

NON-FORMAL HOMOGENEOUS SPACES

MANUEL AMANN

ABSTRACT. Several large classes of homogeneous spaces are known to be formal—in the sense of Rational Homotopy Theory. However, it seems that far fewer examples of non-formal homogeneous spaces are known.

In this article we provide several construction principles and characterisations for non-formal homogeneous spaces, which will yield a lot of examples. This will enable us to prove that, from dimension 72 on, such a space can be found in each dimension.

INTRODUCTION

Homogeneous spaces form a very well-studied and interesting class of manifolds. They appear abundantly in geometry and topology. Our focus will lie on their topological properties adopting the viewpoint of Rational Homotopy Theory. Also from this perspective homogeneous spaces bear remarkable properties, which they share with the larger class of biquotients they are contained in.

The group G will always be taken to be a compact connected Lie group and let $H \subseteq G \times G$ be a closed Lie subgroup. Then H acts on G on the left by $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$. The orbit space of this action is called the *biquotient* $G // H$ of G by H . If the action of H on G is free, then $G // H$ possesses a manifold structure. This is the only case we shall consider. Clearly, the category of biquotients contains the one of homogeneous spaces; in this special case the inclusion of H into the first G factor is trivial.

It is a classical result that biquotients are *rationally elliptic spaces*. Moreover, they admit what is called a *pure model*—see [8], p. 435, for the definition and examples [8].32.2, p. 448 and [8].15.1, p. 218 for the homogeneous case, respectively [16].1, p. 2 for general biquotients. This makes them an interesting yet manageable source of examples within the realm of Rational Homotopy Theory and they definitely do constitute a field of study well-worth the attention it is paid.

The topological aspect this article is centred around is the following question:

Question. *Under which conditions on G, H and the inclusion is a homogeneous space (respectively a biquotient) (non-)formal?*

Recall that a topological space X is called *formal*, if the information contained in its rational cohomology algebra is “the same” as its entire

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rational homotopy type. In particular, the rational homotopy groups $\pi_*(X) \otimes \mathbb{Q}$ of a simply-connected formal X may be computed from its cohomology algebra. Let us give the precise definition:

Definition. The commutative differential graded algebra (A, d) is called *formal* if it is weakly equivalent to the cohomology algebra $(H(A, \mathbb{Q}), 0)$, i.e. if there is a chain of quasi-isomorphisms

$$(A, d) \xrightarrow{\cong} \dots \xleftarrow{\cong} \dots \xrightarrow{\cong} \dots \xleftarrow{\cong} (H(A, \mathbb{Q}), 0)$$

We call a path-connected topological space *formal* if its rational homotopy type is a formal consequence of its rational cohomology algebra, i.e. if $(A_{\text{PL}}(X), d)$, the commutative differential graded algebra of polynomial differential forms on X , is formal

Formality is one of the most important and most discussed topics in Rational Homotopy Theory. Conjecturally, it forms an obstruction to the existence of metrics of positive curvature on manifolds. Besides, it can be used to distinguish Kähler manifolds from symplectic manifolds, and it is an obstruction to *geometric formality*—which also was extensively studied on homogeneous spaces.

An elaborate discussion of the formality of homogeneous spaces was given in the classical book [14]. Amongst others this resulted in long lists of formal homogeneous spaces (cf. [14].XI, p. 492-497). It is well known that symmetric spaces of compact type are formal (cf. [8].12.3, p. 162), N -symmetric spaces are formal (cf. [24], Main Theorem, p. 40, for the precise statement, [17]). (The first mentioned spaces are even geometrically formal; this is not true in general in the latter case.) Moreover, if $\text{rk } G = \text{rk } H$ the space $G//H$ is *positively elliptic* (or F_0), i.e. rationally elliptic with positive Euler characteristic and consequently, formal.

However, it seems that only very few examples of non-formal homogeneous spaces are known. For example consider [14].XI.5, p. 486-491. (One example was generalised to a parametrised family in [25]. A few further examples are cited in example [18].1, p. 158.) Exemplarily, the following homogeneous spaces are known to be non-formal for $p, q \geq 3$ and $n \geq 5$:

$$\frac{\mathbf{SU}(pq)}{\mathbf{SU}(p) \times \mathbf{SU}(q)}, \quad \frac{\mathbf{Sp}(n)}{\mathbf{SU}(n)}$$

In this article we shall discuss mainly three principles of how to construct non-formal homogeneous spaces respectively biquotients. In particular, this will produce uniform proofs for the non-formality of the known examples and it will result in more examples like

$$\frac{\mathbf{SU}(p+q)}{\mathbf{SU}(p) \times \mathbf{SU}(q)}, \quad \frac{\mathbf{SO}(2n)}{\mathbf{SU}(n)}$$

for $p+q \geq 4$, $n \geq 8$ just to mention some simple cases. We refer the reader to theorems 1.6 and 1.10 as well as example 1.11 for several more non-formal homogeneous spaces.

We provide the following tools to construct these spaces.

Proposition A. *Let $H \subseteq G$ and $K \subseteq H \times H$ be compact connected Lie groups. Suppose that the inclusion of H into G induces an injective morphism on rational homotopy groups, i.e. $\pi_*(H) \otimes \mathbb{Q} \hookrightarrow \pi_*(G) \otimes \mathbb{Q}$.*

Then $G//K$ is formal if and only if $H//K$ is formal.

For the next theorem we need to recall the following: A space E possesses the *Hard-Lefschetz property* if there exists a closed 2-form $l \in H^2(E, \mathbb{R})$ such that for all $k \in \mathbb{N}$

$$L^k : H^{n-k}(E, \mathbb{R}) \rightarrow H^{n+k}(E, \mathbb{R}) \quad L^k([\alpha]) = [l^k \wedge \alpha]$$

is an isomorphism. The most prominent examples of *Hard-Lefschetz manifolds*, i.e. manifolds with the Hard-Lefschetz property, are Kähler manifolds. Note that a space is formal (over \mathbb{Q}) if and only if it is so over each field extension of \mathbb{Q} (cf. [8], p. 156 and theorem 12.1, p. 316); thus we need not worry about coefficients here.

The next theorem will provide the main source for finding non-formal homogeneous spaces.

Theorem B. *Let E^{2n+1} be a simply-connected space with finite dimensional rational cohomology and let B^{2n} be a simply-connected Hard-Lefschetz space (for $n \geq 1$). Suppose there is a fibration $\mathbb{S}^1 \hookrightarrow E \xrightarrow{p} B$. Assume further the following to hold true:*

- *The Euler class of p is a non-vanishing multiple of the Kähler class l of B in rational cohomology.*
- *The rational cohomology of B is concentrated in even degrees only.*
- *The rational homotopy groups of B are concentrated in degrees smaller or equal to n .*

Then E is a non-formal space.

We remark that although this theorem requires extremely special conditions, it fits perfectly to the case of homogeneous spaces (and bioquotients). As we shall see, all these requirements can easily be fulfilled in a large class of cases.

This theorem has a “mirror version”, which does give a better characterisation of formality in this situation. Recall that a space is called an *F_0 -space* if it is rationally elliptic with positive Euler characteristic.

Theorem C. *Let B be a positively elliptic simply-connected Hard-Lefschetz space of dimension $2n$. Let*

$$\mathbb{S}^1 \hookrightarrow E \rightarrow B$$

be a fibration with simply-connected total space E of formal dimension $2n+1$. Suppose that the Euler class of the fibration equals the Hard-Lefschetz class (up to non-trivial multiples in the second rational cohomology group).

Then E is formal if and only if E splits rationally as a product with one factor an odd-dimensional rational sphere of dimension greater than or equal to n and with the other factor being an F_0 -space.

We remark that an elliptic space is positively elliptic if and only if its rational cohomology is concentrated in even degrees—cf. proposition [8].32.10, p. 444.

The last criterion will be

Proposition D. *Let G be a rationally elliptic H -space of finite type. Suppose it admits a free group action by a compact connected Lie group H . Then G/H is formal if and only if G/T_H is formal.*

This implies, in particular, that $G//H$ is formal if and only if so is $G//T_H$ —a generalisation of the homogeneous case (cf. [22], p. 212). This principle follows from our much more general result in [3]; yet, we provide a much simpler proof here. (In order to avoid confusion, we remark that the term “ H -space” refers to the existence of a multiplication up to homotopy and not to the action of the Lie group H .)

It is a classical result (see [21]) that simply-connected compact manifolds of dimension at most six are formal spaces. In [11] Fernández and Muñoz pose the question whether there are non-formal simply-connected compact manifolds in dimensions seven and higher. They answer this in the affirmative by constructing seven-dimensional and eight-dimensional examples. The requested example in a certain dimension above dimension seven is then given by a direct product with the corresponding even-dimensional sphere.

A next goal is to find features which usually tend to be favourable to establishing formality and to construct non-formal examples bearing these properties. One direction in this vein is to find highly-connected examples (cf. for example [10]). Note that the methods of constructing these manifolds involve surgery theory, in particular.

We shall use a different approach towards this existence problem of “geometrically nice” non-formal spaces in every large dimension.

Combining the Bott conjecture and the Hopf conjecture, every manifold of positive curvature should be positively elliptic and formal, in particular. The G -invariant metric on homogeneous spaces G/H of compact Lie groups has non-negative curvature. However, we shall see that from dimension 72 on in each dimension there exists a compact manifold of non-negative curvature, which is not formal, but elliptic.

Theorem E. *In every dimension $d \geq 72$ there is an irreducible simply-connected compact homogeneous space which is not formal. In particular, every such space constitutes a rationally elliptic non-formal manifold admitting a metric of non-negative curvature.*

As a reference for Rational Homotopy Theory we draw on the textbook [8] and we shall follow its terminology and notation.

All the Lie groups under consideration will be compact connected. All commutative differential graded algebras—and cohomology, in particular—will be taken with rational coefficients unless stated differently.

Structure of the article. In section 1 we shall prove the first two construction principles—proposition A, theorems B and C—and we provide a large list of non-formal homogeneous examples. Section 2 is devoted to providing the necessary examples for the proof of theorem E. Finally, in section 3 we show proposition D via a discussion of formality in special fibrations.

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1. CONSTRUCTION PRINCIPLES

In theory, given the groups G , H and the inclusion of H into G it is possible to compute whether the space G/H is formal or not. Consider the appendix A for an exemplary computation. However, understanding the topological nature of the inclusion may be a non-trivial task. Moreover, computational complexity theory enters the stage: In [13] the following result was established: Given a rationally elliptic simply-connected space, computing its Betti numbers from its minimal Sullivan model—which serves as an encoding of the space—is an NP-hard problem. In [2] it is proved that computing the rational cup-length and the rational Lusternik–Schnirelmann category of an elliptic space is also NP-hard. So on an elliptic space of large dimension it should be rather challenging to decide whether the space is formal or not. Besides, it is “desirable” (cf. [24], p. 38) to give criteria on invariants of G , H and the embedding $H \hookrightarrow G$ by which a direct identification of the homogeneous space G/H as being formal or not may be achieved. (In the article [24], this was done by identifying N -symmetric spaces as formal. See also [17] for the same result.)

Thus finding a priori arguments which allow to identify infinite non-formal families should be worth-while the effort. Let us begin to do so by proving proposition A.

PROOF OF PROPOSITION A. Compact connected Lie groups are simple spaces. Choose minimal Sullivan models $(\Lambda V_G, 0)$, $(\Lambda V_H, 0)$ and $(\Lambda V_K, 0)$ for G , H and K . Let x_1, \dots, x_n be a homogeneous basis of V_H . Since the inclusion $H \hookrightarrow G$ induces an injective morphism on rational homotopy groups, we may choose x'_1, \dots, x'_k such that $x_1, \dots, x_n, x'_1, \dots, x'_k$ is a homogeneous basis of V_G . (For this we identify generators of homotopy groups with the x_i, x'_i up to duality.)

So we obtain

$$H^*(G \times G) \cong \Lambda \langle x_1, \dots, x_n, x'_1, \dots, x'_k, y_1, \dots, y_n, y'_1, \dots, y'_k \rangle$$

(The y_i, y'_i are constructed just like the x_i, x'_i , i.e. as a “formal copy”).

Hence a model for the biquotient $G//K$ is given by (cf. proposition [16].1, p. 2)

$$(\Lambda V_K^{+1} \otimes \Lambda \langle q_1, \dots, q_n, q'_1, \dots, q'_k \rangle, d)$$

(The degrees of V_K are shifted by +1.)

Denote the inclusion $K \xrightarrow{i_1} H \times H \xrightarrow{i_2} G \times G$ by $\phi = i_2 \circ i_1$. The differential d on the q_i is given by $d(q_i) = H^*(\mathbf{B}\phi)(\bar{x}_i - \bar{y}_i)$, where $\bar{x}_i, \bar{y}_i \in H^*(\mathbf{B}G)$ is the class corresponding to $x_i, y_i \in H^*(G)$. (The analogue holds for the q'_i .) By functoriality the morphism $H^*(\mathbf{B}\phi)$ factors through

$$H^*(\mathbf{B}G \times \mathbf{B}G) \xrightarrow{H^*(\mathbf{B}i_2)} H^*(\mathbf{B}H \times \mathbf{B}H) \xrightarrow{H^*(\mathbf{B}i_1)} H^*(\mathbf{B}K)$$

In particular, $H^*(\mathbf{B}\phi) = H^*(\mathbf{B}i_1) \circ H^*(\mathbf{B}i_2)$ and

$$d(q'_i) = H^*(\mathbf{B}i_1)(H^*(\mathbf{B}i_2)(\bar{x}'_i - \bar{y}'_i))$$

for $1 \leq i \leq k$. The inclusion i_2 is the product $i_2 = i_2|_H \times i_2|_H$ by definition. Thus we obtain a decomposition $H^*(\mathbf{B}i_2) = H^*(\mathbf{B}(i_2|_H)) \otimes H^*(\mathbf{B}(i_2|_H))$ and $H^*(\mathbf{B}i_2)(\bar{x}'_i) \in \Lambda\langle x_1, \dots, x_n \rangle$; i.e. it is a linear combination in products of the x_i . By the splitting of $H^*(\mathbf{B}i_2)$ we obtain that $H^*(\mathbf{B}i_2)(\bar{y}'_i) \in \Lambda\langle y_1, \dots, y_n \rangle$ is the same linear combination with the x_i replaced by the y_i . This implies that

$$d(q'_i) = H^*(\mathbf{B}i_1)(H^*(\mathbf{B}i_2)(\bar{x}'_i - \bar{y}'_i)) \in \text{im } H^*(\mathbf{B}i_1) = d(\Lambda\langle q_1, \dots, q_n \rangle)$$

(In fact, the very same linear combination as above with the x_i now replaced by the q_i will serve as a preimage of $d(q'_i)$ under $d|_{\Lambda\langle q_1, \dots, q_n \rangle}$.)

Thus for each $1 \leq i \leq k$ there is an element $z_i \in \Lambda\langle q_1, \dots, q_n \rangle$ with the property that $d(q'_i - z_i) = 0$. Set $\tilde{q}_i := q'_i - z_i$. We have an isomorphism of commutative differential graded algebras

$$\begin{aligned} \sigma : (\Lambda V_K^{+1} \otimes \Lambda\langle q_1, \dots, q_n, q'_1, \dots, q'_k \rangle, d) \\ \xrightarrow{\cong} (\Lambda V_K^{+1} \otimes \Lambda\langle q_1, \dots, q_n, \tilde{q}_1, \dots, \tilde{q}_k \rangle, d) \end{aligned}$$

induced by the ‘‘identity’’ $q'_i \mapsto \tilde{q}_i + z_i$ for all $1 \leq i \leq k$. (By abuse of notation we now consider the \tilde{q}_i with $d\tilde{q}_i := d(q'_i - z_i) = 0$ as abstract elements in the graded vector space upon which the algebra is built.) This morphism is an isomorphism of commutative graded algebras which commutes with differentials:

$$d(\sigma(q'_i)) = d(\tilde{q}_i + z_i) = d(q'_i) = \sigma(d(q'_i))$$

as $\sigma|_{\Lambda\langle q_1, \dots, q_k \rangle} = \text{id}$.

Thus we obtain a quasi-isomorphism

$$\begin{aligned} \text{APL}(G//K) &\simeq (\Lambda V_K^{+1} \otimes \Lambda\langle q_1, \dots, q_n, q'_1, \dots, q'_k \rangle, d) \\ (1) \quad &\cong (\Lambda V_K^{+1} \otimes \Lambda\langle q_1, \dots, q_n, \tilde{q}_1, \dots, \tilde{q}_k \rangle, d) \\ &= (\Lambda V_K^{+1} \otimes \Lambda\langle q_1, \dots, q_n \rangle, d) \otimes (\Lambda\langle \tilde{q}_1, \dots, \tilde{q}_k \rangle, 0) \end{aligned}$$

The algebra $(\Lambda V_K^{+1} \otimes \Lambda\langle q_1, \dots, q_n \rangle, d)$ is a model for $H // K$, since $x_1, \dots, x_n, y_1, \dots, y_n$ is a basis of V_H and since its differential d corresponds to $H^*(\mathbf{B}i_1)$. Hence the last algebra in (1) is rationally the product of a model of $H//K$ and a formal algebra. Thus it is formal if and only if $H//K$ is formal (cf. theorem [1].1.47, p. 39). \square

Corollary 1.1. *Let K be a compact Lie subgroup of the Lie group G , which itself is a Lie subgroup of a Lie group \tilde{G} . Table 1 gives pairs of groups G and \tilde{G} together with relevant relations and the type of the inclusion such that it holds: The homogenous space G/K is formal if and only if \tilde{G}/K is formal.*

PROOF. For the minimal models of the relevant Lie groups and further reasoning on their inclusions see [8].15, p. 220. Lie groups are formal and so are the depicted inclusions. More precisely, we may identify minimal models with their cohomology algebra and induced maps on minimal models with

TABLE 1. Inclusions of Lie groups

G	\tilde{G}	<i>with</i>	<i>type of embedding</i>
$\mathbf{SO}(n)$	$\mathbf{SO}(N)$	$n \geq 1, n \text{ odd}, N \geq n, N \text{ odd}$	<i>blockwise</i>
$\mathbf{SO}(n)$	$\mathbf{SO}(N)$	$n \geq 1, n \text{ odd}, N \geq n + 1, N \text{ even}$	<i>blockwise</i>
$\mathbf{SU}(n)$	$\mathbf{SU}(N)$	$n > 1, N \geq n$	<i>blockwise</i>
$\mathbf{Sp}(n)$	$\mathbf{Sp}(N)$	$n \geq 1, N \geq n$	<i>blockwise</i>
$\mathbf{SO}(n)$	$\mathbf{SU}(n)$	$n \geq 1, n \text{ odd}$	<i>induced componentwise by $\mathbb{R} \hookrightarrow \mathbb{C}$</i>
$\mathbf{Sp}(n)$	$\mathbf{SU}(2n)$	$n \geq 1$	<i>induced by the identification $\mathbb{H} \hookrightarrow \mathbb{C}^{2 \times 2}$</i>

induced maps in cohomology. Thus it suffices to see that the given inclusions induce surjective morphisms in cohomology.

The blockwise inclusions are surjective in cohomology due to theorem [20].6.5.(4), p. 148.

The inclusion $\mathbf{SO}(n) \hookrightarrow \mathbf{SU}(n)$ induces a surjective morphism by [20].6.7.(2), p. 149. So does the inclusion $\mathbf{Sp}(n) \hookrightarrow \mathbf{SU}(2n)$ by [20].6.7.(1), p. 149. \square

We remark that the chain of inclusions $\mathbf{SO}(n) \subseteq \mathbf{U}(n) \subseteq \mathbf{Sp}(n)$ does *not* induce a surjective morphism in cohomology (cf. [20].5.8.(1), p. 138 and [20].6.7.(2), p. 149). Neither does the chain $\mathbf{Sp}(n) \subseteq \mathbf{U}(2n) \subseteq \mathbf{SO}(4n)$ nor the chain $\mathbf{Sp}(n) \subseteq \mathbf{U}(2n) \subseteq \mathbf{SO}(4n+1)$ in general by a reasoning taking into account theorem [20].6.11, p. 153, which lets us conclude that the transgression in the rationalised long exact homotopy sequence for $\mathbf{U}(n) \hookrightarrow \mathbf{SO}(2n) \rightarrow \mathbf{SO}(2n)/\mathbf{U}(n)$ is surjective. Let us give a first example, which will be largely generalised later.

Example 1.2. • Using the computation from the appendix A and proposition A we see that the space

$$\frac{\mathbf{SU}(n)}{\mathbf{SU}(3) \times \mathbf{SU}(3)}$$

is non-formal for $n \geq 6$.

- We also obtain pretty simple proofs for formality: For example, the spaces

$$\frac{\mathbf{U}(n)}{\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_l)} \quad \text{and} \quad \frac{\mathbf{SO}(2n+1)}{\mathbf{SO}(2k_1) \times \cdots \times \mathbf{SO}(2k_l)}$$

with $\sum_{i=1}^l k_i \leq n$ are formal: In the equal rank case $\sum_{i=1}^l k_i = n'$ the spaces are positively elliptic and formality holds. Then apply proposition A to the inclusions $\mathbf{U}(n') \hookrightarrow \mathbf{U}(n)$ and $\mathbf{SO}(2n'+1) \hookrightarrow \mathbf{SO}(2n+1)$.

\square

Let us now prove theorem B which serves to derive non-formality via additional topological structure. This will simplify proofs for some of the known non-formal examples whilst producing a whole variety of further ones. We cite the following technical lemma from [6], theorem 4.1, p. 261, and from lemma [12].2.7 p. 154.

Lemma 1.3. *A minimal model $(\Lambda V, d)$ is formal if and only if it can be written (up to isomorphism) such that there is in each V^i a complement N^i to the subspace of d -closed elements C^i with $V^i = C^i \oplus N^i$ and such that any closed form in the ideal $I(\bigoplus N^i)$ generated by $\bigoplus N^i$ in ΛV is exact.*

□

PROOF OF THEOREM B. Since the Kähler class and the Euler class rationally are non-trivial multiples, we may formulate the Gysin sequence with rational coefficients for p as follows:

$$\dots \rightarrow H^p(B) \xrightarrow{\cup[l]} H^{p+2}(B) \rightarrow H^{p+2}(E) \rightarrow H^{p+1}(B) \xrightarrow{\cup[l]} \dots$$

(The fibration is oriented, since B is simply-connected.) The Hard-Lefschetz property of B implies that taking the cup-product with $[l]$ is injective in degrees $p < n$. So the sequence splits, yielding

$$H^{p+2}(B) = H^p(B) \oplus H^{p+2}(E)$$

for $-2 \leq p \leq n-2$. Since we assumed the odd-dimensional rational cohomology groups of B to vanish, we obtain in particular that

$$(2) \quad H^p(E) = 0 \quad \text{for odd } p \leq n$$

Let $(\Lambda V_E, d_E)$ be a minimal model of E . As E was supposed to be simply-connected, we clearly have $\pi_p(E) \otimes \mathbb{Q} \cong V_E^p$ up to duality. By the long exact sequence of homotopy groups we obtain

$$\pi_p(E) \cong \pi_p(B) \quad \text{for } p \geq 3$$

As we assumed

$$\pi_p(B) \otimes \mathbb{Q} = 0 \quad \text{for } p > n$$

we see that $V_E^{>n} = 0$, if $n \geq 2$. If $n = 1$, we clearly still have $V_E^{\geq 3} = 0$. So for arbitrary $n \geq 1$ we obtain

$$(3) \quad V_E^p = 0 \quad \text{for odd } p > n$$

Assume now that E is formal. This will lead to a contradiction. By lemma 1.3 we may split (a suitable minimal model $(\Lambda V_E, d_E)$ of E as) $V_E = C_E \oplus N_E$ with $C_E = \ker d_E|_{V_E}$ and with the property that every closed element in $I(N_E)$ is exact.

By observation (2) we know that every closed element of odd degree in $(\Lambda V_E)^{\leq n}$ is exact. Thus by the minimality of the model we directly derive that

$$(4) \quad C_E^{\leq n} = (C_E^{\leq n})^{\text{even}}$$

Together with (3) this implies that

$$(5) \quad C_E = C_E^{\text{even}}$$

is concentrated in even degrees only. Hence so is ΛC_E . Thus, using the splitting

$$\Lambda V_E = \Lambda C_E \oplus I(N_E)$$

we derive that

$$(6) \quad H^{\text{odd}}(E) = \left(\frac{\ker d_E|_{I(N)}}{\text{im } d_E} \right)^{\text{odd}} = 0$$

from (5) and from the formality of E .

However, by assumption, the space E is an odd-dimensional simply-connected rationally elliptic space. Thus its rational cohomology satisfies Poincaré duality by theorem [7].A, p. 70. The space E possesses a volume form which hence generates $H^{2n+1}(E) \cong \mathbb{Q} \neq 0$. This contradicts formula (6). Thus E is non-formal. \square

Remark 1.4.

- Actually, we do not only prove the non-formality of the total space; we can even derive the existence of a non-vanishing Massey product of odd degree. (Note that there are examples of non-formal spaces with vanishing Massey products.)
 - We see that the proof works equally well in the following setting: Assume the spaces E and B not to be simply-connected but to be compact connected manifolds with a finite fundamental group and to be *simple spaces*, i.e. the action of the fundamental group on all homotopy groups (in positive degree) is trivial. Indeed, the spaces then are orientable and satisfy Poincaré duality. Since we may choose the Euler class of the \mathbb{S}^1 -fibration up to (non-trivial) multiples, we then may assume that the fibration is orientable (up to coverings of the fibre), and the Gysin sequence can be applied. Moreover, Rational Homotopy Theory is equally applicable to simple spaces.
- From proposition [22].1.17, p. 84, we cite that the homogeneous space G/H of a compact connected Lie group G with a connected closed Lie subgroup H is simple.
- Kähler manifolds are clearly not the only source of examples of B . For example, the lemma applies to the case of certain Donaldson submanifolds (cf. [12]) or biquotients (cf. [16]).

We may relax the assumptions of the lemma even further: We need not require the rational homotopy groups of B to be concentrated in degrees smaller or equal to n as long as the ones concentrated in odd degrees above n correspond to relations in the minimal model of E via the long exact homotopy sequence; i.e. they are not in the image of the dual of the rationalised Hurewicz homomorphism. Then the proof does not undergo any severe adaptation: We just obtain $C_E^p = 0$ instead of $V_E^p = 0$ for odd $p > n$. This extends the lemma to not necessarily elliptic spaces E and B .

- In theory, the approach of this theorem is not restricted to \mathbb{S}^1 -fibrations. So, for example, using the Kraines form in degree four

instead of the Kähler form in degree two, one may formulate a version for Positive Quaternion Kähler Manifolds—however, this seems to be less fruitful.

□

See the corollary on [22].13, p. 221 and the examples below for a related result in the category of homogeneous spaces.

Via theorem C we cast a little more light on the situation depicted in theorem B and we characterise when formality occurs. So before we begin to use theorem B in order to find several examples of non-formal homogeneous spaces—which clearly provides the motivation for the theorem—let us prove theorem C. We recall the following

Lemma 1.5. *A pure elliptic minimal Sullivan algebra $(\Lambda V, d)$ is formal if and only if it is of the form*

$$(7) \quad (\Lambda V, d) \cong (\Lambda V', d) \otimes (\Lambda \langle z_1, \dots, z_l \rangle, 0)$$

(for maximal such l) with a pure minimal Sullivan algebra $(\Lambda V', d)$ of positive Euler characteristic—which is automatically formal then—and with odd degree generators z_i .

PROOF. Choose a basis z_1, \dots, z_m of V^{odd} with the property that $dz_i = 0$ for $1 \leq i \leq l$ —and some fixed $1 \leq l \leq m$ —and that $d|_{\langle z_{l+1}, \dots, z_m \rangle}$ is injective. From pureness it follows:

$$\begin{aligned} d(\langle z_1, \dots, z_l \rangle) &= 0 \\ d(\langle z_{l+1}, \dots, z_m \rangle) &\in \Lambda V^{\text{even}} \\ d(V^{\text{even}}) &= 0 \end{aligned}$$

Thus we obtain

$$(\Lambda V, d) = (\Lambda(V^{\text{even}} \oplus \langle z_{l+1}, \dots, z_m \rangle), d) \otimes (\Lambda \langle z_1, \dots, z_l \rangle, 0)$$

Set $V' := V^{\text{even}} \oplus \langle z_{l+1}, \dots, z_m \rangle$. The minimality of $(\Lambda V, d)$ enforces the minimality of $(\Lambda V', d)$ and $(\Lambda V', d)$ is again a pure minimal Sullivan algebra.

By theorem [1].1.47, p. 39, the formality of $(\Lambda V, d)$ now is equivalent to the formality of $(\Lambda V', d)$.

If $(\Lambda V', d)$ has positive Euler characteristic, it is positively elliptic and formal—cf. theorem [1].1.52, p. 43. Thus $(\Lambda V, d)$ then is formal, too.

For the reverse implication we use contraposition: Suppose the Euler characteristic of $(\Lambda V', d)$ to vanish. We shall show that the algebra is non-formal. The algebra (V', d) has the property that $d((V')^{\text{even}}) = 0$ and that $d|_{(V')^{\text{odd}}}$ is injective. Thus, setting $C := (V')^{\text{even}}$ and $N := (V')^{\text{odd}}$, yields a decomposition $V' = C \oplus N$ as required in lemma 1.3; this decomposition is uniquely determined up to isomorphism.

Due to $\chi(\Lambda V', d) = 0$ there is a closed and non-exact element x in $(\Lambda V', d)$ that has odd degree. Therefore x necessarily lies in $I(N)$. Thus $(\Lambda V', d)$ is not formal and neither is $(\Lambda V, d)$. □

PROOF OF THEOREM C. Choose minimal models $(\Lambda V, d)$ for B and $(\Lambda \langle s \rangle, 0)$ for S^1 . Then we obtain a model for the fibration—cf. proposition

[8].15.5, p. 198:

$$(\Lambda V \otimes \Lambda\langle s \rangle, d) \xrightarrow{\cong} (A_{\text{PL}}(E), d)$$

with $ds = l \in V^2$ and $[l] \in H^2(Z)$ representing the Hard-Lefschetz class.

Obviously, the model of the fibration is not minimal. However, we may construct its minimal model. For this we write the model as

$$(\Lambda V' \otimes \Lambda\langle s, l \rangle, d)$$

with $V' \subseteq V$ a homogeneous complement of $\langle l \rangle$. This point of view reveals this model as the model of a rational fibration with fibre $\Lambda V'$ and contractible base space $(\Lambda\langle s, l \rangle, d')$. Thus we obtain (quasi-)isomorphisms

$$\begin{aligned} & (\Lambda V \otimes \Lambda\langle s \rangle, d) \\ & \cong (\Lambda V' \otimes \Lambda\langle s, l \rangle, d) \\ & \simeq (\Lambda V' \otimes \Lambda\langle s, l \rangle, d) / (\Lambda\langle s, l \rangle, d') \\ & \simeq (\Lambda V', \bar{d}) \end{aligned}$$

where \bar{d} is the projection of the differential d to V' . Since $(\Lambda V, d)$ is minimal, so is $(\Lambda V', \bar{d})$. Hence we have found a minimal model of E .

Applying the Gysin sequence to the fibration

$$\dots \rightarrow H^k(E) \rightarrow H^{k-1}(B) \xrightarrow{\cup[l]} H^{k+1}(B) \rightarrow H^{k+1}(E) \rightarrow \dots$$

yields

$$(8) \quad H^k(E) = 0$$

for odd $k < n$. This follows from the assumption that B is positively elliptic, i.e. that its rational cohomology is concentrated in even degrees. Moreover, we use the assumption that the Euler class of the fibration is a rational multiple of the Hard-Lefschetz class $[l]$. Due to Hard-Lefschetz, cupping with $[l]$ is injective in degrees smaller than n .

We may now characterise formality. The model $(\Lambda V', \bar{d})$ has odd formal dimension $2n + 1$. Since the space E is elliptic by the long exact homotopy sequence of the fibration, it satisfies Poincaré duality. In particular, there is a non-vanishing cohomology class of odd degree.

Since the space B is positively elliptic, it is pure. Consequently, also the model $(\Lambda V', \bar{d})$ is pure. By lemma 1.5 and the long exact sequence of the fibration we obtain the following: The formality of E is equivalent to the existence of a decomposition

$$(\Lambda V', \bar{d}) \cong (\Lambda V'', \tilde{d}) \otimes (\Lambda\langle \tilde{n} \rangle, 0)$$

for $0 \neq \tilde{n} \in (V')^{\text{odd}}$ and a homogeneous complement V'' of $\langle \tilde{n} \rangle$ in V' . Thus E is formal if and only if its minimal model splits as a product of a formal space $(\Lambda V'', \tilde{d})$ and the model of an odd sphere.

In this case we observe that

$$\dim(V'')^{\text{odd}} = \dim(V'')^{\text{even}}$$

Indeed, this was true for V already, since the base is positively elliptic. The equality above then follows, since V'' results from V by splitting of a class of even degree, l , and a class of odd degree, \tilde{n} . Thus, the base B is again

positively elliptic (cf. proposition [8].32.10, p. 444), since the cohomology $H(\Lambda V'', \tilde{d})$ is finite-dimensional by construction. Hence $(\Lambda V'', \tilde{d})$ is an F_0 -space and consequently formal.

We conclude that this shows that E is formal if and only if it splits as a product of formal spaces, which then are necessarily an F_0 -space and a rational sphere of odd dimension at least n —cf. equation (8). This proves the result. \square

A main step in the construction of our non-formal homogeneous spaces will be to consider certain \mathbb{S}^1 -bundles over homogeneous Hermitian spaces. For this we cite the following simply-connected compact homogeneous Kähler manifolds from [5].8.H, p. 229–234:

$$(9) \quad \begin{array}{l} \frac{\mathbf{Sp}(n)}{\mathbf{U}(p_1) \times \cdots \times \mathbf{U}(p_q) \times \mathbf{Sp}(l)} \quad \text{for } \sum_{i=1}^q p_i + l = n \\ \frac{\mathbf{SO}(2n+1)}{\mathbf{U}(p_1) \times \cdots \times \mathbf{U}(p_q) \times \mathbf{SO}(2l+1)} \quad \text{for } \sum_{i=1}^q p_i + l = n \\ \frac{\mathbf{SO}(2n)}{\mathbf{U}(p_1) \times \cdots \times \mathbf{U}(p_q) \times \mathbf{SO}(2l)} \quad \text{for } \sum_{i=1}^q p_i + l = n \\ \frac{\mathbf{SO}(2n)}{\mathbf{U}(p_1) \times \cdots \times \mathbf{U}(p_{q-1}) \times \tilde{\mathbf{U}}(p_q)} \quad \text{for } \sum_{i=1}^q p_i = n \\ \frac{\mathbf{SU}(n)}{\mathbf{S}(\mathbf{U}(p_1) \times \cdots \times \mathbf{U}(p_q))} \quad \text{for } \sum_{i=1}^q p_i = n \end{array}$$

where $\tilde{\mathbf{U}}(p_q) \subseteq \mathbf{SO}(2p_q)$ is the unitary group with respect to a slightly altered complex structure of \mathbb{R}^{2p_q} (cf. [5].8.113, p. 231).

One important observation will be that the dimension of these spaces grows quadratically in n , whereas the top dimension of their rational homotopy groups grows linearly. This enables us to apply theorem B.

In table 2 we shall use the convention that $\sum_{i=1}^t k_i = p$, $\sum_i l_i = l$ and $p_1 + p_2 = p$. Moreover, we also require $p, p_1, p_2 > 0$ in the table.

Theorem 1.6. *The homogeneous spaces in table 2 are non-formal.*

PROOF. We shall consider a certain subclass of the spaces given in (9). Their top rational homotopy group can easily be computed using the long exact homotopy sequence of the fibration formed by denominator, numerator and quotient. For this we shall use that the top rational homotopy of $\mathbf{Sp}(n)$, $\mathbf{SU}(n)$, $\mathbf{SO}(2n)$, $\mathbf{SO}(2n+1)$ lies in degree $4n-1$, $2n-1$, $4n-5$ for $n \geq 2$ and $4n-1$ respectively. So we are able to give the spaces under consideration with their dimensions and with the largest degree of a non-vanishing rational homotopy group in table 3—as always $p, p_1, p_2 > 0$.

Theses homogeneous spaces have the property that numerator and denominator form an equal rank pair, whence they are positively elliptic and rational cohomology is concentrated in even degrees only.

TABLE 2. Examples of non-formal homogeneous spaces

<i>homogeneous space</i>	<i>with</i>
$\frac{\mathbf{Sp}(N)}{\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_t)) \times \mathbf{Sp}(l_1) \times \cdots \times \mathbf{Sp}(l_r) \times \mathbf{U}(l_{r+1}) \times \cdots \times \mathbf{U}(l_s)}$	$p + l \leq N, r \geq 0, s \geq 0$ $\frac{1}{2}p^2 + (2l - \frac{7}{2})p - 4l + 1 \geq 0$
$\frac{\mathbf{SU}(N)}{\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_t)) \times \mathbf{Sp}(l_1) \times \cdots \times \mathbf{Sp}(l_r) \times \mathbf{U}(l_{r+1}) \times \cdots \times \mathbf{U}(l_s)}$	$2(p + l) \leq N, r \geq 0, s \geq 0$ $\frac{1}{2}p^2 + (2l - \frac{7}{2})p - 4l + 1 \geq 0$
$\frac{\mathbf{SO}(N)}{\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_t)) \times \mathbf{SO}(2l_1) \times \cdots \times \mathbf{SO}(2l_r) \times \mathbf{SO}(2l_{r+1} + 1)}$	$2(p + l) + 1 \leq N, r + 1 \geq 0$ $\frac{1}{2}p^2 + (2l - \frac{7}{2})p - 4l + 1 \geq 0$
$\frac{\mathbf{SU}(N)}{\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_t)) \times \mathbf{SO}(2l_1) \times \cdots \times \mathbf{SO}(2l_r) \times \mathbf{SO}(2l_{r+1} + 1)}$	$p + l = n, N \geq 2n + 1,$ $r + 1 \geq 0$ $\frac{1}{2}p^2 + (2l - \frac{7}{2})p - 4l + 1 \geq 0$
$\frac{\mathbf{SO}(2n)}{\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_t)) \times \mathbf{SO}(2l_1) \times \cdots \times \mathbf{SO}(2l_r)}$	$p + l = n, r \geq 0, n \geq 2, l \neq 1$ $\frac{1}{2}p^2 + (2l - \frac{9}{2})p - 4l + 5 \geq 0$
$\frac{\mathbf{SO}(2n)}{\mathbf{S}(\bar{\mathbf{U}}(k_1) \times \cdots \times \bar{\mathbf{U}}(k_t))}$	$p = n, r \geq 0, n \geq 2$ $\frac{1}{2}n^2 - \frac{9}{2}n + 5 \geq 0$
$\frac{\mathbf{SU}(N)}{\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_s)) \times \mathbf{S}(\mathbf{U}(k_{s+1}) \times \cdots \times \mathbf{U}(k_t))}$	$p_1 + p_2 = n \leq N, p_1, p_2 > 0$ $N \geq 2,$ $p_1 p_2 - 2(p_1 + p_2) + 1 \geq 0$

TABLE 3. Certain Hermitian homogeneous spaces

homogeneous space	for	dimension	top rational homotopy
$\frac{\mathbf{Sp}(n)}{\mathbf{U}(p) \times \mathbf{Sp}(l)}$	$p + l = n$	$p^2 + 4 \cdot l \cdot p + p$	$4n - 1$
$\frac{\mathbf{SO}(2n+1)}{\mathbf{U}(p) \times \mathbf{SO}(2l+1)}$	$p + l = n$	$p^2 + 4 \cdot l \cdot p + p$	$4n - 1$
$\frac{\mathbf{SO}(2n)}{\mathbf{U}(p) \times \mathbf{SO}(2l)}$	$p + l = n$	$p^2 + 4 \cdot l \cdot p - p$	$4n - 5$ for $n \geq 2$
$\frac{\mathbf{SO}(2n)}{\bar{\mathbf{U}}(n)}$		$n^2 - n$	$4n - 5$ for $n \geq 2$
$\frac{\mathbf{SU}(n)}{\mathbf{S}(\mathbf{U}(p_1) \times \mathbf{U}(p_2))}$	$p_1 + p_2 = n$	$2 \cdot p_1 \cdot p_2$	$2n - 1$

All these manifolds are compact homogeneous Kähler manifolds by [5].8.H. Since for the following arguments the spaces one to four behave similarly, we shall do showcase computations for the cases one and five.

The fibre bundle

$$\mathbf{SU}(p) \hookrightarrow \mathbf{U}(p) \xrightarrow{\det} \mathbb{S}^1$$

lets us conclude that $\mathbf{U}(p) = \mathbb{S}^1 \cdot \mathbf{SU}(p) = \mathbb{S}^1 \times_{\mathbb{Z}_p} \mathbf{SU}(p)$, where \mathbb{Z}_p is the (multiplicative) cyclic group of p -th roots of unity acting on each factor by (left) multiplication. So the canonical projection yields a fibre bundle

$$\mathbb{S}^1/\mathbb{Z}_p \hookrightarrow \frac{\mathbf{Sp}(n)}{(\mathbb{Z}_p \times_{\mathbb{Z}_p} \mathbf{SU}(p)) \times \mathbf{Sp}(l)} \rightarrow \frac{\mathbf{Sp}(n)}{(\mathbb{S}^1 \times_{\mathbb{Z}_p} \mathbf{SU}(p)) \times \mathbf{Sp}(l)}$$

of homogeneous spaces, which clearly is no other than

$$\mathbb{S}^1/\mathbb{Z}_p \hookrightarrow \frac{\mathbf{Sp}(n)}{\mathbf{SU}(p) \times \mathbf{Sp}(l)} \rightarrow \frac{\mathbf{Sp}(n)}{\mathbf{U}(p) \times \mathbf{Sp}(l)}$$

since $\mathbb{Z}_p \times_{\mathbb{Z}_p} \mathbf{SU}(p) \xrightarrow{\cong} \mathbf{SU}(p)$ —given by left multiplication of \mathbb{Z}_p on $\mathbf{SU}(p)$ —is an isomorphism.

In case five we obtain a homomorphism

$$\begin{aligned} \mathbb{S}^1 \times \mathbf{SU}(p_1) \times \mathbf{SU}(p_2) &\rightarrow \mathbf{S}(\mathbf{U}(p_1) \times \mathbf{U}(p_2)) \\ (x, A, B) &\mapsto (x^{p_2} \cdot A, x^{-p_1} \cdot B) \end{aligned}$$

with kernel formed by all the elements of the form $(a, a^{-p_2} I_{p_1}, a^{p_1} I_{p_2})$ with $a^{p_1 p_2} = 1$. Thus the kernel is isomorphic to $\mathbb{Z}_{p_1 p_2}$. (It is consequent to use the terminology that $\mathbb{Z}_1 = 1$ be the trivial group.) In particular, this defines an action of $\mathbb{Z}_{p_1 p_2} \subseteq \mathbb{S}^1$ on $\mathbf{SU}(p_1) \times \mathbf{SU}(p_2)$ and we obtain

$$\mathbf{S}(\mathbf{U}(p_1) \times \mathbf{U}(p_2)) = \mathbb{S}^1 \times_{\mathbb{Z}_{p_1 p_2}} (\mathbf{SU}(p_1) \times \mathbf{SU}(p_2))$$

This leads to the fibre bundle

$$\mathbb{S}^1/\mathbb{Z}_{p_1 p_2} \hookrightarrow \frac{\mathbf{SU}(n)}{\mathbb{Z}_{p_1 p_2} \times_{\mathbb{Z}_{p_1 p_2}} (\mathbf{SU}(p_1) \times \mathbf{SU}(p_2))} \rightarrow \frac{\mathbf{SU}(n)}{\mathbb{S}^1 \times_{\mathbb{Z}_{p_1 p_2}} (\mathbf{SU}(p_1) \times \mathbf{SU}(p_2))}$$

of homogeneous spaces, which clearly is no other than

$$\mathbb{S}^1/\mathbb{Z}_{p_1 p_2} \hookrightarrow \frac{\mathbf{SU}(n)}{\mathbf{SU}(p_1) \times \mathbf{SU}(p_2)} \rightarrow \frac{\mathbf{SU}(n)}{\mathbf{S}(\mathbf{U}(p_1) \times \mathbf{U}(p_2))}$$

Thus, in each case identifying $\mathbb{S}^1/\mathbb{Z}_p$ (for each respective p) with its finite covering \mathbb{S}^1 , we obtain the following sphere bundles:

$$\begin{aligned} \mathbb{S}^1 &\hookrightarrow \frac{\mathbf{Sp}(n)}{\mathbf{SU}(p) \times \mathbf{Sp}(l)} \rightarrow \frac{\mathbf{Sp}(n)}{\mathbf{U}(p) \times \mathbf{Sp}(l)} \\ \mathbb{S}^1 &\hookrightarrow \frac{\mathbf{SO}(2n+1)}{\mathbf{SU}(p) \times \mathbf{SO}(2l+1)} \rightarrow \frac{\mathbf{SO}(2n+1)}{\mathbf{U}(p) \times \mathbf{SO}(2l+1)} \\ \mathbb{S}^1 &\hookrightarrow \frac{\mathbf{SO}(2n)}{\mathbf{SU}(p) \times \mathbf{SO}(2l)} \rightarrow \frac{\mathbf{SO}(2n)}{\mathbf{U}(p) \times \mathbf{SO}(2l)} \\ &\mathbb{S}^1 \hookrightarrow \frac{\mathbf{SO}(2n)}{\mathbf{S}\tilde{\mathbf{U}}(n)} \rightarrow \frac{\mathbf{SO}(2n)}{\tilde{\mathbf{U}}(n)} \\ \mathbb{S}^1 &\hookrightarrow \frac{\mathbf{SU}(n)}{\mathbf{SU}(p_1) \times \mathbf{SU}(p_2)} \rightarrow \frac{\mathbf{SU}(n)}{\mathbf{S}(\mathbf{U}(p_1) \times \mathbf{U}(p_2))} \end{aligned}$$

Due to the long exact homotopy sequence, we see that the total space E of each bundle has a finite fundamental group, namely 0 or \mathbb{Z}_2 . Thus the

Euler class of each bundle does not vanish, since otherwise the bundle would be rationally trivial. This would imply $E \simeq_{\mathbb{Q}} B \times \mathbb{S}^1$ —with the respective base space B —and $\pi_1(E) \otimes \mathbb{Q} \neq 0$.

The long exact homotopy sequence associated to $H \hookrightarrow G \rightarrow G/H$ —where G is the numerator and H is the denominator group of the base space $B = G/H$ in each respective case—lets us conclude that

$$H^2(B) \cong \pi_2(G/H) \otimes \mathbb{Q} = \mathbb{Q}$$

for $l \neq 1$ and $n \geq 2$ in case three, for $n \geq 2$ in case four and without further restrictions in the other cases. (In each respective case we then have $\pi_2(G) \otimes \mathbb{Q} = 0$, $\pi_1(H) \otimes \mathbb{Q} = \mathbb{Q}$ and $\pi_1(G) \otimes \mathbb{Q} = 0$.)

This implies that the Kähler class and the Euler class—both contained in $H^2(B)$ —are non-zero multiples in rational cohomology.

In order to apply theorem B we need to demand that the top rational homotopy does not lie above half the dimension of the space. This leads to the following restrictions in the respective cases:

$$\begin{aligned} \frac{1}{2}p^2 + \left(2l - \frac{7}{2}\right)p - 4l + 1 &\geq 0 \\ \frac{1}{2}p^2 + \left(2l - \frac{7}{2}\right)p - 4l + 1 &\geq 0 \\ \frac{1}{2}p^2 + \left(2l - \frac{9}{2}\right)p - 4l + 5 &\geq 0 \\ \frac{1}{2}n^2 - \frac{9}{2}n + 5 &\geq 0 \\ p_1p_2 - 2(p_1 + p_2) + 1 &\geq 0 \end{aligned}$$

Due to the first point in remark 1.4 we may now apply theorem B to the depicted fibre bundles. This yields that the total spaces

$$\begin{aligned} &\frac{\mathbf{Sp}(n)}{\mathbf{SU}(p) \times \mathbf{Sp}(l)}, \quad \frac{\mathbf{SO}(2n+1)}{\mathbf{SU}(p) \times \mathbf{SO}(2l+1)}, \quad \frac{\mathbf{SO}(2n)}{\mathbf{SU}(p) \times \mathbf{SO}(2l)}, \\ &\frac{\mathbf{SO}(2n)}{\mathbf{S}\tilde{\mathbf{U}}(n)}, \quad \frac{\mathbf{SU}(n)}{\mathbf{SU}(p_1) \times \mathbf{SU}(p_2)} \end{aligned}$$

of the respective bundles are non-formal under the given conditions.

Due to proposition D (respectively [22], p. 212), we may replace the stabilisers of the total spaces of the fibrations by maximal rank subgroups sharing the maximal torus. We apply this first to the $\mathbf{SU}(p)$ -factor which arose from the fibre-bundle construction and replace it by a $\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_t))$. Then we also substitute the other factors by suitable other ones.

Finally, we are done by an application of proposition A respectively table 1, which allows us to use the described embeddings of the numerator into larger Lie groups to form new homogeneous spaces. The result of this process is depicted in table 2:

The first example we considered here leads to the first line in table 2 by blockwise inclusion $\mathbf{Sp}(n) \hookrightarrow \mathbf{Sp}(N)$ and to the second line by the inclusion

$$\mathbf{Sp}(n) \hookrightarrow \mathbf{SU}(2n) \hookrightarrow \mathbf{SU}(N)$$

(which is identical to $\mathbf{Sp}(n) \hookrightarrow \mathbf{Sp}(N') \hookrightarrow \mathbf{SU}(2N') \hookrightarrow \mathbf{SU}(N)$).

The second example produces lines three and four by the inclusions $\mathbf{SO}(2n+1) \hookrightarrow \mathbf{SO}(N)$ and

$$\mathbf{SO}(2n+1) \hookrightarrow \mathbf{SU}(2n+1) \hookrightarrow \mathbf{SU}(N)$$

(which is the same as $\mathbf{SO}(2n+1) \hookrightarrow \mathbf{SO}(N') \hookrightarrow \mathbf{SU}(2N') \hookrightarrow \mathbf{SU}(N)$).

For examples three and four we do not use any further inclusions. Example five produces line seven by the inclusion $\mathbf{SU}(n) \hookrightarrow \mathbf{SU}(N)$. \square

The proof guides the way of how to interpret table 2. Nonetheless, let us—exemplarily for the first line of the table—shed some more light upon the used inclusions: The inclusion of the denominator in the first example is given by blockwise inclusions of

$$\begin{aligned} \mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_t)) &\hookrightarrow \mathbf{Sp}(p) \hookrightarrow \mathbf{Sp}(n) \hookrightarrow \mathbf{Sp}(N) \\ \mathbf{Sp}(l_1) \times \cdots \times \mathbf{Sp}(l_r) &\hookrightarrow \mathbf{Sp}\left(\sum_{i=1}^r l_i\right) \hookrightarrow \mathbf{Sp}(n) \hookrightarrow \mathbf{Sp}(N) \\ \mathbf{U}(l_{r+1}) \times \cdots \times \mathbf{U}(l_s) &\hookrightarrow \mathbf{Sp}\left(\sum_{i=r+1}^s l_i\right) \hookrightarrow \mathbf{Sp}(n) \hookrightarrow \mathbf{Sp}(N) \end{aligned}$$

where $n = p + \sum_{i=1}^s l_i$.

In examples three and four it does not matter whether or not we have a stabiliser which has—beside the unitary part—special orthogonal groups of the type $\mathbf{SO}(2l_i)$ only, if there is additionally one factor of the form $\mathbf{SO}(2l_{r+1}+1)$ or if there is the factor $\mathbf{SO}(2l_r+1)$ only: All three cases establish maximal rank subgroups of $\mathbf{SO}(2l+1)$, which therefore share the maximal torus with $\mathbf{SO}(2l+1)$.

Moreover, note that the relations in table 2 are growing quadratically in p . Thus for a constant l we can always find p and N that will satisfy the restrictions.

Example 1.7. Let us describe some of the simplest and most classical consequences that arise out of our reasoning. We have proved that the spaces

$$\frac{\mathbf{Sp}(n)}{\mathbf{SU}(n)} \quad \frac{\mathbf{SO}(2n)}{\mathbf{SU}(n)} \quad \frac{\mathbf{SU}(p+q)}{\mathbf{SU}(p) \times \mathbf{SU}(q)}$$

are non-formal for $n \geq 7$, $n \geq 8$ and $p+q \geq 4$ respectively. \square

The numerical conditions we impose on the examples seem to be quite sharp. The prerequisites in the last example in table 2 for the space to be non-formal are evidently satisfied if $p_1, p_2 \geq 4$ or if $p_1 \geq 3$ and $p_2 \geq 5$. These are sharp bounds:

Proposition 1.8. *The homogeneous space*

$$\frac{\mathbf{SU}(7)}{\mathbf{SU}(3) \times \mathbf{SU}(4)}$$

is formal.

PROOF. Using the model (25) from the appendix, we compute a minimal model for the given homogeneous space as generated by

$$c, c', \tilde{x} \quad \deg c = 4, \deg c' = 6, \deg \tilde{x} = 13$$

with vanishing differentials and by generators representing relations

$$x_1, x_2 \quad \deg x_1 = 9, \deg x_2 = 11$$

with

$$d(x_1) = c \cdot c', \quad d(x_2) = c^3 - (c')^2$$

This model splits as the model of the product of an F_0 -space and an \mathbb{S}^{13} . Thus it is formal. \square

As for the interlink between formality and geometry, we remark that, for example, from the computations in lemma [4].8.2, p. 272, we directly see that all known odd-dimensional examples—cf. proposition [4].8.1, p. 271—of positively curved Riemannian manifolds are formal spaces; so are the newly announced examples in [15] and [23].

The importance of formality issues in geometry is also stressed by the following result in [4]. The non-formality of

$$C := \frac{\mathbf{SU}(6)}{\mathbf{SU}(3) \times \mathbf{SU}(3)}$$

plays a crucial role for the following (cf. [4], theorem 1.4): It gives rise to the existence of a non-negatively curved vector bundle over $C \times T$ (with a torus T of $\dim T \geq 2$) which does not split in a certain sense as a product of bundles over the factors of the base space, but which has the property that its total space admits a complete metric of non-negative sectional curvature with the zero section being a soul.

Remark 1.9. As a generalisation of section [3].4, p. 20, let us recall that it is very easy to construct fibre bundles with interesting properties as far as formality is concerned using non-formal homogeneous space. These bundles illustrate that formality for the base space in general does not have to be related to formality on the total space. Contrast this with a positive result given in theorem [3].A, p. 4.

Homogeneous spaces G/H occur in terms of a fibre bundle

$$H \hookrightarrow G \rightarrow G/H$$

In the case of a non-formal homogeneous space, this is a fibration with formal fibre and formal total space, but with non-formal base space. Moreover, by proposition D we know that G/H is formal if and only if G/T_H is formal, where $T_H \subseteq H$ is a maximal torus of H . Thus we may successively form the spherical fibre bundles

$$(10) \quad \mathbb{S}^1 \hookrightarrow G/T^k \rightarrow G/T^{k+1}$$

for $k \geq 0$ and $T^k \subseteq T_H$ a k -torus. If G/H is non-formal, the space G/T_H is non-formal. Moreover, we see that there is a choice for k with the property that (10) is a fibre bundle with formal total space and non-formal base space.

Conversely, we may construct a spherical fibre bundle with non-formal total space and formal base space by

$$\mathbb{S}^1 \hookrightarrow \frac{\mathbf{SU}(6)}{\mathbf{SU}(3) \times \mathbf{SU}(3)} \rightarrow \frac{\mathbf{SU}(6)}{\mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))}$$

The fibration is induced by the canonical inclusion

$$\mathbf{SU}(3) \times \mathbf{SU}(3) \hookrightarrow \mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))$$

The total space is formal, as it has positive Euler characteristic. \square

Let us finally give another example of how to apply theorem B. This will result in yet another long list of non-formal spaces. In order to shed more light on the motivation for the next theorem—and, in particular, on the inclusions we use—we observe the following: The dimension of the Cartesian product of two spaces clearly is the sum of the dimensions. If both spaces are elliptic, however, the top degree of a non-vanishing rational homotopy group of the product is the maximum over the maximal degrees of non-trivial rational homotopy groups of both factors. So we get the following meta-principle: Taking sufficiently many factors of elliptic Hard-Lefschetz manifolds will result in a Hard-Lefschetz space—with Hard-Lefschetz class the sum of the ones of the factors—to which theorem B applies. So the base space of the fibration we shall then use will just be the Cartesian product of all the factors, whilst the total space will be gained by “factoring out” the Hard-Lefschetz class. This explains, in particular, how the inclusions in the next examples are to be understood.

Theorem 1.10. *Every homogeneous space of the form*

$$\begin{aligned} & \left(\prod_{1 \leq i \leq k_1} \mathbf{Sp}(n_{1,i}) \times \prod_{1 \leq i \leq k_2} \mathbf{SO}(2n_{2,i} + 1) \times \prod_{1 \leq i \leq k_3} \mathbf{SO}(2n_{3,i}) \right. \\ & \quad \left. \times \prod_{1 \leq i \leq k_4} \mathbf{SO}(2n_{4,i}) \times \prod_{1 \leq i \leq k_5} \mathbf{U}(n_{5,i}) \right) / \\ & \left(\mathbf{S} \left(\prod_{1 \leq i \leq k_1} \mathbf{U}(p_{1,i}) \times \prod_{1 \leq i \leq k_2} \mathbf{U}(p_{2,i}) \times \prod_{1 \leq i \leq k_3} \mathbf{U}(p_{3,i}) \times \prod_{1 \leq i \leq k_4} \tilde{\mathbf{U}}(n_{4,i}) \right. \right. \\ & \quad \left. \left. \times \prod_{1 \leq i \leq k_5} (\mathbf{U}(p_{5,i}) \times \mathbf{U}(q_{5,i})) \right) \right) \\ & \times \prod_{1 \leq i \leq k_1} \mathbf{Sp}(l_{1,i}) \times \prod_{1 \leq i \leq k_2} \mathbf{SO}(2l_{2,i} + 1) \times \prod_{1 \leq i \leq k_3} \mathbf{SO}(2l_{3,i}) \end{aligned}$$

satisfying

$$\begin{aligned}
& \frac{1}{2} \left(\sum_{1 \leq i \leq k_1} p_{1,i}^2 + 4l_{1,i}p_{1,i} + p_{1,i} + \right. \\
& \quad \sum_{1 \leq i \leq k_2} p_{2,i}^2 + 4l_{2,i}p_{2,i} + p_{2,i} + \\
& \quad \left. \sum_{1 \leq i \leq k_3} p_{3,i}^2 + 4l_{3,i}p_{3,i} - p_{3,i} \right) \\
& \geq \max \left(4 \max_{1 \leq i \leq k_1} p_{1,i} + l_{1,i} - 1, \right. \\
& \quad 4 \max_{1 \leq i \leq k_2} p_{2,i} + l_{2,i} - 1, \\
& \quad \left. 4 \max_{1 \leq i \leq k_2} p_{2,i} + l_{2,i} - 5 \right)
\end{aligned}$$

is non-formal.

□

Example 1.11. It is obvious that this last method is suited for a variety of generalisations and will produce a myriad of new examples. Let us mention just two more:

- In remark 1.4 we extend theorem C to the situation of suitable \mathbb{S}^3 -fibrations. One source of examples may be a large product of Positive Quaternion Kähler manifolds together with the sum of the respective Kraines forms in degree 4 corresponding to the fibre \mathbb{S}^3 . As a first example of this kind, we see directly that

$$\frac{\mathbf{Sp}(n_1 + 1) \times \mathbf{Sp}(n_2 + 1) \times \mathbf{Sp}(n_3 + 1) \times \dots}{\mathbf{Sp}(n_1) \times \mathbf{Sp}(n_2) \times \mathbf{Sp}(n_3) \times \dots \times \mathbf{S}(\mathbf{Sp}(1) \times \mathbf{Sp}(1) \times \mathbf{Sp}(1) \times \dots)}$$

is a non-formal homogeneous space (provided the number of factors is large). Here, by abuse of notation, $\mathbf{S}(\mathbf{Sp}(1) \times \mathbf{Sp}(1) \times \mathbf{Sp}(1) \times \dots)$ denotes all the elements (a_1, a_2, \dots) in the quaternionic torus $\mathbf{Sp}(1) \times \mathbf{Sp}(1) \times \dots$ with the property that their product $a_1 \cdot a_2 \cdots = 1$ in \mathbb{H} . For this we consider $\mathbf{Sp}(1)$ as the unit quaternions.

- Using a product involving the Eschenburg biquotient $\mathbb{S}^1 \backslash \mathbf{U}(3) / T^2$ it is easy to construct further non-formal genuine biquotients (which are not homogeneous). For this we recall that this biquotient has a Hard-Lefschetz structure, as can be checked easily.

□

We leave it to the reader to enrich the examples in theorem 1.10 and example 1.11 using theorem B and proposition D. Doing so one gains even many more non-formal homogeneous spaces/ biquotients.

2. NON-FORMAL HOMOGENEOUS MANIFOLDS IN EVERY LARGE DIMENSION

Let us now use the constructed examples of non-formal homogeneous spaces in order to give the

PROOF OF THEOREM E. The example

$$\frac{\mathbf{SU}(N)}{\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_s)) \times \mathbf{S}(\mathbf{U}(k_{s+1}) \times \cdots \times \mathbf{U}(k_t))}$$

from table 2 will serve as a main source of further examples. (It is simply-connected as is the numerator group.)

The restrictions in the table will certainly be satisfied if $p_1, p_2 \geq 4$ or if $p_1 \geq 3$ and $p_2 \geq 5$ (respectively $N \geq p_1 + p_2$). By our computations in appendix A we may also use the case $p_1 = p_2 = 3$. We use a slightly different terminology: Set $p := p_1$, $k := p_2$ and $N \geq 0$. Thus whenever $p = k = 3$ or $p, k \geq 4$ or $p \geq 3$ and $k \geq 5$ the spaces

$$\begin{aligned} (11) \quad & \frac{\mathbf{SU}(p+k+N)}{\mathbf{SU}(p) \times \mathbf{SU}(k)} \\ (12) \quad & \frac{\mathbf{SU}(p+k+N)}{\mathbf{SU}(p) \times \mathbf{S}(\mathbf{U}(k-1) \times \mathbf{U}(1))} \\ (13) \quad & \frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-1) \times \mathbf{U}(1)) \times \mathbf{SU}(k)} \\ (14) \quad & \frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-1) \times \mathbf{U}(1)) \times \mathbf{S}(\mathbf{U}(k-1) \times \mathbf{U}(1))} \\ (15) \quad & \frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(2)) \times \mathbf{SU}(k)} \\ (16) \quad & \frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(2)) \times \mathbf{S}(\mathbf{U}(k-1) \times \mathbf{U}(1))} \\ (17) \quad & \frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(2)) \times \mathbf{S}(\mathbf{U}(k-2) \times \mathbf{U}(1) \times \mathbf{U}(1))} \\ (18) \quad & \frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(1) \times \mathbf{U}(1)) \times \mathbf{SU}(k)} \\ (19) \quad & \frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(1) \times \mathbf{U}(1)) \times \mathbf{S}(\mathbf{U}(k-2) \times \mathbf{U}(2))} \\ (20) \quad & \frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(1) \times \mathbf{U}(1)) \times \mathbf{S}(\mathbf{U}(k-2) \times \mathbf{U}(1) \times \mathbf{U}(1))} \end{aligned}$$

will be non-formal. In the given order the spaces have the following dimensions:

$$\begin{aligned}
& 2(k+N)p + (2kN + N^2 + 1) \\
& 2(k+N)p + (2kN + N^2 + 2k - 1) \\
& 2(k+N+2)p + (2kN + N^2 - 1) \\
(21) \quad & 2(k+N+2)p + (2kN + N^2 + 2k - 3) \\
& 2(k+N+4)p + (2kN + N^2 - 7) \\
& 2(k+N+4p) + (2kN + N^2 + 2k - 9) \\
& 2(k+N+4)p + (2kN + N^2 + 4k - 13) \\
& 2(k+N+4)p + (2kN + N^2 - 5) \\
& 2(k+N+4)p + (2kN + N^2 + 4k - 13) \\
& 2(k+N+4)p + (2kN + N^2 + 4k - 11)
\end{aligned}$$

Regard these homogeneous spaces $M_p^{k,N}$ as manifolds parametrised over \mathbb{N} by the variable p . In other words, we have the infinite sequences $(M_p^{k,N})_{p \in \mathbb{N}}^{(11)}, \dots, (M_p^{k,N})_{p \in \mathbb{N}}^{(20)}$. We shall now show that for each congruence class $[m]$ modulo 16, there is a family $(M_p^{k,N})_{p \in \mathbb{N}}^m$ which—for certain numbers p —consists of non-formal manifolds only and which has the property that

$$\dim(M_p^{k,N})_{p \in \mathbb{N}}^m \equiv [m] \pmod{16}$$

For this we fix the coefficient of p in each dimension in (21) to be $2 \cdot 8 = 16$. The following series realise the congruence classes $0, \dots, 15$. In the third column we give the smallest dimension for which there is a $p \in \mathbb{N}$ making the space non-formal. (From this dimension on the series will produce non-formal examples only.) Once we are given p , we may compute this dimension. We determine p by the following rule: If $k = 3$ set $p = 3$, if $k = 4$ set $p = 4$, if $k \geq 5$ set $p = 3$. This guarantees non-formality. The necessary data is given by the subsequent table:

$[m]$	series	starting dimension
0	$(M_p^{3,1})_{p \in \mathbb{N}}^{(15)}$	48
1	$(M_p^{8,0})_{p \in \mathbb{N}}^{(11)}$	49
2	$(M_p^{5,1})_{p \in \mathbb{N}}^{(18)}$	50
3	$(M_p^{4,0})_{p \in \mathbb{N}}^{(19)}$	67
4	$(M_p^{3,1})_{p \in \mathbb{N}}^{(16)}$	52
5	$(M_p^{4,0})_{p \in \mathbb{N}}^{(20)}$	69
6	$(M_p^{3,1})_{p \in \mathbb{N}}^{(17)}$	54
7	$(M_p^{6,2})_{p \in \mathbb{N}}^{(12)}$	87
8	$(M_p^{3,1})_{p \in \mathbb{N}}^{(20)}$	56
9	$(M_p^{4,0})_{p \in \mathbb{N}}^{(15)}$	57
10	$(M_p^{5,1})_{p \in \mathbb{N}}^{(13)}$	58
11	$(M_p^{4,0})_{p \in \mathbb{N}}^{(18)}$	59
12	$(M_p^{7,1})_{p \in \mathbb{N}}^{(12)}$	76
13	$(M_p^{6,2})_{p \in \mathbb{N}}^{(11)}$	77
14	$(M_p^{6,2})_{p \in \mathbb{N}}^{(14)}$	78
15	$(M_p^{6,0})_{p \in \mathbb{N}}^{(13)}$	47

From dimension 72 onwards every congruence class modulo 16 can be realised by one of the given spaces. So we are done. \square

Clearly, this theorem is far from being optimal and can easily be improved by just stepping through table 2 or by doing explicit calculations—see appendix A. Indeed, in the proof we have found non-formal homogeneous spaces in dimensions

$$47, 48, 49, 50, 52, 54, 56, 57, 58, 59, 63, 64, 65, 66, 67, 68, 69, 70$$

and from dimension 72 onwards.

We remark that one clearly may use other non-formal series like $\mathbf{Sp}(n)/\mathbf{SU}(n)$ in order to establish similar results. Since the number of partitions of a natural number $n \in \mathbb{N}$ is growing exponentially in n , one will find many stabilisers sharing their maximal tori with $\mathbf{SU}(n)$.

As we pointed out, compact homogeneous spaces are rationally elliptic. Thus the question that arises naturally is: In which dimensions are there examples of non-formal elliptic simply-connected irreducible compact manifolds?

3. PROOF OF PROPOSITION D AND FORMALITY IN FIBRATIONS

Before we shall provide a direct and simple proof of proposition D, let us first see how the result follows from a much more general approach. Using relative obstruction theory we prove in [3] (see theorem A, p. 4)

Theorem 3.1. *Let*

$$F \hookrightarrow E \xrightarrow{f} B$$

be a fibration of simply-connected topological spaces of finite type. Suppose that F is elliptic, formal and satisfies the Halperin conjecture. Then E is formal if and only if B is formal.

Moreover, if B and E are formal, then the map f is formal.

We may apply this to the fibration

$$H/T_H \hookrightarrow G/T_H \rightarrow G/H$$

which is totally non-cohomologous to zero. Indeed, the rational Leray–Serre spectral sequence of this fibrations degenerates at the E_2 -term, since homogeneous spaces satisfy the Halperin conjecture. (Recall that this conjecture states exactly this degeneration for arbitrary fibrations of simply-connected spaces with an elliptic fibre of positive Euler-characteristic.)

However, the proof of this result draws on several additional results and can be simplified a lot in our special case. Thus it seems worth while to provide a direct argument.

By $T_H \subseteq H$ we denote the maximal torus. We shall consider the inclusions $i_1: H \hookrightarrow G$, $i_2: T_H \hookrightarrow G$ and $i_3: T_H \hookrightarrow H$.

PROOF OF PROPOSITION D. The space G admits a minimal Sullivan model of the form $(\Lambda V, 0)$ with $V = V^{\text{odd}}$ of finite dimension; a minimal model of H is given by $(\Lambda W_H, 0)$ with finite dimensional $W_H = W_H^{\text{odd}}$. Equally, we have a minimal model for T_H given by $(\Lambda W_T, 0)$ with finite dimensional $W_T = W_T^1$. Thus a model of G/H is given by $(\Lambda W_H^{+1} \otimes \Lambda V, d_1)$ and a model for G/T is given by $(\Lambda W_T^{+1} \otimes \Lambda V, d_2)$ —see [8] p. 218. (By the superscript $+1$ we denote a corresponding degree shift, as usual.)

Both in the case of G/H as well as for G/T_H we see that the corresponding Sullivan models are pure algebras: The differential $d \in \{d_1, d_2\}$ in each case has the property that $d|_{\Lambda W^{+1}} = 0$ (as $(\Lambda W^{+1}, 0)$ forms a differential subalgebra) and that $d|_{\Lambda V} \in \Lambda W^{+1}$ by [8] p. 217. This directly yields a pure filtration. Clearly, the corresponding minimal models then are pure, too.

The differentials are given by $d_1 v = H(\mathbf{B}i_1)(v^{+1})$ (for $v \in V$ and in the case of G/H) respectively by $d_2 v = H(\mathbf{B}i_2)(v^{+1})$ (for G/T_H). Since $H^*(\mathbf{B}H) = H^*(\mathbf{B}T_H)^{W(H)}$, i.e. the cohomology of the classifying space of a Lie group consists of the invariant polynomials in the cohomology of the classifying space of the maximal torus under the action of the Weyl group, we derive the following: The map $H^*(\mathbf{B}i_3)$ induced by the inclusion $i_3: T_H \hookrightarrow H$ is also just the inclusion in this setting. Consequently, on the level of differentials we obtain

$$(22) \quad d_2 v = H(\mathbf{B}i_2)(v^{+1}) = H(\mathbf{B}(i_3 \circ i_1))(v^{+1}) = H(\mathbf{B}i_3)(d_1 v^{+1})$$

Thus we may identify the differentials d_1 and d_2 from the models for G/H and for G/T_H . This will allow us to deduce that one space is formal if and only if so is the other:

First assume G/T_H to be formal. By lemma 1.5 and theorem [8].14.9 we obtain the decomposition

$$(23) \quad (\Lambda W_{T_H}^{+1} \otimes V, d_2) \cong (\Lambda V', d) \otimes (\Lambda \langle z_1, \dots, z_l \rangle, 0) \otimes (C, d)$$

where $(\Lambda V', d)$ is minimal of positive Euler characteristic and where (C, d) is a contractible algebra. Due to the formality of G/T_H we obtain that $(V')^{\text{even}} = (V')^{\text{odd}}$. Since we may identify the differentials for G/H and G/T_H as above, we derive a similar splitting

$$(24) \quad (\Lambda W_H^{+1} \otimes V, d_1) \cong (\Lambda V'', d) \otimes (\Lambda \langle z'_1, \dots, z'_l \rangle, 0) \otimes (C', d)$$

Indeed, by the constructions in the proofs of lemma 1.5 and theorem [8].14.9, the right hand side is contained in the left hand side and the isomorphism is rather an equality. Thus decomposition (24) can be established in analogy to decomposition (23) considering the following arguments: It might happen that the image $d_1(a)$ of an element $a \in V$ has an element b from W_H^{+1} as a non-trivial summand. In this case, both elements a, b —with $\deg a$ being odd and $\deg b$ being even—may be taken to lie in C' . (In particular, the element a does not lie in V'' .) Thus the dimension of V'' may be smaller than the dimension of V' . However, the Euler characteristic is preserved, i.e. $\chi(V'') = \chi(V') = 0$.

Note that the free part $(\Lambda \langle z_1, \dots, z_l \rangle, 0) = (\Lambda \langle z'_1, \dots, z'_l \rangle, 0)$ is not altered, since an element with non-trivial differential in $(\Lambda V', d)$ has non-trivial differential in (24) and vice versa due to (22).

Consequently, we have that $\deg(V'')^{\text{even}} = \deg(V'')^{\text{odd}}$ and the minimal model of G/H splits as in lemma 1.5. In other words, the space G/H is formal.

Conversely, given the formality of G/H and the splitting (24), the decomposition (23) can be derived by an analogous inverse process again preserving the Euler characteristic $\chi(V'') = \chi(V') = 0$. This yields the formality of G/T_H . \square

Corollary 3.2. *A biquotient $G//H$ is formal if and only if $G//T_H$ is formal.*

\square

There are several questions which may be stated to further explore the Halperin conjecture in the context of formality and ellipticity. One is

Question 3.3 (Lupton [19], p. 18). *Let $F \hookrightarrow E \rightarrow B$ be a fibration in which F is formal and elliptic and B is formal. If E is formal, then is the fibration totally non-cohomologous to zero?*

A partial positive answer in the case of odd dimensional spheres as a base is given in proposition [19].3.3, p. 11.

Although, in general, this question can be trivially answered in the negative via the example of the universal fibration $G \hookrightarrow \mathbf{E}G \rightarrow \mathbf{B}G$ where G is a non-trivial Lie group together with its classifying space $\mathbf{B}G$, it is

certainly worth while to impose further conditions on B , for example, and to investigate this question and related ones in this context. However, the question can also be answered in the negative if ellipticity on the base space is imposed. Here the Hopf bundles $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2$ or $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \rightarrow \mathbb{S}^4$ (for the simply-connected case) may serve as an example.

Any positive answer to question 3.3 might be considered as a partial converse to theorem 3.1 in a certain sense.

APPENDIX A. A COMPUTATIONAL EXAMPLE

Let us illustrate a showcase computation by which we reprove the non-formality of $\frac{\mathbf{SU}(6)}{\mathbf{SU}(3) \times \mathbf{SU}(3)}$ via a direct computation. This serves the following purposes:

- We illustrate how direct computations as in the proof of proposition 1.8 work.
- We show that any sort of sufficiently “small” respectively low-dimensional example can easily be provided. And most important of all:
- We illustrate how fast the complexity of computations increases making it absolutely necessary to provide general principles as we did.

We construct a model for the homogeneous space according to theorem [9].3.50. A minimal Sullivan model for the Lie group $\mathbf{SU}(n)$ is given by $\Lambda(\langle x_2, x_3, \dots, x_n \rangle, 0)$ with $\deg x_i = 2i - 1$; a Sullivan model for the classifying space $\mathbf{BSU}(n_k)$ (with $n_k \geq 2$) is given by the polynomial algebra $(\Lambda\langle c_2, \dots, c_{n_k} \rangle, 0)$ with $\deg c_i = 2i$. We obtain a model

$$(25) \quad \text{A}_{\text{PL}} \left(\frac{\mathbf{SU}(n)}{\mathbf{SU}(n_1) \times \dots \times \mathbf{SU}(n_k)}, d \right) \simeq \Lambda(\langle c_2^1, \dots, c_{n_1}^1, c_2^2, \dots, c_{n_2}^2, \dots, c_2^k, \dots, c_{n_k}^k \rangle \otimes \langle x_2, \dots, x_n \rangle, d)$$

with $\deg c_i^j = 2i, \deg x_i = 2i - 1$. The differential vanishes on the c_i^j and $dx_i = H^*(\mathbf{B}(\phi))y_i$, where y_i is the generator of $\mathbf{BSU}(n)$ corresponding to x_i and ϕ is the blockwise inclusion map $\mathbf{SU}(n_1) \times \dots \times \mathbf{SU}(n_k) \hookrightarrow \mathbf{SU}(n)$.

Note that we may take the c_i^j for the i -th universal Chern classes of $\mathbf{BSU}(n_j)$. Thus blockwise inclusion of the $\mathbf{SU}(n_j)$ yields the following equations

$$\begin{aligned} d(x_2) &= \sum_{i=1}^k c_2^i \\ d(x_3) &= \sum_{i=1}^k c_3^i \\ d(x_4) &= \sum_{i=1}^k c_4^i + \sum_{i \neq j} c_2^i c_2^j \\ &\vdots \\ d(x_i) &= (c^1 \cdots c^k)|_{2i} \\ &\vdots \\ d(x_n) &= c_{n_1}^1 \cdots c_{n_k}^k \end{aligned}$$

which become obvious when writing the Chern classes as elementary symmetric polynomials in elements that generate the cohomology of the classifying spaces of the maximal tori. By c^j we denote the total Chern class $1 + c_2^j + c_3^j + \cdots + c_{n_j}^j$ of $\mathbf{SU}(n_j)$ and by $|_{2i}$ the projection to degree $2i$.

In the case $k = 2$, $n_1 = n_2 = 3$ we obtain a model $(\Delta V, d)$, where the graded vector space V is generated by

$$\begin{array}{ll} c, c', d, d' & \deg c = \deg d = 4, \deg c' = \deg d' = 6 \\ x_2, x_3, x_4, x_5, x_6 & \deg x_2 = 3, \deg x_3 = 5, \deg x_4 = 7, \deg x_5 = 9, \\ & \deg x_6 = 11 \end{array}$$

The differential is given by

$$dc = dc' = dd = dd' = 0$$

$$dx_2 = c + d, \quad dx_3 = c' + d', \quad dx_4 = c \cdot d, \quad dx_5 = c' \cdot d + d' \cdot c, \quad dx_6 = c' \cdot d'$$

Thus the cohomology algebra of this algebra is given by generators

$$e, e', y, y' \quad \deg e = 4, \deg e' = 6, \deg y = 13, \deg y' = 15$$

and by relations

$$e^2 = e \cdot e' = e'^2 = y \cdot e = y^2 = y \cdot y' = y'^2 = y' \cdot e' = 0, \quad y \cdot e' = -y' \cdot e$$

Hence Betti numbers are as follows:

$$b_0 = b_4 = b_6 = b_{13} = b_{15} = b_{19} = 1$$

with the remaining ones equal to zero.

Using the construction principle from [8].12, p. 144–145, we compute the minimal model for this algebra. On generators it is given by

$$e, e', x, x', x'' \quad \deg e = 4, \deg e' = 6, \deg x = 7, \deg x' = 9, \deg x'' = 11$$

The differential d is given by

$$de = de' = 0, \quad dx = e^2, \quad dx' = e \cdot e', \quad dx'' = e'^2$$

It is easy to check that the Massey product $\langle e, e, e' \rangle$ is non-trivial. So $\frac{\mathbf{SU}(6)}{\mathbf{SU}(3) \times \mathbf{SU}(3)}$ is not formal—see proposition [9].2.90, p. 94.

We remark that the spaces $\mathbf{SU}(4)/(\mathbf{SU}(2) \times \mathbf{SU}(2))$ and $\mathbf{SU}(5)/(\mathbf{SU}(2) \times \mathbf{SU}(3))$ are formal as a straight-forward computation shows.

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MANUEL AMANN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PENNSYLVANIA
DAVID RITTENHOUSE LABORATORY
209 SOUTH 33RD STREET
PHILADELPHIA, PENNSYLVANIA
PA 19104-6395
USA

mamann@uni-muenster.de
<http://hans.math.upenn.edu/~mamann/>