(K),\quad \forall g\in \mathrm{SL}(V,\mathbb{R}),$$

where V^* denotes the dual space of V and g^{-*} denotes the inverse of the dual map of g.

A valuation $Z:\mathcal{K}(V)\to\mathcal{K}(V)$ is $\mathrm{SL}(V,\mathbb{R})$-*covariant* if

$$Z(gK)=gZ(K),\quad \forall g\in \mathrm{SL}(V,\mathbb{R}).$$

An example of a continuous, translation invariant Minkowski valuations which is $\mathrm{SL}(V,\mathbb{R})$-contravariant is the projection body operator. For $K\in\mathcal{K}(V)$ the *projection body* ΠK *of* K has support function

$$h(\Pi K,u)=\frac{n}{2}V(K,\dots,K,[-u,u]),\quad u\in V,$$

where $V(K,\dots,K,[-u,u])$ denotes the mixed volume with $(n-1)$ copies of K and one copy of the segment joining u and $-u$.

The projection body was introduced in the 19th century by Minkowski and since then it has been widely studied (see, for instance, the books [15, 24, 28, 40, 44]). In the framework of the classification results of Minkowski valuations, Ludwig proved in [29] that the projection body operator is the only (up to a positive constant) continuous Minkowski valuation which is translation invariant and $\mathrm{SL}(V,\mathbb{R})$-contravariant. In [1] a complex version of this result was shown. The result is as follows

Theorem 1.1 ([1]). *Let W be a complex vector space of complex dimension $m\geq 3$. A map $Z:\mathcal{K}(W)\to\mathcal{K}(W^*)$ is a continuous, translation invariant and $\mathrm{SL}(W,\mathbb{C})$-contravariant Minkowski valuation if and only if there exists a convex body $C\subset\mathbb{C}$ such that $Z=\Pi_C$, where $\Pi_C K\in\mathcal{K}(W)$ is the convex body with support function*

$$h(\Pi_C K,w)=V(K[2m-1],Cw),\quad \forall w\in W,\tag{1}$$

where $Cw:=\{cw\,|\,c\in C\subset\mathbb{C}\}$. Moreover, C is unique up to translations.

For the covariant case, Ludwig proved in [31] that the difference body is the unique (up to a positive constant) continuous Minkowski valuation which is translation invariant and $\mathrm{SL}(V,\mathbb{R})$-covariant. In fact, she classified the continuous, $\mathrm{SL}(V,\mathbb{R})$-covariant Minkowski valuations (not necessarily translation invariant). The *difference body of a convex body* $K \in \mathcal{K}(V)$ is defined by

$$\mathrm{D}K = K + (-K),$$

where $-K$ denotes the reflection of K at the origin.

In this paper we study the continuous Minkowksi valuations in a complex vector space W which are translation invariant and $\mathrm{SL}(W,\mathbb{C})$-covariant. Our main result gives a classification of these valuations.

Theorem 1. *Let W be a complex vector space of complex dimension $m \geq 3$. A map $Z : \mathcal{K}(W) \to \mathcal{K}(W)$ is a continuous, translation invariant and $\mathrm{SL}(W,\mathbb{C})$-covariant Minkowski valuation if and only if there exists a convex body $C \subset \mathbb{C}$ such that $Z = \mathrm{D}_C$, where $\mathrm{D}_C K \in \mathcal{K}(W)$ is the convex body with support function*

$$h(\mathrm{D}_C K, \xi) = \int_{S^1} h(\alpha K, \xi) dS(C, \alpha), \quad \forall \xi \in W^*, \tag{2}$$

where $dS(C,\cdot)$ denotes the area measure of C, and $\alpha K = \{\alpha k : k \in K \subset W\}$ with $\alpha \in S^1 \subset \mathbb{C}$. Moreover, C is unique up to translations.

The hypothesis $m \geq 3$ in Theorem 1 cannot be omitted. In Section 4 we give for $m = 2$ another family of valuations satisfying all the properties and we characterize the continuous, translation invariant Minkowski valuations which are $\mathrm{SL}(W,\mathbb{C})$-covariant and have fixed degree of homogeneity. We also show that the continuous, translation invariant, $\mathrm{SL}(W,\mathbb{C})$-contravariant Minkowski valuations with degree of homogeneity 1 are precisely the ones introduced in [1, Proposition 3.3].

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2. Background and conventions

We denote by V a real vector space of dimension n and by W a complex vector space of complex dimension m. The space of compact convex bodies in V (resp. in W) is denoted by $\mathcal{K}(V)$ (resp. $\mathcal{K}(W)$). The dual vector space of V (resp. W) is denoted by V^* (resp. W^*).

For more information about the notions introduced here we refer to [15, 17, 40].

2.1. **Support function.** Let $K \in \mathcal{K}(V)$. The *support function of K* is given by

$$h_K : V^* \to \mathbb{R}, \qquad \xi \mapsto \sup_{x \in K} \langle \xi, x \rangle,$$

where $\langle \xi, x \rangle$ denotes the pairing of $\xi \in V^*$ and $x \in V$.

The support function is 1-homogeneous (i.e. $h_K(t\xi) = th_K(\xi)$ for all $t \geq 0$) and subadditive (i.e. $h_K(\xi+\eta) \leq h_K(\xi) + h_K(\eta)$ for all $\xi, \eta \in V^*$). Moreover, if a function on V^* is 1-homogeneous and subadditive, then it is the support function of a unique compact convex set $K \in \mathcal{K}(V)$ (cf. [40, Theorem 1.7.1]). We also write $h(K,\xi)$ for $h_K(\xi)$.

The support function is also linear with respect to the Minkowski sum on $\mathcal{K}(V)$ and has the following important property

$$h(gK,\xi) = h(K, g^*\xi), \quad \forall \xi \in V^*,\, g \in \mathrm{GL}(V,\mathbb{R}). \tag{3}$$

In a complex vector space W this equality holds for $g \in \mathrm{GL}(W,\mathbb{C})$. In particular, for $\alpha \in \mathbb{C}$ and $K \in \mathcal{K}(W)$ we can interpret $\alpha K = gK$ with $g = \alpha \mathrm{Id} \in \mathrm{GL}(W,\mathbb{C})$, where Id denotes the identity matrix. Hence, we have

$$h(\alpha K, \xi) = h(K, \alpha^* \xi),$$

where α^* denotes $g^* = \overline{\alpha}\mathrm{Id}$.

The vector space spanned by all support functions has the following density property (cf. [40, Lemma 1.7.9]).

Lemma 2.1 ([40])**.** *Every twice-differentiable function on the sphere is the difference of two support functions.*

In particular, the real vector space spanned by the differences of support functions (restricted to S^{n-1}) is dense in the space $C(S^{n-1})$ of continuous functions on the sphere (with the maximum norm).

2.2. **Surface area measure and Minkowski's theorem.** Let $K \in \mathcal{K}(V)$, V endowed with a scalar product, and $\omega \subset S^{n-1}$ a Borel subset of S^{n-1}. The *surface area measure of K* is given by

$$S(K,\omega) = \mathrm{Vol}_{n-1}(\{x \in \partial K \,:\, \text{an outward unit normal of } x \text{ is in } \omega\}).$$

Note that if $K \in \mathcal{K}(V)$ is a polytope, then the surface area measure is a discrete measure: the sum of point masses at the outward unit normal vectors to the facets of K, with weight the surface area of the corresponding facet.

Minkowski's existence theorem gives necessary and sufficient conditions for a positive measure on S^{n-1} to be the surface area measure of some convex body (cf. [40, Theorem 7.1.2]).

Theorem 2.2 (Minkowski's existence theorem)**.** *Let μ be a positive finite Borel measure on S^{n-1}. Then, μ is the surface area measure of some convex body $K \subset V$ with non-empty interior if and only if μ is not concentrated on any great subsphere of S^{n-1} and*

$$\int_{S^{n-1}} u\, d\mu(u) = 0. \tag{4}$$

2.3. **Translation invariant valuations.** Let Val denote the Banach space of real-valued, translation invariant, continuous valuations on V.

A valuation $\phi \in \mathrm{Val}$ is called *homogeneous of degree* k if $\phi(tK) = t^k\phi(K)$ for all $t \geq 0$; *even* if $\phi(-K) = \phi(K)$ for all K; and *odd* if $\phi(-K) = -\phi(K)$. The subspace of even (resp. odd) valuations of degree k is denoted by Val_k^+ (resp. Val_k^-).

Theorem 2.3 (McMullen [35])**.**

$$\mathrm{Val} = \bigoplus_{\substack{k=0,\dots,n\\ \varepsilon=+,-}} \mathrm{Val}_k^\varepsilon . \tag{5}$$

In [22] Klain (see also [23]) gives the following description of even translation invariant valuations. For simplicity, we fix a Euclidean scalar product on V. Let $\phi \in \mathrm{Val}_k^+$ and let E be a k-dimensional subspace of V. Klain proved that $\phi|_E$ is a multiple of the volume on E, i.e.

$$\phi(K) = \mathrm{Kl}_\phi(E)\,\mathrm{Vol}(K), \quad \forall K \in \mathcal{K}(E).$$

The function $\mathrm{Kl}_\phi : \mathrm{Gr}_k(V) \to \mathbb{R}$, where $\mathrm{Gr}_k(V)$ the Grassmannian manifold of all k-dimensional subspaces in V, is called the *Klain function of* ϕ.

Theorem 2.4 (Klain's injectivity theorem [22])**.** *Let $\phi \in \mathrm{Val}_k^+$. Then ϕ is uniquely determined by its Klain function $\mathrm{Kl}_\phi \in C(\mathrm{Gr}_k V)$.*

The group $\mathrm{GL}(V)$ acts naturally on Val by

$$g\mu(K) = \mu(g^{-1}K), \quad g \in \mathrm{GL}(V, \mathbb{R}),\ K \in \mathcal{K}(V).$$

A valuation $\mu \in \mathrm{Val}$ is called *smooth* if the map $g \mapsto g\mu$ from the Lie group $\mathrm{GL}(V, \mathbb{R})$ to the Banach space Val is smooth. The subspace of smooth valuations is denoted by Val^{sm}, it is a dense subspace in Val. We will use that if $\mu \in \mathrm{Val}_k^{sm,+}$, then the Klain function of μ is a smooth function on $\mathrm{Gr}_k V$. See [6, 8, 11] for more information on smooth valuations.

2.4. **Valuations and distributions.** Let $\mathcal{E}$ denote the space of continuous 1-homogeneous functions defined on V^*. Let $K \subset V^*$ be a compact convex body containing the origin in its interior. Let us endow $\mathcal{E}$ with the supremum norm restricted to K in V^*, i.e. $\|f\|_K = \sup\{|f(\xi)| : \xi \in K\}$. Then, for every K, L compact convex bodies containing the origin in its interior, the norms $\|\cdot\|_K$, $\|\cdot\|_L$ are equivalent in $\mathcal{E}$ and it becomes a Banach space.

Let $\mathcal{D}$ denote the space of the functions in $\mathcal{E}$ which are smooth on $V^* \setminus \{0\}$.

Goodey and Weil [16] give a representation of a continuous, translation invariant, real-valued valuation of homogeneity degree one in terms of a distribution on the sphere S^{n-1}. We need the following special case.

Theorem 2.5 ([16])**.** *Let $\phi : \mathcal{K}(V) \to \mathbb{R}$ be a continuous, translation invariant valuation which is homogeneous of degree 1. Then, there exists a unique distribution T on $\mathcal{D}$ which can be extended to the Banach subspace* *of $\mathcal{E}$ generated by the support functions h_K for every $K \in \mathcal{K}(V)$ in such a way that*

$$\phi(K) = T(h_K).$$

3. Proof of Theorem 1

Lemma 3.1. *Let W be a complex vector space of complex dimension $m \geq 3$. Let $Z : \mathcal{K}(W) \to \mathcal{K}(W)$ be a continuous, translation invariant, $\mathrm{SL}(W,\mathbb{C})$-covariant Minkowski valuation with degree of homogeneity k, $1 < k \leq 2m-1$. Then $ZK = \{0\}$, $\forall K \in \mathcal{K}(W)$.*

Proof. Let Z be a Minkowski valuation of degree k satisfying the hypothesis of the lemma. Define the operator $\tilde{Z} : \mathcal{K}(W) \to \mathcal{K}(W)$ by

$$\tilde{Z}(K) := \int_{S^1}\int_{S^1} q_1 Z(q_2 K) dq_1 dq_2.$$

It satisfies $\tilde{Z}(qK) = \tilde{Z}(K)$ and $q\tilde{Z}(K) = \tilde{Z}(K)$ for all $q \in S^1$ and $K \in \mathcal{K}(W)$. We say that $\tilde{Z}$ is an S^1-*bi-invariant* valuation.

$\tilde{Z}$ inherits all the desired properties from Z and it turns out to be a continuous, translation invariant and $\mathrm{SL}(W,\mathbb{C})$-covariant Minkowski valuation of degree k. In order to prove the lemma, it suffices to show that there cannot exist a non-trivial S^1-bi-invariant valuation satisfying the hypothesis. We denote again this valuation by Z.

Let $g \in \mathrm{GL}(W,\mathbb{C})$ and write $g = g_0 t q$ with $g_0 \in \mathrm{SL}(W,\mathbb{C})$, $t \in \mathbb{R}_{>0}$, $q \in S^1$. Using the S^1-bi-invariance and the homogeneity of degree k of Z we have

$$Z(gK) = Z(g_0 t q K) = t^k g_0 q Z(K) = t^{k-1} g Z K,$$

and it follows that

$$Z(gK) = |\det g|^{\frac{k-1}{m}} g Z K, \quad \forall g \in \mathrm{GL}(W,\mathbb{C}). \tag{6}$$

We distinguish two cases.

Case $k = m+1$. Let $e_1, \dots, e_m$ be a complex basis of W and $e^1, \dots, e^m$ its dual basis. We denote by E the $(m+1)$-dimensional real subspace generated by $e_1, \dots, e_m, ie_1$.

Let $g \in \mathrm{GL}(W,\mathbb{C})$ be defined by $ge_j = \lambda_j e_j$ with $\lambda_1, \dots, \lambda_m \in \mathbb{R}_{>0}$. Note that g fixes E. Let $D = \lambda_1^2 \prod_{j=2}^m \lambda_j$ be the determinant of the restriction of g to E (considered as an element of $\mathrm{GL}(E,\mathbb{R})$). Let $j \in \{2, \dots, m\}$ and $\xi = e^j$ or $\xi = ie^j$. Using (6) we get

$$h(ZgK, \xi) = h(ZK, g^*\xi)|\det g| = h(ZK, \xi)\lambda_j |\det g|.$$

On the other hand, by Klain's result, the restriction of $h(Z(\cdot), \xi)$ to E is a multiple of the $(m+1)$-dimensional volume. Thus, for every $K \in \mathcal{K}(E)$

$$h(ZgK, \xi) = \mathrm{Vol}(gK)\,\mathrm{Kl}(E) = D\,\mathrm{Vol}(K)\,\mathrm{Kl}(E) = Dh(ZK, \xi).$$

Consequently,

$$h(ZK,\xi)\lambda_1\prod_{j=2}^{m}\lambda_j = h(ZK,\xi)\lambda_j\prod_{j=2}^{m}\lambda_j, \quad \forall\lambda_1,\dots,\lambda_m,$$

which implies

$$h(ZK,e^j)=h(ZK,ie^j)=0, \quad \forall j\neq 1,\, K\in\mathcal{K}(E).$$

Hence, the support function $h := h_{ZK}$ vanishes on all lines $\mathbb{R}\cdot e^j$, $\mathbb{R}\cdot ie^j$, $j = 2,\dots,m$. Since $ZK = -ZK$, this implies that ZK is a two-dimensional convex body contained in the space generated by $\{e_1, ie_1\}$.

Let now $g \in \mathrm{GL}(W,\mathbb{C})$ be defined by $ge_1 = \alpha e_1$, $\alpha = x+iy \in \mathbb{C}$, $ge_j = \lambda_j e_j$, $\lambda_j \in \mathbb{R}_{>0}$, $j = \{2,\dots,m\}$. The determinant D of the restriction of g to E is

$$D = (x^2+y^2)\lambda_2\dots\lambda_m,$$

and

$$|\det g| = |\alpha|\lambda_2\dots\lambda_m = \sqrt{x^2+y^2}\lambda_2\dots\lambda_m.$$

Choosing α with $|\alpha|=1$ we get, for every $K\in\mathcal{K}(E)$

$$h(Z(gK),e^1) = |\det g| h(ZK,\overline{\alpha}e^1) = \lambda_2\dots\lambda_m h(ZK,\overline{\alpha}e^1),$$

$$h(Z(gK),e^1) = Dh(ZK,e^1) = \lambda_2\dots\lambda_m h(ZK,e^1).$$

Thus,

$$h(ZK,e^1) = h(ZK,\alpha e^1), \quad \forall\alpha\in S^1,\, K\in\mathcal{K}(E), \tag{7}$$

and ZK is a disc of radius $r(K)$ contained in the complex line generated by e_1.

Let $K_0 \subset E$ be the parallelotope $[0,e_1]+[0,ie_1]+[0,e_2]+\dots+[0,e_m]$ which we denote by $[e_1, ie_1, e_2,\dots,e_m]$, and let $K=[w_1, iw_1, w_2\dots,w_m]$ be a parallelotope with $w_1 = \alpha e_1$, $\alpha\in S^1$. We claim that

$$h(ZK,e^1) = c|\det(w_1,\dots,w_m)|, \tag{8}$$

where $c = h(ZK_0,e^1)$. Indeed, using the continuity of both sides of (8) it is enough to prove it when $w_1,\dots,w_m$ are linearly independent over $\mathbb{C}$. In this case, we can define $g\in\mathrm{GL}(W,\mathbb{C})$ by $ge_j = w_j$, $j=1,\dots,m$, and from (6) and (7) we have

$$h(ZK,e^1) = h(Z(gK_0),e^1) = |\det g| h(ZK_0, g^*e^1) = c|\det(w_1,\dots,w_m)|.$$

Let us fix a Hermitian scalar product on W such that $e_1,\dots,e_m$ constitutes an orthonormal basis.

Let W_0 be the $(m-1)$-dimensional complex subspace of W generated by $\{e_2,\dots,e_m\}$. Now, let us define a valuation $\phi : \mathcal{K}(W_0)\to\mathbb{R}$ by

$$\phi(K') = h(Z[e_1, ie_1, K'], e^1),$$

where $[e_1, ie_1, K']$ denotes the product of the parallelotope $[e_1, ie_1]$ and $K'\subset W_0$. Note that both convex sets lie in orthogonal spaces.

Define $H \subset SU(W)$ as the stabilizer of $SU(W)$ at e_1. We have $H \cong SU(W_0) \cong SU(m-1)$. If $m \geq 3$, then H acts transitively on the unit sphere of W_0.

By (8), ϕ is $SU(W_0)$-invariant. Alesker established in [6, Proposition 2.6] that if G is a compact subgroup of the orthogonal group acting transitively on the unit sphere of a vector space, then each G-invariant translation invariant continuous valuation is smooth. Thus, ϕ is a smooth valuation. In particular, the Klain function of ϕ is a smooth function.

Let us consider the smooth curve $\gamma : \mathbb{R} \to \mathcal{K}(W_0)$ given by

$$\gamma(t) = [\cos t e_2 + \sin t i e_3, e_3, \ldots, e_m].$$

For these convex sets,

$$\begin{aligned}\phi(\gamma(t)) &= h(Z[e_1, ie_1, \cos t e_2 + \sin t i e_3, e_3, \ldots, e_m], e^1) \\ &= c|\det(e_1, \cos t e_2 + \sin t i e_3, e_3, \ldots, e_m)| = c|\cos t|,\end{aligned}$$

which is smooth only if $c = 0$.

Hence, we get $h(ZK, e^1) = 0$ and from (7) we have $h(ZK, \alpha e^1) = 0$ for all $K \in \mathcal{K}(E)$ and $\alpha \in S^1$. Thus, $r(K) = 0$ (the radius of ZK) and by Klain's injectivity theorem we have $Z \equiv \{0\}$.

Case $1 < k \leq m$ or $m + 1 < k \leq 2m - 1$. The proof of this case is completely analogous to the proof of the contravariant case in [1, Lemma 3.2] and we do not reproduce it here. The main idea of the proof was to use the same matrices $g \in \mathrm{GL}(W, \mathbb{C})$ defined in the previous case. Using (6) and the fact that the power of $|\det g|$ is not an integer one obtains that Z must be the trivial valuation. □

Remark 3.2. If $Z : \mathcal{K}(W) \to \mathcal{K}(W)$ is a continuous, translation invariant, $\mathrm{SL}(W, \mathbb{C})$-covariant Minkowski valuation of degree $2m$ (resp. 0), then the support function of the image is a multiple – depending on the direction – of the volume (resp. the Euler characteristic) and it can be proved as before that it must be the trivial valuation.

Proof of Theorem 1. We assume first that D_C is defined as in (2) and we prove that it satisfies all the stated properties.

The function on the right hand side of (2) is a support function since $h(\alpha K, \cdot)$ is a support function for every α and $dS(C, \cdot)$ is a positive measure. Hence $\mathrm{D}_C K$ is a convex body on W for every $C \in \mathcal{K}(\mathbb{C})$.

In order to show that D_C is a Minkowski valuation we use the additivity of the support function in its first argument. Let $K, L \in \mathcal{K}(W)$ with $K \cup L \in$

$\mathcal{K}(W)$. Then, $K \cup L + K \cap L = K + L$ (cf. [40, Lemma 3.1.1]) and it follows

$$\begin{aligned} h(\mathrm{D}_C(K \cup L) + \mathrm{D}_C(K \cap L), \xi) &= h(\mathrm{D}_C(K \cup L), \xi) + h(\mathrm{D}_C(K \cap L), \xi) \\ &= \int_{S^1} h(K \cup L + K \cap L, \alpha^* \xi) dS(C, \alpha) \\ &= \int_{S^1} h(K + L, \alpha^* \xi) dS(C, \alpha) \\ &= h(\mathrm{D}_C K, \xi) + h(\mathrm{D}_C L, \xi), \end{aligned}$$

which implies the valuation property of D_C.

The continuity of D_C follows from the continuity of the support function.

To prove that D_C is translation invariant we use the only if part of Theorem 2.2. Indeed, for $u \in W$ it follows

$$\begin{aligned} h(\mathrm{D}_C(K + u), \xi) &= h(\mathrm{D}_C K, \xi) + \int_{S^1} \langle \alpha u, \xi \rangle dS(C, \alpha) \\ &= h(\mathrm{D}_C K, \xi) + \left\langle u \int_{S^1} \alpha dS(C, \alpha), \xi \right\rangle \\ &= h(\mathrm{D}_C K, \xi). \end{aligned}$$

Finally, the $\mathrm{SL}(W, \mathbb{C})$-covariance is obtained from (3). For each $g \in \mathrm{SL}(W, \mathbb{C})$ we have

$$\begin{aligned} h(\mathrm{D}_C(gK), \xi) &= \int_{S^1} h(\alpha g K, \xi) dS(C, \alpha) = \int_{S^1} h(\alpha K, g^* \xi) dS(C, \alpha) \\ &= h(\mathrm{D}_C K, g^* \xi) = h(g\, \mathrm{D}_C K, \xi). \end{aligned}$$

It follows that $\mathrm{D}_C(gK) = g\, \mathrm{D}_C K$, hence D_C has all the required properties.

Let us now show the uniqueness of C up to translations. As the area measure $S(C, \cdot)$ is invariant under translations, we can assume that the Steiner point of C is the origin. (Recall that the *Steiner point of a convex body* $K \in \mathcal{K}(V)$ is defined by, see [40, p. 42]

$$s(K) = \frac{1}{\mathrm{Vol}(B^n)} \int_{S^{n-1}} h(K, u) u du,$$

where B^n denotes the unit ball in V.)

Let C_1, C_2 be convex bodies in $\mathbb{C}$ with $s(C_1) = s(C_2) = 0$ and $\mathrm{D}_{C_1} = \mathrm{D}_{C_2}$, i.e.

$$h(\mathrm{D}_{C_1} K, \xi) = h(\mathrm{D}_{C_2} K, \xi), \quad \forall K \in \mathcal{K}(W), \xi \in W^*.$$

Fix $\xi \in W^*$ and $u \in W$ such that $\xi(u) = 1$. Consider $\overline{C_1} u \subset W$ and $\overline{C_2} u \subset W$. For these convex sets and $i, j \in \{1, 2\}$ we have

$$\begin{aligned} h(\mathrm{D}_{C_i}(\overline{C_j} u), \xi) &= \int_{S^1} h(\alpha \overline{C_j} u, \xi) dS(C_i, \alpha) = \int_{S^1} h(\overline{C}_j, \overline{\alpha}) dS(C_i, \alpha) \\ &= \int_{S^1} h(C_j, \alpha) dS(C_i, \alpha) = V_2(C_i, C_j), \end{aligned}$$

where V_2 denotes the mixed volume in $\mathbb{C}$. Hence, we have $V_2(C_1, C_1) = V_2(C_2, C_2) = V_2(C_1, C_2)$.

In particular, either C_1 and C_2 both have empty interior or both have non-empty interior.

Assume that C_1 and C_2 have non-empty interior. The Minkowski inequality in dimension 2 states that (see [40, Theorem 6.2.1])

$$V_2(C_1, C_2)^2 \geq V_2(C_1, C_1)V_2(C_2, C_2),$$

with equality if and only if C_1 and C_2 are homothetic. Thus, we can write $C_1 = rC_2 + z$ with $r \in \mathbb{R}_{>0}$, $z \in \mathbb{C}$. But, from $V(C_1, C_1) = V(C_2, C_2)$ we get $r = 1$ and from $s(C_1) = s(C_2)$, we get $z = 0$. That is, $C_1 = C_2$.

Assume now that C_1, C_2 have empty interior. Then, $C_1 = [-z_1, z_1]$ and $C_2 = [-z_2, z_2]$ with $C_1, C_2 \in \mathbb{C}$. In this case, the area measure of C_1 is given by

$$S([-z_1, z_1], \cdot) = \delta_{iz_1}(\cdot) + \delta_{-iz_1}(\cdot),$$

and

$$h(\mathrm{D}_{C_1}K, \xi) = h(K, i\overline{z}_1\xi) + h(K, -i\overline{z}_1\xi).$$

Then, for every $K = [-zu, zu]$, $z \in \mathbb{C}$, we have

$$h(\mathrm{D}_{C_1}K, \xi) = h(K, i\overline{z}_1\xi) + h(K, -i\overline{z}_1\xi) = 2|\mathrm{Re}(i\overline{z}_1\xi(zu))| = 2|\mathrm{Re}(i\overline{z}_1 z)|,$$

and similarly

$$h(\mathrm{D}_{C_1}K, \xi) = h(\mathrm{D}_{C_2}K, \xi) = 2|\mathrm{Re}(i\overline{z}_2 z)|.$$

It follows that $z_1 = z_2$, and $C_1 = C_2$.

Conversely, let us suppose that Z is a translation invariant continuous Minkowski valuations which is $\mathrm{SL}(W, \mathbb{C})$-covariant. We want to show that there exists some compact convex $C \subset \mathbb{C}$ with $Z = \mathrm{D}_C$ and $s(C) = 0$.

First, we prove that Z must be homogeneous of degree one. McMullen's decomposition (5) applied to Z gives the decomposition

$$h(ZK, \cdot) = \sum_{k=0}^{2m} f_k(K, \cdot),$$

with $f_k(K, \cdot)$ a 1-homogeneous function. In general, f_k is not subadditive as was recently proved in [38]. For the minimal index k_0 and the maximal index k_1 with $f_k \neq 0$, it was proved in [41] that f_{k_0} and f_{k_1} are support functions.

By Lemma 3.1 and Remark 3.2 there is no non-trivial, continuous, translation invariant and $\mathrm{SL}(W, \mathbb{C})$-covariant Minkowski valuation Z of degree $k \neq 1$, if $\dim W \geq 3$. We thus get $k_0 = k_1 = 1$, and Z is of degree 1.

For every $\xi \in W^*$, $h(Z\cdot, \xi)$ is a real-valued valuation, which is also continuous, translation invariant and homogeneous of degree 1. Thus, by Theorem 2.5, there exists a distribution T_ξ defined on W^* such that

$$h(ZK, \xi) = T_\xi(h_K).$$

In order to derive the result for the 1-homogeneous case, we divide the proof in several steps. The first step is to show that, in our case, the distribution T_ξ can be interpreted as a distribution on S^1. Then, using the $\mathrm{SL}(W,\mathbb{C})$-covariance, we show that this distribution on S^1 is independent of ξ. The fourth step is to prove that this distribution is given by a measure defined on S^1. In the last two steps we find that this measure must be positive and the surface area measure of a convex set in $\mathbb{C}$.

STEP 1: *Let $\xi_0 \in W^*$. We claim that there exists a distribution T on S^1 satisfying $(m_{\xi_0})_*T = T_{\xi_0}$, where $m_{\xi_0} : S^1 \to W^*$, $m_{\xi_0}(\alpha) = \alpha^*\xi_0$, and $(m_{\xi_0})_*T(f) := T(f \circ m_{\xi_0})$ for every $f \in \mathcal{D}$, i.e. a continuous, 1-homogeneous function on W^*, smooth on $W^* \setminus \{0\}$.*

Let $E \subset W^*$ be the 1-dimensional complex subspace spanned by ξ_0. Let f be a function defined on W^* such that $f|_E \equiv 0$.

Let us suppose first that f is the support function of $K \in \mathcal{K}(W)$. Then, the condition $f|_E = h_K|_E \equiv 0$ implies that the convex body K lies in the complex subspace $F = \ker \xi_0 \subset W$. Let $g_\lambda \in \mathrm{GL}(W,\mathbb{C})$ with $g_\lambda^*\xi_0 = \xi_0$ and $g_\lambda(v) = \lambda v$, $\lambda \in \mathbb{R}_{>0}$, for every $v \in F$. As g_λ has real entries and $\det g_\lambda > 0$, there exist $t > 0$ and $g_0 \in \mathrm{SL}(W,\mathbb{C})$ (with real entries) such that $g_\lambda = tg_0$. From the 1-homogeneity and the $\mathrm{SL}(W,\mathbb{C})$-covariance of Z, it easily follows that

$$Z(g_\lambda K) = g_\lambda ZK.$$

From the properties of Z and the above equality, we get

$$h(ZK,\xi_0) = h(ZK, g_\lambda^*\xi_0) = h(Z(g_\lambda K),\xi_0) = h(Z(\lambda K),\xi_0) = \lambda h(ZK,\xi_0).$$

As the above equation holds for every $\lambda \in \mathbb{R}_{>0}$, it follows that $T_{\xi_0}(h_K) = h(ZK,\xi_0) = 0$.

Let now $f = h_K - h_L$ with $K, L \in \mathcal{K}(W)$ and $f|_E \equiv 0$, that is, $h(K,\alpha^*\xi_0) = h(L,\alpha^*\xi_0)$ for every $\alpha \in S^1$. Let $g_\lambda \in \mathrm{GL}(W,\mathbb{C})$ be as above. Then, $h(g_\lambda K,\alpha^*\xi_0) = h(g_\lambda L,\alpha^*\xi_0)$ for all $\lambda > 0$. Thus,

$$\lim_{\lambda\to 0} g_\lambda K = \lim_{\lambda \to 0} g_\lambda L,$$

and from the continuity of Z we get on one hand

$$\lim_{\lambda\to 0} Z(g_\lambda K) = \lim_{\lambda\to 0} Z(g_\lambda L).$$

On the other hand, we have for every $\lambda \in \mathbb{R}_{>0}$,

$$h(ZK,\xi_0) - h(ZL,\xi_0) = h(Z(g_\lambda K),\xi_0) - h(Z(g_\lambda L),\xi_0).$$

Taking limits on both sides we get $h(ZK,\xi_0) = h(ZL,\xi_0)$ and $T_{\xi_0}(f) = 0$.

As every function $f \in \mathcal{D}$ can be written as the difference of two support functions, that is, $f = h_K - h_L$ for some $K, L \in \mathcal{K}(W)$ (cf. Lemma 2.1), we get $T_{\xi_0}(f) = 0$ for every $f \in \mathcal{D}$.

Thus, we get that the value of $T_{\xi_0}(f)$ only depends on $f|_E$. We define the distribution T on S^1 by $T(g) := T_{\xi_0}(\tilde{g})$, where $\tilde{g}$ denotes an extension on $\mathcal{D}$ of g (satisfying $\tilde{g}(\alpha^*\xi_0) = g(\alpha)$ and hence, T is well-defined). By definition of m_{ξ_0} we have $T(f\circ m_{\xi_0})=T_{\xi_0}(f)$ since f is an extension of $f\circ m_{\xi_0}$, and the claim follows.

STEP 2: *Let $g\in \mathrm{SL}(W,\mathbb{C})$ and $\xi\in W^*$. The distribution T_ξ satisfies $T_{g^*\xi}=(g^*)_*T_\xi$, where $(g^*)_*T_\xi(f)=T_\xi(f\circ g^*)$ for every $f\in\mathcal{D}$.*

We first prove the equality for a support function h_K, $K\in\mathcal{K}(W)$. Using property (3) of support functions, and that Z is an $\mathrm{SL}(W,\mathbb{C})$-covariant valuation, we get

$$\begin{aligned}(g^*)_*T_\xi(h_K)&=T_\xi(h_K\circ g^*)=T_\xi(h_{gK})\\&=h(Z(gK),\xi)=h(ZK,g^*\xi)=T_{g^*\xi}(h_K).\end{aligned}$$

The general case follows by linearity and Lemma 2.1.

STEP 3: *The distribution T on S^1 given in Step 1 satisfies $(m_\xi)_*T=T_\xi$, for every $\xi\in W^*$.*

Let $\xi_0\in W^*$ as in Step 1 and $\xi\in W^*$. There exists $g\in \mathrm{SL}(W,\mathbb{C})$ such that $g^*\xi_0=\xi$. Using Steps 2 and 1, it follows that

$$T_\xi=T_{g^*\xi_0}=(g^*)_*T_{\xi_0}=(g^*)_*(m_{\xi_0})_*T=(g^*\circ m_{\xi_0})_*T=(m_\xi)_*T.$$

STEP 4: *The distribution T defined in Step 1 is given by a signed measure μ. That is,*

$$(m_\xi)_*T(f)=\int_{S^1}f(\alpha^*\xi)d\mu(\alpha).$$

Schneider obtained in [39] a classification of continuous, Minkowski valuations Φ on a 2-dimensional vector space V which satisfy $\Phi b=b\Phi$ for every b in $\mathrm{SO}(V,\mathbb{R})$ or b a translation in V. The general expression for such a Φ is

$$h(\Phi(K),\alpha)=\int_0^{2\pi}h(K-s(K),u(\alpha+\beta))d\nu(\beta)+\langle u(\alpha),s(K)\rangle,$$

where $s(K)$ denotes the Steiner point of K, $u(\alpha)=\cos(\alpha)e^1+\sin(\alpha)e^2$ with $\{e^1,e^2\}$ a given basis on V^*, and $\langle\,,\rangle$ the pairing of V with its dual space. The signed measure ν is unique up to a linear measure l defined by $\omega\mapsto\int_\omega\langle u(\alpha),a\rangle d\alpha$ with $a\in V$ a constant vector.

Schneider's proof can be easily adapted to our situation. Indeed, let $\xi\in W^*$, $\{e^1,\dots,e^m\}$ a basis of W^* with $e^1=\xi$ and $\{e_1,\dots,e_m\}$ the basis of W dual to $\{e^1,\dots,e^m\}$. Define E as the 1-dimensional complex space in W spanned by e_1. We can identify E^* with $\mathrm{span}_{\mathbb{C}}(\xi)\subset W^*$, and write $\alpha\in E^*$ as a multiple of ξ, which we denote again by α.

Let $\phi:E\to E$ be the restriction of Z to E, i.e. $h(\phi(K),\alpha)=h(ZK,\alpha\xi)$ with $K\in\mathcal{K}(E)$.

The operator ϕ inherits the properties of Z, that is, it is a continuous, translation invariant Minkowski valuation which is covariant with respect to $\mathrm{SL}(E,\mathbb{C})\cong S^1$. Now, ϕ satisfies all the hypothesis of Schneider's result except the covariance with respect to translations (i.e. $\Phi(K+t)=\Phi(K)+t$), which is replaced by the invariance (i.e. $\phi(K+t) = \phi(K)$). However, the first step in Schneider's proof is to construct a translation invariant valuation from the translation covariant one via $\Phi - s$ (where s denotes the Steiner point), and the same argument can be used in our situation.

STEP 5: *The measure μ is positive.*

By assumption, the function $F(\xi) = h(ZK, \xi)$ must be convex. Thus, the second differential of F at each ξ must be a positive semi-definite bilinear form (cf. [40, p. 108 and Theorem 1.5.10]). Note that, by the representation of $h(ZK, \xi)$ obtained in the previous step, if $h(K, \cdot)$ is a smooth function, then $h(ZK, \cdot)$ is also smooth.

For simplicity, we fix a scalar product on W. Fix $\xi \in W$ and let $S^1 \cdot \xi \subset W$ be a circle contained in the complex line spanned by ξ. Let $\epsilon > 0$ and $\mathring{B}$ the open $2m$-dimensional ball in W. We have that $S^1 \cdot \xi \times \epsilon \mathring{B}$ is a neighborhood of $S^1 \cdot \xi$ in W. Geometrically, it can be interpreted as the open tube of radius ϵ along $S^1 \cdot \xi$.

Let $f : S^1 \to \mathbb{R}_{>0}$ be a positive smooth function. Attach to each point $p_\theta := e^{i\theta}\xi \in S^1 \cdot \xi$ a spherical cap of a $(2m-2)$-dimensional sphere S_θ with radius $f(\theta)$, center on the segment $[0, p_\theta]$ and tangent plane at p_θ orthogonal to $\{\xi, i\xi\}$.

For ϵ small enough the intersection between $S^1 \cdot \xi \times \epsilon \mathring{B}$ and the set described in the previous paragraph is a smooth hypersurface. Denote it by K_ϵ.

The principal curvatures of K_ϵ at p_θ are $f^{-1}(\theta) > 0$ (the inverse of the radius of the attached $(2m-2)$-dimensional sphere at p_θ) with multiplicity $2m-2$, and 1, corresponding to the principal direction $ie^{i\theta}\xi$, tangent to $S^1 \cdot \xi$ at p_θ. Since the function f is smooth, strictly positive, and defines the hypersurface in a neighborhood of $S^1 \cdot \xi$, all principal curvatures at any point of K_ϵ are strictly positive, provided ϵ is small enough.

Set $K := \mathrm{conv}\overline{K_\epsilon}$, where $\mathrm{conv}\overline{K_\epsilon}$ denotes the convex hull of $\overline{K_\epsilon}$, the closure of K_ϵ, at W. As K_ϵ is a convex hypersurface, we have $\partial K \cap (S^1 \cdot \xi \times \epsilon B) = K_\epsilon \cap (S^1 \cdot \xi \times \epsilon B)$, and K is smooth in a neighborhood of $S^1 \cdot \xi$. The second differential of $h(ZK, \cdot)$ at ξ must be positive semi-definite (cf. [40, Theorem 1.5.10]), that is,

$$(d^2 h(ZK, \xi))(a, a) = \int_{S^1} (d^2 h(\alpha K, \xi))(a, a) d\mu(\alpha) \geq 0, \, \forall a \in W.$$

The eigenvalues of the second differential of the support function of a convex body $K \subset W$ in a direction $\xi \in W$ are the radii of curvature of K at the corresponding supporting point with eigenvector the principal directions at this supporting point (cf. [40, Corollary 2.5.2]). By the construction of K, the support point of $\overline{\alpha}K$ in direction ξ is $p_{-\theta}$, with $e^{-i\theta} = \overline{\alpha}$.

Take $a = b = u$ with u a principal direction of K at $p_0 = \xi$ different from $i\xi$. Then, u is also a principal direction of K at every point p_θ, with principal radius of curvature $f(e^{i\theta})$, the radius of the attached sphere at p_θ.

Hence,

$$(d^2h(\alpha K,\xi))(u,u)=f(\overline{\alpha}).$$

Therefore, for every strictly positive smooth function on S^1 we get

$$\int_{S^1} f(\overline{\alpha})d\mu(\alpha)\geq 0,$$

and from the density of smooth positive functions on the space of continuous positive functions, we get that the measure μ is positive.

STEP 6: *The measure μ is the surface area measure for some convex body in $\mathbb{C}$.*

Using the translation invariance of Z, we have that μ satisfies condition (4) of Minkowski's existence theorem (see Theorem 2.2).

If μ is not concentrated on two antipodal points of S^1, then from Theorem 2.2 we get the existence of a 2-dimensional convex body $C\subset\mathbb{C}$ with $dS(C,\cdot)=\mu(\cdot)$.

Otherwise, if μ is concentrated on two antipodals points $\pm\alpha$ of S^1, then μ coincides with the surface area of a centered interval with normal vector given by the direction α. Thus, μ is the surface area of a (1-dimensional) convex body. □

4. THE CASE $\dim W=2$

In order to have a complete classification of continuous, translation invariant, $\mathrm{SL}(W,\mathbb{C})$-covariant or $\mathrm{SL}(W,\mathbb{C})$-contravariant Minkowski valuations with a fixed homogeneity degree, it just remains to study the case of $\mathrm{SL}(W,\mathbb{C})$-covariant valuations of degree 3 and $\mathrm{SL}(W,\mathbb{C})$-contravariant valuations of degree 1 in a complex 2-dimensional space W.

Fix a basis of W and consider the determinant map

$$\begin{array}{rccl}\det: & W\times W & \longrightarrow & \mathbb{C}\\ & (u,v) & \mapsto & \det(u,v).\end{array}\tag{9}$$

This map induces an identification Φ between W and its dual space W^*, which satisfies $\Phi(gu)=(\det g)g^{-*}\Phi(u)$, for every $g\in\mathrm{GL}(W,\mathbb{C})$, $u\in W$.

Then, every $\mathrm{SL}(W,\mathbb{C})$-contravariant (resp. covariant) Minkowksi valuation Z of degree k is in correspondence with an $\mathrm{SL}(W,\mathbb{C})$-covariant (resp. contravariant) Minkowski valuation $\Phi^{-1}\circ Z$ (resp. $\Phi\circ Z$) also of degree k. Thus, the following classification results follow directly from Theorem 1.1 and Theorem 1.

Proposition 4.1. *Let $\dim_{\mathbb{C}} W=2$ and $Z:\mathcal{K}(W)\to\mathcal{K}(W^*)$ a continuous, translation invariant and $\mathrm{SL}(W,\mathbb{C})$-contravariant Minkowski valuation of degree k. If $k\neq 1,3$, then $Z\equiv\{0\}$. If $k=3$, then Z is of the form (1). If $k=1$, then there exists a convex body $C\subset\mathbb{C}$ such that*

$$h(ZK,w)=\int_{S^1} h(\det(K,w),\alpha)dS(C,\alpha),\quad K\in\mathcal{K}(W),\ w\in W,$$

where $\det(K,w) := \{\det(k,w) \,|\, k \in K\}$. *Moreover,* C *is unique up to translations.*

Proposition 4.2. *Let* $\dim_{\mathbb{C}} W = 2$ *and* $Z : \mathcal{K}(W) \to \mathcal{K}(W)$ *a continuous, translation invariant and* $\mathrm{SL}(W,\mathbb{C})$*-covariant Minkowski valuation of degree* k. *If* $k \neq 1,3$, *then* $Z \equiv \{0\}$. *If* $k = 1$, *then* Z *is of the form* (2). *If* $k = 3$, *then there exists a convex body* $C \subset \mathbb{C}$ *such that*

$$h(ZK,\xi) = V(K,K,K,C\cdot w),$$

where $w \in W$ *is the corresponding vector to* ξ *given by the identification* Φ^{-1} *between* W^* *and* W *determined by* (9). *Moreover,* C *is unique up to translations.*

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