HEAT EQUATION FOR WEIGHTED BANACH SPACE VALUED FUNCTION SPACES

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Abstract. We study the homogeneous equation (*) $u' = \Delta u$, $t > 0$, $u(0) = f \in wX$, where wX is a weighted Banach space, $w(x) = (1+||x||)^k$, $x \in \mathbb{R}^n$ with $k \geq 0$, Δ is the Laplacian, Y a complex Banach space and X one of the spaces $BUC(\mathbb{R}^n, Y)\}$, $C_0(\mathbb{R}^n, Y)$, $L^p(\mathbb{R}^n, Y)$, $1 \leq p < \infty$. It is shown that the mild solutions of (*) are still given by the classical Gauss-Poisson formula, a holomorphic C_0-semigroup.

§1. Introduction, Notation and Preliminaries

In this note[1] Example 3.7.6 of [1, p. 154] about solutions of the heat equation via holomorphic C_0-semigroups is extended to weighted function spaces and Banach space valued functions. Our treatment is different from [1, p. 154]: instead of using Fourier transforms, direct methods are used.

Let $w(x) := w_k(x) = (1 + ||x||)^k$ with $k \in \mathbb{R}_+ = [0, \infty)$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $||x|| = (\sum_{k=1}^n x_k^2)^{1/2}$. Then $w \in C(\mathbb{R}^n)$ and

$$(1.1) \qquad 1 \leq w(x+y) \leq w(x)w(y),\ w(y) \leq w(x-y)w(x),\ w(0) = 1,$$
$$|w(x+y)/w(x) - 1| \leq w(y)(w(y) - 1), \qquad x, y \in \mathbb{R}^n.$$

Let Y be a complex Banach space and

$$(1.2) \qquad wX = \{wg : g \in X\} \text{ with } X \text{ one of the spaces}$$
$$BUC(\mathbb{R}^n, Y),\ C_0(\mathbb{R}^n, Y),\ L^p(\mathbb{R}^n, Y),\ 1 \leq p < \infty.$$

Then wX is a Banach space with norm $||f||_{wX} = ||f/w||_X$ and a linear subset of $\mathcal{S}'(\mathbb{R}^n, Y)$, wX is translation invariant, since X is and, with $f = wg$, $g \in X$,

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$f_h(x) := f(x+h)$, one has $f_h/w = g_h w_h/w$ with $w_h/w \in BUC(\mathbb{R}^n, \mathbb{R})$, using (1.1).

For any $f : \mathbb{R}^n \to Y$, $|f|(x) = ||f(x)||$, $x \in \mathbb{R}^n$.

For $f \in wX$ and $\zeta \in \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Re}\zeta > 0\}$ define (see Lemma 1.3)

$$(1.3) \qquad (G(\zeta)f)(x) := (4\pi\zeta)^{-n/2} \int_{\mathbb{R}^n} f(x-y) e^{-||y||^2/4\zeta}\, dy,\ x \in \mathbb{R}^n.$$

Let $\chi_\zeta(x) = (4\pi\zeta)^{-n/2} e^{-||x||^2/4\zeta}$, $\zeta \in \mathbb{C}^+$, $x \in \mathbb{R}^n$. Then $\chi_\zeta \in \mathcal{S}(\mathbb{R}^n)$ if $\zeta \in \mathbb{C}^+$,

$$(1.4) \qquad (G(\zeta)f) = \chi_\zeta * f,\ \ \zeta \in \mathbb{C}^+,\ G(0)f = f, \qquad f \in wX.$$

The function χ_ζ is defined and $\chi'_\zeta = \frac{d\chi_\zeta}{d\zeta}$ exists for each $\zeta \in \mathbb{C}^+$, thus holomorphic on $\mathbb{C}^+$. Moreover, $\chi_\zeta^{(k)} = \frac{d^k\chi_\zeta}{d\zeta^k} \in \mathcal{S}(\mathbb{R}^n)$ for each $\zeta \in \mathbb{C}^+$, $k \in \mathbb{N}_0$.

$$(1.5) \qquad I = I(\zeta) = ((4\pi\zeta)^{-n/2} \int_{\mathbb{R}^n} e^{-(||x||^2/4\zeta)}\, dx = 1 \text{ for each } \zeta \in \mathbb{C}^+.$$

Indeed, $I(\zeta)$ is holomorphic on $\mathbb{C}^+$ with $I = 1$ on $(0, \infty)$. It follows $I = 1$ on $\mathbb{C}^+$ by the identity theorem for complex valued holomorphic functions.

Also, for $\zeta = re^{i\phi}$, $0 \le |\phi| < \alpha < \pi/2$, $r > 0$, for any $x \in \mathbb{R}^n$

$$(1.6) \qquad |\chi_\zeta(x)| = (4\pi r)^{-n/2} e^{-(||x||^2 \cos\phi)/4r} < (4\pi r)^{-n/2} e^{-(||x||^2 \cos\alpha)/4r}.$$

$$(1.7) \qquad \text{Fourier transform } \widehat{\chi_\zeta}(x) = e^{-\zeta\, ||x||^2},\ x \in \mathbb{R}^n,\ \zeta \in \mathbb{C}^+.$$

Indeed, it is enough to prove the case $n = 1$. We have

$\widehat{\chi_\zeta}(y) = e^{-\zeta y^2} I(\zeta, y)$, where

$I(\zeta, y) = (4\pi\zeta)^{-1/2} \int_{\mathbb{R}} e^{-(x+2i\zeta y)^2/4\zeta}\, dx$, $y \in \mathbb{R}$, $\zeta \in \mathbb{C}^+$.

With $F(x, y) := e^{-(x+2i\zeta y)^2/4\zeta}$,

$$\frac{\partial}{\partial y} \int_{\mathbb{R}} F(x,y)\, dx = \int_{\mathbb{R}} \frac{\partial}{\partial y} F(x,y)\, dx = \int_{\mathbb{R}} 2i\zeta \frac{\partial}{\partial x} F(x,y) dx =$$
$$= 2i\zeta \lim_{N\to\infty} (F(N,y) - F(-N,y)) = 0,$$

so $I(\zeta, y) = I(\zeta, 0), = 1$ for ζ real > 0 (e.g. [3, p. 274, Beispiel 1]), then for $\zeta \in \mathbb{C}^+$ since I is holomorphic there.

Lemma 1.1. *If $f \in wX$ respectively $wf \in L^p(\mathbb{R}^n, \mathbb{C})$ with $1 \le p < \infty$, then $||(f_y - f)/w||_X \to 0$ respectively $||w(f_y - f)||_{L^p} \to 0$ as $y \to 0$.*

Proof. Let $f = wg$, where $g \in X$. Then $||f_y - f||_{wX} = ||w_y g_y - wg||_{wX} = ||(w_y - w)g_y + wg_y - wg||_{wX} \leq ||(w_y - w)g_y||_{wX} + ||wg_y - wg||_{wX} = ||(w_y/w - 1)g_y||_X + ||g_y - g||_X \to 0$ as $y \to 0$ with (1.1) since $||g_y - g||_X \to 0$ as $y \to 0$. The second case follows similarly. □

Lemma 1.2. *(A) If $f \in wL^p(\mathbb{R}^n, Y)$, $wg \in L^q(\mathbb{R}^n, \mathbb{C})$ with $1/p + 1/q = 1$ and $1 \leq p \leq \infty$, then $(g * f)(x)$ exists as a Bochner integral for all $x \in \mathbb{R}^n$, and $g * f \in wBUC(\mathbb{R}^n, Y)$; if additionally $1 < p < \infty$ or $f \in wC_0(\mathbb{R}^n, Y)$ and $q = 1$, then $g * f \in wC_0(\mathbb{R}^n, Y)$.*

(B) If $f \in wL^p(\mathbb{R}^n, Y)$, $wg \in L^1(\mathbb{R}^n, \mathbb{C})$ with $1 \leq p \leq \infty$, then $g * f(x)$ exists as Bochner integral almost everywhere in $\mathbb{R}^n$ and $g * f \in wL^p(\mathbb{R}^n, Y)$.

Proof. (A) Since

$$(1.8) \qquad ||f(y)g(x-y)|| = ||(f/w)(y)|||(wg)(x-y)|(w(y)/w(x-y)) \leq |f/w|(y)\ |wg|(x-y)w(x),$$

(1.1), $|f/w| \in L^p(\mathbb{R}^n)$ and $|wg|(x - \cdot) \in L^q(\mathbb{R}^n)$, with the Hölder inequality [5, p. 34, Proposition 2] one has $f(\cdot)g(x - \cdot) \in L^1(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$,

$$(1.9) \qquad ||g * f(x)|| \leq w(x)||wg||_{L^q}||f/w||_{L^p}.$$

With this

$$||g * f(x + y) - g * f(x)|| \leq w(x)||f/w||_{L^p}||w(g_y - g)||_{L^q},$$
$$||g * f(x + y) - g * f(x)|| \leq w(x)||(f_y - f)/w||_{L^p}||wg||_{L^q}.$$

By Lemma 1.1, $||w(g_y - g)||_{L^q} \to 0$ respectively $||(f_y - f)/w||_{L^p} \to 0$ as $y \to 0$ if $1 \leq q < \infty$ respectively $1 \leq p < \infty$. It follows $g*f \in wBUC(\mathbb{R}^n, Y)$ if $1 \leq p, q \leq \infty$. If $p > 1, q < \infty$ or $f \in wC_0(\mathbb{R}^n, Y)$ and $q = 1$, then $(w|g|) * (|f|/w) \in C_0(\mathbb{R}^n)$ by [1,Proposition 1.3.2 b), d), p. 22]. It follows $g * f \in wC_0(\mathbb{R}^n, Y)$.

(B) By Young's inequality [5, p. 29], $(w|g|) * (|f|/w) \in L^p(\mathbb{R}^n)$. So, $(w|g|) * (|f|/w)(x))$ is finite almost everywhere on $\mathbb{R}^n$. This, measurability of $g(x - \cdot)f(\cdot)$ and (1.8) imply $g * f(x)$ exists as a Bochner integral almost everywhere on $\mathbb{R}^n$. The above $(w|g|) * (|f|/w) \in L^p(\mathbb{R}^n, \mathbb{C})$ and (1.8) give $g * f \in wL^p(\mathbb{R}^n, Y)$. □

Lemma 1.3. *Let $f \in wX$, $G(\zeta)$ defined by (1.3n) and $g = \chi_\zeta$ or $\chi'_\zeta := \frac{d\chi_\zeta}{d\zeta}$, $\zeta \in \mathbb{C}^+$.*

(i) $g * f(x)$ exist as a Bochner integral for all $x \in \mathbb{R}^n$ and $g * f \in wBUC(\mathbb{R}^n, Y) \cap wX$; if additionally $1 < p < \infty$ or $f \in wC_0(\mathbb{R}^n, Y)$, then $g * f \in wC_0(\mathbb{R}^n, Y) \cap wX$.

(ii) $G(\zeta) \in L(wX)$.

(iii) If $0 < \alpha < \pi/2$, then

$$\text{(1.10)} \qquad \lim_{0 \neq \zeta \to 0, |\arg \zeta| < \alpha} ||\chi_\zeta * f - f||_{wX} = 0.$$

Proof. (i) Since $wg \in L^q(\mathbb{R}^n, \mathbb{C})$ for each $1 \leq q \leq \infty$, (i) follows by Lemma 1.2.

(ii) The operator $G(\zeta) : wX \to wX$ defined by $G(\zeta)f := \chi_\zeta * f$ is linear and bounded by (1.9).

(iii) With $y = |\zeta|^{1/2} z$ and $\theta = \frac{\zeta}{|\zeta|}$, it follows by (1.5)

$$\chi_\zeta * f(x) - f(x) = \int_{\mathbb{R}^n} [f(x-y) - f(x)]\chi_\zeta(y)\, dy = \int_{\mathbb{R}^n} [f(x - |\zeta|^{1/2} z) - f(x)]\chi_\theta(z)\, dz.$$

Case $X = BUC(\mathbb{R}^n, Y)$, $C_0(\mathbb{R}^n, Y)$. Let $\varepsilon > 0$. Since $w\chi_\theta \in L^1(\mathbb{R}^n)$, then using (1.1), for $0 < |\zeta| \leq 1$, $|\arg \zeta| < \alpha$ there exists $c = c(\varepsilon, \alpha) > 0$ independent of ζ, such that

$$I_1 = \sup_{x \in \mathbb{R}^n} \frac{1}{w(x)} \int_{||z|| \geq c} ||f(x - |\zeta|^{1/2} z) - f(x)|| |\chi_\theta(z)|\, dz \leq 2||f||_{wX} \times$$
$$\int_{||z|| \geq c} w(z)|\chi_\theta(z)|\, dz \leq 2||f||_{wX} (4\pi)^{-n/2} \int_{||z|| > c} w(z) e^{-||z||^2 (\cos\, \alpha)/4} dz < \varepsilon.$$

Then for the above ζ

$$||\chi_\zeta * f(x) - f(x)||_{wX} \leq I_1 + \sup_{x \in \mathbb{R}^n} \frac{1}{w(x)} \int_{||z|| \leq c} ||f(x - |\zeta|^{1/2} z) - f(x)|| |\chi_\theta(z)|\, dz \leq$$
$$I_1 + \sup_{x \in \mathbb{R}^n, ||z|| \leq c} \frac{||f(x - |\zeta|^{1/2} z) - f(x)||}{w(x)} (4\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-||z||^2 (\cos \alpha)/4}\, dz = I_1 + I_2.$$

Using Lemma 1.1, there is $\delta > 0$ such that $I_2 \leq \varepsilon$ if $|\zeta|^{1/2} c < \delta$. It follows $I_1 + I_2 \leq 2\varepsilon$ if $0 < |\zeta|^{1/2} < \delta/c$ and $|\arg \zeta| \leq \alpha$.

Case $X = L^p$: By (i) $\chi_\zeta * f \in wL^p(\mathbb{R}^n, Y) \cap BUC(\mathbb{R}^n, Y)$. For $\zeta \in \mathbb{C}^+$ with $y = |\zeta|^{1/2} z$ using the Minkowski inequality [3, p. 251, A 92]

$$||\chi_\zeta * f - f||_{wL^p} = [\int_{\mathbb{R}^n} \frac{|| \int_{\mathbb{R}^n} [f(x-y) - f(x)]\chi_\zeta(y)\, dy||^p}{w_k^p(x)} dx]^{\frac{1}{p}} \leq$$
$$\int_{\mathbb{R}^n} [\int_{\mathbb{R}^n} \frac{||f(x-y) - f(x)||^p}{w_k^p(x)}\, dx]^{\frac{1}{p}} |\chi_\zeta(y)|\, dy = \int_{\mathbb{R}^n} [\int_{\mathbb{R}^n} \frac{||f(x - |\zeta|^{1/2} z) - f(x)||^p}{w_k^p(x)}\, dx]^{\frac{1}{p}} |\chi_\theta(z)|\, dz.$$

By Lemma 1.1, $\int_{\mathbb{R}^n} \frac{||f(x - |\zeta|^{1/2} z) - f(x)||^p}{w_k^p(x)}\, dx]^{\frac{1}{p}} \to 0$ as $|\zeta| \to 0$ for each $z \in \mathbb{R}^n$. So, by the dominated convergence theorem as in Lemma 1.3.3 (b) of [1, p. 23] we get the statement since

$|\chi_\theta(z)| < (4\pi)^{-n/2}e^{-||z||^2(\cos\alpha)/4} =: F(z)$ by (1.6), $||f_{-|\zeta|^{1/2}z}||_{wX} \leq w(z)||f||_{wX}$ and $wF \in L^1(\mathbb{R}^n)$, if $z \in \mathbb{R}^n$, $|arg\zeta| < \alpha$, $0 < |\zeta| \leq 1$. □

§2. Main results

Theorem 2.1. *For wX of (1.2), the G of (1.3) is a holomorphic C_0-semigroup of angle $\pi/2$ on wX. Its generator is the Laplacian $\Delta_{wX} := \Delta$ on wX with domain:*

$D(\Delta_{wX}) = \{f \in wX : \textit{distribution-}\Delta f \in wX\}$, $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$

where we identify wX with a subspace of $\mathcal{S}'(\mathbb{R}^n, Y)$.

Proof. (a): We have $\chi_\zeta \in \mathcal{S}(\mathbb{R}^n)$ for $\zeta \in \mathbb{C}^+$ and

$\frac{d\chi_\zeta}{d\zeta}(x) = \Delta\chi_\zeta(x)$ for $\zeta \in \mathbb{C}^+$, $x \in \mathbb{R}^n$.

Moreover, by Lemma 1.3 $G(\zeta)f = \chi_\zeta * f \in wX$, $||\chi_\zeta * f - f||_{wX} \to 0$ as in (1.10) for all $\zeta \in \mathbb{C}^+$, $f \in wX$ and $G(\zeta) \in L(wX)$. Then $\widehat{G(\zeta)f} = \widehat{\chi_\zeta} \cdot \widehat{f}$ follows as in [1, p. 154]. By (1.7), $\widehat{\chi_{\zeta_1+\zeta_2}} = \widehat{\chi_{\zeta_1}}\widehat{\chi_{\zeta_2}}$. So, $G(\zeta_1+\zeta_2) = G(\zeta_1)G(\zeta_2)$, $\zeta_1, \zeta_2 \in \mathbb{C}^+$. This means that G is a C_0-semigroup on wX.

(b) Holomorphy of $G : \mathbb{C}^+ \to L(wX)$. By [1, Proposition A.3, (ii) ⇒ (i)], it is enough to show that for any $f \in wX$ with $U(\zeta) = G(\zeta)f$ the U is holomorphic on $\mathbb{C}^+$. Now, again by [1, Proposition A.3], holomorphy of the function $\zeta \to w\chi_\zeta$ defined on $\mathbb{C}^+$ with values in $L^1(\mathbb{R}^n)$ follows, since the complex valued $F(\zeta) = \int_{\mathbb{R}^n} w(x)\chi_\zeta(x)g(x)\,dx$ is continuous for each $g \in L^\infty(\mathbb{R}^n)$ and by Morera's theorem [4, p.75], Fubini and (1.6) it is holomorphic. So to fixed z there exists ψ in $L^1(\mathbb{R}^n)$ with $w(\frac{\Delta\chi_\zeta}{\Delta\zeta}) \to \psi$ in $L^1(\mathbb{R}^n)$; so there are $\zeta_n \to \zeta$ with $\frac{\chi_{\zeta_n} - \chi_\zeta}{\zeta_n - \zeta} \to \psi/w$ almost everywhere on $\mathbb{R}^n$; with the holomorphy of $\chi_\zeta(x)$ for each $x \in \mathbb{R}^n$ one gets $\psi/w = \chi'_\zeta$ almost everywhere and

$$(2.1) \qquad ||(\frac{\Delta\chi_\zeta}{\Delta\zeta} - \chi'_\zeta)w||_{L^1} = \int_{\mathbb{R}^n} |\frac{\Delta\chi_\zeta(x)}{\Delta\zeta} - \chi'_\zeta(x)|w(x)\,dx \to 0 \text{ as } 0 \neq \Delta\zeta \to 0.$$

Since $w\chi'_\zeta \in L^q(\mathbb{R}^n)$ for all $q \geq 1$, $\chi'_\zeta * f(x)$ exists with Hölder's inequality as a Bochner integral for all $x \in \mathbb{R}^n$. By Lemma 1.3, $\frac{\Delta U(\zeta)}{\Delta\zeta}, \chi'_\zeta * f \in wX$, $\zeta, \zeta + \Delta\zeta \in \mathbb{C}^+$, $\Delta\zeta \neq 0$. We have

$\frac{\Delta U(\zeta)}{\Delta\zeta} - \chi'_\zeta * f = (\frac{\Delta\chi_\zeta}{\Delta\zeta} - \chi'_\zeta) * f$ and using Young's inequality [5, p. 29]

$||(\frac{\Delta\chi_\zeta}{\Delta\zeta} - \chi'_\zeta) * f||_{wX} = ||(1/w)\int_{\mathbb{R}^n}(\frac{\Delta\chi_\zeta}{\Delta\zeta} - \chi'_\zeta)(\cdot - y)f(y)\,dy||_X \leq$

$||\int_{\mathbb{R}^n}|(\frac{\Delta\chi_\zeta}{\Delta\zeta} - \chi'_\zeta)(\cdot - y)|w(\cdot - y)(||f(y)||/w(y))\,dy||_X \leq$

$||(\frac{\Delta\chi_\zeta}{\Delta\zeta} - \chi'_\zeta)w||_{L^1}||f/w||_X$.

With (2.1), holomorphy of U on $\mathbb{C}^+$ follows, and

(2.2) $\quad G'(\zeta)f = (G(\zeta)f)' = \chi'_\zeta * f,\ \zeta \in \mathbb{C}^+,\ f \in wX.$

(c) Let f, distribution $\Delta f \in wX$. We have $\frac{\partial\chi_t}{\partial t} = \Delta_x\chi_t$ on $(0,\infty)\times\mathbb{R}^n$, $\Delta_x = \sum_{j=1}^n(\frac{\partial}{\partial x_j})^2$. So by (2.2), in $\mathcal{S}'(\mathbb{R}^n, Y)$, $t > 0$,

(2.3) $\quad \frac{dG(t)f}{dt} = \frac{d(\chi_t * f)}{dt} = \frac{d\chi_t}{dt} * f =$

$(\Delta\chi_t) * f = \Delta G(t)f = \chi_t * (\Delta f) = G(t)\Delta f.$

Let $A|D(A)$ be the generator of the C_0- semigroup $G : \mathbb{R}_+ \to L(wX)$, defined by Proposition 3.1.9 g) of [1, p. 115]; let Δ be the Laplace operator applied to $S \subset \mathcal{D}' := \mathcal{D}'(\mathbb{R}^n, Y)$; with $wX \subset \mathcal{D}'(\mathbb{R}^n, Y)$,

$D := \{f \in wX : \Delta_{wX} f \in wX\}$ and $\Delta_{wX} := \Delta|D$ are well defined. We show

(2.4) $\quad D(A) = D,\ A = \Delta_{wX}.$

(c.1) $\quad D \subset D(A),\ A = \Delta_{wX}$ on D:

If $f \in D$, $G(\cdot)f \in C([0,\infty), wX)$ by (a), with (2.3) and $g := \Delta_{wX} f \in wX$ one has $G(f)f - f = \int_0^t(\frac{d}{ds})(G(s)f)\,ds = \int_0^t G(s)g\,ds$, $t \in \mathbb{R}_+$. With Proposition 3.1.9 f) of [1, p. 115] one gets $f \in D(A)$ and $Af = g = \Delta_{wX} f$.

(c.2) $\quad D(A) \subset D$:

With $F(t) := (1/t)\int_0^t G(s)f\,ds$, $t > 0$, $f \in D(A)$, one has $F(t) \to f$ in wX as $t \to 0$, since $G(t)f \to f$ in wX by (a). $F(t) \to f$ in wX implies $F(t) \to f$ in $L^1_{loc}(\mathbb{R}^n, Y)$, so $(\Delta F(t))(\varphi) = F(t)(\Delta_x\varphi) \to f(\Delta_x\varphi) = (\Delta f)(\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{C})$.

Now by (2.5) below one has $\Delta F(t) = (1/t)(G(t)f - f)$; by definition of $D(A)$ and Proposition 3.1.9 g) [A., p. 115], $(1/t)(G(t)f - f) \to$ some g in wX, so in $L^1_{loc}(\mathbb{R}^n, Y)$, so $(1/t)(G(t)f - f)(\varphi) \to f(\varphi)$. together one gets $\Delta f = g, \in wX$, that is $f \in D$. With (c.1) this gives (2.4). It remains to show

$$(2.5) \qquad \Delta \int_0^t G(s) f ds = G(t)f - f,\ f \in wX.$$

For this, with $f \in wX$, with Lemma 1.3 define $\beta(t,x) := (\chi_t * f)(x)$, $(t,x) \in M := (0,\infty) \times \mathbb{R}^n$. With Lebesgue's Dominated Convergence theorem and analogs of (1.6) for the derivatives of χ_t one gets inductively $\beta \in C^\infty(M, Y)$, with

$$(2.6) \qquad \partial\beta/\partial t = (\chi_t') * f = (\Delta_x \chi_t) * f = \Delta_x \beta.$$

If $0 < \varepsilon < t$, $\Psi_\varepsilon(t,x) := \int_\varepsilon^t \beta(s,x) ds, x \in \mathbb{R}^n$, is well defined with $\Psi_\varepsilon \in C((\varepsilon,\infty) \times \mathbb{R}^n, Y)$, $\Psi_\varepsilon(t) := \Psi_\varepsilon(t,\cdot) \in C(\mathbb{R}^n, Y) \subset \mathcal{D}'(\mathbb{R}^n, Y)$ if $t > \varepsilon$. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, all the following integrals exist (even as Riemann integrals), with twice Fubini, partial integration and (2.6) one has

$$(\Delta\Psi_\varepsilon(t))(\varphi) = \int_{\mathbb{R}^n} \Psi_\varepsilon(t,x)(\Delta_x\varphi)(x)\,dx = \int_{\mathbb{R}^n} \int_\varepsilon^t \beta(s,x)(\Delta_x\varphi)(x) ds\,dx =$$
$$\int_\varepsilon^t \int_{\mathbb{R}^n} \Delta_x\beta(s,x)\varphi(x) dx ds = \int_\varepsilon^t \int_{\mathbb{R}^n} (\partial/\partial s)\beta(s,x)\varphi(x) dx\,ds =$$
$$\int_{\mathbb{R}^n} (\int_\varepsilon^t (\partial/\partial s)\beta(s,x) ds)\varphi(x) dx = \int_{\mathbb{R}^n} (\beta(t,x) - \beta(\varepsilon,x))\varphi(x) dx.$$

This implies

$$(2.7) \qquad \Delta\Psi_\varepsilon = G(t)f - G(\varepsilon)f, \in wX.$$

$G(\cdot)f : \mathbb{R}_+ \to wX$ is continuous, so $\int_\varepsilon^t G(s)f\,ds \to \int_0^t G(s)f\,ds$ as $\varepsilon \to 0$. Furthermore, the Riemann sums $\Sigma_m := \sum_1^m (G(s_j)f)\,(s_j - s_{j-1}) \to \int_\varepsilon^t G(s)f\,ds$ in wX as $m \to \infty$, $s_j = \varepsilon + j(t-\varepsilon)/m$. Similarly $\Sigma_m(x) := \sum_1^m \beta(s_j,x)(s_j - s_{j-1}) \to \int_\varepsilon^t \beta(s,x)\,ds = \Psi_\varepsilon(t,x)$ in Y as $m \to \infty$, for each $x \in \mathbb{R}^n$.

If K is compact $\subset \mathbb{R}^n$, then $\sup \{||\Sigma_m(x)|| : m \in \mathbb{N}, x \in K\} < \infty$, so

$$\int_{\mathbb{R}^n} \Sigma_m(x)\varphi(x)\,dx \to \int_{\mathbb{R}^n} \Psi_\varepsilon(t,x)\varphi(x)\,dx \text{ for } \varphi \in \mathcal{D}(\mathbb{R}^n).$$

As above, $\int_\varepsilon^t G(s)f\,ds = \Psi_\varepsilon(t)$ follows, and then $\Psi_\varepsilon(t) \to \int_0^t G(s)f\,ds$ in wX. Therefore $(\Delta\Psi_\varepsilon(t))(\varphi) = \int_{\mathbb{R}^n} \Psi_\varepsilon(t)\Delta_x\varphi\,dx \to \int_{\mathbb{R}^n}(\int_0^t G(s)f\,ds)\Delta_x\varphi\,dx =$ $(\Delta \int_0^t G(s)f\,ds)(\varphi)$ as $\varepsilon \to 0$; since $G(\varepsilon)f \to f$, (2.7) implies (2.5). □

Corollary 2.2. *All mild solutions $u : [0,\infty) \to wX$ of*

(a) $u' = \Delta_{wX} u$ are given by $u(t) = G(t)f$ with $f \in wX$, they are C^1-solutions

on $(0,\infty)$, $\in C^\infty(M,Y)$ *and classical solutions of*

(b) $\frac{\partial u(t,x_1,\cdots,x_n)}{\partial t} = \sum_1^n \frac{\partial^2 u(t,x_1,\cdots,x_n)}{\partial x_j^2}$ *on* $M := (0,\infty)\times\mathbb{R}^n$ *with*

(c) $u(t,\cdot)\to f$ *in* wX *as* $t\to 0$.

Conversely, any classical solution of (b) with (c) defines a mild solution of (a) on $[0,\infty)$.

For the proofs of most of this see [1, Corollary 3.7.21].

Remark 2.3. Since the Gauss-Poisson formula (1.4) for G defines by Theorem 2.1 a holomorphic C_0-semigroup with generator $A = \Delta_{wX}$, with Corollary 2.2 the results of [2, Theorems 5.2/6.3, Examples 6.2] can be applied to the heat equation.

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