MARKOV TRACE ON FUNAR ALGEBRA

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1. Introduction

Let B_n be the braid group with n strings and $\sigma_1,\dots,\sigma_{n-1}$ its standard generators. Let k be a commutative ring with $1\ne 0$. Given $\alpha,\beta\in k$, we define the k-algebra $K_n=K_n(\alpha,\beta)=K_n(\alpha,\beta;k)$ as the quotient of the group algebra kB_n by the relations

$$\sigma_1^3-\alpha\sigma_1^2+\beta\sigma_1-1=0 \tag{1}$$

and

$$\begin{aligned} y\bar{x}y =&2\alpha-\beta^2-(x+y)-(\alpha^2-\beta)(\bar{x}+\bar{y})+\beta(xy+yx)+\alpha(x\bar{y}+y\bar{x}+\bar{x}y+\bar{y}x)\\ &+(\alpha\beta-1)(\bar{x}\bar{y}+\bar{y}\bar{x})-\alpha xyx-(\bar{x}yx+x\bar{y}x+xy\bar{x})-\beta(\bar{x}\bar{y}x+x\bar{y}\bar{x})\\ &+(\alpha-\beta^2)\bar{x}\bar{y}\bar{x}. \end{aligned} \tag{2}$$

where x, $\bar{x}$, y, $\bar{y}$ in (2) stand for σ_1, σ_1^{-1}, σ_2, σ_2^{-1} respectively. Up to a change of the sign of β (for the sake of symmetricity), our definition of K_n is equivalent to the definition given by Bellingeri and Funar in [1]. Our relation (2) is much shorter than the corresponding relation in [1] (see [1; (2) and Table 1]) because we use σ_i^{-1} instead of σ_i^2. Multiplying (2) by σ_1 from the left or from the right, and simplifying the result using (1) and the braid group relations, we obtain

$$\bar{y}x\bar{y}=2\beta-\alpha^2-(\bar{x}+\bar{y})-\dots \qquad (\text{swap } x\leftrightarrow\bar{x},\ y\leftrightarrow\bar{y},\ \alpha\leftrightarrow\beta \text{ in (2)}) \tag{3}$$

Using (1) – (3) together with the braid relations, it is easy to see that K_n are finitely generated k-modules. Following [2], we denote the image of σ_i in K_n by s_i.

Set $K_\infty=\lim K_n$ (in contrary to the case of Hecke or BMW algebras, the morphisms $K_n\to K_{n+1}$ induced by the standard embeddings $B_n\subset B_{n+1}$ are not injective in general). We say that $t: K_\infty\otimes k[u,v]\to M$ is a *Markov trace* on K_∞ if M is a $k[u,v]$-module and t is a morphism of $k[u,v]$-modules such that $t(xy)=t(yx)$, $t(xs_n)=ut(x)$, $t(xs_n^{-1})=vt(x)$, $x,y\in K_n$, $n=1,2,\dots$.

It is claimed in [3] and [1] that a nontrivial Markov trace is constructed on K_n. About 2004–2005 I indicated a gap in the proof of its well-definedness (see Remark 2.8 below). As it is explained in [2], the gap was really serious: formally, the main result of [3] is wrong in the form it is stated. However, we show in this paper that the main idea in [1, 3] is correct: to construct a Markov trace on K_n, it suffices to check a finite number of identities though the number of them is much bigger than in [1, 3]. Theoretically, this approach allows to compute the universal Markov

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trace on K_∞, i. e., the projection of $K_\infty(\alpha,\beta;\mathbb{Z}[\alpha,\beta,u,v])$ onto its quotient by the submodule $\bar R$ generated by

$$xy-yx,\ \ xs_n-ux,\ \ xs_n^{-1}-vx,\quad x,y\in K_n,\ n=1,2,\dots \tag{4}$$

but in practice, the volume of computations is so huge that we did them only in some cases including the case of the universal Markov trace on $K_\infty(0,0)$. It appears that it takes its values in $\mathbb{Z}[u,v]/I$ where $I=(16,4u^2+4v,4v^2+4u,u^3+v^3+uv-3)$. Note, that it was checked in [2] that $K_5(0,0;\mathbb{Z})/(K_5\cap\bar R)=\mathbb{Z}[u,v]/I$.

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2. Definitions and statement of results

2.1. ***K*-reductions.** Let F_n^+ be the free monoid on generators $x_1^{\pm1},\dots,x_{n-1}^{\pm1}$ (the set of all not necessarily reduced words in $x_i^{\pm1}$) and $F_\infty^+=\bigcup F_n^+$. We denote the empty word by 1. Let kF_n^+ and kF_∞^+ be the corresponding free associative algebras over k (as k-modules, they are freely generated by F_n^+ and by F_∞^+ respectively).

We call *basic replacements* the pairs (U,V) with $U\in F_\infty^+$, $V\in kF_\infty^+$ (which we denote by $U\to V$) from the following list:

- (i) $x_ix_i^{-1}\longrightarrow 1,\ x_i^{-1}x_i\longrightarrow 1,\ i\ge 1;$
- (ii) $x_i^2\longrightarrow \alpha x_i-\beta+x_i^{-1},\ i\ge1;$
- (iii) $x_i^{-2}\longrightarrow \beta x_i^{-1}-\alpha+x_i,\ i\ge1;$
- (iv) $x_{i+1}^{\varepsilon_1}x_i^{\varepsilon_2}x_{i+1}^{\varepsilon_3}\longrightarrow x_i^{\varepsilon_3}x_{i+1}^{\varepsilon_2}x_i^{\varepsilon_1},\ \varepsilon_2\in\{\varepsilon_1,\varepsilon_3\}\subset\{-1,1\},\ i\ge1;$
- (v) $x_{i+1}x_i^{-1}x_{i+1}\longrightarrow$ (the right hand side of (2) with $x=x_i$, $y=x_{i+1}$), $i\ge1$;
- (vi) $x_{i+1}^{-1}x_ix_{i+1}^{-1}\longrightarrow$ (the right hand side of (3) with $x=x_i$, $y=x_{i+1}$), $i\ge1$;
- (vii) $x_{i+1}^{\varepsilon_1}x_i^{\varepsilon_2}Wx_{i+1}^{\varepsilon_3}\longrightarrow VW$ where $x_{i+1}^{\varepsilon_1}x_i^{\varepsilon_2}x_{i+1}^{\varepsilon_3}\longrightarrow V$ is one of (iv)–(vi) and W is a word in $x_1^{\pm1},\dots,x_{i-1}^{\pm1}$;
- $(viii)$ $x_j^{\varepsilon_1}x_i^{\varepsilon_2}\longrightarrow x_i^{\varepsilon_2}x_j^{\varepsilon_1},\ \{\varepsilon_1,\varepsilon_2\}\subset\{-1,1\},\ j-1>i\ge1;$

An *elementary K-reduction of a monomial* is $AUB\to AVB$ where $AUB\in F_\infty^+$ and $U\to V$ is a basic replacement. An *elementary K-reduction of an element of* kF_∞^+ is $\sum_{j=1}^m c_jW_j\to c_1W_1'+\sum_{j=2}^m c_jW_j$ where $c_1,\dots,c_m\in k$, $W_1,\dots W_m$ are *pairwise distinct* elements of F_∞^+, and $W_1\to W_1'$ is an elementary K-reduction of a monomial.

An element of F_∞^+ (resp. of kF_∞^+) is *K-reduced* if no K-reduction can be applied to it. We denote the set of such elements by $F_\infty^{\rm red}$ (resp. $kF_\infty^{\rm red}$). We set also $F_n^{\rm red}=F_n^+\cap F_\infty^{\rm red}$ and $kF_n^{\rm red}=kF_n^+\cap kF_\infty^{\rm red}$. Then $kF_\infty^{\rm red}$ is a submodule (not a subalgebra) of kF_∞^+. We denote $\pi:kF_\infty^+\to K_\infty$ and $\pi_n:kF_n^+\to K_n$ the morphisms of k-algebras induced by $x_i\mapsto s_i$.

We say that an element X of F_∞^+ is *almost K-reduced* if there exists a sequence $X=X_1\to X_2\to\dots\to X_m$ of elementary K-reductions of type $(viii)$ such that X_m is K-reduced.

For $X=x_{i_1}^{\varepsilon_1}\dots x_{i_m}^{\varepsilon_m}\in F_\infty^+$, $\varepsilon_j=\pm1$, we define the *weight* $\operatorname{wt}X=\sum_j i_j$ and the *auxiliary weight* $\operatorname{wt}'X=\sum_j ji_j$. It is clear that the set of all monomials of a given weight is finite. For $X\in kF_\infty^+$ we set $\operatorname{wt}X=\max_i\operatorname{wt}X_i$ if $X=\sum_i c_iX_i$ with $c_i\in k$ and $X_1,X_2,\dots$ pairwise distinct elements of F_∞^+.

The following statement is easy and we omit its proof.

Proposition 2.1.

a). If $X \to X'$ *is an elementary* K*-reduction, then* $\pi(X) = \pi(X')$ *and* $\operatorname{wt} X \geq \operatorname{wt} X'$. *If, moreover,* X *is a monomial, then* $\operatorname{wt} X = \operatorname{wt} X'$ *if and only if* $X \to X'$ *is a* K*-reduction of type* $(viii)$ *and in this case we have* $\operatorname{wt}'(X) < \operatorname{wt}'(X')$.

b). $\pi(F_\infty^{\rm red})$ *generates* K_∞ *as a* k*-module.*

c). $kF_\infty^{\rm red}$ *is a free* k*-module and* $F_\infty^{\rm red}$ *is a free base of* $kF_\infty^{\rm red}$.

d). $F_\infty^{\rm red}$ *is the set of all words* $X_1X_2\dots X_m$ *where* $X_\nu = x_{i_\nu}^{\pm1}x_{i_\nu-1}^{\pm1}\dots x_{j_\nu}^{\pm1}$, $i_\nu \geq j_\nu$ $(1 \leq \nu \leq m)$, $i_1 < \dots < i_m$, *and all the signs are mutually independent.*

e). (Proven in [3]) π_3 *is an isomorphism of* k*-modules* $kF_3^{\rm red}$ *and* K_3.

Remark 2.2. Let

$$S_{i,j} = \{x_i^{\pm1}x_{i-1}^{\pm1}\dots x_j^{\pm1}\} \qquad \text{and} \qquad S_i = \{1\} \cup S_{i,i} \cup S_{i,i-1} \cup \dots \cup S_{i,1}. \tag{5}$$

Then Part (d) of Proposition 2.1 can be stated as follows: each element of $F_n^{\rm red}$ can be represented in a unique way as a product $X_1X_2\dots X_{n-1}$ with $X_i \in S_i$. Since $|S_i| = 1 + 2 + \dots + 2^i = 2^{i+1} - 1$, we obtain $|F_n^{\rm red}| = \prod_{i=1}^n (2^i - 1)$, in particular,

$$|F_2^{\rm red}| = 3,\ \ |F_3^{\rm red}| = 3{\cdot}7 = 21,\ \ |F_4^{\rm red}| = 3{\cdot}7{\cdot}15 = 315,\ \ |F_5^{\rm red}| = 3{\cdot}7{\cdot}15{\cdot}31 = 9765.$$

Remark 2.3. In basic replacements (vii), it is enough to consider only words W belonging to S_{i-1} (see (5) for the definition of S_{i-1}).

We define a k-linear mapping $\mathbf{r} : kF_\infty^+ \to kF_\infty^{\rm red}$ as follows. For each $X \in F_\infty^+$ we fix an arbitrary sequence of elementary K-reductions $X = X_1 \to X_2 \to \dots \to X_m \in kF_\infty^{\rm red}$ and we set $\mathbf{r}(X) = X_m$. Then we extend the mapping to kF_∞^+ by linearity.

2.2. Markov trace. Let $A = k[u,v]$ and $AK_n = K_n(\alpha,\beta;A)$. Let $M = M(\alpha,\beta;k)$ be the quotient of AK_∞ by the relations (4) and let $t : AK_\infty \to M$ be the quotient map. We call t *the universal Markov trace* on K_∞ over k. It is indeed universal in the sense that any Markov trace on $K_\infty(\alpha,\beta;A)$ with values in an A-module M' is $f \circ t$ for some $f \in \operatorname{Hom}_A(M,M')$.

We define A-linear mappings $\tau_n : AF_n^+ \to AF_{n-1}^{\rm red}$ called *Markov reductions* as follows. By Proposition 2.1(d), we have $F_n^{\rm red} \subset F_{n-1}^{\rm red} \cup (F_{n-1}^{\rm red}x_{n-1}F_{n-1}^{\rm red}) \cup (F_{n-1}^{\rm red}x_{n-1}^{-1}F_{n-1}^{\rm red})$. So, we set $\tau_n(X) = X$, $\tau_n(Xx_{n-1}Y) = u\mathbf{r}(XY)$, and $\tau_n(Xx_{n-1}^{-1}Y) = v\mathbf{r}(XY)$ for $X,Y \in F_{n-1}^+$ and then we extend τ_n to $AF_n^{\rm red}$ by linearity and to AF_n^+ by setting $\tau(X) = \tau(\mathbf{r}(X))$. Finally, we define $\tau : F_\infty^{\rm red} \to AF_1^+ = A$ by setting $\tau(X) = \tau_2 \circ \dots \circ \tau_n(X)$ for $X \in AF_n^{\rm red}$.

By definition of t and τ, we have $t(\pi(X)) = t(\pi(\tau(X)))$, thus $M = t(K_\infty)$ is generated by $t(1)$. Let $I = I(\alpha,\beta;k)$ be the annihilator of M. Thus we have $M \cong A/I$.

2.3. Statement of the main result. Let $\operatorname{sh}^n : AF_\infty^+ \to AF_\infty^+$, $n \in \mathbb{Z}$, be the A-algebra endomorphism (the *n-shift*) induced by

$$\operatorname{sh}^n x_i = \begin{cases} x_{i+n}, & i+n > 0, \\ 0, & i+n \leq 0. \end{cases}$$

We set $\mathrm{sh} = \mathrm{sh}^1$.

For $X \in F_5^+$, we define $\rho_X \in \mathrm{End}_A(AF_4^{\mathrm{red}})$ by setting $\rho_X(Y) = \tau_5(X\,\mathrm{sh}\,Y)$. Let $J_4 = J_4(\alpha,\beta;k)$ be the minimal submodule of AF_4^{red} satisfying the following properties (recall that the sets $S_{i,j}$ and S_i are defined in (5)):

(J1) $\mathbf{r}(\mathbf{r}(X_3X_2)X_1) - \mathbf{r}(X_3\mathbf{r}(X_2X_1)) \in J_4$ for any $X_j \in \mathrm{sh}^{3-j}\,S_j\setminus\{1\}$, $j = 1,2,3$;

(J2) $\rho_X(J_4) \subset J_4$ for any $X \in S_4$.

In a similar way we define a module L. Let $N = AF_2^{\mathrm{red}} \otimes_A AF_2^{\mathrm{red}}$. We define A-linear mappings $\tau_N : N \to A$ and $\rho_\delta : N \to N$, $\delta = (\delta_1,\delta_2) \in \{-1,0,1\}$, by setting for any $Y = x_1^{\varepsilon_1} \otimes x_1^{\varepsilon_2}$ ($\varepsilon_1,\varepsilon_2 \in \{-1,0,1\}$)

$$\tau_N(Y) = \tau(x_1^{\varepsilon_1}x_1^{\varepsilon_2}), \qquad \rho_\delta(Y) = x_1^{\delta_1} \otimes \tau_3(x_2^{\varepsilon_1}x_1^{\delta_2}x_2^{\varepsilon_2})$$

and we define L as the minimal submodule of N satisfying the conditions:

(L1) $\tau_3(x_2^{\varepsilon_1}x_1^{\varepsilon_2}x_2^{\varepsilon_3}) \otimes x_1^{\varepsilon_4} - x_1^{\varepsilon_2} \otimes \tau_3(x_2^{\varepsilon_3}x_1^{\varepsilon_4}x_2^{\varepsilon_1}) \in L$ for any $\varepsilon_1,\varepsilon_3 \in \{-1,1\}$ and for any $\varepsilon_2,\varepsilon_4 \in \{-1,0,1\}$;

(L2) $\rho_\delta(L) \subset L$ for any $\delta \in \{-1,0,1\}^2$.

Theorem 2.4. (Main Theorem). *$I = \tau(J_4) + \tau_N(L)$.*

It is proven in §3 (see Corollary 3.2 for "$\supset$" and Corollary 3.7 for "$\subset$").

This result allows (at least theoretically) to compute I. Indeed, we start with the A-module $J_4^{(0)}$ generated by the elements in (J1) and compute its Gröbner base $G^{(0)}$. Set $\bar{G}^{(0)} = \bigcup_{X\in S_4}\rho_X(G^{(0)})$. Let $J_4^{(1)}$ be the A-module generated by $G^{(0)} \cup \bar{G}^{(0)}$ and $G^{(1)}$ its Gröbner base. Continuing this process, we construct an increasing sequence of submodules $J_4^{(0)} \subset J_4^{(1)} \subset \dots$. Since the subring of A generated by α,β,u,v is noetherian, there exists m_0 such that $J_4^{(m_0)} = J_4^{(m_0+1)} = \dots$ (m_0 is determined by the condition $G^{(m_0)} = G^{(m_0+1)}$). Then we have $J_4 = J_4^{(m_0)}$. The module L can be computed in a similar way as the limit of $L^{(0)} \subset L^{(1)} \subset \dots$ where $L^{(0)}$ is generated by the elements in (L1) and $L^{(i+1)} = \sum_\delta \rho_\delta(L^{(i)})$.

Performing in practice this computation for $\alpha = \beta = 0$, $k = \mathbb{Z}$ (the case considered in [3] and [2]) and in some other special cases, we obtain the following results. To compute Gröbner bases, we use `Singular 3-1-3` software.

Corollary 2.5. *a). $I(0,0;\mathbb{Z}) = (16,\ 4u^2+4v,\ 4v^2+4u,\ u^3+v^3+uv-3)$.*

b). $I(\alpha,0;\mathbb{F}_2[\alpha]) = (\alpha^4,\ \alpha^2(u^2+v),\ \alpha^2(v^2+u+\alpha),\ u^3+\alpha uv^2+v^3+uv+\alpha^2u+\alpha v+1)$;

c). $I(\alpha,0;\mathbb{F}_3[\alpha]) = \big(\alpha^3-1,\ (u^2-\alpha^2)(u^2-\alpha u-\alpha^2),\ v+u^2\big)$;

d). If $k = \mathbb{Q}$ or $k = \mathbb{F}_p$ for $p = 5,7,11,13,17,19$, then $I(\alpha,0;k[\alpha]) = (f_1,\dots,f_5)$ where

$$\begin{aligned}
f_1 &= \gamma_1\gamma_2\gamma_3, \qquad \gamma_1 = \alpha^3+8, \quad \gamma_2 = 2\alpha^3+1, \quad \gamma_3 = 3\alpha^3+8,\\
f_2 &= \gamma_1\gamma_3(u-\alpha),\\
f_3 &= \gamma_3(6u^3-3\alpha^2u+\alpha^3+2),\\
f_4 &= 336u^4-792\alpha u^3+12(15\alpha^3+106)\alpha^2u^2+6(141\alpha^3+544)u\\
&\quad -114\alpha^7-1405\alpha^4-3152\alpha,\\
f_5 &= 288v+336\alpha^2u^3+72(3\alpha^3+28)u^2-48(9\alpha^3+44)\alpha u-6a^8+53a^5+472a^2.
\end{aligned}$$

The reduced Gröbner base of I with respect to the lexicographic order ($v > u > \alpha$) is $\{f_1, \dots, f_5\}$ except the case $k = \mathbb{F}_7$ when it is $\{f_1, f_2, g, f_5\}$ with

$$g = 3\alpha^2 f_4 - f_3 + 2\alpha(u+\alpha)f_2 + 2f_1 = u^3 + 2\gamma_1\alpha u^2 + 2\gamma_3\alpha^2 u + 3\alpha^3(\alpha^3 - 1),$$

Remark 2.6. The Markov trace t over k defines a link invariant $P(L) = P_{\alpha,\beta,k}(L) = u^{(-n-e)/2}v^{(-n+e)/2}t(b) \in k[u^{\pm 1/2}, v^{\pm 1/2}]/I(\alpha,\beta;k)$ where b is a representation of a link L by a braid with n strings and e is the sum of exponents of b. It is shown in [2] that $P_{0,0;\mathbb{F}_2}$ and $P_{0,0;\mathbb{Z}/4\mathbb{Z}}$ depend on HOMFLY polynomial. A computation shows that $P_{\alpha,0;\mathbb{Q}[\alpha]}$ and $P_{\alpha,0;\mathbb{F}_3[\alpha]}$ detect the chirality of the knot 10_{71} (if one believes in Corollary 2.5, then $P_{\alpha,0;\mathbb{F}_3[\alpha]}(10_{71})$ can be computed by hand very fast). Thus, in general $P_{\alpha,\beta;k}$ is independent of both HOMFLY and Kauffman polynomials. Also we read in [1; §7.3]: "The 2-cabling of HOMFLY does not detect the chirality of 10_{71} (this result was kindly communicated by H. R. Morton)".

Remark 2.7. In the computed cases, u and v are not zero divisors in A/I. If they are for some $(\alpha,\beta;k)$, then (due to [4]) it is a priori possible that t could distinguish transversal links which are isotopic and have equal Bennequin numbers. In fact, this was my main motivation in 2004 to study in detail [1] and [3].

Remark 2.8. The main mistake in [3] (which was repeated also in [1]) is that the modules $J_4^{(0)}$ and $L^{(0)}$ were considered instead of J_4 and L.

3. Proof of Main Theorem

3.1. Easy part: $\tau(J_4) + \tau_N(L) \subset I$.

Let $J_4^{(0)} \subset J_4^{(1)} \subset \dots$ and $L^{(0)} \subset L^{(1)} \subset \dots$ be as defined in §2.3.

For $n \ge 4$ and $a \in AK_n$, we define $t_{n,a} \in \operatorname{Hom}_A(F_4^{\rm red}, A)$ by setting $t_{n,a}(X) = t(a\,\pi(\operatorname{sh}^{n-4} X))$. Similarly, for $n \ge 1$ and $a, b \in AK_n$, we define $t_{n,a,b} \in \operatorname{Hom}_A(N, A)$ by setting $t_{n,a,b}(X \otimes Y) = t\big(\pi(\operatorname{sh}^{n-1} X)\,a\,\pi(\operatorname{sh}^{n-1} Y)\,b\big)$.

Lemma 3.1.

a). $J_4 \subset \ker t_{n,a}$ for any $n \ge 4$ and any $a \in K_n$.

b). $L \subset \ker t_{n,a,b}$ for any $n \ge 1$ and any $a, b \in K_n$.

Proof. We prove by induction that a) $J_3^{[i]} \subset \ker t_{n,a}$ and b) $L^{[i]} \subset \ker t_{n,a,b}$. For $i = 0$, the statement is evident. Suppose that it is true for $i-1$ and let us prove it for i. Note that we have

$$t\big(a\,\pi(\operatorname{sh}^p \tau_{n-p}(X))\,b\big) = t\big(a\,\pi(\operatorname{sh}^p X)\,b\big) \qquad \text{for} \quad a, b \in K_{n-1},\ X \in AF_n^+ \tag{6}$$

a). It is enough to check that $\rho_X(Y) \in \ker t_{n,a}$ for any $Y \in J_4^{(i-1)}$, $X \in S_4$. $n \ge 4$, $a \in K_n$. Indeed,

$$\begin{aligned}
t_{n,a}(\rho_X(Y)) &= t\big(a\,\pi(\operatorname{sh}^{n-4} \rho_X(Y))\big) && \text{by definition of } t_{n,a}\\
&= t\big(a\,\pi(\operatorname{sh}^{n-4} \tau_5(X \operatorname{sh} Y))\big) && \text{by definition of } \rho_X\\
&= t\Big(a\,\pi\big((\operatorname{sh}^{n-4} X)(\operatorname{sh}^{n-3} Y)\big)\Big) && \text{by (6)}\\
&= t_{n+1,a'}(Y) && \text{for } a' = a\,\pi(\operatorname{sh}^{n-4} X) \in K_{n+1}\\
&= 0 && \text{by the induction hypothesis}
\end{aligned}$$

b). It is enough to check that $\rho_\delta(Y)\in\ker t_{n,a,b}$ for any $Y\in L^{(i-1)}$, $\delta=(\delta_1,\delta_2)\in\{-1,0,1\}^2$, $n\ge 1$, $a,b\in K_n$. Indeed, let $Y=\sum_j c_j x_1^{\varepsilon_1(j)}\otimes x_1^{\varepsilon_2(j)}$. Then

$$\begin{aligned}
t_{n,a,b}(\rho_\delta(Y)) &= t_{n,a,b}\Big(\sum c_j x_1^{\delta_1}\otimes\tau_3(x_2^{\varepsilon_1(j)}x_1^{\delta_2}x_2^{\varepsilon_2(j)})\Big) && \text{def. of } \rho_\delta\\
&= \sum c_j t\Big(\pi(\mathrm{sh}^{n-1}x_1^{\delta_1})\,a\,\pi\big(\mathrm{sh}^{n-1}\tau_3(x_2^{\varepsilon_1(j)}x_1^{\delta_2}x_2^{\varepsilon_2(j)})\big)\,b\Big) && \text{def. of } t_{n,a,b}\\
&= \sum c_j t\big(s_n^{\delta_1}\,a\,s_{n+1}^{\varepsilon_1(j)}s_n^{\delta_2}s_{n+1}^{\varepsilon_2(j)}\,b\big) && \text{by (6)}\\
&= \sum c_j t\big(s_{n+1}^{\varepsilon_1(j)}s_n^{\delta_2}s_{n+1}^{\varepsilon_2(j)}\,b\,s_n^{\delta_1}\,a\big) && t(xy)=t(yx)\\
&= \sum c_j t\Big(\pi\big(\mathrm{sh}^n x_1^{\varepsilon_1(j)}\big)\,s_n^{\delta_2}\pi\big(\mathrm{sh}^n x_1^{\varepsilon_2(j)}\big)\,b\,s_n^{\delta_1}\,a\Big)\\
&= t_{n+1,a',b'}(Y) && a'=s_n^{\delta_2},\ b'=bs_n^{\delta_1}a\\
&= 0 && \text{by induction hypothesis}
\end{aligned}$$

Corollary 3.2. *$\tau(J_4)+\tau_N(L)\subset I$.*

Proof. Indeed, by Lemma 3.1, we have $t(\tau(X))=t_{4,1}(X)=0$ for any $X\in J_4$ and $t(\tau_N(X))=t_{1,1,1}(X)=0$ for any $X\in L$. Thus $\tau(J_4)+\tau_N(L)\subset\ker(t|_A)=I$.

3.2. Difficult part: $I\subset\tau(J_4)+\tau_N(L)$.

Let, as above, $\bar R$ be the submodule of K_∞ generated by the elements (4). Set $R=\pi^{-1}(\bar R)$. Then we have $I=A\cap\bar R=A\cap R$. Let $\mathrm{wt}:AF_\infty^+\to\mathbb{Z}_{\ge0}$ be the weight function defined in §2.1. It defines a filtration on AF_∞^+, namely, $A=AF_{[0]}^+\subset AF_{[1]}^+\subset AF_{[2]}^+\subset\dots$ where $AF_{[w]}^+=\{X\in AF_\infty^+\mid \mathrm{wt}\,X\le w\}$.

We shall work with the following set of generators $\mathcal{R}=\mathcal{R}_T\cup\mathcal{R}_M\cup\mathcal{R}_N\cup\mathcal{R}_H$ of R as an A-module (we set here $u_+=u$, $u_-=v$):

$$\begin{aligned}
\mathcal{R}_T &= \{XY-YX\mid X,Y\in F_\infty^+\}, && \text{trace relations;}\\
\mathcal{R}_M &= \{x_n^{\pm1}X-u_\pm X\mid X\in F_n^+, n\ge1\}, && \text{Markov relations;}\\
\mathcal{R}_N &= \{UX-VX\mid X,U\in F_\infty^+, U\overset{(i)-(vi)}{\longrightarrow}V\}, && \text{nonhomogeous } K\text{-relations;}\\
\mathcal{R}_H &= \{UX-VX\mid X,U\in F_\infty^+, U\overset{(viii)}{\longrightarrow}V\}, && \text{homogeous } K\text{-relations.}
\end{aligned}$$

Let $\mathcal{R}_{[w]}=\mathcal{R}\cap AF_{[w]}^+$ and let $R_{[w]}$ be the A-submodule of R generated by $\mathcal{R}_{[w]}$ and let H be the submodule generated by $\mathcal{R}_T\cup\mathcal{R}_H$ (the elements of H are wt-homogeneous). Note, that by Proposition 2.1(a) we have

$$X\equiv\mathbf{r}(X)\equiv\tau_n(X)\equiv\tau(X)\mod R_{[\mathrm{wt}\,X]}\qquad\text{for } X\in AF_n^+. \tag{7}$$

In what follows, a notation like $X_1\equiv X_2\equiv X_3\equiv\dots$ means that $X_i\equiv X_{i+1}\mod R_{[\mathrm{wt}\,X_i]}$ and $\mathrm{wt}\,X_{i+1}\ge\mathrm{wt}\,X_i$, in particular, in this case we always have $X_1\equiv X_2\equiv X_3\equiv\dots\mod R_{[\mathrm{wt}\,X_1]}$.

Lemma 3.3. *Let $Z=X\,\mathrm{sh}^{n-4}Y$ for $X\in AF_\infty^+$, $Y\in J_4\cap\mathrm{sh}^{4-n}AF_\infty^+$, $n\ge1$. Then $Z\in R_{[w]}+\tau(J_4)$ where $w=\mathrm{wt}\,Z$.*

Proof. We denote $\mathrm{sh}^{n-4}Y$ by Y_n. If $X\in AF_m^+$ with $m>n$, then

$$XY_n\equiv\tau_m(X)Y_n\equiv\tau_{m-1}(\tau_m(X))Y_n\equiv\dots\equiv\tau_{n+1}\circ\dots\circ\tau_{m-1}\circ\tau_m(X)Y_n,$$

hence it is enough to prove the statement of the lemma under the additional hypothesis $X \in AF_n^+$. We prove it by induction.

If $n = 1$, then $X \in AF_1^+ = A$ and $Y \in J_4 \cap \operatorname{sh}^3 AF_\infty^+ = J_4 \cap A \subset \tau(J_4)$, so, the statement is trivial.

Suppose that $n \geq 2$, the statement is true for $n-1$, and let us prove it for n. By linearity, it is enough to consider the case when $X \in F_n^+$ and since $X \equiv \mathbf{r}(X)$, we may assume that $X \in F_n^{\mathrm{red}}$. Let $X = X_1 X_2 \dots X_{n-1}$, $X_i \in S_i$ (see Remark 2.2). We have $X_{n-1} = (\operatorname{sh}^{n-5} X_4')X_{n-5}''$ with $X_4' \in S_4 \cap \operatorname{sh}^{5-n} AF_\infty^+$ and $X_{n-5}'' \in S_{n-5}$ (we assume here that $S_i = \{1\}$ when $i \leq 0$). Note that Y_n may involve only $x_{n-4+i}^{\pm 1}$, $i = 1, 2, 3$, whereas X_{n-5}'' may involve only $x_i^{\pm 1}$, $i \leq n-5$, hence they commute. Therefore, denoting $X_1 \dots X_{n-2}$ by X_{n-2}''', we obtain

$$\begin{aligned} Z &= X_{n-2}'''(\operatorname{sh}^{n-5} X_4')X_{n-5}''(\operatorname{sh}^{n-4} Y) \equiv X_{n-2}'''(\operatorname{sh}^{n-5} X_4')(\operatorname{sh}^{n-4} Y)X_{n-5}'' \\ &\equiv X_{n-5}''X_{n-2}'''(\operatorname{sh}^{n-5} X_4')(\operatorname{sh}^{n-4} Y) = X_{n-5}''X_{n-2}''' \operatorname{sh}^{n-5}(X_4' \operatorname{sh} Y) \\ &\equiv X_{n-5}''X_{n-2}''' \operatorname{sh}^{n-5}\big(\tau_5(X_4' \operatorname{sh} Y)\big) = X' \operatorname{sh}^{n-5} Y' \end{aligned}$$

where $X' = X_{n-5}''X_{n-2}''' \in AF_{n-1}^+$ and $Y' = \tau_5(X_4' \operatorname{sh} Y) = \rho_{X_4'}(Y) \in J_4$.

To complete the proof, it remains to check that $Y' \in \operatorname{sh}^{5-n} AF_\infty^+$. Indeed, we have $X_4' \in \operatorname{sh}^{5-n} AF_\infty^+$, $Y \in \operatorname{sh}^{4-n} AF_\infty^+$, hence $\operatorname{sh} Y \in \operatorname{sh}^{5-n} AF_\infty^+$ and we obtain $X_4' \operatorname{sh} Y \in \operatorname{sh}^{5-n} AF_\infty^+$ whence $Y' = \tau_5(X_4' \operatorname{sh} Y) \in \operatorname{sh}^{5-n} AF_\infty^+$. $\square$

The next lemma is similar. For $n \geq 1$ and $X_1, X_2 \in AF_n^+$ we define $\varphi_{n,X_1,X_2} \in \operatorname{Hom}_A(N, AF_{n+1}^+)$ by setting $\varphi_{n,X_1,X_2}(Y_1 \otimes Y_2) = X_1(\operatorname{sh}^{n-1} Y_1)X_2(\operatorname{sh}^{n-1} Y_2)$.

Lemma 3.4. *Let $Z = \varphi_{n,X_1,X_2}(Y)$ for $n \geq 1$, $X_1, X_2 \in AF_n^+$, $Y \in L$. Then $Z \in R_{[w]} + \tau_N(L)$ where $w = \operatorname{wt} Z$.*

Proof. It is enough to consider the case when $X_1, X_2 \in F_n^{\mathrm{red}}$. Then there exist $X_i', X_i'' \in F_{n-1}^{\mathrm{red}}$ and $\delta_i \in \{-1, 0, 1\}$ such that $X_i = X_i' x_{n-1}^{\delta_i} X_i''$ $(i = 1, 2)$. Let

$$Y = \sum_j c_j x_1^{\varepsilon_1(j)} \otimes x_1^{\varepsilon_2(j)}. \tag{8}$$

Then we have

$$\begin{aligned} Z &= \sum c_j\, X_1 x_n^{\varepsilon_1(j)} X_2 x_n^{\varepsilon_2(j)} = \sum c_j\, X_1' x_{n-1}^{\delta_1} X_1'' x_n^{\varepsilon_1(j)} X_2' x_{n-1}^{\delta_2} X_2'' x_n^{\varepsilon_2(j)} \\ &\equiv \sum c_j\, X_2'' X_1' x_{n-1}^{\delta_1} X_1'' X_2' x_n^{\varepsilon_1(j)} x_{n-1}^{\delta_2} x_n^{\varepsilon_2(j)} \\ &= \sum c_j\, X_2'' X_1' x_{n-1}^{\delta_1} X_1'' X_2' \operatorname{sh}^{n-2}\big(x_2^{\varepsilon_1(j)} x_1^{\delta_2} x_2^{\varepsilon_2(j)}\big) \\ &\equiv \sum c_j\, X_2'' X_1' x_{n-1}^{\delta_1} X_1'' X_2' \operatorname{sh}^{n-2} \tau_3\big(x_2^{\varepsilon_1(j)} x_1^{\delta_2} x_2^{\varepsilon_2(j)}\big) = \varphi_{n-1,\bar X_1,\bar X_2}(\bar Y) \end{aligned}$$

where $\bar X_1 = X_2'' X_1'$, $\bar X_2 = X_1'' X_2'$, $\bar Y = \rho_\delta(Y)$. So, we have $Z \equiv \bar Z = \varphi_{n-1,\bar X_1,\bar X_2}(\bar Y)$ where $\bar X_1, \bar X_2 \in AF_{n-1}^+$, $\bar Y \in L$.

Thus, by induction we reduce the problem to the case $n = 1$. In this case we have $X_1, X_2 \in AF_1^+ = A$, hence, for Y as in (8), we have $Z = \varphi_{1,X_1,X_2}(Y) = \sum c_j x_1^{\varepsilon_1(j)} x_1^{\varepsilon_2(j)}$, hence $Z \equiv \tau_2(Z) = \tau_N(Y) \in \tau_N(L)$. $\square$

The next statement is the Pentagon Lemma from [3] adapted for our setting.

Lemma 3.5 (Pentagon Lemma). *Let $Z_2, Z_2 \in \mathcal{R}_N \cup \mathcal{R}_M$ be such that $Z_1 - Z_2 \in H + AF^+_{[w-1]}$ where $w = \operatorname{wt} Z_1 = \operatorname{wt} Z_2$. Then $Z_1 - Z_2 \in R_{[w-1]} + \tau(J_4) + \tau_N(L)$.*

Proof. Let $X_i \in F^+_\infty$ be the leading monomial of Z_i, $i = 1, 2$, i. e. $\operatorname{wt} X_i = \operatorname{wt} Z_i$ and $\operatorname{wt}(Z_i - X_i) \le w - 1$. Then $X_1 - X_2 \in H$, hence there exists a sequence of words $X_1 = W_1, \dots, W_m = X_2$ such that W_{i+1} is obtained from W_i either by a cyclic permutation or by exchanging two consecutive commuting letters. By definition of $\mathcal{R}_M$ and $\mathcal{R}_N$ we have $X_i = U_i X'_i$ and $Z_i = (U_i - V_i)X'_i$, $i = 1, 2$, where $U_i \to V_i$ is an elementary K-reduction of types (i)–(vi) if $Z_i \in \mathcal{R}_N$ and $U_i = x_n^{\pm 1}$, $V_i = u_\pm$ if $Z_i \in \mathcal{R}_M$.

Following [3] and [1], we represent such sequences $W_1, \dots, W_m$ by diagrams. A *diagram* is a union of mutually transversal curves in the cylinder $S^1 \times [0, 1]$, each curve being labeled by a letter $x_i^{\pm 1}$. In pictures we represent the cylinder by a rectangle whose vertical sides are supposed to be identified, so, the fibers of the projection $\mathrm{pr}_2 : S^1 \times [0, 1] \to [0, 1]$ we call *horizontal circles*. Each curve is *monotone*, i. e., its projection onto $[0, 1]$ is bijective. We say that a diagram is *admissible* if two curves labeled by $x_i^{\pm 1}$ and $x_j^{\pm 1}$ may cross only if $|i - j| \ge 2$. The words W_i (up to cyclic permutation) are read on horizontal circles.

We say that curves $\Gamma_1, \dots, \Gamma_m$ form a *bunch of parallel curves* or just a *bunch* if the curves are pairwise disjoint and all the crossings lying on $\bigcup \Gamma_i$ can be covered by disks whose intersections with the diagram are as in Figure 1.

In our case, the first and the last word of the sequence are X_1 and X_2. So, on the boundary of the cylinder we indicate (by a bold line) segments corresponding to U_1 and U_2. As in [3] and [1], a diagram is called *interactive* if it contains a curve which joins the bold segments. We also say that a curve is *active* if it meets at least one bold segment.

Step 1. *If all active curves form a single bunch, then $Z_1 - Z_2 \in H$.*

In this case we have $U_1 = U_2$. Let $V_1 = \mathbf{r}(U_1) = \sum c_j W_j$, $c_j \in A$, $W_j \in F^+_\infty$. For each j we consider the diagram obtained from the initial diagram by replacing the bunch of active curves by a bunch of curves labeled by W_j. If a curve crosses the bunch, its label commutes with all letters occurring in U_1, hence it commutes with all letters in W_j, i. e., the new diagram is admissible and it defines a congruence $W_j X'_1 \equiv W_j X'_2 \mod H$. Hence (recall that $X_1 - X_2 \in H$) we have $Z_1 - Z_2 = (X_1 - V_1 X'_1) - (X_2 - V_1 X'_2) \equiv V_1 X'_2 - V_1 X'_1 = \sum c_j W_j (X'_2 - X'_1) \equiv 0 \mod H$.

Step 2. *If $Z_1, Z_2 \in \mathcal{R}_M$, then $Z_1 - Z_2 \in H$.*

In this case there is only one active curve, so we apply the result of Step 1.

Step 3. *If the diagram is non-interactive, then $Z_1 - Z_2 \in H + R_{[w-1]}$.*

Due to Step 2, we may suppose that $Z_1 \in \mathcal{R}_N$. Then $U_1 = x_n^{\varepsilon_1} x_{n-1}^{\varepsilon_2} x_n^{\varepsilon_3}$ with $\varepsilon_1, \varepsilon_3 \in \{-1, 1\}$ and $\varepsilon_2 \in \{-1, 0, 1\}$.

Let A and B be the points on the lower bold segment that correspond to the letters $x_n^{\varepsilon_1}$ and $x_n^{\varepsilon_3}$ of U_1 and let AD and BC be the corresponding active curves (see Figure 2). They cut the cylinder into two halves. Let Q be that half whose side AB is contained in the bold segment (the quadrangle $ABCD$ in Figure 2).

Let Γ be the curve outcoming from U_1 and labeled by $x_{n-1}^{\varepsilon_2}$ if $\varepsilon_2 \ne 0$ or a generic monotone curve in Q if $\varepsilon_2 = 0$. Let us choose a horizontal circle (the dashed line in

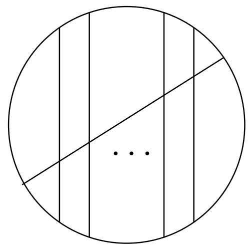

FIGURE 1

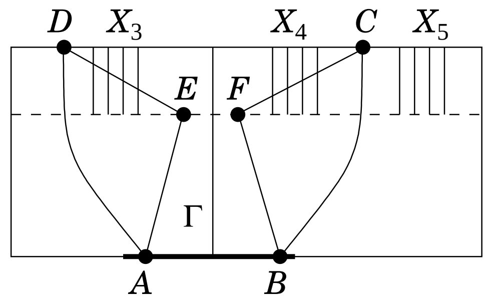

FIGURE 2

Figure 2) so that all crossings are below it and let us choose points E and F on it so that the segment EF which crosses Γ has no other intersections with the diagram. We may suppose that the intersection of the diagram with the upper half-cylinder (beyond EF) is a union of segments of vertical lines.

Let Δ be the diagram obtained by replacing AD and BC with monotone curves AED and BFC where ED, FC are straight line segments and AE, BF are curves in Q which are chosen so close to Γ that the active curves outcoming from U_1 form a bunch in the lower half-cylinder (below EF). The label of any curve $\Gamma' \neq \Gamma$ entering Q is not $x_i^{\pm 1}$ with $|n-i| \le 1$ (indeed, since Γ' attains the lower boundary outside the bold segment, it crosses AD or BC). Hence Δ is admissible.

Let Y be the word read from Δ along the circle EF. The bunch of active curves in the lower half-cylinder ensures that $Y = U_1 Y'$ and the result of Step 1 yields

$$Z_1 \equiv (U_1 - \mathbf{r}(U_1))Y' \mod H. \tag{9}$$

Now, let us study the upper part of Δ (beyond EF). All possible crossing in this part are on ED and FC. Hence, up to cyclic permutation, we have $X_2 = x_n^{\varepsilon_1} X_3 x_{n-1}^{\varepsilon_2} X_4 x_n^{\varepsilon_3} X_5$ and $Y = U_1 Y' = U_1 X_4 X_5 X_3$ (see Figure 2). Since the diagram is not interactive, U_2 is a subword of one of X_3, X_4, X_5, hence the active curves outcoming from U_2 form a bunch and $Y' = Y_1 U_2 Y_2$, i. e., $Y = U_1 Y_1 U_2 Y_2$, $Y_1, Y_2 \in F_\infty^+$. Hence, by Step 1, we have

$$Z_2 \equiv U_1 Y_1 (U_2 - \mathbf{r}(U_2)) Y_2 \mod H. \tag{10}$$

We have also

$$U_1 Y_1 \mathbf{r}(U_2) Y_2 \equiv \mathbf{r}(U_1) Y_1 \mathbf{r}(U_2) Y_2 \equiv \mathbf{r}(U_1) Y_1 U_2 Y_2 \mod R_{[w-1]}.$$

Combining this with (9) and (10), we obtain

$$Z_1 \equiv (U_1 - \mathbf{r}(U_1)) Y_1 U_2 Y_2 \equiv U_1 Y_1 (U_2 - \mathbf{r}(U_2)) Y_2 \equiv Z_2 \mod H + R_{[w-1]}.$$

Step 4. *Consider the open intervals obtained after removing of all endpoints of all active curves. If at least one of the words corresponding to these intervals is not almost K-reduced (see the definition in §2.1), then $Z_1 - Z_2 \in H + R_{[w-1]}$.*

Suppose that the word which is not almost K-reduced is a subword Y of X_2. Since it is disjoint from the active curves, we can write $X_2 = U_2 X_3 Y X_4$. The fact that Y is not almost K-reduced means that there exists a sequence $Y = Y_0 \to Y_1 \to$

$\dots \to Y'U_3Y''$ of exchanges of commuting letters such that U_3 is the left hand side of an elementary replacement of type (i)–(vi). The fact that Y does not meet any active curve means that the diagrams corresponding to the both chains

$$X_1 \to \dots \to X_2 = U_2X'Y_0X'' \to U_2X'Y_1X'' \to \dots \to U_2X'(Y'U_3Y'')X'',$$
$$X_2 = U_2X'Y_0X'' \to U_2X'Y_1X'' \to \dots \to U_2X'(Y'U_3Y'')X''$$

are non-interactive. By Step 3 this implies $Z_1 \equiv Z_3 \equiv Z_2 \mod H + R_{[w-1]}$ where $Z_3 = U_2X'Y'(U_3 - \mathbf{r}(U_3))Y''X''$.

Step 5. *If $Z_1 \in \mathcal{R}_N$ and the diagram is interactive, then the active curves are arranged up to symmetry either as in Figure 3.1 or as in Figure 3.2 where each of the dashed lines may or may not be included into the diagram, $n \geq 1$.*

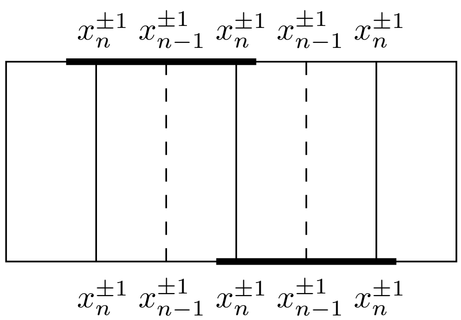

FIGURE 3.1

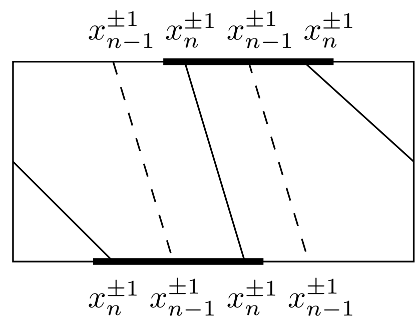

FIGURE 3.2

Indeed, we draw the curves adjacent to one of the bold segments and we try all the ways to complete the picture to an admissible diagram. Easy to see that only these two pictures can be obtained.

Step 6. *If the active curves are as in Figure 3.1, then $Z_1 - Z_2 \in H + R_{[w-1]} + \tau(J_4)$.*

Suppose that the active curves are as in Figure 3.1 (the bottom boundary corresponds to X_1). Then $U_1 = x_n^{\varepsilon_4}x_{n-1}^{\varepsilon_5}x_n^{\varepsilon_6}$, $U_2 = x_n^{\varepsilon_1}x_{n-1}^{\varepsilon_2}x_n^{\varepsilon_4}$, $X_2 = U_2Yx_{n-1}^{\varepsilon_5}X_3x_n^{\varepsilon_6}X_4$ where $\varepsilon_1, \varepsilon_4, \varepsilon_6 = \pm1$ and $\varepsilon_2, \varepsilon_5 \in \{-1, 0, 1\}$.

We begin as in Step 3. Let Q be the curvilinear quadrangle adjacent to the lower bold segment and bounded by the active x_n-curves outcoming from U_1. Let C_1 be a horizontal circle such that the part of the diagram beyond C_1 is a union of segments of vertical lines. Let Γ be either the (x_{n-1})-curve outcoming from U_1 (if it exists) or just a generic monotone curve in Q. Then we push the x_n-curves inside the domain Q from its boundary so that they form (together with Γ) a bunch below C_1, and so that the portions of the pushed curves beyond C_1 are segments of straight lines (see Figure 4.1).

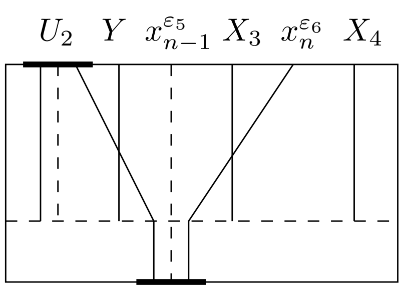

FIGURE 4.1

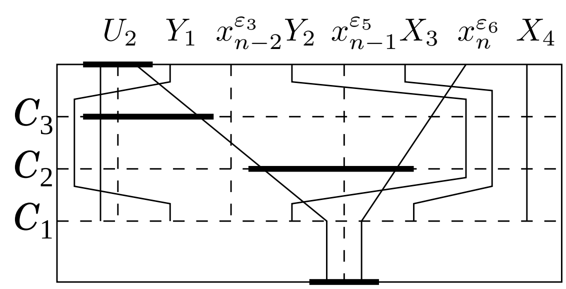

FIGURE 4.2

Since all curves outcoming from Y cross an x_n-curve, Y does not contain $x_i^{\pm 1}$ for $n-1 \le i \le n+1$. By Step 4, we may suppose that Y is almost K-reduced, hence Y has at most one occurrence of $x_{n-2}^{\pm 1}$, i. e., $Y = Y_1 x_{n-2}^{\varepsilon_3} Y_2$ with $\varepsilon_3 \in \{-1,0,1\}$ and Y_1, Y_2 do not contain $x_i^{\pm 1}$ for $n-2 \le i \le n+1$.

We choose horizontal circles C_2 and C_3 so that the intersection point of the x_n-curve and the (x_{n-2})-curve (if it exists) is between them and we modify the diagram as it is shown in Figure 4.2. If we apply the result of Step 1 to the part of the diagram which is below C_2 and to that which it beyond C_3, we obtain:

$$Z_1 \equiv Y_1 x_n^{\varepsilon_1} x_{n-1}^{\varepsilon_2} x_{n-2}^{\varepsilon_3} \big(U_1 - \mathbf{r}(U_1)\big) Y_2 X_3 X_4 \mod H \qquad (\text{below } C_2),$$
$$Z_2 \equiv Y_1 \big(U_2 - \mathbf{r}(U_2)\big) x_{n-2}^{\varepsilon_3} x_{n-1}^{\varepsilon_5} x_n^{\varepsilon_6} Y_2 X_3 X_4 \mod H. \qquad (\text{beyond } C_3).$$

Hence $Z_1 - Z_2 \equiv X' \operatorname{sh}^{n-3} Y' \mod H$ where $X' = Y_2 X_3 X_4 Y_1$ and

$$Y' = \mathbf{r}(x_3^{\varepsilon_1} x_2^{\varepsilon_2} x_3^{\varepsilon_4})\, x_1^{\varepsilon_3} x_2^{\varepsilon_5} x_3^{\varepsilon_6} - x_3^{\varepsilon_1} x_2^{\varepsilon_2} x_1^{\varepsilon_3}\, \mathbf{r}(x_3^{\varepsilon_4} x_2^{\varepsilon_5} x_3^{\varepsilon_6}).$$

If $\varepsilon_2 \ne 0$, then $\mathbf{r}(Y') \in J_4$ by Condition (J1) of the definition of J_4. Thus, using Lemma 3.3 and observing that $\operatorname{wt}(X' \operatorname{sh}^{n-3} Y') < w$, we obtain

$$Z_1 - Z_2 \equiv X' \operatorname{sh}^{n-3} Y' \equiv X' \operatorname{sh}^{n-3} \mathbf{r}(Y') \equiv 0 \mod H + R_{[w-1]} + \tau(J_4).$$

If $\varepsilon_2 = 0$, then $X' \operatorname{sh}^{n-3} Y' \equiv X' \operatorname{sh}^{n-3}(x_1^{\varepsilon_3} Y'') \mod H$ where $\mathbf{r}(Y'') \in J_4$, thus

$$Z_1 - Z_2 \equiv X' \operatorname{sh}^{n-3}(x_1^{\varepsilon_3} Y'') \equiv X' x_{n-2}^{\varepsilon_3} \operatorname{sh}^{n-3} \mathbf{r}(Y'') \equiv 0 \mod H + R_{[w-1]} + \tau(J_4).$$

Step 7. *If the active curves are as in Figure 3.2, then* $Z_1 - Z_2 \in H + R_{[w-1]} + \tau_N(L)$.

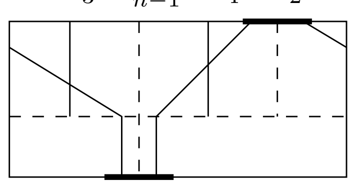

FIGURE 5

Again, as in the beginning of Steps 3 and 6, we transform the diagram as in Figure 5 and we obtain

$$Z_1 \equiv X_3(U_1 - \mathbf{r}(U_1)) X_4 x_{n-1}^{\varepsilon_4} \quad \text{and} \quad Z_2 \equiv X_3 x_{n-1}^{\varepsilon_2} X_4 (U_2 - \mathbf{r}(U_2)) \mod H$$

where $U_1 = x_n^{\varepsilon_1} x_{n-1}^{\varepsilon_2} x_n^{\varepsilon_3}$, $U_2 = x_n^{\varepsilon_3} x_{n-1}^{\varepsilon_4} x_n^{\varepsilon_1}$, $\varepsilon_1, \varepsilon_3 = \pm 1$, $\varepsilon_2, \varepsilon_4 \in \{-1,0,1\}$. Hence

$$Z_1 - Z_2 \equiv X_3 x_{n-1}^{\varepsilon_2} X_4 \mathbf{r}(U_2) - X_3 \mathbf{r}(U_1) X_4 x_{n-1}^{\varepsilon_4} \mod H \qquad (11)$$

Note that $x_i^{\pm 1}$ for $n-1 \le i \le n+1$ does not occur in X_1 and X_2. Indeed, if it does, then the diagram curve starting in it cannot attain the opposite side of the cylinder outside the bold segment because it cannot cross the x_n-curves. Thus,

$$X_3 x_{n-1}^{\varepsilon_2} X_4 \mathbf{r}(U_2) \equiv X_3' x_{n-1}^{\varepsilon_2} X_4' \mathbf{r}(U_2) X_5 \equiv X_3' x_{n-1}^{\varepsilon_2} X_4' \mathbf{r}(U_2) \tau(X_5) \mod H + R_{[w-1]}$$

where $X_3', X_4' \in F_{n-1}^+$ and $X_5 \in \mathrm{sh}^{n+1} F_\infty^+$ (the same for other term in (11)). So, replacing, if necessary, X_j by X_j' $(j = 3, 4)$, we may assume that $X_3, X_4 \in F_{n-1}^+$. Then we may pass from (11) to

$$\begin{aligned} Z_1 - Z_2 &\equiv X_3 x_{n-1}^{\varepsilon_2} X_4 \tau_{n+1}(U_2) - X_3 \tau_{n+1}(U_1) X_4 x_{n-1}^{\varepsilon_4} \mod H + R_{[w-1]} \\ &= \varphi_{n-1,X_3,X_4}(Y) \end{aligned}$$

where $Y = x_1^{\varepsilon_2} \otimes \tau_3(x_2^{\varepsilon_3} x_1^{\varepsilon_4} x_2^{\varepsilon_1}) - \tau_3(x_2^{\varepsilon_1} x_1^{\varepsilon_2} x_2^{\varepsilon_3}) \otimes x_1^{\varepsilon_4} \in L$. Thus if $n > 1$, then the result follows from Lemma 3.4. If $n = 1$, then $\varepsilon_2 = \varepsilon_4 = 0$, $X_3 = X_4 = 1$, and (11) yields $Z_1 - Z_2 \equiv 0$.

Step 8. We proved that $Z_1 - Z_2 \subset H + R_{[w-1]} + \tau(J_4) + \tau_N(L)$. Hence $Z_1 - Z_2 \subset R_{[w-1]} + \tau(J_4) + \tau_N(L)$ because H is homogeneous and $Z_1 - Z_2 \in AF_{[w-1]}^+$. The Pentagon Lemma is proven. ☐

Lemma 3.6. $R_{[w]} \cap AF_{[w-1]}^+ \subset R_{[w-1]} + \tau(J_4) + \tau_N(L)$

Proof. For $Z \in R_{[w]} \cap AF_{[w-1]}^+$, let $m = m(Z)$ be the minimal number such that $Z \equiv c_1 Z_1 + \dots + c_m Z_m \mod H + R_{[w-1]}$ with $c_i \in A$, $Z_i \in \mathcal{R}$. To prove that $Z \in R_{[w-1]} + \tau(J_4) + \tau_N(L)$, we use the induction by m. The statement is trivial for $m = 0$ because H is homogeneous.

Suppose that $m > 0$ and the statement is true for any smaller m. Let X_i be the leading monomial of Z_i, i. e., $X_i \in F_\infty^+$ and $\mathrm{wt}\, Z_i - X_i < w$. Then $\sum c_i X_i \equiv 0 \mod H$. The term $c_m X_m$ of this congruence cancels. Hence there exists $j < m$ such that $X_m - X_j \in H$. Then $c_m(Z_m - Z_j) \in R_{[w-1]} + \tau(J_4) + \tau_N(L)$ by Lemma 3.5 and $Z - c_m(Z_m - Z_j) \in R_{[w-1]} + \tau(J_4) + \tau_N(L)$ by the induction hypothesis. ☐

Corollary 3.7. $I \subset \tau(J_4) + \tau_N(L)$.

Proof. Let $I' = \tau(J_4) + \tau_N(L)$. Since $I = R \cap A$ and $R = \bigcup_w R_{[w]}$, it is enough to prove that $R_{[w]} \cap A \subset I'$ for any w. For $w = 0$ we have $R_{[0]} = 0$, hence $R_{[0]} \cap A \subset I'$. Suppose that $R_{[w-1]} \cap A \subset I'$. Let $Z \in R_{[w]} \cap A$. Since $R_{[w]} \cap A \subset R_{[w]} \cap AF_{[w-1]}^+$, by Lemma 3.6 we have $Z \in R_{[w-1]} + I'$, i. e., $Z = Z' + Z_0$ with $Z' \in R_{[w-1]}$, $Z_0 \in I'$. Since $Z' = Z - Z_0 \in A$, we have $Z' \in R_{[w-1]} \cap A$ and by the induction hypothesis we obtain $Z' \in I'$ whence $Z = Z' + Z_0 \in I'$. Thus $R_{[w]} \cap A \subset I'$. ☐

Main Theorem is proven (see Corollary 3.2 and Corollary 3.7).

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