Determination of Integral Cayley Graphs on Finite Abelian Groups

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Abstract A graph is integral means that all its eigenvalues are integers. In this note, we determine all the integral Cayley graphs on finite abelian groups. Moreover, we calculate the the number of integral Cayley graphs on a given finite abelian group.

Keywords: abelian group, character, cayley graph, integral graph,

1 Introduction and Main Results

A graph ,whose eigenvalues are integers, is called integral. Since 1974, Harary and Schwenk (see [HS]) began to research the question. There are lots of literature studying integral graphs(see e.g., [A][BC][Bu][EH][LL][So][Z]). In this note we focus on the collection of Cayley graphs.

Let G be a group and let S be a subset of G that is closed under taking inverses and does not contain the identity. Then the Cayley graph (see [GR])$D(G, S)$ is the

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graph with vertex set G and edge set $E(D(G,S)) = \{gh \mid hg^{-1} \in S\}$.

Our purpose in this note is to characterize the integral Cayley graphs among the finite abelian groups and compute the the number of integral Cayley graphs on a given finite abelian group.

Let G be a abelian group with order n. By the theorem of finite abelian group, G is isomorphic to the direct sum of some cyclic groups. So, in the following, we assume that $G = C_1 \bigoplus C_2 \bigoplus \cdots \bigoplus C_m$, where $C_i = \mathbb{Z}/n_i\mathbb{Z}$ (n_i not necessary be a power of a prime number)for some integer $n_i, 1 \leq i \leq m$. The main results in the present note are stated as follows.

A. (see Theorem 1.) The Cayley graph $D(G,S)$ is integral if and only if S is a union of some orbits Hg_i's.

B. (see Corollary 1.) For $G = (\mathbb{Z}/n_1\mathbb{Z}) \bigoplus (\mathbb{Z}/n_2\mathbb{Z}) \bigoplus \cdots \bigoplus (\mathbb{Z}/n_m\mathbb{Z})$, there are at most $2^{r(G)}$ integral Cayley graphs on G.

2 Proof of Main Theorem

Let $\mathbb{Q}(\zeta_n)$ be the n-th cyclotomic field, and $H = Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ be the Galois group of $\mathbb{Q}(\zeta_n)$ over the rational number field $\mathbb{Q}$. Now n_1 is a positive divisor of n with $n = n_1 n_2 \cdots n_m = n_1 t$. We define a group operation π_1 of H on $\mathbb{Z}/n_1\mathbb{Z}$, $\pi_1 : H \times (\mathbb{Z}/n_1\mathbb{Z}) \longrightarrow \mathbb{Z}/n_1\mathbb{Z}$. Since $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$, for each $\sigma \in H$, there is an element $a \in (\mathbb{Z}/n\mathbb{Z})^*$ such that $\sigma(\zeta_n) = \zeta_n^a$. Then, for $x \in \mathbb{Z}/n_1\mathbb{Z}$, we have $\sigma(\zeta_{n_1}^x) = \zeta_n^{atx} = \zeta_{n_1}^{ax} = \zeta_{n_1}^y$, with $y \equiv ax (mod n_1), 0 \leq y < n_1$. The operation π_1 is defined by $\pi_1(\sigma, x) = \sigma(x) = y$. Then we extend the operation to G by $\sigma(x_1, x_2, \cdots, x_m) = (\sigma(x_1), \sigma(x_2), \cdots, \sigma(x_m))$, for each $(x_1, x_2, \cdots, x_m) \in G$. Using the orbit decomposition formula(see [L]), $G \setminus \{0\}$ is the disjoint union of the distinct orbits, and we can write $G \setminus \{0\} = \bigsqcup_{i\in I} Hg_i$, where I is some indexing set, and the g_i are elements of distinct orbits.

We obtain the necessary and sufficient condition for a Cayley graph $D(G,S)$ to be integral as in the following theorem.

Theorem 1. The Cayley graph $D(G,S)$ is integral if and only if S is a union of some orbits Hg_i's.

In the following, we give the proof Theorem 1 in the case of $m = 2$, i.e., G is a direct sum of two summands.

Let $\widehat{G}$ denote the group of multiplicative homomorphisms from G to $\mathbb{C}^*$ (see [Se] or [W]), where $\mathbb{C}$ is the complex number field. Then we have isomorphism $G \cong \widehat{G}$, since G is abelian. For each $(a,b) \in G$, there is an element $\chi_{(a,b)} \in \widehat{G}$ such that $\chi_{(a,b)}(s,t) = \zeta_{n_1}^{as} \times \zeta_{n_2}^{bt}, (s,t) \in G$. The spectrum of Cayley graph $D(G,S)$ is given by [Ba] and [Bi],

$$spec(D(G,S)) = \{\lambda_{(0,0)}, \lambda_{(0,1)}, \cdots, \lambda_{(0,n_2-1)}, \lambda_{(1,0)}, \cdots, \lambda_{(n_1-1,0)}, \cdots, \lambda_{(n_1-1,n_2-1)}\},$$

where $\lambda_{(a,b)} = \sum_{(s,t)\in S} \chi_{(a,b)}(s,t) = \sum_{(s,t)\in S} \zeta_{n_1}^{as} \cdot \zeta_{n_2}^{bt}$.

Proposition 1. If S is a union of some orbits Hg_i's, the Cayley graph $D(G,S)$ is integral.

Proof. For each orbit Hg_i, we have $\sigma Hg_i = \{\sigma(hg_i) \mid h \in H\} = Hg_i, \sigma \in H$.

So $\sigma S = S$, since S is a union of some orbits Hg_i's. Thus

$$\begin{aligned}\sigma(\lambda_{(a,b)}) &= \sum_{(s,t)\in S} \sigma(\zeta_{n_1}^{as} \cdot \zeta_{n_2}^{bt}) = \sum_{(s,t)\in S} \sigma(\zeta_{n_1}^{as}) \cdot \sigma(\zeta_{n_2}^{bt}) \\ &= \sum_{(s,t)\in S} \zeta_{n_1}^{a\sigma(s)} \cdot \zeta_{n_2}^{b\sigma(t)} = \lambda_{(a,b)},\end{aligned}$$

for every element $\sigma \in H$. By the Galois theory for finite Galois extensions, we get $\lambda_{(a,b)} \in \mathbb{Q}$. Notice that $\lambda_{(a,b)}$ are algebraic integers, hence $\lambda_{(a,b)} \in \mathbb{Z}$, which shows that the Cayley graph $D(G, S)$ is integral. $\blacksquare$

In another hand, we have the following proposition.

Proposition 2. If the Cayley graph $D(G, S)$ is integral, then S is a union of some orbits Hg_i's .

Before the proof, we need some preparation and the following two lemmas.

Let Γ be a $(n-1)$-order square matrix with the row and column index in $G \setminus \{(0, 0)\}$. Namely, we suppose $\Gamma = (\gamma_{(a,b)(\alpha,\beta)})$, where $\gamma_{(a,b)(\alpha,\beta)}$ is the entry of the (a, b)-th row and (α, β)-th column, with $(a, b), (\alpha, \beta) \in G \setminus \{(0, 0)\}$. And $\gamma_{(a,b)(\alpha,\beta)} = \chi_{(a,b)}(\alpha, \beta) = \zeta_{n_1}^{a\alpha} \times \zeta_{n_2}^{b\beta}$. For the matrix Γ, we have the following lemma.

Lemma 1. The matrix Γ is nonsingular, i.e., $det(\Gamma) \neq 0$.

Proof. Let χ be a nontrivial character of finite abelian group G, we know that $\sum_{g\in G} \chi(g) = 0$. Then the sum of all entries in each row of Γ is -1. Hence, to prove this lemma, it suffice to show the $n \times n$ block matrix $\Gamma' = \begin{pmatrix} 1 & 1 \\ 1 & \Gamma \end{pmatrix}$ is invertible.

Let K be the space of complex valued class function on G. Then all the elements of $\widehat{G}$ form an orthonormal basis of K(see [Se]). Suppose there exist n complex numbers $k_{(a,b)}, (a, b) \in G$ such that $\sum_{(a,b)\in G} k_{(a,b)} R_{(a,b)} = (0, 0, \cdots, 0)$, where $R_{(a,b)}$ is the (a,b)-th row of the matrix Γ'. Then we have the class function $\sum_{(a,b)\in G} k_{(a,b)}\chi_{(a,b)} = 0$. So $k_{(a,b)} = 0$ for all $(a,b) \in G$, which shows that the row vectors of Γ' are linearly independent and so Γ' is invertible. ■

Let τ be such a $(n-1)$-dimension column vector as

$$\tau = (v_{(0,1)}, v_{(0,2)}, \cdots, v_{(0,n_2-1)}, v_{(1,0)}, \cdots, v_{(n_1-1,0)}, \cdots, v_{(n_1-1,n_2-1)})^T$$

with $v_{(a,b)} = 1$ for $(a,b) \in S$ and 0 otherwise. It is easy to see that

$$\Gamma\tau = (\lambda_{(0,1)}, \lambda_{(0,2)}, \cdots, \lambda_{(0,n_2-1)}, \lambda_{(1,0)}, \cdots, \lambda_{(n_1-1,0)}, \cdots, \lambda_{(n_1-1,n_2-1)})^T.$$

Let τ_i be the $(n-1)$-dimension column vector for the orbit Hg_i just as τ for S. We denote W the vector space $\{\omega \in \mathbb{Q}^{n-1} \mid \Gamma\omega \in \mathbb{Q}^{n-1}\}$ and $V \subset \mathbb{Q}^{n-1}$ the the vector space spanned by the vectors $\{\tau_i, i \in I\}$. We obtain the following lemma for W and V.

Lemma 2. we have that W and V are the same vector space, i.e., $W = V$.

Proof. By Proposition 1, $\Gamma\tau_i \in \mathbb{Z}^{n-1}, i \in I$. So we get $V \subset W$. Let $\omega \in W$ and $\omega = (\omega_{(0,1)}, \omega_{(0,2)}, \cdots, \omega_{(0,n_2-1)}, \omega_{(1,0)}, \cdots, \omega_{(n_1-1,0)}, \cdots, \omega_{(n_1-1,n_2-1)})^T, u = \Gamma\omega = (u_{(0,1)}, u_{(0,2)}, \cdots, u_{(0,n_2-1)}, u_{(1,0)}, \cdots, u_{(n_1-1,0)}, \cdots, u_{(n_1-1,n_2-1)})^T$. First, we show that $u_{(a,b)} = u_{(c,d)}$ if $(a,b), (c,d)$ in the same orbit Hg_i for some $i \in I$. Because $(a,b), (c,d) \in Hg_i$, there exist an element $\sigma \in H$ such that $\sigma(a,b) = (c,d)$, namely, $\sigma(a) = c$, and

$\sigma(b) = d$. In fact,

$$\begin{aligned}
u_{(a,b)} = \sigma(u_{(a,b)}) = \sigma(\sum_{(k,l)\in G\backslash\{0\}} \omega_{(k,l)}\chi_{(a,b)}(k,l)) &= \sum_{(k,l)\in G\backslash\{0\}} \omega_{(k,l)}\sigma(\chi_{(a,b)}(k,l)) \\
= \sum_{(k,l)\in G\backslash\{0\}} \omega_{(k,l)}\sigma(\zeta_{n_1}^{ak} \cdot \zeta_{n_2}^{bl}) &= \sum_{(k,l)\in G\backslash\{0\}} \omega_{(k,l)}\zeta_{n_1}^{\sigma(a)k} \cdot \zeta_{n_2}^{\sigma(b)l} \\
= \sum_{(k,l)\in G\backslash\{0\}} \omega_{(k,l)}\zeta_{n_1}^{ck} \cdot \zeta_{n_2}^{dl} = u_{(c,d)},
\end{aligned}$$

which implies that $\Gamma(W) \subset V$. Notice that the matrix Γ is nonsingular by Lemma 1 and $V \subset W$. Hence $dimW = dimV$, and $W = V$. ■

Now it is time to prove Proposition 2.

Proof of Proposition 2. Since the Cayley graph $D(G, S)$ is integral, then $\Gamma\tau \in \mathbb{Q}^{n-1}$. We have $\tau \in W$. By Lemma 2, $\tau \in V$ and $\tau = \sum_{i\in I} c_i\tau_i$ for some coefficients $c_i \in \mathbb{Q}$. By the construction of τ and τ_i's, we conclude that S is the union of the Hg_i's with $c_i = 1$. The proof is completed. ■

Remark Merging Proposition 1 and Proposition 2 together, we have Theorem 1 in the case of $m = 2$. For the general case, by [Ba] and [Bi], $spec(D(G, S)) = \{\lambda_g \mid g = (g_1, g_2, \cdots, g_m) \in G\}$, where $\lambda_g = \sum_{(s_1,s_2,\cdots,s_m)\in S} \zeta_{n_1}^{g_1s_1}\zeta_{n_1}^{g_2s_2} \cdot \zeta_{n_2}^{g_ms_m}$. So far, we have given the ideas and methods to the proof of Theorem 1. It is not hard to find these ideas and methods can be applied to the general case. In another words, with the formula of spectrum, we can prove the Theorem 1 for any $m \in \mathbb{Z}^+$ ($\mathbb{Z}^+$, the set of all positive integers) in the same way as above. So we obtain the result of Theorem 1.

3 Computation the number of integral Cayley graphs

Let $d \mid n_0, 0 < d < n_0$, and denote $G_{n_0}(d)$ the set $\{k \mid 0 < k < n_0, gcd(k, n_0) = d\}$. So we get a collection G_{n_0} of such $G_{n_0}(d)$'s, i.e., $G_{n_0} = \{G_{n_0}(d) \mid d \mid n_0, 0 < d <$

$n_0\}$. In the case of $m = 1, G = \mathbb{Z}/n\mathbb{Z}$, it is easy to check that all the orbits Hg_i's is the collection G_n. Thus Theorem 7.1 in [So] can be deduced from Theorem 1.

Denote P_G the collection of Cartesian product $\{p_1 \times p_1 \times \cdots \times p_m \mid p_i \in G_{n_i} \text{ or } p_i = 0, \text{ not all } p_i \text{ are zeros }\}$. For a Cartesian product $P \in P_G$, choose one element $\rho \in P, \rho = (a_1, a_2, \cdots, a_m)$. Denote $\mathbb{Q}(P)$ the cyclotomic field

$$\mathbb{Q}(\zeta_{n_1}^{a_1}, \zeta_{n_2}^{a_2}, \cdots, \zeta_{n_m}^{a_m}) = \mathbb{Q}(\zeta_{n_1}^{a_1}) \cdot \mathbb{Q}(\zeta_{n_2}^{a_2}) \cdots \mathbb{Q}(\zeta_{n_m}^{a_m}),$$

$[\mathbb{Q}(P) : \mathbb{Q}]$ the dimension of $\mathbb{Q}(P)$ as vector space over $\mathbb{Q}$ and $|P|$ the cardinal number of P. We have the following Lemma.

Lemma 3. Under the operation of H restricted on the Cartesian product $P \in P_G, P$ is divided into $\frac{|P|}{[\mathbb{Q}(P):\mathbb{Q}]}$ orbits.

To prove Lemma 3, we need the following proposition.

Proposition 3. Let $K_1, K_2, \cdots, K_m (m > 2)$, be Galois extensions of a field k, with Galois group $H_1, H_2, \cdots, H_m$, respectively. Assume $K_1, K_2, \cdots, K_m$ are subfields of some field. Then $K_1 K_2 \cdots K_m$ is Galois over k. Let map

$$Gal(K_1 K_2 \cdots K_m / k) \to H_1 \times H_2 \times \cdots \times H_m$$

by restriction, namely, $\sigma \mapsto (\sigma\mid_{K_1}, \sigma\mid_{K_2}, \cdots, \sigma\mid_{K_m})$. This map is injective.

Proof. By induction, it is easy to see Proposition 3 is a extension of Theorem 1.14 in Chapter 6 of [L]. We omit the details. ■

Proof of Lemma 2. Notice that $\mathbb{Q}(P) \subset \mathbb{Q}(\zeta_n)$. If $\sigma \in H$, then the restriction of σ to $\mathbb{Q}(P)$ is in $Gal(\mathbb{Q}(P)/\mathbb{Q})$. So by Proposition 3 above, every orbit contained in some Cartesian product P has $[\mathbb{Q}(P):\mathbb{Q}]$ elements, which implies that P can be divided into $\frac{|P|}{[\mathbb{Q}(P):\mathbb{Q}]}$ orbits under group action of H. ■

Let $r(G) = \Sigma_{P \in P_G} \frac{|P|}{[\mathbb{Q}(P):\mathbb{Q}]}$. By Lemma 3, $r(G)$ is the orbits number of group operation of H on G. So we obtain the following Corollary by Theorem 1.

Corollary 1. For $G = (\mathbb{Z}/n_1\mathbb{Z}) \bigoplus (\mathbb{Z}/n_2\mathbb{Z}) \bigoplus \cdots \bigoplus (\mathbb{Z}/n_m\mathbb{Z})$, there are at most $2^{r(G)}$ integral Cayley graphs on G.

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