# DERIVATIONS AND PROJECTIONS ON JORDAN TRIPLES
## An introduction to nonassociative algebra, continuous cohomology, and quantum functional analysis

Bernard Russo

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This paper is an elaborated version of the material presented by the author in a three hour minicourse at *V International Course of Mathematical Analysis in Andalusia*, at Almeria, Spain September 12-16, 2011. The author wishes to thank the organizing committees for the opportunity to present the course and for their hospitality. The author also thanks Antonio Peralta for suggesting my name to the scientific commitee.

The minicourse on which this paper is based had its genesis in a series of talks the author had given to undergraduates at Fullerton College in Fullerton, California and at the University of California, Irvine. Using only the product rule for differentiation as a starting point, these enthusiastic students were introduced to some aspects of the esoteric subject of non associative algebra, including triple systems as well as algebras. Slides of these talks as well as other related material, including this paper, can be found at the author's website (www.math.uci.edu/∼brusso). Conversely, these undergraduate talks were motivated by the author's past and recent joint works on derivations of Jordan triples ([91],[92],[156]), which are among the many results discussed here.

Parts I and II of this paper were covered in the three lectures making up the minicourse. For lack of time, Part III of this paper was not covered in the lectures, although there was some implicit reference to it in part I.

Part I is devoted to an exposition of the properties of derivations on various algebras and triple systems in finite and infinite dimensions, the primary questions addressed being whether the derivation is automatically continuous and to what extent it is an inner derivation. One section in Part I is devoted to the subject of contractive projection, which plays an important role in the structure theory of Jordan triples and in Part III.

Part II discusses cohomology theory of algebras and triple systems, in both finite and infinite dimensions. Although the cohomology of associative and Lie algebras is substantially developed, in both finite and infinite dimensions ([40],[76],[123]; see [95] for a review of [40]), the same could not be said for Jordan algebras. Moreover, the cohomology of triple systems has a rather sparse literature which is essentially non-existent in infinite dimensions. Thus, one of the goals of this paper is to encourage the study of continuous cohomology of some Banach triple systems. Occasionally, an idea for a research project is mentioned (see subsection 8.2). Readers are invited to contact the author, at brusso@uci.edu, if they share this goal.

Part III discusses three topics, two very recent, which involve the interplay between Jordan theory and operator space theory (quantum functional analysis). The first one, a joint work of the author [145], discusses the structure theory of contractively complemented Hilbertian operator spaces, and is instrumental to the third topic, which is concerned with some recent work on enveloping TROs and K-theory for JB*-triples [35],[36],[30]. The second topic presents some very recent joint work by the author concerning quantum operator algebras [149].

A few proofs have been included but for the most part, theorems and other results are just stated and a reference for the proof and in some cases for a definition, is provided. Brief biographical information on some mathematicians, obtained primarily from Wikipedia, is provided in footnotes.

As part of the undergraduate lectures mentioned above, a set of exercises was developed, proceeding from elementary verifications to nontrivial theorems in the literature, with appropriate references. These occur at the end of sections 1 and 3.

Since this paper is meant to introduce a large number of topics, many of which are purely algebraical in nature, and others which the author has not studied yet but fully intends to study, he has taken the liberty occasionally to rely on Mathematical Reviews, as well as authors' introductions for some of the summaries.

# Contents

# Part I
# Derivations

## 1 Algebras

Let $\mathcal{C}$ denote the algebra of continuous complex valued functions on a locally compact Hausdorff space.

**Definition 1.1** *A derivation on $\mathcal{C}$ is a linear mapping $\delta : \mathcal{C} \to \mathcal{C}$ satisfying the "product" rule:*

$$\delta(fg) = \delta(f)g + f\delta(g).$$

**Theorem 1.2** *There are no (non-zero) derivations on $\mathcal{C}$. In other words, every derivation of $\mathcal{C}$ is identically zero.*

The proof of this theorem can be broken up into two parts of independent interest. The first theorem, due to Singer[1] and Wermer[2], is valid for commutative semisimple Banach algebras. The second one is due to Sakai[3] and is valid for all $C^*$-algebras.

**Theorem 1.3 (Singer and Wermer 1955 [185])** *Every continuous derivation on $\mathcal{C}$ is zero.*

**Theorem 1.4 (Sakai 1960 [175])** *Every derivation on $\mathcal{C}$ is continuous.*

### 1.1 Derivations on finite dimensional algebras

#### 1.1.1 Matrix multiplication—Associative algebras

Let $M_n(\mathbf{C})$ denote the algebra of all $n$ by $n$ complex matrices, or more generally any finite dimensional semisimple associative algebra.

**Definition 1.5** *A derivation on $M_n(\mathbf{C})$ with respect to matrix multiplication is a linear mapping $\delta$ which satisfied the product rule*

$$\delta(AB) = \delta(A)B + A\delta(B).$$

**Proposition 1.6** *Fix a matrix $A$ in $M_n(\mathbf{C})$ and define*

$$\delta_A(X) = AX - XA.$$

*Then $\delta_A$ is a derivation with respect to matrix multiplication.*

[1] **Isadore Singer** (b. 1924). Isadore Manuel Singer is an Institute Professor in the Department of Mathematics at the Massachusetts Institute of Technology. He is noted for his work with Michael Atiyah in 1962, which paved the way for new interactions between pure mathematics and theoretical physics
[2] **John Wermer** (b. 1925)
[3] **Soichiro Sakai** (b. 1928)

A more general form of the following theorem is due to Hochschild[4] [93]. For an interesting summary of Hochschild's career see [178] and [177]. Theorem 1.7 is probably due to either Wedderburn[5] or Noether[6]. The proof we present here is due to Jacobson [100].

**Theorem 1.7** *Every derivation on $M_n(\mathbf{C})$ with respect to matrix multiplication is of the form $\delta_A$ for some $A$ in $M_n(\mathbf{C})$.*

*Proof.* If $\delta$ is a derivation, consider the two representations of $M_n(\mathbf{C})$

$$z \mapsto \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \text{ and } z \mapsto \begin{bmatrix} z & 0 \\ \delta(z) & z \end{bmatrix}.$$

The first is a direct sum of two copies of the identity representation; but so is the second, since

$$\begin{bmatrix} 0 & 0 \\ \delta(z) & z \end{bmatrix} \text{ is equivalent to } \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}.$$

So $\begin{bmatrix} z & 0 \\ \delta(z) & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$. Thus $az = za,\ bz = zb$ and

$$\delta(z)a = cz - zc \text{ and } \delta(z)b = dz - zd.$$

Since $a$ and $b$ are multiples of $I$ they can't both be zero.

### 1.1.2 Bracket multiplication—Lie algebras

We next consider a multiplication on matrices (or associative algebras) which leads to the theory of Lie algebras[7].

**Definition 1.8** *A <u>derivation</u> on $M_n(\mathbf{C})$ with respect to bracket multiplication*

$$[X, Y] = XY - YX$$

*is a linear mapping $\delta$ which satisfies the product rule*

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)].$$

**Proposition 1.9** *Fix a matrix $A$ in $M_n(\mathbf{C})$ and define*

$$\delta_A(X) = [A, X] = AX - XA.$$

*Then $\delta_A$ is a derivation with respect to bracket multiplication.*

An algebra $\mathcal{L}$ with multiplication $(x, y) \mapsto [x, y]$ is a Lie algebra if

$$[x, x] = 0$$

and

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

[4] **Gerhard Hochschild** (1915–2010) Gerhard Paul Hochschild was an American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory

[5] **Joseph Henry Maclagan Wedderburn** (1882–1948). Scottish mathematician, who taught at Princeton University for most of his career. A significant algebraist, he proved that a finite division algebra is a field, and part of the Artin–Wedderburn theorem on simple algebras. He also worked on group theory and matrix algebra

[6] **Amalie Emmy Noether** (1882–1935) Amalie Emmy Noether was an influential German mathematician known for her groundbreaking contributions to abstract algebra and theoretical physics. Described as the most important woman in the history of mathematics, she revolutionized the theories of rings, fields, and algebras. In physics, Noether's theorem explains the fundamental connection between symmetry and conservation laws

[7] **Sophus Lie** (1842–1899) Marius Sophus Lie was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations

Left multiplication in a Lie algebra is denoted by $\mathrm{ad}(x)$: $\mathrm{ad}(x)(y) = [x, y]$. An associative algebra $A$ becomes a Lie algebra $A^-$ under the product, $[x, y] = xy - yx$.

The first axiom implies that $[x, y] = -[y, x]$ and the second (called the *Jacobi identity*) implies that $x \mapsto \mathrm{ad}x$ is a homomorphism of $\mathcal{L}$ into the Lie algebra $(\mathrm{End}\,\mathcal{L})^-$, that is, $\mathrm{ad}\,[x, y] = [\mathrm{ad}\,x, \mathrm{ad}\,y]$.

Assuming that $\mathcal{L}$ is finite dimensional, the Killing form is defined by $\lambda(x, y) = \mathrm{tr}\,\mathrm{ad}(x)\mathrm{ad}(y)$.

**Theorem 1.10 (CARTAN criterion [140])** *A finite dimensional Lie algebra $\mathcal{L}$ over a field of characteristic 0 is semisimple if and only if the Killing form is nondegenerate.*

The proof is not given in Meyberg's notes [140], but can be found in many places, for example [104].

A linear map $D$ is a derivation if $D \cdot \mathrm{ad}(x) = \mathrm{ad}(Dx) + \mathrm{ad}(x) \cdot D$. Each $\mathrm{ad}(x)$ is a derivation, called an inner derivation. Let $\Theta(\mathcal{L})$ be the set of all derivations on $\mathcal{L}$.

A more general form of the following theorem is due to Hochschild [93]. Theorem 1.11 has been attributed to Zassenhaus[8] in [140], and to E. Cartan[9] in [101]. We have taken the proof from [140, p.42].

**Theorem 1.11** *If the finite dimensional Lie algebra $\mathcal{L}$ over a field of characteristic 0 is semisimple (that is, its Killing form is nondegenerate), then every derivation is inner.*

*Proof.* Let $D$ be a derivation of $\mathcal{L}$. Since $x \mapsto \mathrm{tr}\,D \cdot \mathrm{ad}\,(x)$ is a linear form, there exists $d \in \mathcal{L}$ such that $\mathrm{tr}\,D \cdot \mathrm{ad}\,(x) = \lambda(d, x) = \mathrm{tr}\,\mathrm{ad}\,(d) \cdot \mathrm{ad}\,(x)$. Let $E$ be the derivation $E = D - \mathrm{ad}\,(d)$ so that

$$\mathrm{tr}\,E \cdot \mathrm{ad}\,(x) = 0. \tag{1}$$

Note next that

$$\begin{aligned} E \cdot [\mathrm{ad}\,(x), \mathrm{ad}\,(y)] &= E \cdot \mathrm{ad}\,(x) \cdot \mathrm{ad}\,(y) - E \cdot \mathrm{ad}\,(y) \cdot \mathrm{ad}\,(x) \\ &= (\mathrm{ad}\,(x) \cdot E + [E, \mathrm{ad}\,(x)]) \cdot \mathrm{ad}\,(y) - E \cdot \mathrm{ad}\,(y) \cdot \mathrm{ad}\,(x) \end{aligned}$$

so that

$$\begin{aligned} [E, \mathrm{ad}\,(x)] \cdot \mathrm{ad}\,(y) &= E \cdot [\mathrm{ad}\,(x), \mathrm{ad}\,(y)] - \mathrm{ad}\,(x) \cdot E \cdot \mathrm{ad}\,(y) + E \cdot \mathrm{ad}\,(y) \cdot \mathrm{ad}\,(x) \\ &= E \cdot [\mathrm{ad}\,(x), \mathrm{ad}\,(y)] + [E \cdot \mathrm{ad}\,(y), \mathrm{ad}\,(x)] \end{aligned}$$

and

$$\mathrm{tr}\,[E, \mathrm{ad}\,(x)] \cdot \mathrm{ad}\,(y) = \mathrm{tr}\,E \cdot [\mathrm{ad}\,(x), \mathrm{ad}\,(y)].$$

However, since $E$ is a derivation

$$\begin{aligned} [E, \mathrm{ad}\,(x)] \cdot \mathrm{ad}\,(y) &= E \cdot \mathrm{ad}\,(x) \cdot \mathrm{ad}\,(y) - \mathrm{ad}\,(x) \cdot E \cdot \mathrm{ad}\,(y) \\ &= (\mathrm{ad}\,(Ex) + \mathrm{ad}\,(x) \cdot E) \cdot \mathrm{ad}\,(y) - \mathrm{ad}\,(x) \cdot E \cdot \mathrm{ad}\,(y) \\ &= \mathrm{ad}\,(Ex) \cdot \mathrm{ad}\,(y). \end{aligned}$$

Thus

$$\begin{aligned} \lambda(Ex, y) &= \mathrm{tr}\,\mathrm{ad}\,(Ex) \cdot \mathrm{ad}\,(y) \\ &= \mathrm{tr}\,[E, \mathrm{ad}\,(x)] \cdot \mathrm{ad}\,(y) \\ &= \mathrm{tr}\,E \cdot [\mathrm{ad}\,(x), \mathrm{ad}\,(y)] = 0 \text{ by } (1)). \end{aligned}$$

Since $x$ and $y$ are arbitrary, $E = 0$ and so $D - \mathrm{ad}\,(d) = 0$.

[8] **Hans Zassenhaus** (1912–1991) Hans Julius Zassenhaus was a German mathematician, known for work in many parts of abstract algebra, and as a pioneer of computer algebra

[9] **Elie Cartan 1869–1951.** Elie Joseph Cartan was an influential French mathematician, who did fundamental work in the theory of Lie groups and their geometric applications. He also made significant contributions to mathematical physics, differential geometry, and group theory. He was the father of another influential mathematician, Henri Cartan.

### 1.1.3 Circle multiplication—Jordan algebras

We next consider a multiplication on matrices (or associative algebras) which leads to the theory of Jordan algebras[10]

**Definition 1.12** *A* <u>*derivation*</u> *on* $M_n(\mathbf{C})$ *with respect to circle multiplication*

$$X \circ Y = (XY + YX)/2$$

*is a linear mapping* $\delta$ *which satisfies the product rule*

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

**Proposition 1.13** *Fix a matrix* $A$ *in* $M_n(\mathbf{C})$ *and define*

$$\delta_A(X) = AX - XA.$$

*Then* $\delta_A$ *is a derivation with respect to circle multiplication.*

The first part of the following theorem of Sinclair[11] is valid for all semisimple Banach algebras (and continuous Jordan derivations).

**Theorem 1.14 (Sinclair 1970 [181])** *Every derivation on* $M_n(\mathbf{C})$ *with respect to circle multiplication is also a derivation with respect to matrix multiplication, and is therefore of the form* $\delta_A$ *for some* $A$ *in* $M_n(\mathbf{C})$.

According to the pioneering work of Jacobson[12] the above proposition and theorem need to be modified for the real subalgebra (with respect to circle multiplication) of Hermitian matrices.

**Proposition 1.15** *Fix two matrices* $A, B$ *in* $M_n(\mathbf{C})$ *and define*

$$\delta_{A,B}(X) = A \circ (B \circ X) - B \circ (A \circ X).$$

*Then* $\delta_{A,B}$ *is a derivation with respect to circle multiplication.*

**Theorem 1.16 (1949 Jacobson [101])** *Every derivation of a finite dimensional semisimple Jordan algebra* $J$ *(in particular* $M_n(\mathbf{C})$ *with circle multiplication) is a sum of derivations of the form* $\delta_{A,B}$ *for some* $A'$*s and* $B'$*s elements of* $J$.

*Proof.* First note that for any algebra, $D$ is a derivation if and only if $[R_a, D] = R_{Da}$, where $R_a$ denotes the operator of multiplication (on the right) by $a$. If you polarize the Jordan axiom $(a^2b)a = a^2(ba)$, you get $[R_a, [R_b, R_c]] = R_{A(b,a,c)}$ where $A(b, a, c) = (ba)c - b(ac)$ is the "associator". From the commutative law $ab = ba$, you get $A(b, a, c) = [R_b, R_c]a$ and so $[R_b, R_c]$ is a derivation, sums of which are called **inner**, forming an ideal in the Lie algebra of all derivations.

The **Lie multiplication algebra** $L$ of the Jordan algebra $A$ is the Lie algebra generated by the multiplication operators $R_a$. It is given by

$$L = \{R_a + \sum_i [R_{b_i}, R_{c_i}] : a, b_i, c_i \in A\}$$

so that $L$ is the sum of a Lie triple system and the <u>ideal</u> of inner derivations.

Now let $D$ be a derivation of a semisimple finite dimensional unital Jordan algebra $A$. Then $\tilde{D} : X \mapsto [X, D]$ is a derivation of $L$. It is well known to algebraists that $L = L' + C$ where $L'$ (the derived algebra $[L, L]$) is semisimple and $C$ is the center of $L$. Also $\tilde{D}$ maps $L'$ into itself and $C$ to zero.

[10] **Pascual Jordan** (1902–1980) Pascual Jordan was a German theoretical and mathematical physicist who made significant contributions to quantum mechanics and quantum field theory

[11] **Alan M. Sinclair** (retired)

[12] **Nathan Jacobson** (1910–1999) Nathan Jacobson was an American mathematician who was recognized as one of the leading algebraists of his generation, and he was also famous for writing more than a dozen standard monographs

By Theorem 1.11, $\tilde{D}$ is an inner derivation of $L'$ and hence also of $L$, so there exists $U \in L$ such that $[X, D] = [X, U]$ for all $X \in L$ and in particular $[R_a, D] = [R_a, U]$. Then $Da = R_{Da}1 = [R_a, D]1 = [R_a, U]1 = (R_aU - UR_a)1 = a \cdot U1 - Ua$ so that $D = R_{U1} - U \in L$. Thus, $D = R_a + \sum [R_{b_i}, R_{c_i}]$ and so

$$0 = D1 = a + 0 = a.$$

We summarize the previous three theorems, insofar as they concern $M_n(\mathbf{C})$, in the following table. Since $M_n(\mathbf{C})$ is not a semisimple Lie algebra, its derivations must be taken modulo the center (See 2.3).

**Table 1**

$M_n(\mathbf{C})$ (SEMISIMPLE ALGEBRAS)

| matrix | bracket | circle |
|---|---|---|
| $ab = a \times b$ | $[a, b] = ab - ba$ | $a \circ b = ab + ba$ |
| Th. 1.7 | Th. 1.11 | Th. 1.14 |
| $\delta_a(x)$ $=$ $ax - xa$ | $\delta_a(x)$ $=$ $ax - xa$ | $\delta_a(x)$ $=$ $ax - xa$ |

The following table shows some properties of the three kinds of multiplication considered and thus provides the axioms for various kinds of algebras which we will consider. These two tables also provide a model for the discussion of various kinds of triple systems which will be considered later.

**Table 2**

ALGEBRAS

**commutative algebras**

$ab = ba$

**associative algebras**

$a(bc) = (ab)c$

**Lie algebras**

$a^2 = 0$

$(ab)c + (bc)a + (ca)b = 0$

**Jordan algebras**

$ab = ba$

$a(a^2b) = a^2(ab)$

## 1.2 Derivations on $C^*$-algebras

The algebra $M_n(\mathbf{C})$, with matrix multiplication, as well as the algebra $\mathcal{C}$, with ordinary multiplication, are examples of $C^*$-algebras (finite dimensional and commutative respectively). The following theorem, which is due to Kadison[13] and Sakai, and was preceded by some preliminary results of Kaplansky[14] thus explains Theorems 1.2 and 1.7.

**Theorem 1.17 (Sakai, Kadison 1966 [176],[113])** *Every derivation of a $C^*$-algebra is of the form $x \mapsto ax - xa$ for some $a$ in the weak closure of the $C^*$-algebra.*

[13]**Richard Kadison** (b. 1925) Richard V. Kadison is an American mathematician known for his contributions to the study of operator algebras

[14] **Irving Kaplansky** (1917–2006) Kaplansky made major contributions to group theory, ring theory, the theory of operator algebras and field theory

### 1.3 Exercises (Gradus ad Parnassum)—Algebras

1. Prove the analog of Proposition 1.6 for associative algebras: Fix an element $a$ in an associative algebra $A$ and define
$$\delta_a(x) = [a, x] = ax - xa.$$
Then $\delta_a$ is a derivation with respect to the product of $A$.

2. Prove the analog of Proposition 1.9 for associative algebras: Fix an element $a$ in an associative algebra $A$ and define
$$\delta_a(x) = [a, x] = ax - xa.$$
Then $\delta_a$ is a derivation with respect to bracket multiplication $[x, y] = xy - yx$.

3. Prove the analog of Proposition 1.13 for associative algebras: Fix an element $a$ in an associative algebra $A$ and define
$$\delta_a(x) = [a, x] = ax - xa.$$
Then $\delta_a$ is a derivation with respect to circle multiplication $x \circ y = xy + yx$.

4. Let $a$ and $b$ be two fixed elements in an associative algebra $A$. Show that the linear mapping
$$\delta_{a,b}(x) = a \circ (b \circ x) - b \circ (a \circ x)$$
is a derivation of $A$ with respect to circle multiplication $x \circ y = xy + yx$.
(cf. the paragraph following Theorem 1.14)

5. Show that an associative algebra is a Lie algebra with respect to bracket multiplication.

6. Show that an associative algebra is a Jordan algebra with respect to circle multiplication.

7. Let us write $\delta_{a,b}$ for the linear mapping $\delta_{a,b}(x) = a(bx) - b(ax)$ in a Jordan algebra. Show that $\delta_{a,b}$ is a derivation of the Jordan algebra by following the outline below. (cf. problem 4 above.)

   (a) In the Jordan algebra axiom
$$u(u^2v) = u^2(uv),$$
replace $u$ by $u + w$ to obtain the equation
$$2u((uw)v) + w(u^2v) = 2(uw)(uv) + u^2(wv) \tag{2}$$

   (b) In (1), interchange $v$ and $w$ and subtract the resulting equation from (1) to obtain the equation
$$2u(\delta_{v,w}(u)) = \delta_{v,w}(u^2). \tag{3}$$

   (c) In (2), replace $u$ by $x + y$ to obtain the equation
$$\delta_{v,w}(xy) = y\delta_{v,w}(x) + x\delta_{v,w}(y),$$
which is the desired result.

# 2 Automatic Continuity

The automatic continuity of various algebraic mappings plays important roles in the general theory of Banach algebras and in particular in operator algebra theory. In this section, to set the stage for later work, we highlight some of the theory using [54] as our source, to which the reader can refer for details and references.

## 2.1 Uniqueness of Norm and automatic continuity. A short survey

A Banach algebra $(A, \|\cdot\|)$ has a unique norm if each norm with respect to which $A$ is a normed algebra is equivalent to the given norm. In this case, the topological and algebraic structures of $A$ are intimately linked.

**Proposition 2.1** *A commutative semisimple Banach algebra has a unique norm.*

Proposition 2.1 follows from

**Theorem 2.2 (Silov 1947)** *Any homomorphism of a Banach algebra $A$ to a commutative semisimple Banach algebra is continuous.*

The following two theorems are certainly milestone results in the theory of automatic continuity in Banach algebras. The second one was proved before Banach algebras were invented.

**Theorem 2.3 (Johnson 1967)** *Each semisimple Banach algebra has a unique norm.*

**Theorem 2.4 (Eidelheit 1940)** *$B(E)$ has a unique norm for any Banach space $E$,*

Johnson's[15] 1967 milestone is a consequence of the following theorem, which is called "the seed from which automatic continuity theory has grown" and has lead to several basic unsolved problems

**Theorem 2.5 (Johnson 1967)** *A homomorphism of a Banach algebra $A$* **ONTO** *a semisimple Banach algebra $B$ is continuous.*

**Question 2.6** *In Johnson's theorem, can you replace* **onto** *by* **dense range***? What if $A$ is a $C^*$-algebra? What if $A$ and $B$ are both $C^*$-algebras?*

In this connection, we have the following proposition.

**Proposition 2.7 (Esterle 1980)** *Any homomorphism from $C(\Omega)$ onto a Banach algebra is continuous.*

Theorem 2.3 has been extended to semisimple Jordan algebras (Rodriquez-Palacios 1985) and more recently to arbitrary non associative semisimple Banach algebras in [136].

The next two propositions show some relations between automatic continuity and uniqueness of norm.

**Proposition 2.8** *If $(A, \|\cdot\|)$ is a Banach algebra with the property that for any algebra norm $\|\cdot\|_1$ there is a constant $C$ such that $\|a\| \le C\|a\|_1$, then $A$ has a unique norm if and only if every homomorphism from $A$ to any Banach algebra is continuous.*

**Proposition 2.9** *Let $A$ be a commutative Banach algebra, $E$ a Banach $A$-module. Then $A \oplus E$ is a commutative algebra via $(a, x)(b, y) = (ab, a \cdot y + b \cdot x)$. If $D : A \to E$ is a derivation, define two norms $\|(a, x)\|_1 = \|a\| + \|x\|$ and $\|(a, x)\|_2 = \|a\| + \|Da - x\|$. These norms are equivalent if and only if $D$ is continuous.*

Some other examples of Banach algebras with a unique norm are $L^1(\mathbf{R}^+, \omega)$ (**Jewell and Sinclair 1976**) and all finite dimensional Banach algebras.

## 2.2 Cohomology of Banach algebras

The automatic continuity problem for derivations can be stated as follows: Under what conditions on a Banach algebra $A$ are all derivations from $A$ into some or all Banach $A$-modules automatically continuous? It is of interest to note that if all homomorphisms from a Banach algebra $A$ into any Banach algebra are continuous, then all derivations from $A$ into any Banach $A$-module $E$ are continuous. The converse is false however, as shown by Ringrose.

[15] **Barry Johnson** (1937–2002) cf. [182]

Why are derivations important? One answer is given by cohomology, which is introduced here and revisited in later sections as the principal purpose of this paper.

Let $M$ be a Banach algebra and $X$ a Banach $M$-module. For $n \geq 1$, let $L^n(M,X) =$ all <u>continuous</u> $n$-linear maps ($L^0(M,X) = X$). The coboundary operator is $\partial : L^n \to L^{n+1}$ (for $n \geq 1$), defined by

$$\partial\phi(a_1, \cdots, a_{n+1}) = a_1\phi(a_2, \cdots, a_{n+1}) + \sum(-1)^j\phi(a_1, \cdots, a_{j-1}, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1}\phi(a_1, \cdots, a_n)a_{n+1}$$

For $n = 0$, $\partial : X \to L(M,X)$ $\quad \partial x(a) = ax - xa$ so Im$\partial =$ the space of inner derivations. Since $\partial \circ \partial = 0$, Im$(\partial : L^{n-1} \to L^n) \subset \ker(\partial : L^n \to L^{n+1})$, $H^n(M,X) = \ker\partial/\mathrm{Im}\partial$ is a vector space. For $n = 1$, $\ker\partial = \{\phi : M \to X : a_1\phi(a_2) - \phi(a_1a_2) + \phi(a_1)a_2 = 0\} =$ the space of continuous derivations from $M$ to $X$ Thus,

$$H^1(M,X) = \frac{\text{derivations from } M \text{ to } X}{\text{inner derivations from } M \text{ to } X}$$

measures how close continuous derivations are to inner derivations.

Later we shall discuss what the spaces $H^2(M,X)$, $H^3(M,X)$,... measure. In anticipation of this, we present some miscellaneous remarks.

1. One of the earliest papers on continuous cohomology is the pioneering [116] in which it is shown that $H^1(C(\Omega), E) = H^2(C(\Omega), E) = 0$
2. "The major open question in the theory of derivations on C*-algebras" is whether $H^1(A, B(H)) = 0$ $(A \subset B(H))$
3. A derivation from $A$ into $B(H)$ is inner if and only if it is completely bounded. ([42, Theorem 3.1])
4. The origin of continuous cohomology is the memoir of Barry Johnson [107]

## 2.3 Lie derivations of operator algebras and Banach algebras

Later we shall discuss derivations on operator algebras and more generally the continuous cohomology of operator algebras. In this subsection, we record some results on derivations of the Lie algebra structure of operator algebras, which anticipates said discussion as well as introduces Lie triple structure, which is one of the main themes of this survey paper.

**Theorem 2.10 (Miers 1973 [141])** *If $M$ is a von Neumann algebra, $[M,M]$ the Lie algebra linearly generated by $\{[X,Y] = XY - YX : X, Y \in M\}$ and $L : [M,M] \to M$ a Lie derivation, i.e., $L$ is linear and $L[X,Y] = [LX,Y] + [X,LY]$, then $L$ has an extension $D : M \to M$ that is a derivation of the associative algebra.*

The proof involves matrix-like computations. Using Theorem 1.17 leads to

**Theorem 2.11 (Miers 1973 [141])** *If $L : M \to M$ is a Lie derivation, then $L = D + \lambda$, where $D$ is an associative derivation and $\lambda$ is a linear map into the center of $M$ vanishing on $[M,M]$.*

A Lie triple derivation of an associative algebra $M$ is a linear map $L : M \to M$ such that

$$L[[X,Y],Z] = [[L(X),Y],Z] + [[X,L(Y)],Z] + [[X,Y],L(Z)]$$

for all $X, Y, Z \in M$.

**Theorem 2.12 (Miers 1978 [142])** *If $M$ is a von Neumann algebra with no central Abelian summands and $L$ is a Lie triple derivation, then there exists an operator $A \in M$ such that $L(X) = [A,X] + \lambda(X)$ where $\lambda : M \to Z_M$ is a linear map which annihilates brackets of operators in $M$.*

Notice that no assumption of continuity was made in these theorems. Continuous Lie derivations of associative Banach algebras have also been studied outside the operator algebra context, as evidenced in the following.

**Theorem 2.13 (Johnson 1996 [109])** *Every continuous Lie derivation of a symmetrically amenable Banach algebra A into a Banach A-module X is the sum of an associative derivation and a "trivial" derivation, that is, a linear map which vanishes on commutators and maps into the "center" of the module. The same holds if A is a $C^*$-algebra.*

As shown in [137] and [5], the continuity assumption can be dropped if $X = A$ and $A$ is a C*-algebra or a semisimple symmetrically amenable Banach algebra.

According to a review by Garth Dales [55]: "It remains an open question whether an analogous result (automatic continuity) for Lie derivations from A into a Banach A-bimodule holds when A is an arbitrary C*-algebra and when A is an arbitrary symmetrically amenable Banach algebra. It is also an interesting open question whether or not every Lie derivation on a semisimple Banach algebra to itself has this form."

As pointed out to the author by Martin Mathieu, a negative answer has been given, by Read [165], to the last question raised by Dales in his review.

# 3 Triple systems

In this section, all algebras and triple systems will generally be finite dimensional. The first subsection parallels subsection 1.1. Infinite dimensional cases and references for the proofs of the theorems in that subsection will be given in section 5.

## 3.1 Derivations on finite dimensional triple systems

### 3.1.1 Triple matrix multiplication

Let $M_{m,n}(\mathbf{C})$ denote the linear space of rectangular $m$ by $n$ complex matrices and note that it is close under the operation $A, B, C \mapsto AB^*C$, which we will call triple matrix multiplication.

**Definition 3.1** *A <u>derivation</u> on $M_{m,n}(\mathbf{C})$ with respect to triple matrix multiplication is a linear mapping $\delta$ which satisfied the triple product rule $\delta(AB^*C) = \delta(A)B^*C + A\delta(B)^*C + AB^*\delta(C)$*

**Proposition 3.2** *For two matrices, $A \in M_m(\mathbf{C}), B \in M_n(\mathbf{C})$, with $A^* = -A, B^* = -B$, define $\delta_{A,B}(X) = AX + XB$. Then $\delta_{A,B}$ is a derivation with respect to triple matrix multiplication.*

**Theorem 3.3** *Every derivation on $M_{m,n}(\mathbf{C})$ with respect to triple matrix multiplication is a sum of derivations of the form $\delta_{A,B}$.*

The proof of Theorem 3.3 is obtained by applying the result of Theorem 3.10 below to the symmetrized product $AB^*C + CB^*A$.

**Remark 3.4** *These results hold true and are of interest for the case $m = n$.*

### 3.1.2 Triple bracket multiplication

Let's go back for a moment to square matrices and the bracket multiplication. Motivated by the last remark, we define the triple bracket multiplication to be $X, Y, Z \mapsto [[X, Y], Z]$.

**Definition 3.5** *A <u>derivation</u> on $M_n(\mathbf{C})$ with respect to triple bracket multiplication is a linear mapping $\delta$ which satisfies the triple product rule $\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$*

**Proposition 3.6** *Fix two matrices $A, B$ in $M_n(\mathbf{C})$ and define $\delta_{A,B}(X) = [[A, B], X]$. Then $\delta_{A,B}$ is a derivation with respect to triple bracket multiplication.*

$M_n(\mathbf{C})$ is not a semisimple Lie algebra or semisimple Lie triple system, so a modification needs to be made when considering Lie derivations or Lie triple derivations on it, as in Theorems 2.11 and 2.12. The proof of the following theorem is taken from [140, Chapter 6].

**Theorem 3.7** *Every derivation of a finite dimensional semisimple Lie triple system $F$ is a sum of derivations of the form $\delta_{A,B}$, for some $A$'s and $B$'s in the triple system. These derivations are called inner derivations and their set is denoted* $\mathrm{Inder}\, F$.

*Proof.* Let $F$ be a finite dimensional semisimple Lie triple system (over a field of characteristic 0) and suppose that $D$ is a derivation of $F$. Let $L$ be the Lie algebra $(\mathrm{Inder}\, F) \oplus F$ with product

$$[(H_1, x_1), (H_2, x_2)] = ([H_1, H_2] + L(x_1, x_2), H_1 x_2 - H_2 x_1).$$

A derivation of $L$ is defined by $\delta(H \oplus a) = [D, H] \oplus Da$. Together with the definition of semisimple Lie triple system, it is proved in [140] that $F$ semisimple implies $L$ semisimple. Thus there exists $U = H_1 \oplus a_1 \in L$ such that $\delta(X) = [U, X]$ for all $X \in L$. Then $0 \oplus Da = \delta(0 \oplus a) = [H_1 + a_1, 0 \oplus a] = L(a_1, a) \oplus H_1 a$ so $L(a_1, a) = 0$ and $D = H_1 \in \mathrm{Inder}\, F$.

### 3.1.3 Triple circle multiplication

Let's now return to rectangular matrices and form the triple circle multiplication $(AB^*C + CB^*A)/2$. For sanity's sake, let us write this as

$$\{A, B, C\} = (AB^*C + CB^*A)/2$$

**Definition 3.8** *A* <u>*derivation*</u> *on $M_{m,n}(\mathbf{C})$ with respect to triple circle multiplication is a linear mapping $\delta$ which satisfies the triple product rule $\delta(\{A, B, C\}) = \{\delta(A), B, C\} + \{A, \delta(B), C\} + \{B, A, \delta(C)\}$*

**Proposition 3.9** *Fix two matrices $A, B$ in $M_{m,n}(\mathbf{C})$ and define*

$$\delta_{A,B}(X) = \{A, B, X\} - \{B, A, X\}.$$

*Then $\delta_{A,B}$ is a derivation with respect to triple circle multiplication.*

The following theorem is a special case of Theorem 3.17.

**Theorem 3.10** *Every derivation of $M_{m,n}(\mathbf{C})$ with respect to triple circle multiplication is a sum of derivation of the form $\delta_{A,B}$.*

We summarize Theorems 3.3,3.7 and 3.10, insofar as they concern $M_n(\mathbf{C})$, in the following table. Since $M_n(\mathbf{C})$ is not a semisimple Lie triple system, its derivations must be taken modulo the center (See 2.3).

**Table 3**

$M_{m,n}(\mathbf{C})$ (SS TRIPLE SYSTEMS)

| triple matrix | triple bracket | triple circle |
|---|---|---|
| $ab^*c$ | $[[a,b],c]$ | $ab^*c + cb^*a$ |
| Th. 3.3 | Th. 3.7 | Th. 3.10 |
| $\delta_{a,b}(x)$ $=$ $ab^*x$ $+xb^*a$ $-ba^*x$ $-xa^*b$ | $\delta_{a,b}(x)$ $=$ $abx$ $+xba$ $-bax$ $-xab$ | $\delta_{a,b}(x)$ $=$ $ab^*x$ $+xb^*a$ $-ba^*x$ $-xa^*b$ |
| (sums) | (sums) $(m = n)$ | (sums) |

### 3.2 Axiomatic approach for triple systems

**Definition 3.11** *A* <u>*triple system*</u> *is defined to be a vector space with one ternary operation called triple multiplication. Addition is denoted by $a+b$ and is commutative and associative:*

$$a+b=b+a, \quad (a+b)+c=a+(b+c).$$

Triple multiplication is denoted (simply and temporarily) by $abc$ and is required to be linear in each of its three variables:

$$(a+b)cd=acd+bcd, \quad a(b+c)d=abd+acd, \quad ab(c+d)=abc+abd.$$

Simple but important examples of triple systems can be formed from any algebra: if $ab$ denotes the algebra product, just define a triple multiplication to be $(ab)c$

Let's see how this works in the algebras we introduced in subsection 1.1.

$\mathcal{C}$; $fgh=(fg)h$, OR $fgh=(f\overline{g})h$

$(M_n(\mathbf{C}),\times)$; $abc=abc$ OR $abc=ab^*c$

$(M_n(\mathbf{C}),[,])$; $abc=[[a,b],c]$

$(M_n(\mathbf{C}),\circ)$; $abc=(a\circ b)\circ c$

All of these except the last one are useful. It turns out that the appropriate form of the triple product for Jordan algebras is $abc=(a\circ b)\circ c+(c\circ b)\circ a-(a\circ c)\circ b$, since then you obtain a Jordan triple system. Also for the second example, by taking the symmetrized triple product, in each case you also obtain a Jordan triple system.

**Definition 3.12** *A triple system is said to be* <u>*associative*</u> *if the triple multiplication is associative in the sense that $ab(cde)=(abc)de=a(bcd)e$ (OR $ab(cde)=(abc)de=a(dcb)e$); and* <u>*commutative*</u> *if $abc=cba$.*

#### 3.2.1 Associative triple systems

The axiom which characterizes triple matrix multiplication is

$$(abc)de=ab(cde)=a(dcb)e.$$

The triple systems so defined are called **associative triple systems** or **Hestenes[16] algebras**

**Theorem 3.13 (Lister 1971 [132])** *Every derivation of a finite dimensional semisimple associative triple system (of the first kind) is inner.*

**Theorem 3.14 (Carlsson 1976 [38])** *Every derivation of a finite dimensional semisimple associative triple system (first or second kind) into a module, is inner.*

The following theorem really belongs in section 5 since it concerns Banach associative triple systems.

**Theorem 3.15 (Zalar 1995 [198])** *Let $W\subset B(H,K)$ be a TRO which contains all the compact operators. If $D$ is a derivation of $W$ with respect to the associative triple product $ab^*c$ then there exist $a=-a^*\in B(K)$ and $b=-b^*\in B(H)$ such that $Dx=ax+xb$.*

This result has been extended to $B(X,Y)$ ($X,Y$ Banach spaces) in [194].

#### 3.2.2 Lie triple systems

The axioms which characterize triple bracket multiplication are

$$aab=0, \quad abc+bca+cab=0,$$

and

$$de(abc)=(dea)bc+a(deb)c+ab(dec).$$

The triple systems so defined are called **Lie triple systems** and were developed by Jacobson and Koecher[17].

[16]**Magnus Hestenes (1906–1991)** Magnus Rudolph Hestenes was an American mathematician. Together with Cornelius Lanczos and Eduard Stiefel, he invented the conjugate gradient method

[17]**Max Koecher (1924–1990)** Max Koecher was a German mathematician. His main research area was the theory of Jordan algebras, where he introduced the Kantor-Koecher-Tits construction

**Theorem 3.16 (Lister 1952 [131])** *Every derivation of a finite dimensional semisimple Lie triple system is inner.*

### 3.2.3 Jordan triple systems

The axioms which characterize triple circle multiplication are

$$abc = cba,$$

and

$$de(abc) = (dea)bc - a(edb)c + ab(dec).$$

The triple systems so defined are called **Jordan triple systems** and were developed principally by **Kurt Meyberg, Ottmar Loos and Erhard Neher**, among many others.

**Theorem 3.17** *Every derivation of a finite dimensional semisimple Jordan triple system is inner.*

*Proof.* We outline a proof following [140, Chapter 11] and [49]. For simplicity, we assume non-degeneracy of a Jordan triple system [49, p. 25].

Let $V$ be a Jordan triple and let $\mathcal{L}(V)$ be its TKK Lie algebra (**Tits-Kantor-Koecher**). $\mathcal{L}(V) = V \oplus V_0 \oplus V$ and the Lie product is given by

$$[(x,h,y),(u,k,v)] = (hu - kx,\ [h,k] + x \,\square\, v - u \,\square\, y,\ k^\natural y - h^\natural v).$$

Here, $a \,\square\, b$ is the left multiplication operator $x \mapsto \{abx\}$ (also called the box operator), $V_0 = \operatorname{span}\{V \,\square\, V\}$ is a Lie subalgebra of $\mathcal{L}(V)$ and for $h = \sum_i a_i \,\square\, b_i \in V_0$, the map $h^\natural : V \to V$ is defined by

$$h^\natural = \sum_i b_i \,\square\, a_i.$$

We can show the correspondence of derivations $\delta : V \to V$ and $D : \mathcal{L}(V) \to \mathcal{L}(V)$ for Jordan triple $V$ and its TKK Lie algebra $\mathcal{L}(V)$. Let $\theta : \mathcal{L}(V) \to \mathcal{L}(V)$ be the main involution $\theta(x \oplus h \oplus y) = y \oplus -h^\natural \oplus x$

**Lemma 3.18** *Let $\delta : V \to V$ be a derivation of a Jordan triple $V$, with TKK Lie algebra $(\mathcal{L}(V), \theta)$. Then there is a derivation $D : \mathcal{L}(V) \to \mathcal{L}(V)$ satisfying*

$$D(V) \subset V \quad \text{and} \quad D\theta = \theta D.$$

*Proof.* Given $a, b \in V$, we define

$$\begin{aligned}
D(a,0,0) &= (\delta a, 0, 0)\\
D(0,0,b) &= (0,0,\delta b)\\
D(0, a \,\square\, b, 0) &= (0,\ \delta a \,\square\, b + a \,\square\, \delta b,\ 0)
\end{aligned}$$

and extend $D$ linearly on $\mathcal{L}(V)$. Then $D$ is a derivation of $\mathcal{L}(V)$ and evidently, $D(V) \subset V$.

It is readily seen that $D\theta = \theta D$, since

$$\begin{aligned}
D\theta(0, a \,\square\, b, 0) &= D(0, -b \,\square\, a, 0)\\
&= (0,\ -\delta b \,\square\, a - b \,\square\, \delta a,\ 0)\\
&= \theta(0,\ \delta a \,\square\, b + a \,\square\, \delta b,\ 0)\\
&= \theta D(0, a \,\square\, b, 0).
\end{aligned}$$

**Lemma 3.19** *Let $V$ be a Jordan triple with TKK Lie algebra $(\mathcal{L}(V), \theta)$. Given a derivation $D : \mathcal{L}(V) \to \mathcal{L}(V)$ satisfying $D(V) \subset V$ and $D\theta = \theta D$, the restriction $D|_V : V \to V$ is a triple derivation.*

**Theorem 3.20** *Let $V$ be a Jordan triple with TKK Lie algebra $(\mathcal{L}(V), \theta)$. There is a one-one correspondence between the triple derivations of $V$ and the Lie derivations $D : \mathcal{L}(V) \to \mathcal{L}(V)$ satisfying $D(V) \subset V$ and $D\theta = \theta D$.*

**Lemma 3.21** *Let $V$ be a Jordan triple with TKK Lie algebra $(\mathcal{L}(V), \theta)$. Let $D : \mathcal{L}(V) \to \mathcal{L}(V)$ be a Lie inner derivation such that $D(V) \subset V$. Then the restriction $D|_V$ is a triple inner derivation of $V$.*

**Corollary 3.22** *Let $\delta$ be a derivation of a finite dimensional semisimple Jordan triple $V$. Then $\delta$ is a triple inner derivation of $V$.*

*Proof.* The TKK Lie algebra $\mathcal{L}(V)$ is semisimple. Hence the result follows from the Lie result and Lemma 3.21.

The proof of Lemma 3.21 is instructive. The steps are as follows.

1. $D(x, k, y) = [(x, k, y), (a, h, b)]$ for some $(a, h, b) \in \mathcal{L}(V)$
2. $D(x, 0, 0) = [(x, 0, 0), (a, h, b)] = (-h(x), x \,\square\, b, 0)$
3. $\delta(x) = -h(x) = -\sum_i \alpha_i \,\square\, \beta_i(x)$
4. $D(0, 0, y) = [(0, 0, y), (a, h, b)] = (0, -a \,\square\, y, h^\natural(y))$
5. $\delta(x) = -h^\natural(x) = \sum_i \beta_i \,\square\, \alpha_i(x)$
6. $\delta(x) = \frac{1}{2} \sum_i (\beta_i \,\square\, \alpha_i - \alpha_i \,\square\, \beta_i)(x)$

We summarize the previous three definitions in the following table

**Table 4**

TRIPLE SYSTEMS

**associative triple systems**

$$(abc)de = ab(cde) = a(dcb)e$$

**Lie triple systems**

$$aab = 0$$
$$abc + bca + cab = 0$$
$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

**Jordan triple systems**

$$abc = cba$$
$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

## 3.3 Exercises (Gradus ad Parnassum)—Triple systems

1. Prove the analog of Proposition 3.2 for associative algebras $A$ with involution: For two elements $a, b \in A$ with $a^* = -a, b^* = -b$, define $\delta_{a,b}(x) = ax + xb$. Then $\delta_{a,b}$ is a derivation with respect to the triple multiplication $ab^*c$. (Use the notation $\langle abc \rangle$ for $ab^*c$)

2. Prove the analog of Proposition 3.6 for associative algebras $A$: Fix two elements $a, b \in A$ and define $\delta_{a,b}(x) = [[a, b], x]$. Then $\delta_{a,b}$ is a derivation with respect to the triple multiplication $[[a, b], c]$. (Use the notation $[abc]$ for $[[a, b], c]$)

3. Prove Proposition 3.9: Fix two matrices $a, b$ in $M_{m,n}(\mathbf{C})$ and define $\delta_{a,b}(x) = \{a, b, x\} - \{b, a, x\}$. Then $\delta_{A,B}$ is a derivation with respect to triple circle multiplication. (Use the notation $\{abc\}$ for $ab^*c + cb^*a$)

4. Show that $M_n(\mathbf{C})$ is a Lie triple system with respect to triple bracket multiplication. In other words, show that the three axioms for Lie triple systems in Table 4 are satisfied if $abc$ denotes $[[a, b], c] = (ab - ba)c - c(ab - ba)$ ($a, b$ and $c$ denote matrices). (Use the notation $[abc]$ for $[[a, b], c]$)

5. Show that $M_{m,n}(\mathbf{C})$ is a Jordan triple system with respect to triple circle multiplication. In other words, show that the two axioms for Jordan triple systems in Table 4 are satisfied if $abc$ denotes $ab^*c + cb^*a$ ($a, b$ and $c$ denote matrices). (Use the notation $\{abc\}$ for $ab^*c + cb^*a$)

6. Let us write $\delta_{a,b}$ for the linear process
$$\delta_{a,b}(x) = abx$$
in a Lie triple system. Show that $\delta_{a,b}$ is a derivation of the Lie triple system by using the axioms for Lie triple systems in Table 4. (Use the notation $[abc]$ for the triple product in any Lie triple system, so that, for example, $\delta_{a,b}(x)$ is denoted by $[abx]$)

7. Let us write $\delta_{a,b}$ for the linear process

$$\delta_{a,b}(x) = abx - bax$$

in a Jordan triple system. Show that $\delta_{a,b}$ is a derivation of the Jordan triple system by using the axioms for Jordan triple systems in Table 4. (Use the notation $\{abc\}$ for the triple product in any Jordan triple system, so that, for example, $\delta_{a,b}(x) = \{abx\} - \{bax\}$)

8. On the Jordan algebra $M_n(\mathbf{C})$ with the circle product $a \circ b = ab + ba$, define a triple product

$$\{abc\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b.$$

Show that $M_n(\mathbf{C})$ is a Jordan triple system with this triple product.

Hint: show that $\{abc\} = 2abc + 2cba$

9. On the vector space $M_n(\mathbf{C})$, define a triple product $\langle abc \rangle = abc$ (matrix multiplication without the adjoint in the middle). Formulate the definition of a derivation of the resulting triple system, and state and prove a result corresponding to Proposition 3.2. Is this triple system associative?

10. In an associative algebra, define a triple product $\langle abc \rangle$ to be $abc$. Show that the algebra becomes an associative triple system with this triple product.

11. In an associative triple system with triple product denoted $\langle abc \rangle$, define a binary product $ab$ to be $\langle aub \rangle$, where $u$ is a fixed element. Show that the triple system becomes an associative algebra with this product. Suppose further that $\langle auu \rangle = \langle uua \rangle = a$ for all $a$. Show that we get a unital involutive algebra with involution $a^\sharp = \langle uau \rangle$.

12. In a Lie algebra with product denoted by $[a, b]$, define a triple product $[abc]$ to be $[[a, b], c]$. Show that the Lie algebra becomes a Lie triple system with this triple product. ([140, ch. 6, ex. 1, p. 43])

13. Let $A$ be an algebra (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \mathrm{Der}\,(A)$ of all derivations of $A$ is a Lie subalgebra of $\mathrm{End}\,(A)$. That is, $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on $A$, and if $D_1, D_2 \in \mathcal{D}$, then $D_1D_2 - D_2D_1 \in \mathcal{D}$.

14. Let $A$ be a triple system (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \mathrm{Der}\,(A)$ of derivations of $A$ is a Lie subalgebra of $\mathrm{End}\,(A)$. That is, $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on $A$, and if $D_1, D_2 \in \mathcal{D}$, then $D_1D_2 - D_2D_1 \in \mathcal{D}$.

## 3.4 Supplemental exercises (Gradus ad Parnassum)—Algebras and triple systems

1. In an arbitrary Jordan triple system, with triple product denoted by $\{abc\}$, define a triple product by

$$[abc] = \{abc\} - \{bac\}.$$

Show that the Jordan triple system becomes a Lie triple system with this new triple product.
([140, ch. 11, Th. 1, p. 108])

2. In an arbitrary associative triple system, with triple product denoted by $\langle abc \rangle$, define a triple product by

$$[xyz] = \langle xyz \rangle - \langle yxz \rangle - \langle zxy \rangle + \langle zyx \rangle.$$

Show that the associative triple system becomes a Lie triple system with this new triple product.
([140, ch. 6, ex. 3, p. 43])

3. In an arbitrary Jordan algebra, with product denoted by $xy$, define a triple product by $[xyz] = x(yz) - y(xz)$. Show that the Jordan algebra becomes a Lie triple system with this new triple product.
([140, ch. 6, ex. 4, p. 43])

4. In an arbitrary Jordan triple system, with triple product denoted by $\{abc\}$, fix an element $y$ and define a binary product by
$$ab = \{ayb\}.$$
Show that the Jordan triple system becomes a Jordan algebra with this (binary) product.
([140, ch. 10, Th. 1, p. 94]—using different language; see also [193, Prop. 19.7, p. 317])

5. In an arbitrary Jordan algebra with multiplication denoted by $ab$, define a triple product
$$\{abc\} = (ab)c + (cb)a - (ac)b.$$
Show that the Jordan algebra becomes a Jordan triple system with this triple product. ([140, ch. 10, p. 93]—using different language; see also [193, Cor. 19.10, p. 320])

6. Show that every Lie triple system, with triple product denoted $[abc]$ is a subspace of some Lie algebra, with product denoted $[a, b]$, such that $[abc] = [[a, b], c]$.
([140, ch. 6, Th. 1, p. 45])

7. Find out what a semisimple associative algebra is and prove that every derivation of a finite dimensional semisimple associative algebra is inner, that is, of the form $x \mapsto ax - xa$ for some fixed $a$ in the algebra.
([93, Theorem 2.2])

8. Find out what a semisimple Lie algebra is and prove that every derivation of a finite dimensional semisimple Lie algebra is inner, that is, of the form $x \mapsto [a, x]$ for some fixed $a$ in the algebra.
([140, ch. 5, Th. 2, p. 42]; see also [93, Theorem 2.1])

9. Find out what a semisimple Jordan algebra is and prove that every derivation of a finite dimensional semisimple Jordan algebra is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(a_i(b_i x) - b_i(a_i x))$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the algebra. [105, c. p. 320] and [32, c.p. 270]

10. In an associative triple system with triple product $\langle xyz\rangle$, show that you get a Jordan triple system with the triple product $\{xyz\} = \langle xyz\rangle + \langle zyx\rangle$. Then use Theorem 3.10 to prove Theorem 3.3.

11. Find out what a semisimple associative triple system is and prove that every derivation of a finite dimensional semisimple associative triple system is inner (also find out what inner means in this context). ([38])

12. Find out what a semisimple Lie triple system is and prove that every derivation of a finite dimensional semisimple Lie triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}[a_i b_i x]$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the Lie triple system.
([140, ch. 6, Th.10, p. 57])

13. Find out what a semisimple Jordan triple system is and prove that every derivation of a finite dimensional semisimple Jordan triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(\{a_i b_i x\} - \{b_i a_i x\})$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the Jordan triple system.
([140, ch. 11, Th. 8, p.123 and Cor. 2, p. 124])

# 4 Contractive projections

The purpose of this section is to describe the role which contractive projections have played in the theory of $JB^*$-triples, which is the primary algebraic structure of interest in this survey. We shall return to the subject of contractive projections in section 9.2.

Contractive projections on Banach spaces have been the focus of much study. For a survey see [163]. We are concerned here with the interplay between the algebraic structure and contractive projections.

## 4.1 Projective stability

A well-known and useful result in the structure theory of operator triple systems is the "contractive projection principle," that is, the fact that the range of a contractive projection on a $JB^*$-triple is linearly isometric in a natural way to another $JB^*$-triple. The genesis of the role of contractive projections in Banach triple systems lies in the following theorem. This was preceded by a similar principle in a geometric context [186], which was unknown to the authors of [73].

**Theorem 4.1 (Friedman and Russo 1985 [73])** *The range of a contractive projection on a $C^*$-algebra is linearly isometric to a $JC^*$-triple, that is, a linear subspace of $B(H)$ which is closed under the symmetric triple product $xy^*z + zy^*x$.*

Thus, as shown later and described below, the category of $JB^*$-triples and contractions is stable under contractive projections. To put this result in proper prospective, let $\mathcal{B}$ be the category of Banach spaces and contractions. We say that a sub-category $\mathcal{S}$ of $\mathcal{B}$ is **projectively stable** if it has the property that whenever $A$ is an object of $\mathcal{S}$ and $X$ is the range of a morphism of $\mathcal{S}$ on $A$ which is a projection, then $X$ is isometric (that is, isomorphic in $\mathcal{S}$) to an object in $\mathcal{S}$.

Examples of projectively stable categories are the following.

- $L_1$, contractions (Grothendieck 1955 [81])
- $L^p$, $1 \leq p < \infty$, contractions
  (Douglas 1965 [58], Ando 1966 [3], Bernau-Lacey 1974 [22], Tzafriri 1969 [190])
- $C^*$-algebras, completely positive unital maps (Choi-Effros 1977 [41])
- $\ell_p$, $1 \leq p < \infty$, contractions
  (Lindenstrauss-Tzafriri 1977 [130])
- $JC^*$-algebras, positive unital maps
  (Effros-Stormer 1979 [67])
- $TROs$ (ternary rings of operators),
  complete contractions (Youngson 1983 [197])
- $JB^*$-triples, contractions
  (Kaup 1984 [119], Stacho 1982 [186], Friedman-Russo 1985 [73])
- $\ell^p$-direct sums of $C_p(H)$, $1 \leq p < \infty$, $H$ Hilbert space, contractions
  (Arazy-Friedman, 1978 [10], 2000 [11])
- $\ell^p$-direct sums of $L^p(\Omega, H)$, $1 \leq p < \infty$, $H$ Hilbert space, contractions
  (Raynaud 2004 [164])
- $\ell^p$-direct sums of $C_p(H)$, $1 \leq p \neq 2 < \infty$, $H$ Hilbert space, complete contractions
  (LeMerdy-Ricard-Roydor 2009 [128])

It follows immediately that if $\mathcal{S}$ is projectively stable, then so is the category $\mathcal{S}_*$ of spaces whose dual spaces belong to $\mathcal{S}$. It should be noted that $TROs$, $C^*$-algebras and $JC^*$-algebras are not stable under contractive projections and $JB^*$-triples are not stable under bounded projections.

### 4.1.1 More about JB*-triples

JB*-triples are generalizations of JB*-algebras and C*-algebras. The axioms can be said to come geometry in view of Kaup's Riemann mapping theorem [118] . Kaup showed in 1983 that JB*-triples are exactly those Banach spaces whose open unit ball is a bounded symmetric domain. Kaup's holomorphic characterization of JB*-triples directly led to the proof of the projective stability of JB*-triples mentioned above.

Many authors have studied the interplay between JB*-triples and infinite dimensional holomorphy. Contractive projections have proved to be a valuable tool for the study of problems on JB*-triples (Gelfand-Naimark theorem [75], structure of inner ideals [64], operator space characterization of TROs [146], to name a few) They are justified both as a natural generalization of operator algebras as well as because of their connections with complex geometry.

Preduals of JBW*-triples have been called pre-symmetric spaces [59] and have been proposed as mathematical models of physical systems [69]. In this model the operations on the physical system are represented by contractive projections on the pre-symmetric space.

JB*-triples first arose in Koecher's proof [124], [133] of the classification of bounded symmetric domains in $\mathbf{C}^n$. The original proof of this fact, done in the 1930's by Cartan, used Lie algebras and Lie groups, techniques which do not extend to infinite dimensions. On the other hand, to a large extent, the Jordan algebra techniques do so extend, as shown by Kaup and Upmeier.[18]

#### 4.1.2 Application: Gelfand Naimark theorem for JB*-triples

**Theorem 4.2 (Friedman and Russo 1986 [75])** *Every JB*-triple is isometically isomorphic to a subtriple of a direct sum of Cartan factors.*

The theorem was not unexpected. However, the proof required new techniques because of the lack of an order structure on a JB*-triple.

- **Step 1: February 1983 Friedman-Russo [73]**

  Let $P : A \to A$ be a linear projection of norm 1 on a JC*-triple $A$. Then $P(A)$ is a JB*-triple under $\{xyz\}_{P(A)} = P(\{xyz\})$ for $x, y, z \in P(A)$.

- **Step 2: April 1983 Friedman-Russo [72]**

  Same hypotheses. Then $P$ is a conditional expectation in the sense that

  $$P\{PaPbPc\} = P\{Pa, b, Pc\}$$

  and

  $$P\{PaPbPc\} = P\{aPbPc\}.$$

- **Step 3: May 1983 Kaup [119]**

  Let $P : U \to U$ be a linear projection of norm 1 on a JB*-triple $U$. Then $P(U)$ is a JB*-triple under $\{xyz\}_{P(U)} = P(\{xyz\}_U)$ for $x, y, z \in P(U)$. Also, $P\{PaPbPc\} = P\{Pa, b, Pc\}$ for $a, b, c \in U$, which extends one of the formulas in the previous step.

- **Step 4: February 1984 Friedman-Russo [74]**

  Every JBW*-triple splits into atomic and purely non-atomic ideals.

- **Step 5: August 1984 Dineen [56]**

  The bidual of a JB*-triple is a JB*-triple.

- **Step 6: October 1984 Barton-Timoney [21]**

  The bidual of a JB*-triple is a JBW*-triple, that is, the triple product is separately weak*-continuous.

- **Step 7: December 1984 Horn [97]**

  Every JBW*-triple factor of type I is isomorphic to a Cartan factor. More generally, every JBW*-triple of type I is isomorphic to a direct sum of $L^\infty$ spaces with values in a Cartan factor.

[18] The opposite is true concerning cohomology. Lie algebra cohomology is well developed, Jordan algebra cohomology is not

- **Step 8: March 1985 Friedman-Russo [75]**
  **Putting it all together**

$$\pi : U \to U^{**} = A \oplus N = (\oplus_\alpha C_\alpha) \oplus N = \sigma(U^{**}) \oplus N$$

  implies that $\sigma \circ \pi : U \to A = \oplus_\alpha C_\alpha$ is an isometric isomorphism.

**Consequences of the Gelfand-Naimark theorem**

- Every JB*-triple is isomorphic to a subtriple of a JB*-algebra.
- In every JB*-triple, $\|\{xyz\}\| \leq \|x\|\|y\|\|z\|$
- Every JB*-triple contains a unique norm-closed ideal $J$ such that $U/J$ is isomorphic to a JC*-triple and $J$ is purely exceptional, that is, every homomorphism of $J$ into a C*-algebra is zero.

#### 4.1.3 Preservation of type

The following results are due to Chu-Neal-Russo [50], and simultaneously, to Bunce-Peralta [34]. The corresponding (classical) result for von Neumann algebras, due to Tomiyama [189], required, of necessity, that the range of the projection be a subalgebra, since the category of von Neumann algebras and positive normal contractions is not projectively stable.

**Theorem 4.3** *Let $P$ be a normal contractive projection on a $JBW^*$-triple $Z$ of type I. Then $P(Z)$ is of type I.*

**Theorem 4.4** *Let $P$ be a normal contractive projection on a semifinite $JBW^*$-triple $Z$. Then $P(Z)$ is a semifinite $JW^*$-triple.*

## 4.2 Projective rigidity

By considering the converse of projective stability, one is lead to the following definition.

A sub-category $\mathcal{S}$ of $\mathcal{B}$ is **projectively rigid** if it has the property that whenever $A$ is an object of $\mathcal{S}$ and $X$ is a subspace of $A$ which is isometric to an object in $\mathcal{S}$, then $X$ is the range of a morphism of $\mathcal{S}$ on $A$ which is a projection.

Examples of projectively rigid categories are the following.

- $\ell_p$, $1 < p < \infty$, contractions
  (Pelczynski 1960 [154])
- $L^p$, $1 \leq p < \infty$, contractions
  (Douglas 1965 [58], Ando 1966 [3], Bernau-Lacey 1974 [22])
- $C_p, 1 \leq p < \infty$, contractions
  (Arazy-Friedman 1977 [9])
- Preduals of von Neumann algebras,
  contractions (Kirchberg 1993 [122])
- Preduals of $TROs$, complete contractions (Ng-Ozawa 2002 [150])
- Preduals of JBW*-triples, contractions[19] (Neal-Russo 2008 [148])
- $\ell^p$-direct sums of $C_p(H)$, $1 \leq p \neq 2 < \infty$, $H$ Hilbert space, complete contractions
  (LeMerdy-Ricard-Roydor 2009 [128])

**Theorem 4.5 (Neal-Russo)** *The category of preduals of $JBW^*$-triples with no summands of the form $L^1(\Omega, H)$ where $H$ is a Hilbert space of dimension at least two, is projectively rigid.*

[19] with one exception, see Theorem 4.5

## 4.3 Structural Projections

### 4.3.1 Structure of inner ideals

In a series of papers, mainly by Edwards and Rüttimann (cf. [63]), the ideal structure of C*-algebras and JB*-algebras has been thoroughly studied, and a surprising geometric characterization of the closed inner ideals among the closed subtriples of a JB*-triple has been established: a norm-closed subtriple B of a JB*-triple A is an inner ideal if and only if every bounded linear functional on B has a unique norm-preserving linear extension to A.

Loos and Neher ([134],[135]) have introduced the notions of complementation and structural projection in the purely algebraic setting of Jordan pairs and Jordan *-triples. Let A be an anisotropic Jordan *-triple; then for every element a in A the quadratic mapping Q(a) is defined by Q(a)x: = {axa}. The kernel Ker(B) of a subset B in A is the subspace of A consisting of all elements annihilated by the mappings Q(b) as b ranges over B. A subtriple B is said to be complemented if $A = B \oplus \mathrm{Ker}(B)$. A linear projection P on A is said to be structural if it satisfies Q(Pa) = PQ(a)P for all a in A, which occurs if and only if its range B = P(A) is a complemented subtriple of A.

In [64], the study of structural projections on JBW*-triples is continued. It is shown that a structural projection on a JBW*-triple is necessarily contractive and weak* continuous, and that every weak* closed inner ideal in a JBW*-triple A is a complemented subtriple of A and therefore the range of a unique structural projection.

### 4.3.2 Geometric characterization

Since structural projections are contractive and weak* continuous, every structural projection is the adjoint of a contractive linear projection on the predual of A. The contractive projections P on $A_*$ that arise in this way also have the property that they are neutral, in that, if x is an element of for which $\|Px\|$ and $\|x\|$ coincide then Px and x also coincide. It follows from the results of [62] that the mapping $P \mapsto P^*(A)$ is a bijection from the family of neutral projections on $A_*$ for which $P^*A$ is a subtriple of A onto the complete lattice of weak*-closed inner ideals in A.

The L-orthogonal complement $G^\diamond$ of a subset G of a complex Banach space E is the set of elements x in E such that, for all elements y in G, $\|x \pm y\| = \|x\| + \|y\|$: A contractive projection P on E is said to be a GL-projection if the L-orthogonal complement $PE^\diamond$ of the range PE of P is contained in the kernel of P. It is shown in [61] that, for a $JBW^*$-triple $A$ with pre dual $A_*$, a linear projection $R$ on $A$ is structural if and only if it is the adjoint of a neutral GL-projection on $A$, thereby giving a purely geometric characterization of structural projections.

### 4.3.3 Physical interpretation

The predual of a JBW*-triple A, also called a prosymmetric space, has been proposed as a model of the state space of a statistical physical system, in which contractive linear mappings on $A_*$ represent operations or filters on the system. For a discussion of this model we refer to [61] and for an application to decoherent states we refer to [60].

In the theory of Banach spaces and in the study of state spaces of physical systems, much effort has been devoted to the investigation of when particular closed subspaces of a Banach space are the ranges of contractive projections. Since the range of a contractive projection on a pre-symmetric space is itself a pre-symmetric space, this problem is particularly relevant in the case of pre-symmetric spaces. The main results of [60] are used to determine under what circumstances the closed subspace of $A_*$ generated by a family $\{x_j : j \in \Lambda\}$ of elements of $A_*$ of norm one which are pairwise decoherent (see [60] for the definition) is the range of a contractive projection. (Is Theorem 4.5 relevant here?)

# 5 Derivations on operator algebras and operator triple systems

In this section we shall review the literature about derivations on associative and non associative operator algebras and operator triple systems, with the main focus being on some new results concerning the notion of weak amenability.

There are two basic questions concerning derivations of Banach algebras and Banach triple systems, namely, whether they are automatically continuous, and to what extent are they all inner, in the appropriate context. These questions are significant in the case of a derivation of a space into itself, or into an appropriate module. The contexts which we shall consider are

**(i)** C*-ALGEBRAS (more generally, associative Banach algebras)

**(ii)** JC*-ALGEBRAS (more generally, Jordan Banach algebras)

**(iii)** JC*-TRIPLES (more generally, Banach Jordan triples)

One could and should also consider:

**(i′)** Banach associative triple systems

**(ii′)** Banach Lie algebras

**(iii′)** Banach Lie triple systems

## 5.1 C*-algebras

In the context of associative algebras, a derivation is a linear mapping satisfying the property $D(ab) = a \cdot Db + Da \cdot b$, and an inner derivation is a derivation of the form ad $\mathrm{x}(a) = x \cdot a - a \cdot x$ $(x \in M)$.

The main automatic continuity results for $C*$-algebra are due to Kaplansky (commutative and type I cases), Sakai, and Ringrose[20].

The main inner derivation results are due primarily to Sakai, Kadison, Connes[21] , and Haagerup[22]. We won't define the terms amenable and nuclear here. We refer to the excellent notes by Runde [173] for these definitions and much other information. Weak amenability will be defined shortly, as it is the focus of our discussion.

**Theorem 5.1 (Sakai 1960 [175])** *Every derivation from a C*-algebra into itself is continuous.*

**Theorem 5.2 (Ringrose 1972 [166])** *Every derivation from a C*-algebra into a Banach A-***bimodule** *is continuous.*

**Theorem 5.3 (Sakai [176], Kadison [113])** *Every derivation of a $C^*$-algebra is of the form $x \mapsto ax - xa$ for some a in the weak closere of the $C^*$-algebra*

**Theorem 5.4 (Connes 1976 [51])** *Every amenable $C^*$-algebra is nuclear.*

**Theorem 5.5 (Haagerup 1983 [82])** *Every nuclear $C^*$-algebra is amenable.*

**Theorem 5.6 (Haagerup 1983 [82])** *Every $C^*$-algebra is weakly amenable.*

[20] **John Ringrose (b. 1932)** John Ringrose is a leading world expert on non-self-adjoint operators and operator algebras. He has written a number of influential texts including Compact non-self-adjoint operators (1971) and, with R V Kadison, Fundamentals of the theory of operator algebras in four volumes published in 1983, 1986, 1991 and 1992

[21] **Alain Connes b. 1947** Alain Connes is the leading specialist on operator algebras. In his early work on von Neumann algebras in the 1970s, he succeeded in obtaining the almost complete classification of injective factors. Following this he made contributions in operator K-theory and index theory, which culminated in the Baum-Connes conjecture. He also introduced cyclic cohomology in the early 1980s as a first step in the study of noncommutative differential geometry. Connes has applied his work in areas of mathematics and theoretical physics, including number theory, differential geometry and particle physics

[22] **Uffe Haagerup b. 1950** Haagerup's research is in operator theory, and covers many subareas in the subject which are currently very active - random matrices, free probability, C*-algebras and applications to mathematical physics

## 5.2 A bridge to Jordan algebras

A *Jordan derivation* from a Banach algebra $A$ into a Banach $A$-module is a linear map $D$ satisfying $D(a^2) = aD(a) + D(a)a$, $(a \in A)$, or equivalently, $D(ab + ba) = aD(b) + D(b)a + D(a)b + bD(a)$, $(a, b \in A)$. Sinclair proved in 1970 [181] that a bounded Jordan derivation from a semisimple Banach algebra to itself is a derivation, although this result fails for derivations of semisimple Banach algebras into a Banach bi-module.

Nevertheless, a celebrated result of B.E. Johnson in 1996 [109] states that every bounded Jordan derivation from a C*-algebra $A$ to a Banach $A$-bimodule is an associative derivation. In view of the intense interest in automatic continuity problems in the past half century, it is therefore somewhat surprising that the following problem has remained open for fifteen years.

**Problem 5.7** *Is every Jordan derivation from a C*-algebra A to a Banach A-bimodule automatically continuous (and hence a derivation, by Johnson's theorem)?*

In 2004, J. Alaminos, M. Brešar and A.R. Villena [4] gave a positive answer to the above problem for some classes of C*-algebras including the class of abelian C*-algebras. Combining a theorem of Cuntz from 1976 [53] with the theorem just quoted yields

**Theorem 5.8** *Every Jordan derivation from a C*-algebra A to a Banach A-module is continuous.*

In the same way, using the solution in 1996 by Hejazian-Niknam [88] in the commutative case we have

**Theorem 5.9** *Every Jordan derivation from a C*-algebra A to a Jordan Banach A-module is continuous.*

(Jordan module will be defined below)

These last two results will also be among the consequences of some work of Peralta and Russo [156], described below, on automatic continuity of derivations into Jordan triple modules.

## 5.3 JC*-algebras

In the context of Jordan algebras, a derivation is a linear mapping satisfying the property $D(a \circ b) = a \circ Db + Da \circ b$, and an inner derivation is a derivation of the form $\sum_i [L(x_i)L(a_i) - L(a_i)L(x_i)]$, $(x_i \in M, a_i \in A)$ $b \mapsto \sum_i [x_i \circ (a_i \circ b) - a_i \circ (x_i \circ b)]$

The main automatic continuity results for $JC^*$-algebras are due to Upmeier[23], Hejazian-Niknam, and Alaminos-Bresar-Villena. The main inner derivation results are due to Jacobson and Upmeier. We have already mentioned the results of Hejazian-Niknam and Alaminos-Bresar-Villena in this context.

**Theorem 5.10 (Upmeier 1980 [191])** *Every derivation of a reversible $JC^*$-algebra extends to a derivation of its enveloping $C^*$-algebra.*

In particular, this theorem improves Theorem 1.14 in the case of a $C^*$-algebra.

**Theorem 5.11 (Jacobson 1951 [105])** *Every derivation of a finite dimensional semisimple Jordan algebra into a (Jordan) module is inner.*

The proof of this theorem uses the theory of Lie algebras and Lie triple systems.

Because of the structure theory for real Jordan operator algebras, the following theorem completely answers the question of when all derivations are inner on a JBW-algebra.

**Theorem 5.12 (Upmeier 1980 [191])**
1. *Purely exceptional JBW-algebras have the inner derivation property*
2. *Reversible JBW-algebras have the inner derivation property*
3. $\oplus L^\infty(S_j, U_j)$ *has the inner derivation property if and only if* $\sup_j \dim U_j < \infty$ *($U_j$ spin factors).*

[23] **Harald Upmeier (b. 1950)**

## 5.4 JC*-triples

In the context of Jordan triple systems, a derivation is a linear or conjugate linear mapping satisfying the property $D\{a,b,c\} = \{Da.b,c\} + \{a,Db,c\} + \{a,b,Dc\}$, and an inner derivation is a derivation of the form $\sum_i [L(x_i,a_i) - L(a_i,x_i)]$ $(x_i \in M, a_i \in A)$ $b \mapsto \sum_i [\{x_i,a_i,b\} - \{a_i,x_i,b\}]$.

For reasons related to the definition of Jordan triple module in the complex case, and which are discussed in both [92] and [156], although a derivation from a Jordan triple system to itself is linear, nevertheless, in the complex case, a derivation of a Jordan triple system into a Jordan triple module (other than the triple itself) is declared to be a conjugate linear map.

The main automatic continuity results for $JC^*$-triples, where the triple product is given by $\{x,y,z\} = (xy^*z + zy^*x)/2$ are due to Barton[24] and Friedman[25] and Peralta[26] and Russo[27], and the main inner derivation results are due to Ho-Martinez-Peralta-Russo, Meyberg, Kühn-Rosendahl, and Ho-Peralta-Russo..

Before stating these results, we must acknowledge the pioneering efforts of Harris[28] to the study of Jordan operator triple systems, especially for the papers [86] and [87] devoted respectively to infinite dimensional holomorphy and spectral and ideal theory.

**Theorem 5.13 (Barton-Friedman 1990 [20])** *Every derivation of a $JB^*$-triple is continuous.*

**Theorem 5.14 (Peralta-Russo 2010 [156])** *Necessary and sufficient conditions under which a derivation of a $JB^*$-triple into a Jordan triple module is continuous.*

(JB*-triple and Jordan triple module are defined below)

The proofs of the following two theorems rely heavily on the theories of Lie algebras and Lie triple systems.

**Theorem 5.15 (Meyberg 1972 [140])** *Every derivation of a finite dimensional semisimple Jordan triple system is inner*

**Theorem 5.16 (Kühn-Rosendahl 1978 [126])** *Every derivation of a finite dimensional semisimple Jordan triple system into a Jordan triple module is inner.*

The proofs of the following two theorems rely on operator theory and functional analysis.

**Theorem 5.17 ((Ho-Martinez-Peralta-Russo 2002 [91])** *Cartan factors of type $I_{n,n}$, II (even or $\infty$), and III have the inner derivation property.*

**Theorem 5.18 ((Ho-Martinez-Peralta-Russo 2002 [91])** *Infinite dimensional Cartan factors of type $I_{m,n}, m \neq n$, and IV do not have the inner derivation property.*

### 5.4.1 Some consequences of Peralta-Russo work [156] on automatic continuity

For JB*-triples:

1. Automatic continuity of derivation on $JB^*$-triple (Barton-Friedman)
2. Automatic continuity of derivation of $JB^*$-triple into its dual (Suggests ternary weak amenability)
3. Automatic continuity of derivation of $JB^*$-algebra into a Jordan module (Hejazian-Niknam)

For C*-algebras:

1. Automatic continuity of derivation of $C^*$-algebra into a module (Ringrose)
2. Automatic continuity of Jordan derivation of $C^*$-algebra into a module (Johnson)
3. Automatic continuity of Jordan derivation of $C^*$-algebra into a Jordan module (Hejazian-Niknam)

[24] **Tom Barton (b. 1955)** Tom Barton is Senior Director for Architecture, Integration and CISO at the University of Chicago. He had similar assignments at the University of Memphis, where he was a member of the mathematics faculty before turning to administration

[25] **Yaakov Friedman (b. 1948)** Yaakov Friedman is director of research at Jerusalem College of Technology

[26] **Antonio Peralta (b. 1974)**

[27] **Bernard Russo (b. 1939)**

[28] **Lawrence A. Harris (PhD 1969)**

### 5.4.2 Summary of Ho-Peralta-Russo work [92] on ternary weak amenability

Recall that a Banach algebra $A$ is said to be *amenable* if every bounded derivation of $A$ into a dual $A$-module is inner, and *weakly amenable* if every (bounded) derivation from $A$ to $A^*$ is inner.

A Jordan Banach triple $E$ is said to be *weakly amenable* or *ternary weakly amenable* if every continuous triple derivation from $E$ into its dual space is necessarily inner.

The main results of [92] are:

1. Commutative $C^*$-algebras are ternary weakly amenable.
2. Commutative $JB^*$-triples are approximately weakly amenable, that is, every derivation into its dual is approximated in norm by inner derivations.
3. $B(H), K(H)$ are ternary weakly amenable if and only if they are finite dimensional.
4. Cartan factors of type $I_{m,n}$ of finite rank with $m \neq n$, and of type IV are ternary weakly amenable if and only if they are finite dimensional.

We provide some examples now of the details of the results of [92] just mentioned.

**Lemma 5.19** *The $C^*$-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space $H$ is not Jordan weakly amenable, that is, not every Jordan derivation into the dual is inner.*

*Proof:* We shall identify $A^*$ with the trace-class operators on $H$. Supposing that $A$ were Jordan weakly amenable, let $\psi \in A^*$ be arbitrary. Then $D_\psi$ $(= \mathrm{ad}\,\psi)$ is an associative derivation and hence a Jordan derivation, so by assumption would be an inner Jordan derivation. Thus there would exist $\varphi_j \in A^*$ and $b_j \in A$ such that

$$D_\psi(x) = \sum_{j=1}^{n} [\varphi_j \circ (b_j \circ x) - b_j \circ (\varphi_j \circ x)]$$

for all $x \in A$.

For $x, y \in A$, a direct calculation yields

$$\psi(xy - yx) = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)(xy - yx).$$

It is known (Pearcy-Topping 1971 [153]) that every compact operator on a separable (which we may assume WLOG) infinite dimensional Hilbert space is a finite sum of commutators of compact operators.

By the just quoted theorem of Pearcy and Topping, every element of $K(H)$ can be written as a finite sum of commutators $[x, y] = xy - yx$ of elements $x, y$ in $K(H)$.

Thus, it follows that the trace-class operator

$$\psi = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)$$

is a finite sum of commutators of compact and trace-class operators, and hence has trace zero. This is a contradiction, since $\psi$ was arbitrary, completing the proof.

**Proposition 5.20** *The $JB^*$-triple $A = M_n(\mathbf{C})$ is ternary weakly amenable.*

*Proof:* Let us denote by $\mathcal{D}_t(A, A^*)$ the triple derivations of $A$ into $A^*$, by $\mathcal{I}nn_b^*(A, A^*)$ the inner associative derivations of $A$ into $A^*$ and by $\mathcal{I}nn_t(A, A^*)$ the inner triple derivations from $A$ into $A^*$. Then by a Proposition which is a step in the proof that commutative C*-algebras are ternary weakly amenable,

$$\mathcal{D}_t(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*),$$

so it suffices to prove that $\mathcal{I}nn_b^*(A, A^*) \circ * \subset \mathcal{I}nn_t(A, A^*)$.

As in the proof of the Lemma, if $D \in \mathcal{I}nn_b^*(A, A^*)$ so that $Dx = \psi x - x\psi$ for some $\psi \in A^*$, then

$$\psi = [\varphi_1, b_1] - [\varphi_2, b_2] + \frac{\mathrm{Tr}\,(\psi)}{n} I,$$

where $b_1, b_2$ are self adjoint elements of $A$ and $\varphi_1$ and $\varphi_2$ are self adjoint elements of $A^*$.

It is easy to see that, for each $x \in A$, we have

$$D(x^*) = \{\varphi_1, 2b_1, x\} - \{2b_1, \varphi_1, x\} - \{\varphi_2, 2b_2, x\} + \{2b_2, \varphi_2, x\},$$

so that $D \circ * \in \mathcal{I}nn_t(A, A^*)$, as required.

## 5.5 Main automatic continuity result

We shall now give some details about the paper [156] which has been mentioned several times already. This involves Jordan triples, Jordan triple modules, quadratic annihilator, and separating spaces

### 5.5.1 Jordan triples

A complex (resp., real) *Jordan triple* is a complex (resp., real) vector space $E$ equipped with a non-trivial triple product

$$E \times E \times E \to E$$

$$(x, y, z) \mapsto \{xyz\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one satisfying the so-called *"Jordan Identity"*:

$$L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y), \tag{4}$$

for all $a, b, x, y$ in $E$, where $L(x, y)z := \{xyz\}$.

A JB$^*$-algebra is a complex Jordan Banach algebra $A$ equipped with an algebra involution $^*$ satisfying $\|\{a, a^*, a\}\| = \|a\|^3$, $a \in A$. (Recall that $\{a, a^*, a\} = 2(a \circ a^*) \circ a - a^2 \circ a^*$).

A *(complex) JB$^*$-triple* is a complex Jordan Banach triple $E$ satisfying the following axioms:

(a) For each $a$ in $E$ the map $L(a, a)$ is an hermitian operator on $E$ with non negative spectrum.

(b) $\|\{a, a, a\}\| = \|a\|^3$ for all $a$ in $A$.

Every C$^*$-algebra (resp., every JB$^*$-algebra) is a JB$^*$-triple with respect to the product $\{abc\} = \frac{1}{2}\,(ab^*c + cb^*a)$ (resp., $\{abc\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$).

### 5.5.2 Jordan triple modules

If $A$ is an associative algebra, an *A-bimodule* is a vector space $X$, equipped with two bilinear products $(a, x) \mapsto ax$ and $(a, x) \mapsto xa$ from $A \times X$ to $X$ satisfying the following axioms:

$$a(bx) = (ab)x, \quad a(xb) = (ax)b, \text{ and, } (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$.

If $J$ is a Jordan algebra, a *Jordan J-module* is a vector space $X$, equipped with two bilinear products $(a, x) \mapsto a \circ x$ and $(x, a) \mapsto x \circ a$ from $J \times X$ to $X$, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every $a, b \in J$ and $x \in X$

If $E$ is a complex Jordan triple, a *Jordan triple $E$-module* (also called *triple $E$-module*) is a vector space $X$ equipped with three mappings

$\{.,.,.\}_1 : X \times E \times E \to X$
$\{.,.,.\}_2 : E \times X \times E \to X$
$\{.,.,.\}_3 : E \times E \times X \to X$

satisfying:

1. $\{x, a, b\}_1$ is linear in $a$ and $x$ and conjugate linear in $b$, $\{abx\}_3$ is linear in $b$ and $x$ and conjugate linear in $a$ and $\{a, x, b\}_2$ is conjugate linear in $a, b, x$

2. $\{x, b, a\}_1 = \{a, b, x\}_3$, and $\{a, x, b\}_2 = \{b, x, a\}_2$ for every $a, b \in E$ and $x \in X$.

3. Denoting by $\{.,.,.\}$ any of the products $\{.,.,.\}_1$, $\{.,.,.\}_2$ and $\{.,.,.\}_3$, the identity $\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\}$, holds whenever one of the elements $a, b, c, d, e$ is in $X$ and the rest are in $E$.

It is a little bit laborious to check that the dual space, $E^*$, of a complex (resp., real) Jordan Banach triple $E$ is a complex (resp., real) triple $E$-module with respect to the products:

$$\{a, b, \varphi\}(x) = \{\varphi, b, a\}(x) := \varphi\{b, a, x\} \tag{5}$$

and

$$\{a, \varphi, b\}(x) := \overline{\varphi\{a, x, b\}}, \forall \varphi \in E^*, a, b, x \in E. \tag{6}$$

For each submodule $S$ of a triple $E$-module $X$, we define its *quadratic annihilator*, $\mathrm{Ann}_E(S)$, as the set $\{a \in E : Q(a)(S) = \{a, S, a\} = 0\}$.

#### 5.5.3 Separating spaces

Separating spaces have been revealed as a useful tool in results of automatic continuity.

Let $T : X \to Y$ be a linear mapping between two normed spaces. The *separating space*, $\sigma_Y(T)$, of $T$ in $Y$ is defined as the set of all $z$ in $Y$ for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$.

A straightforward application of the closed graph theorem shows that a linear mapping $T$ between two Banach spaces $X$ and $Y$ is continuous if and only if $\sigma_Y(T) = \{0\}$

#### 5.5.4 Main result

**Theorem 5.21** *Let $E$ be a complex JB$^*$-triple, $X$ a Banach triple $E$-module, and let $\delta : E \to X$ be a triple derivation. Then $\delta$ is continuous if and only if $Ann_E(\sigma_X(\delta))$ is a (norm-closed) linear subspace of $E$ and*

$$\{Ann_E(\sigma_X(\delta)), Ann_E(\sigma_X(\delta)), \sigma_X(\delta)\} = 0.$$

**Corollary 5.22** *Let $E$ be a real or complex JB$^*$-triple. Then*
*(a) Every derivation $\delta : E \to E$ is continuous.*
*(b) Every derivation $\delta : E \to E^*$ is continuous.*

# Part II
# Cohomology

## 6 Cohomology of finite dimensional algebras

### 6.1 Associative algebras

The starting point for the cohomology theory of algebras is the paper of Hochschild from 1945 [94]. The standard reference of the theory is [40]. Two other useful references are due to Weibel ([195],[196]). We review here the definitions of the cohomology groups $H^n(M,X)$ and the interpretation of them in the cases $n = 1, 2$.

Let $M$ be an associative algebra and $X$ a two-sided $M$-module. For $n \geq 1$, let $L^n(M,X)$ be the linear space of all $n$-linear maps from $M$ to $X$ ($L^0(M,X) = X$). The coboundary operator is the linear mapping $\partial : L^n \to L^{n+1}$ (for $n \geq 1$) defined by

$$\partial\phi(a_1, \cdots, a_{n+1}) = a_1\phi(a_2, \cdots, a_{n+1}) + \sum(-1)^j\phi(a_1, \cdots, a_{j-1}, a_j a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1}\phi(a_1, \cdots, a_n)a_{n+1}$$

For $n = 0$, $\partial : X \to L(M,X)$ is defined by $\partial x(a) = ax - xa$. Since $\partial \circ \partial = 0$, $\mathrm{Im}(\partial : L^{n-1} \to L^n) \subset \ker(\partial : L^n \to L^{n+1})$, and $H^n(M,X) = \ker\partial/\mathrm{Im}\partial$ is a vector space.

For $n = 1$, $\ker\partial = \{\phi : M \to X : a_1\phi(a_2) - \phi(a_1a_2) + \phi(a_1)a_2 = 0\}$ which is the space of derivations from $M$ to $X$; and $\partial : X \to L(M,X)$ is given by $\partial x(a) = ax - xa$ so that $\mathrm{Im}\,\partial$ is the space of inner derivations

Thus $H^1(M,X)$ measures how close derivations are to inner derivations.

An associative algebra $B$ is an extension of associative algebra $A$ if there is a homomorphism $\sigma$ of $B$ onto $A$. The extension splits if $B = \ker\sigma \oplus A^*$ where $A^*$ is an algebra isomorphic to $A$, and is singular if $(\ker\sigma)^2 = 0$.

**Proposition 6.1** *There is a one to one correspondence between isomorphism classes of singular extensions of $A$ and $H^2(A,A)$*

### 6.2 Lie algebras

Shortly after the introduction of the cohomology for associative algebras was introduced, there appeared a corresponding theory for Lie algebras. We follow [104] for the definitions and initial results. Applications can be found in [76] and [123].

If $L$ is a Lie algebra, then an $L$-module is a vector space $M$ and a mapping of $M \times L$ into $M$, $(m,x) \mapsto mx$, satisfying $(m_1 + m_2)x = m_1x + m_2x$, $\alpha(mx) = (\alpha m)x = m(\alpha x)$, and $m[x_1,x_2] = (mx_1)x_2 - (mx_2)x_1$.

Let $L$ be a Lie algebra, $M$ an $L$-module. If $i \geq 1$, an *$i$-dimensional $M$-cochain* for $L$ is a skew symmetric $i$-linear mapping $f$ of $L \times L \times \cdots \times L$ into $M$. Skew symmetric means that if two arguments in $f(x_1, \cdots, x_i)$ are interchanged, the value of $f$ changes sign. A 0-dimensional cochain is a constant function from $L$ to $M$.

The coboundary operator $\delta$ (for $i \geq 1$) is:

$$\delta(f)(x_1, \cdots, x_{i+1}) = \sum_{q=1}^{i+1}(-1)^{i+1}f(x_1, \cdots, \hat{x}_q, \cdots, x_{i+1})x_q + \sum_{q<r=1}^{i+1}(-1)^{r+q}f(x_1, \cdots, \hat{x}_q, \cdots, \hat{x}_r, \cdots, x_{i+1}, [x_q, x_r]).$$

and for $i = 0$, $\delta(f)(x) = ux$ (module action), if $f$ is the constant $u \in M$.

One verifies that $\delta^2 = 0$ giving rise to cohomology groups

$$H^i(L,M) = Z^i(L,M)/B^i(L,M)$$

If $i = 0$ we take $B^i = 0$ and $H^0(L,M) = Z^0(L,M) = \{u \in M : ux = 0, \forall x \in L\}$.

**Theorem 6.2 (Whitehead lemmas)** *If $L$ is a finite dimensional semisimple Lie algebra over a field of characteristic 0, then*

$$H^1(L,M) = H^2(L,M) = 0$$

*for every finite dimensional module $M$ of $L$.*

**Theorem 6.3 (Whitehead)** *If $L$ is a finite dimensional semisimple Lie algebra over a field of characteristic 0, then*

$$H^i(L,M) = 0 \ (\forall i \geq 0)$$

*for every finite dimensional* <u>*irreducible*</u> *module $M$ of $L$ such that $ML \neq 0$.*

## 6.3 Jordan algebras

The cohomology theory for Jordan algebras is less well developed than for associative and Lie algebras. A starting point would seem to be the papers of Gerstenhaber in 1964 [77] and Glassman in 1970 [80], which concern arbitrary nonassociative algebras. A study focussed primarily on Jordan algebras is [79].

Let $A$ be an algebra defined by a set of identities and let $M$ be an $A$-module. A singular extension of length 2 is, by definition, a null extension of $A$ by $M$. A null extension is a short exact sequence

$$0 \to M \xrightarrow{\alpha} E \xrightarrow{\beta} A \to 0$$

where, provisionally, $M$ is an algebra (rather than an $A$-module) with $M^2 = 0$. If $n > 2$, a singular extension of length $n$ is an exact sequence of bimodules

$$0 \to M \to M_{n-1} \to \cdots \to M_2 \to E \to A \to 0$$

Morphisms, equivalences, addition, and scalar multiplication of equivalence classes of singular extensions can be defined. Then for $n \geq 2$, $H^n(A, M) :=$ equivalence classes of singular extensions of length $n$. These definitions are equivalent to the classical ones in the associative and Lie cases.

It is shown in [77] (using generalized projective resolutions) showed that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $A$-bimodules, then there are natural homomorphisms $\delta^n$ so that the long sequence

$$0 \to \mathrm{Der}\,(A, M') \to \mathrm{Der}\,(A, M) \to \mathrm{Der}\,(A, M'') \xrightarrow{\delta^1}$$

$$H^2(A, M') \to H^2(A, M) \to H^2(A, M'') \xrightarrow{\delta^2}$$
$$H^3(A, M') \to H^3(A, M) \to H^3(A, M'') \to$$
$$\cdots \to H^n(A, M'') \xrightarrow{\delta^n} H^{n+1}(A, M') \to \cdots$$

is exact. In particular,

$$H^n(A, M) = \ker \delta^n / \mathrm{im}\, \delta^{n-1} \quad (n \geq 2)$$

What about $H^0(A, M)$ and $H^1(A, M)$ and Jordan algebras? For this we turn first to Glassman's thesis of 1968, embodied in [79] and [80]. Given an algebra $A$, consider the functor $\mathcal{T}$ from the category $\mathcal{C}$ of $A$-bimodules and $A$-homomorphisms to the category $\mathcal{V}$ of vector spaces and linear maps:

$$M \in \mathcal{C} \mapsto \mathrm{Der}\,(A, M) \in \mathcal{V} \quad , \quad \eta \mapsto \tilde{\eta}$$

where $\eta \in \mathrm{Hom}_A(M_1, M_2)$ and $\tilde{\eta} \in \mathrm{Hom}(\mathrm{Der}\,(A, M_1), \mathrm{Der}\,(A, M_2))$ is given by $\tilde{\eta} = \eta \circ D$. $\left(A \xrightarrow{D} M_1 \xrightarrow{\eta} M_2\right)$

An <u>inner derivation functor</u> is a subfunctor $\mathcal{J}$ which respects epimorphisms, that is,

$$M \in \mathcal{C} \mapsto \mathcal{J}(A, M) \subset \mathrm{Der}\,(A, M) \in \mathcal{V}$$

and if $\eta \in \mathrm{Hom}_A(M_1, M_2)$ is onto, then so is $\mathcal{J}(A, \eta) := \tilde{\eta}|\mathcal{J}(A, M)$

Relative to the choice of $\mathcal{J}$ one defines $H^1(A, M) = \mathrm{Der}\,(A, M)/\mathcal{J}(A, M)$. (The definition of $H^0(A, M)$ is more involved so is omitted here). Glassman then proves that

$$0 \to H^0(A, M') \to H^0(A, M) \to H^0(A, M'') \xrightarrow{\delta^0}$$

$$H^1(A, M') \to H^1(A, M) \to H^1(A, M'') \xrightarrow{\delta^1}$$
$$H^2(A, M') \to H^2(A, M) \to H^2(A, M'') \xrightarrow{\delta^2}$$
$$H^3(A, M') \to H^3(A, M) \to H^3(A, M'') \to$$
$$\cdots \to H^n(A, M'') \xrightarrow{\delta^n} H^{n+1}(A, M') \to \cdots$$

is exact.

We mention just one other result from [79]. In a Jordan algebra, recall that $\{xby\} = (xb)y + (by)x - (xy)b$. The $b$-homotope of $J$, written $J^{(b)}$ is the Jordan algebra structure on the vector space $J$ given by the multiplication $x \cdot_b y = \{xby\}$. If $M$ is a bimodule for $J$ and $b$ is invertible, the corresponding bimodule $M^{(b)}$ for $J^{(b)}$ is the vector space $M$ with action $a \cdot_b m = m \cdot_b a = \{abm\}$.

**Lemma 6.4** $H^n(J, M) \sim H^n(J^{(b)}, M^{(b)})$

We next record the Jordan analogs of the first and second Whitehead lemmas as described in [103]. The first one has already been mentioned in Theorem 5.11.

**Theorem 6.5 (Jordan analog of first Whitehead lemma [102])** *Let $J$ be a finite dimensional semisimple Jordan algebra over a field of characteristic 0 and let $M$ be a $J$-module. Let $f$ be a linear mapping of $J$ into $M$ such that*

$$f(ab) = f(a)b + af(b).$$

*Then there exist $v_i \in M, b_i \in J$ such that*

$$f(a) = \sum_i ((v_i a)b - v_i(ab_i)).$$

**Theorem 6.6 (Jordan analog of second Whitehead lemma [155])** *Let $J$ be a finite dimensional semisimple (separable?) Jordan algebra and let $M$ be a $J$-module. Let $f$ be a bilinear mapping of $J \times J$ into $M$ such that*

$$f(a, b) = f(b, a)$$

*and*

$$f(a^2, ab) + f(a, b)a^2 + f(a, a)ab = f(a^2b, a) + f(a^2, b)a + (f(a, a)b)a$$

*Then there exist a linear mapping $g$ from $J$ into $M$ such that*

$$f(a, b) = g(ab) - g(b)a - g(a)b$$

A study of low dimensional cohomology for quadratic Jordan algebras is given in [138]. Since quadratic Jordan algebras (which we do not define here) can be considered a bridge from Jordan algebras to Jordan triple systems, this would seem to be a good place to look for exploring cohomology theory for Jordan triples. Indeed, this is hinted at in [139]

# 7 Cohomology of operator algebras

We are now going to summarize the main points of this theory using the two survey articles [167], [184] as our primary guide.

## 7.1 Continuous Hochschild cohomology

As we saw in section 6, Hochschild cohomology involves an associative algebra $A$ and $A$-bimodules $X$ and gives rise to

- $n$-cochains $L^n(A, X)$,
- coboundary operators $\Delta_n$,
- $n$-coboundaries $B^n$,
- $n$-cocycles $Z^n$ and
- cohomology groups $H^n(A, X)$.

If $A$ is a Banach algebra and $X$ is a Banach $A$-bimodule ( that is, a Banach space with module actions jointly continuous) we have the continuous versions of the above concepts $L_c^n$ , $B_c^n$ , $Z_c^n$ , $H_c^n(A,X)$. Warning: $B_c^n$ is not always closed, so $H_c^n$ is still only a vector space.

Let $A$ be a C*-algebra of operators acting on a Hilbert space $H$ and let $X$ be a dual normal $A$-module ($X$ is a dual space and the module actions are separately ultra weakly-weak*-continuous). Further, we now have

- normal $n$-cochains $L_w^n(A,X)$, that is, bounded and separately weakly continuous $n$-cochains
- coboundary operators $\Delta_n$,
- normal $n$-coboundaries $B_w^n$,
- normal $n$-cocycles $Z_w^n$ and
- normal cohomology groups $H_w^n(A,X)$.

Thus, for a C*-algebra acting on a Hilbert space we have three possible cohomology theories:

- the purely algebraic Hochschild theory $H^n$
- the bounded theory $H_c^n$
- the normal theory $H_w^n$

**Theorem 7.1 (Johnson,Kadison,Ringrose 1972 [111])** $H_w^n(A,X) \sim H_w^n(R,X)$
*($R$ =ultraweak closure of $A$)*

**Theorem 7.2 (Johnson,Kadison,Ringrose 1972 [111])** $H_w^n(A,X) \sim H_w^c(A,X)$

By Theorems 7.1 and 7.2, all four cohomology groups

$$H_w^n(A,X)\ ,\ H_w^n(R,X)\ ,\ H_c^n(R,X)\ ,\ H_w^n(R,X)$$

are pairwise isomorphic.

**Theorem 7.3 (Johnson,Kadison,Ringrose 1972 [111])** $H_c^n(R,X) = 0\ \forall n \geq 1$
*($R$ =hyperfinite von Neumann algebra)*

**Theorem 7.4 (Connes 1978 [52])** *If $R$ is a von Neumann algebra with a separable predual, and $H_c^1(R,X) = 0$ for every dual normal $R$-bimodule $X$, then $R$ is hyperfinite.*

At this point, there were two outstanding problems of special interest;

**Problem 7.5** $H_c^n(R,R) = 0\ \forall n \geq 1$ *for every von Neumann algebra $R$?*

**Problem 7.6** $H_c^n(R,B(H)) = 0\ \forall n \geq 1$ *for every von Neumann algebra $R$ acting on a Hilbert space $H$?*

In the words of John Ringrose, "The main obstacle to advance was a paucity of information about the general bounded linear (or multilinear) mapping between operator algebras. The major breakthrough, leading to most of the recent advances, came through the development of a rather detailed theory of completely bounded mappings."

This development, based on the following definitions, occurred about a decade later.

Let $A$ be a C*-algebra and let $S$ be a von Neumann algebra, both acting on the same Hilbert space $H$ with $A \subset S$. We can view $S$ as a dual normal $A$-module with $A$ acting on $S$ by left and right multiplication. We now have

- completely bounded $n$-cochains $L_{cb}^n(A,S)$
- coboundary operators $\Delta_n$,
- completely bounded $n$-coboundaries $B_{cb}^n$,
- completely bounded $n$-cocycles $Z_{bc}^n$
- completely bounded cohomology groups $H_{cb}^n(A,S)$.

## 7.2 Completely bounded maps

In this subsection, we shall give an introduction to the category of operator spaces from a point of view advantageous to the goals of this paper. The immediate purpose is to give context to completely bounded maps for use when we return to cohomology in the next subsection. Besides the standard monoraphs which appeared after the beginning of the 21st century [68], [152],[160],[25],[90], there is quite a good bit of the basic theory in the lecture notes of Runde [173]. Proofs of all the results in this subsection can be found in one or more of these references.

### 7.2.1 Banach spaces

Why are normed spaces important? One answer is that $\mathbf{R}^n$ is a vector space with a norm, so you can take derivatives and integrals. Thus, normed spaces are important because you can do calculus.

Why is completeness important? Three basic principles of functional analysis are based in some way on completeness:

- Hahn-Banach theorem ( which depends on Zorn's lemma)
- Open mapping theorem (which depends on Baire category)
- Uniform boundedness theorem (which depends on Baire category)

Some examples which will be of interest to us are:

- $C[0,1],\ L^p,\ \ell^p,\ c_0,\ C(\Omega),\ L^p(\Omega,\mu),\ C_0(\Omega)$
- $B(X,Y)$, Banach algebras, Operator algebras $\mathcal{A} \subset B(X)$
- Operator spaces; C*-algebras, Ternary rings of operators

The Hahn-Banach theorem states that every bounded linear functional on a subspace of a Banach space has a bounded extension to the larger space with the same norm.

**Definition 7.7** *A Banach space is* **injective** *if every bounded linear <u>operator</u> from a subspace of a Banach space into it has a bounded extension to the larger space with the same norm.*

**Theorem 7.8 (Kelley 1952 [121])** *A Banach space is injective if and only if it is isometric to $C(\Omega)$, where $\Omega$ is extremely disconnected.*

Injectivity in a category of Banach spaces is intimately related to contractive projections. Our first example of this is the following easily verified fact.

**Remark 7.9** *A Banach space is injective if and only if there is an injective Banach space containing it and a contractive projection of that space onto it.*

### 7.2.2 Operator spaces

Every Banach space $X$ is commutative, by which we mean that by the Hahn-Banach theorem, $X \subset C(X_1^*)$, a linear subspace of a **commutative** C*-algebra. Hence: $C(\Omega)$ is the "mother of all Banach spaces."

**Definition 7.10** *An* **operator space** *(or noncommutative Banach space, or quantized Banach space) is a linear subspace of a C*-algebra (or $B(H)$).*

Hence: $B(H)$ is the "mother of all operator spaces."

An operator space may be viewed as a C*-algebra for which the multiplication and involution have been ignored. You must replace these by some other structure. What should this structure be and what can you do with it? The answer lies in the morphisms. In the category of Banach spaces, the morphisms are the bounded linear maps. In the category of operator spaces, the morphisms are the **completely bounded maps**.

Operator spaces are intermediate between Banach spaces and C*-algebras. The advantage of operator spaces over C*-algebras is that they allow the use of finite dimensional tools and isomorphic invariants ("local theory"). C*-algebras are too rigid: morphisms are contractive, norms are unique. Operator space theory opens the door to a massive transfer of technology coming from Banach space theory. This quantization process has benefitted operator algebra theory mainly (as opposed to Banach space theory).

**Theorem 7.11 (Arveson-Wittstock-Paulsen Hahn-Banach)** *$B(H)$ is injective in the category of operator spaces and completely contractive maps.*

Our second example of the connection of contractive projections to injectivity is the following early verified fact.

**Remark 7.12** *$X \subset B(H)$ is injective if and only if there is a completely contractive projection $P : B(H) \to B(H)$ with $P(B(H)) = X$.*

### 7.2.3 Injective and mixed injective operator spaces

**Theorem 7.13** *An operator space is injective if and only if it is completely isometric to a corner of a C*-algebra. In particular, it is completely isometric to a ternary ring of operators (TRO).*

**Theorem 7.14** *Every operator space has an injective envelope.*

There are also the related notions of ternary envelope and Shilov boundary, which are useful but are not needed here. The following concept will appear later in part 3.

**Definition 7.15** *An operator space is a* **mixed injective** *if every completely contractive linear operator from a subspace of an operator space into it has a bounded contractive extension to the larger space.*

**Remark 7.16** *$X \subset B(H)$ is a mixed injective if and only if there is a contractive projection $P : B(H) \to B(H)$ with $P(B(H)) = X$.*

### 7.2.4 Applications of operator space theory

The following theorems highlight the profound applications that operator space theory has had to operator algebra theory.

- **Similarity problems**. The Halmos problem is solved in the following theorem. For progress on and a description of the Kadison problem, see [159].

  **Theorem 7.17 (Pisier 1997 [157])** *There exists a polynomially bounded operator which is not similar to a contraction.*

- **Tensor products**

  **Theorem 7.18 (Junge-Pisier 1995 [112])** *$B(H)$ is not a nuclear pair. That is, $B(H) \otimes B(H)$ does not have a unique C*-norm.*

  To quote Pisier [158]: "This is a good case study. The proof is based on the solution to a problem which would be studied for its own sake: whether the set of finite dimensional operator spaces is separable (it is not)."

- **Operator amenable groups**. A well-known theorem of Johnson, which is false for the Fourier algebra, states that a group is amenable if and only if its group algebra is an amenable Banach algebra under convolution.

**Theorem 7.19 (Ruan 1995 [172])** *A group is amenable if and only if its Fourier algebra is amenable in the operator space formulation.*

- Operator local reflexivity. A well known theorem of Lindenstrauss-Rosenthatl with many applications states that every Banach space is locally reflexive (Finite dimensional subspaces of the second dual can be approximated by finite dimensional subspaces of the space itself). One application occurs in the proof that the second dual of a JB*-triple is a JB*-triple, with the consequence being the Gelfand-Naimark for JB*-triples. Not all operator spaces are locally reflexive (in the appropriate operator space sense). However, there is the following theorem.

  **Theorem 7.20 (Effros-Junge-Ruan 2000 [65])** *The dual of a C*-algebra is a locally reflexive operator space.*

## 7.3 Completely bounded cohomology

We begin by recalling the relevant definitions.

Let $A$ be a C*-algebra and let $S$ be a von Neumann algebra, both acting on the same Hilbert space $H$ with $A \subset S$. We can view $S$ as a dual normal $A$-module with $A$ acting on $S$ by left and right multiplication. We now have

- completely bounded $n$-cochains $L^n_{cb}(A, S)$
- coboundary operators $\Delta_n$,
- completely bounded $n$-coboundaries $B^n_{cb}$,
- completely bounded $n$-cocycles $Z^n_{bc}$
- completely bounded cohomology groups $H^n_{cb}(A, S)$.

For a C*-algebra $A$ and a von Neumann algebra $S$ with $A \subset S \subset B(H)$ we thus have two new cohomology theories:

- the completely bounded theory $H^n_{cb}$
- the completely bounded normal theory $H^n_{cbw}$

By straightforward analogues of Theorems 7.1 and 7.2, the four cohomology groups

$$H^n_{cb}(A, S)\ ,\ H^n_{cbw}(A, S)\ ,\ H^n_{cb}(R, S)\ ,\ H^n_{cbw}(R, S)$$

are mutually isomorphic, where $R$ is the ultraweak closure of $A$.

**Theorem 7.21 (Christensen-Effros-Sinclair 1987 [44])** $H^n_{cb}(R, B(H)) = 0\ \forall n \geq 1$ *($R =$any von Neumann algebra acting on $H$)*

**Theorem 7.22 (Christensen-Sinclair 1987 [48])** $H^n_{cb}(R, R) = 0\ \forall n \geq 1$ *($R =$any von Neumann algebra)*

A noteworthy quote from [184] is the following: "Cohomology and complete boundedness have enjoyed a symbiotic relationship where advances in one have triggered progress in the other."

Theorems 7.23 and 7.24 are also due to Christensen-Effros-Sinclair [44].

**Theorem 7.23** $H^n_c(R, R) = 0\ \forall n \geq 1$ *($R =$ von Neumann algebra of type $I$, $II_\infty$, $III$, or of type $II_1$ and stable under tensoring with the hyperfinite factor)*

**Theorem 7.24** $H^n_c(R, B(H)) = 0\ \forall n \geq 1$ *($R =$ von Neumann algebra of type $I$, $II_\infty$, $III$, or of type $II_1$ and stable under tensoring with the hyperfinite factor, acting on a Hilbert space $H$)*

The following theorem is the culmination of earlier work in [161] and [45].

**Theorem 7.25 (Sinclair-Smith,[183])** $H_c^n(R,R) = 0 \ \forall n \geq 1$ *($R$ = von Neumann algebra of type $II_1$ with a Cartan subalgebra and a separable*[29] *predual)*

The following theorem was proved in [46] and [47] and was new only for $n \geq 3$. For $n = 1$, the two cases are in [113],[176] and [45], while for $n = 2$ one can refer to [48] and [43].

**Theorem 7.26 (Christensen-Pop-Sinclair-Smith $n \geq 3$ 2003)** $H_c^n(R,R) = H_c^n(R,B(H)) = 0 \ \forall n \geq 1$ *($R$ = von Neumann algebra factor of type $II_1$ with property $\Gamma$, acting on a Hilbert space $H$)*

We can now add a third problem to our previous two (Problems 7.5,7.6). A candidate for a counterexample is the factor arising from the free group on 2 generators.

**Problem 7.27** $H_c^n(R,R)) = 0 \ \forall n \geq 2$*? ($R$ is a von Neumann algebra of type $II_1$)*

### 7.3.1 Another approach (Paulsen)

A different approach, which is popular among algebraists is taken in the paper [151]. We quote from the review of this paper in Mathematical Reviews [169].

"Let A be a subalgebra in the algebra B(H) of bounded operators in a Hilbert space H, and X be a linear subspace in B(H) such that AXA⊂X. Then X can be considered as an A-bimodule, and the Hochschild cohomology groups for the pair (A,X) can be constructed in the usual way, but with the cocycles being n-linear maps from A to X satisfying certain extra 'complete boundedness' conditions."

"Continuing the study of A. Ya. Khelemskii [89], himself and others, Paulsen presents two new alternative presentations of these completely bounded Hochschild cohomologies, one of them as a relative Yoneda cohomology, i.e. as equivalence classes of relatively split resolutions, and the second as a derived functor, making it similar to EXT. These presentations make clear the importance of the notions of relative injectivity, projectivity and amenability, which are introduced and studied."

"Using the relative injectivity of von Neumann algebras, the author proves the triviality of completely bounded Hochschild cohomologies for all von Neumann algebras. The Yoneda representation makes the proofs of a number of classical results more transparent."

## 7.4 Perturbation of Banach algebras

One of the most striking applications of the cohomology theory of Banach algebras lies in the area of perturbation theory.

### 7.4.1 Theorem of Johnson and Raeburn-Taylor

**Definition 7.28** *A Banach algebra is* **stable** *if any other product making it into a Banach algebra, which is sufficiently close to the original product gives rise to a Banach algebra which is topologically algebraically isomorphic to the original Banach algebra..*

Let's make this more precise. If $m$ is a Banach algebra multiplication on $A$, then its norm is defined by $\|m(x,y)\| \leq \|m\|\|x\|\|y\|$. The following theorem was proved simultaneously and independently by Johnson [108] and the team of Raeburn and Taylor [162].

**Theorem 7.29** *For any Banach algebra $A$ with multiplication $m$, if $H^2(A,A) = H^3(A,A) = 0$, then there exists $\epsilon > 0$ such that if $m_1$ is another Banach algebra multiplication on $A$ such that $\|m_1 - m\| < \epsilon$ then $(A,m_1)$ and $(A,m)$ are topologically algebraically isomorphic.*

For a stimulating discussion of this topic and many others, we recommend [89]. For a similar problem on operator algebras one can look into [115].

[29] The separability assumption was removed in 2009 [37]

#### 7.4.2 Perturbation for nonassociative Banach algebras

The origin of perturbation theory for general non associative algebras is deformation theory. Let $c_{ij}^k$ be the structure constants of a finite dimensional Lie algebra $L$. Let $c_{ij}^k(\epsilon) \to c_{ij}^k$ Stability in this context means $(L, c_{ij}^k(\epsilon))$ is isomorphic to $(L, c_{ij}^k)$ if $\epsilon$ is sufficiently small.

**Theorem 7.30 (Gerstenhaber 1964 [78])** *Finite dimensional semisimple Lie algebras are stable.*

Perturbation results similar to Theorem 7.29 have been proved recently for some non associative Banach algebras in [57].

**Theorem 7.31 (Dosi 2009)** *If $L$ is a Banach Lie algebra and $H^2(L,L) = H^3(L,L) = 0$, then $L$ is a stable Banach Lie algebra.*

An entirely similar result for Banach Jordan algebras is also proved in [57]. However, the definition of cohomology for Jordan algebras in [57] is made only in dimensions 3 or less.

# 8 Cohomology of triple systems

In this section we shall describe the results which the author learned about in preparing this mini course. Since the author of this paper is not (yet) an expert in the purely algebraic side of cohomology theory, he is going to rely on information obtained from reviews of these papers in Mathematical Reviews.

## 8.1 Cohomology of finite dimensional triple systems

### 8.1.1 Cohomology of Lie triple systems

The earliest work on cohomology of triple systems seems to be [85], of which the following description is taken from its review in Mathematical Reviews [127]. Four decades later, the second paper on the subject appeared [96].

A Lie triple system $T$ is a subspace of a Lie algebra $L$ closed under the ternary operation $[xyz] = [x, [y, z]]$ or, equivalently, it is the subspace of $L$ consisting of those elements $x$ such that $\sigma(x) = -x$, where $\sigma$ is an involution of $L$. A $T$-module $M$ is a vector space such that the vector-space direct sum $T \oplus M$ is itself a Lie triple system in such a way that

1. $T$ is a subsystem
2. $[xyz] \in M$ if any of $x, y, z$ is in $M$
3. $[xyz] = 0$ if two of $x, y, z$ are in $M$.

A universal Lie algebra $L_u(T)$ and an $L_u(T)$-module $M_s$ can be constructed in such a way that both are operated on by an involution $\sigma$ and so that $T$ and $M$ consist of those elements of $L_u(T)$ and $M_s$ which are mapped into their negatives by $\sigma$.

Now suppose $L$ is a Lie algebra with involution $\sigma$ and $N$ is an $L$-$\sigma$ module. Then $\sigma$ operates on $H^n(L,N)$ so that

$$H^n(L,N) = H^n_+(L,N) \oplus H^n_-(L,N)$$

with both summands invariant under $\sigma$. The cohomology of the Lie triple system is defined by $H^n(T,M) = H^n_+(L_u(T), M_s)$. Harris investigates these groups for $n = 0, 1, 2$.

1. $H^0(T,M) = 0$ for all $T$ and $M$
2. $H^1(T,M) =$ derivations of $T$ into $M$ modulo inner derivations
3. $H^2(T,M) =$ factor sets of $T$ into $M$ modulo trivial factor sets.

Turning to the case of finite-dimensional simple $T$ and ground field of characteristic 0, one has the Whitehead lemmas $H^1(T,M) = 0 = H^2(T,M)$ and Weyl's theorem: Every finite-dimensional module is semi-simple. Also, if in addition, the ground field $\Phi$ is algebraically closed, then $H^3(T,\Phi)$ is 0 or not 0, according as $L_u(T)$ is simple or not.

The following is the verbatim review [125] of [96].

"This is a study of representations of Lie triple systems, both ordinary and restricted. The theory is based on the connection between Lie algebras and Lie triple systems. In addition, the authors begin the study of the cohomology theory for Lie triple systems and their restricted versions. They also sketch some future applications and developments of the theory."

### 8.1.2 Cohomology of associative triple systems

The following is the verbatim review [180] of [38].

"A cohomology for associative triple systems is defined, with the main purpose to get quickly the cohomological triviality of finite-dimensional separable objects over fields of characteristic $\neq 2$, i.e., in particular the Whitehead lemmas and the Wedderburn principal theorem.

This is achieved by embedding an associative triple system $A$ in an associative algebra $U(A)$ and associating with every trimodule $M$ for $A$ a bimodule $M_u$ for $U(A)$ such that the cohomology groups $H^n(A,M)$ are subgroups of the classical cohomology groups $H^n(U(A),M_u)$.

Since $U(A)$ is chosen sufficiently close to $A$, in order to inherit separability, the cohomological triviality of separable $A$ is an immediate consequence of the associative algebra theory.

The paper does not deal with functorialities, nor with the existence of a long exact cohomology sequence."

### 8.1.3 Wedderburn decomposition

The classical Wedderburn decomposition is the following theorem, which we state in its Banach algebra form from [18]. It has been generalized, not only to non associative algebras, see for example [104] and [105], but to some classes of Banach algebras, see for example [54].

**Theorem 8.1** *Let $A$ be a Banach algebra with radical $R$. If the dimension of $A$ is finite, there is a subalgebra $B$ of $A$ such that $A = B + R$ and $B \cap R = 0$. Moreover, if $A$ is commutative, then $B$ is necessarily unique.*

The following definition is the model to use in extending this concept to infinite dimensional triples.

**Definition 8.2** *If $A$ is a Banach algebra with radical $R$, then $A = B \oplus R$ is a strong Wedderburn decomposition of $A$ if $B$ is a closed subalgebra of $A$.*

The author knows of only two papers dealing with this concept in the context of triple systems, namely, [39] and [126].

The following, including Theorem 8.3 is the verbatim review [29] of [39].

The Wedderburn principal theorem, known for Lie triple systems, is proved for alternative triple systems and pairs. If $i$ is an involution of an alternative algebra $B$, then $\langle xyz\rangle := (x \cdot i(y)) \cdot z$ is an alternative triple $(x, y, z \in B)$. A polarisation of an alternative triple $A$ is a direct sum of two submodules $A^1 \oplus A^{-1}$ with

$$\langle A^1 A^{-1} A^1\rangle \subset A^1, \langle A^{-1} A^1 A^{-1}\rangle \subset A^{-1}$$

and

$$\langle A^1 A^1 A^1\rangle = \langle A^{-1} A^{-1} A^{-1}\rangle = \langle A^1 A^1 A^{-1}\rangle = \langle A^{-1} A^{-1} A^1\rangle = \langle A^{-1} A^1 A^1\rangle = \langle A^1 A^{-1} A^{-1}\rangle = \{0\}.$$

An alternative pair is an alternative triple with a polarisation.

**Theorem 8.3** *If $A$ is a finite-dimensional alternative triple system (or an alternative pair) over a field $K, R$ the radical and $A/R$ separable, then $A = B \oplus R$, where $B$ is a semisimple subtriple (subpair) of $A$ with $B = A/R$.*

The following description of [126] is taken from the abstract and the review [188].

This paper summarizes some properties of Jordan pairs, states some results about some groups defined by Jordan pairs, and constructs a Lie algebra to a Jordan pair. This construction is a generalization of the well-known Koecher-Tits-construction. The radical of this Lie algebra is calculated in terms of the given Jordan pair and a Wedderburn decomposition theorem for Jordan pairs (and triples) in the characteristic zero case is proved.

More precisely, an observation of Koecher that the theorem of Levi for Lie algebras of characteristic 0 implies the Wedderburn principal theorem for Jordan algebras is extended to Jordan pairs (and Jordan triples) V over a field of characteristic 0. In addition, the authors show that any two Wedderburn splittings of V are conjugate under a certain normal subgroup of the automorphism group of V.

#### 8.1.4 Cohomology of algebras and triple systems

The reader may be wondering about the cohomology of Jordan triple systems. As noted in 6.3, there is a hint of this in the paper of McCrimmon [139]. Building on the 1 and 2 dimensional cohomology for quadratic Jordan algebras, [139] broaches the analogous construction for Jordan triple systems.

There is also a more general approach in the paper of Seibt [179]. The following is from the introduction to [179].

"The classical cohomologies of unital associative algebras and of Lie algebras have both a double algebraic character: They are embedded in all of the machinery of derived functors, and they allow full extension theoretic interpretations (**Yoneda**-interpretation) of the higher cohomology groups—which seems natural since the coefficient category for cohomology is actually definable in terms of singular extension theory.

If one wants to define a uniform cohomology theory for (linear) nonassociative algebras and triple systems which "generalizes" these two classical cohomologies one may proceed either via derived functors [19] or via singular extension theory [77], [80]

The purpose of this paper which adopts the first point of view is to discuss compatibility questions with the second one."

## 8.2 Cohomology of Banach triple systems—Prospectus

In this subsection we propose, albeit quite vaguely and briefly, some places to look for extending some of the preceding material to infinite dimensional Banach triples of various kinds. Also included, with or without comment, are some papers related to topics discussed in this survey which the author has downloaded and thinks might be worth exploring.

#### 8.2.1 Lie derivations into a module; automatic continuity

The main challenge in this area is to attack the automatic continuity questions noted in subsection 2.3.

With reference to subsection 2.3, a search for "Lie derivation" in the "review text" option in MathSciNet turns up 148 entries. A good number of these concern Lie derivations on nonselfadjoint operator algebras, for example nest algebras, reflexive algebras, $CSL$ algebras, $UHF$-algebras. We quote only the latest one.

**Theorem 8.4 (Ji and Qi 2011 Corollary 2.1 in [106])** *Let $N$ be an arbitrary nest on a Hilbert space $H$ of dimension greater than 2, and $AlgN$ be the associated nest algebra. Let $\delta$ be a linear mapping on $AlgN$ which satisfies the Lie derivation property on commutators $[a, b]$ with $ab = 0$. Then there exists an operator $r \in AlgN$ and a linear center valued map $\tau$ vainshing on such commutators such that $\delta(a) = ra - ar + \tau(a)$.*

It is worth mentioning in connection with section 7, that there are corresponding results for the cohomology of nonselfadjoint operator algebras. See references [14]-[17],[38],[39] in [167] as well as the recent papers [98] and [99]

We also mention two papers concerning Lie derivations on von Neumann algebras.

**Theorem 8.5 (Zhuraev 2011 [199])** *For a von Neumann algebra $M$ on a Hilbert space $H$ denote by $S(M)$ the unital $^*$-algebra of all measurable operators affiliated with $M$. If $M$ is a homogeneous von Neumann* *algebra of type $I_n$, then very Lie derivation on $S(M)$ can be uniquely decomposed into the sum of a derivation and a centre-valued trace.*

Let $M$ be a von Neumann algebra of operators on a Hilbert space. An $n$-Lie derivation on $M$ is a linear mapping $L$ on $M$ such that, for every element $a \in M$ of the form $a := [\cdots[a_0, a_1], \cdots, a_{n-1}]$, the image $L(a)$ is the sum over $j$ of elements $[\cdots[a_0, a_1], \cdots, L(a_j)], \cdots]$ $(j = 0, \cdots, n-1)$, where $[u, v] = uv - vu$.

**Theorem 8.6 (Abdullaev 1992 [1])** *Every $n$-Lie derivation $L$ on some von Neumann algebra $M$ (or on its skew-adjoint part) can be decomposed as $L = D + E$, where $D$ is an ordinary derivation on $M$ and $E$ is a linear map from $M$ into its center which annihilates the Lie products.*

### 8.2.2 Cohomology of commutative JB*-triples and TROs

The first three cohomology groups of a commutative $C^*$-algebra with values in a Banach module were studied in the pioneering paper [116]. The cohomology groups of a finite dimensional associative triple system with values in the relevant module were studied in the [38]. By some combination of the techniques in these two papers, it should be possible to develop the cohomology of a commutative $JB^*$-triple with values in a module.

More generally, the cohomology of an arbitrary TRO (ternary ring of operators) should be pursued. Once this has been worked out for TROs, the author feels that the enveloping TRO of a $JC^*$-triple, discussed in section 11 may be helpful in working out the theory for $JC^*$-triples, as it seems to be for working out the $K$-theory of $JB^*$-triples, studied in [31].

### 8.2.3 Local derivations on $JB^*$-triples

Linear maps which agree with a derivation at each point are called local derivations. These have been studied in the Banach setting by Kadison [114], Johnson [110], Ajupov et. al [16],[17],[6], among others. Kadison proved that a continuous local derivation from a von Neumann algebra into a Banach module is a derivation.

Michael Mackey gave a talk on this topic at the conference in honor of Cho-Ho Chu's 65th birthday in May 2012 in Hong Kong. He proved that every continuous local derivation on a $JBW^*$-triple is a derivation, and he suggested some problems, among them whether every local derivation on a $JB^*$-triple into itself or into a Banach module is automatically continuous, and if so, whether it is a derivation. There are other problems in this area, some involving nonlinear maps.

### 8.2.4 Some other avenues to pursue

- Wedderburn decompositions for JB*-triples (Kühn-Rosendahl [126])
- Low dimensional cohomology for JBW*-triples and algebras-perturbation (Dosi [57], McCrimmon [139])
- Structure group of JB*-triple (McCrimmon [139], Meyberg [140])
- Alternative Banach triples (Carlsson [39], Braun [33])
- Completely bounded triple cohomology (Bunce-Feely-Timoney [35], Bunce-Timoney [36], Christensen-Effros-Sinclair [44])
- Lie algebraic techniques. Applications of the Koecher-Kantor-Tits construction([49])

# Part III
# Quantum functional analysis

## 9 Hilbertian operator spaces

We are going to quote some definitions and theorems from [160]. Operator space theory may also be considered as noncommutative Banach space theory. Although the subject had its genesis in the papers [187] and [13] (announcement in [14]), the first impetus appeared in the thesis of Ruan [170]. An operator space is a Banach space together with an isometric embedding in $B(H)$. While the objects of this category are Banach spaces, it is the morphisms, namely, the completely bounded maps, which are more important.

An operator space is a subspace $X$ of $B(H)$. Its *operator space structure* is given by the sequence of norms on the set of matrices $M_n(X)$ with entries from $X$, determined by the identification $M_n(X) \subset M_n(B(H)) = B(H \oplus H \oplus \cdots \oplus H)$. A linear mapping $\varphi : X \to Y$ between two operator spaces is *completely bounded* if the induced mappings $\varphi_n : M_n(X) \to M_n(Y)$ defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ satisfy $\|\varphi\|_{\mathrm{cb}} := \sup_n \|\varphi_n\| < \infty$.

Some simple examples of completely bounded maps are restrictions of $^*$-homomorphisms and multiplication operators.

**Theorem 9.1** *Every completely bounded map is the product of the above two mentioned maps.*

Some special types of completely bounded maps which we will define and consider are complete contractions, complete isometries, complete isomorphisms, complete semi-isometries

**Remark 9.2** *The space of completely bounded maps with the completely bounded norm $CB(X,Y)$, $\|\cdot\|_{cb}$ is a Banach space.*

**Definition 9.3** *The completely bounded Banach-Mazur distance is defined by*

$$d_{cb}(E,F) = \inf\{\|u\|_{cb} \cdot \|u^{-1}\|_{cb} : u : E \to F \textit{ complete isomorphism } \}.$$

Besides the pioneering work of Stinespring and Arveson mentioned above, many tools were developed in the 1970s and 1980s by a number of operator algebraists. As noted above, an abstract framework was developed in 1988 in the thesis of Ruan. Besides [160] we have already mentioned four other monographs on the subject ([68],[152],[25],[90]).

### 9.1 Classical operator spaces

"Neoclassical" and "modern" operator spaces will appear later.

Two important examples of Hilbertian operator spaces (:= operator spaces isometric to Hilbert space) are the row and column spaces $R$, $C$, and their finite-dimensional versions $R_n$, $C_n$. In $B(\ell_2)$, *column Hilbert space* $C := \overline{\mathrm{sp}}\{e_{i1} : i \geq 1\}$ and *row Hilbert space* $R := \overline{\mathrm{sp}}\{e_{1j} : j \geq 1\}$. $R$ and $C$ are Banach isometric, but not completely isomorphic. ($\mathrm{d}_{\mathrm{cb}}(R,C) = \infty$) $R_n$ and $C_n$ are completely isomorphic, but not completely isometric. $\mathrm{d}_{\mathrm{cb}}(R_n, C_n) = n$

$R$, $C$, $R_n$, $C_n$ are examples of *homogeneous* operator spaces, that is, operator spaces $E$ for which $\forall u : E \to E$, $\|u\|_{\mathrm{cb}} = \|u\|$. Another important example of an Hilbertian homogeneous operator space is $\Phi(I)$. $\Phi(I) = \overline{\mathrm{sp}}\{V_i : i \in I\}$, where the $V_i$ are bounded operators on a Hilbert space satisfying the canonical anti-commutation relations. In some special cases, the notations $\Phi_n := \Phi(\{1,2,\ldots,n\})$, and $\Phi = \Phi(\{1,2,\ldots\})$ are used.

If $E_0 \subset B(H_0)$ and $E_1 \subset B(H_1)$ are operator spaces whose underlying Banach spaces form a compatible pair in the sense of interpolation theory, then the Banach space $E_0 \cap E_1$, with the norm

$$\|x\|_{E_0 \cap E_1} = \max(\|x\|_{E_0}, \|x\|_{E_1})$$

equipped with the operator space structure given by the embedding $E_0 \cap E_1 \ni x \mapsto (x,x) \in E_0 \oplus E_1 \subset B(H_0 \oplus H_1)$ is called the *intersection* of $E_0$ and $E_1$ and is denoted by $E_0 \cap E_1$ Examples are $R \cap C$, $\Phi =$

$\cap_0^\infty H_\infty^{m,L}$, $\Phi_n = \cap_1^n H_n^k$. The definition of intersection extends easily to arbitrary families of compatible operator spaces

We already noted that $R,\ C,\ R_n,\ C_n$ are homogenous operator spaces. So are $\min(E)$ and $\max(E)$ and $\Phi(I)$, where $\min(E)$ is defined by $E \subset C(T) \subset B(H)$, $\|(a_{ij})\|_{M_n(\min(E))} = \sup_{\xi\in B_{E^*}} \|(\xi(a_{ij}))\|_{M_n}$ and $\max(E)$ is defined by $\|(a_{ij})\|_{M_n(\max(E))} = \sup\{\|(u(a_{ij}))\|_{M_n(B(H_u))} : u : E \to B(H_u),\ \|u\| \leq 1\}$. In the following diagram, the identity map is completely contractive $F \xrightarrow{u} \min(E) \to E \to \max(E) \xrightarrow{v} G$, $(\|u\|_{\text{cb}} = \|u\|, \quad \|v\|_{\text{cb}} = \|v\|)$

The following is an application of the Russo-Dye theorem [174]

**Proposition 9.4** *Let $E$ be a Hilbertian operator space. Then $E$ is homogeneous if and only if $\|U\|_{cb} = 1$ $\forall$ unitary $U : E \to E$.*

The following are considered to be among the "classical" Banach spaces $\ell_p,\ c_0,\ L_p,\ C(K)$. A "second generation" of such spaces would contain Orlicz , Sobolev, and Hardy spaces, the disk algebra, and the Schatten $p$-classes. By analogy, some classical operator spaces are $R,\ C,\ \min(\ell_2),\ \max(\ell_2),\ OH,\ \Phi$ and their finite dimensional versions $R_n,\ C_n,\ \min(\ell_2^n),\ \max(\ell_2^n),\ OH_n,\ \Phi_n$. Note that the classical operator spaces are all Hilbertian.

**Proposition 9.5** *The classical operator spaces are mutually completely non-isomorphic. If $E_n, F_n$ are $n$-dimensional version, then $d_{cb}(E_n, F_n) \to \infty$*

## 9.2 Neoclassical operator spaces

The "neoclassical" operator spaces $H_n^k$ appeared in [145] (announcement in [144]), from which the following three theorems were taken.

**Theorem 9.6** *There is a family of 1-mixed injective Hilbertian operator spaces $H_n^k$, $1 \leq k \leq n$, of finite dimension $n$, with the following properties:*

**(a)** *$H_n^k$ is a subtriple of the Cartan factor of type 1 consisting of all $\binom{n}{k}$ by $\binom{n}{n-k+1}$ complex matrices.*

**(b)** *Let $Y$ be a $JW^*$-triple of rank 1 (necessarily atomic ).*

- **(i)** *If $Y$ is of finite dimension $n$ then it is isometrically completely contractive to some $H_n^k$.*
- **(ii)** *If $Y$ is infinite dimensional then it is isometrically completely contractive to $B(H, \mathbf{C})$ or $B(\mathbf{C}, K)$.*

**(c)** *$H_n^n$ (resp. $H_n^1$) coincides with $R_n$ (resp. $C_n$).*

**(d)** *For $1 < k < n$, $H_n^k$ is not completely semi-isometric to $R_n$ or $C_n$.*

Example 1: $H_3^2$ is the subtriple of $B(\mathbf{C}^3)$ consisting of all matrices of the form

$$\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$$

and hence in this case $Y$ is actually completely semi-isometric to the Cartan factor $A(\mathbf{C}^3)$ of 3 by 3 anti-symmetric complex matrices.

Example 2: $H_4^3$ is the subtriple of $B(\mathbf{C}^6, \mathbf{C}^4)$ consisting of all matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 & -d & c & -b \\ 0 & d & -c & 0 & 0 & a \\ -d & 0 & b & 0 & -a & 0 \\ c & -b & 0 & a & 0 & 0 \end{bmatrix}.$$

**Theorem 9.7** *Let $X$ be a 1-mixed injective operator space which is atomic. Then $X$ is completely semi-isometric to a direct sum of Cartan factors of types 1 to 4 and the spaces $H_n^k$.*

**Theorem 9.8** *Let $Y$ be an atomic $w^*$-closed $JW^*$-subtriple of a $W^*$-algebra.*

**(a)** *If $Y$ is irreducible and of rank at least 2, then it is completely isometric to a Cartan factor of type 1–4 or the space*
*$Diag(B(H,K), B(K,H))$.*

**(b)** *If $Y$ is of finite dimension $n$ and of rank 1, then it is completely isometric to*
*$Diag(H_n^{k_1}, \ldots, H_n^{k_m})$, for appropriately chosen bases, and where $k_1 > k_2 > \cdots > k_m$.*

**(c)** *$Y$ is completely semi-isometric to a direct sum of the spaces in (a) and (b). If $Y$ has no infinite dimensional rank 1 summand, then it is completely isometric to a direct sum of the spaces in (a) and (b).*

We now describe a more detailed description of the spaces $H_n^k$, including the next two theorems from [147].

A frequently mentioned result of Friedman and Russo states that if a subspace $X$ of a C*-algebra $A$ is the range of a contractive projection on $A$, then $X$ is isometric to a JC*-triple, that is, a norm closed subspace of $B(H,K)$ stable under the triple product $ab^*c + cb^*a$. If $X$ is atomic (in particular, finite-dimensional), then it is isometric to a direct sum of Cartan factors of types 1 to 4. This latter result fails, as it stands, in the category of operator spaces.

Nevertheless, there exists a family of $n$-dimensional Hilbertian operator spaces $H_n^k$, $1 \le k \le n$, generalizing the row and column Hilbert spaces $R_n$ and $C_n$ such that, in the above result, if $X$ is atomic, the word "isometric" can be replaced by "completely semi-isometric," provided the spaces $H_n^k$ are allowed as summands along with the Cartan factors. The space $H_n^k$ is contractively complemented in some $B(K)$, and for $1 < k < n$, is not completely (semi-)isometric to either of the Cartan factors $B(\mathbf{C}, \mathbf{C}^n) = H_n^1$ or $B(\mathbf{C}^n, \mathbf{C}) = H_n^n$. These spaces appeared in a slightly different form and context in a memoir of Arazy and Friedman [10].

The construction of $H_n^k$ is as follows. Let $I$ denote a subset of $\{1, 2, \ldots, n\}$ of cardinality $|I| = k-1$. The number of such $I$ is $q := \binom{n}{k-1}$. Let $J$ denote a subset of $\{1, 2, \ldots, n\}$ of cardinality $|J| = n-k$. The number of such $J$ is $p := \binom{n}{n-k}$. The space $H_n^k$ is the linear span of matrices $b_i^{n,k}$, $1 \le i \le n$, given by

$$b_i^{n,k} = \sum_{I \cap J = \emptyset, (I \cup J)^c = \{i\}} \epsilon(I, i, J) e_{J,I},$$

where $e_{J,I} = e_J \otimes e_I = e_J e_I^t \in M_{p,q}(\mathbf{C}) = B(\mathbf{C}^q, \mathbf{C}^p)$, and $\epsilon(I, i, J)$ is the signature of the permutation taking $(i_1, \ldots, i_{k-1}, i, j_1, \ldots, j_{n-k})$ to $(1, \ldots, n)$. Since the $b_i^{n,k}$ are the image under a triple isomorphism (actually ternary isomorphism) of a rectangular grid in a JC*-triple of rank one, they form an orthonormal basis for $H_n^k$.

**Theorem 9.9** *$H_n^k$ is a homogeneous operator space.*

**Theorem 9.10** *$d_{cb}(H_n^k, H_n^1) = \sqrt{\frac{kn}{n-k+1}}$, for $1 \le k \le n$.*

The latter theorem is proved using the relation with creation operators. Let $C_h^{n,k}$ denote the wedge (or creation) operator from $\wedge^{k-1}\mathbf{C}^n$ to $\wedge^k \mathbf{C}^n$ given by

$$C_h^{n,k}(h_1 \wedge \cdots \wedge h_{k-1}) = h \wedge h_1 \wedge \cdots \wedge h_{k-1}.$$

Letting $\mathcal{C}^{n,k}$ denote the space $\mathrm{sp}\{C_{e_i}^{n,k}\}$, we have the following.

**Proposition 9.11** *$H_n^k$ is completely isometric to $\mathcal{C}^{n,k}$.*

*Proof.* There exist unitaries such that

$$W_k^n U_{n-k}^n b_i^{n,k} = V_k^n C_{e_i}^{n,k} U_{k-1}^n.$$

$$
\begin{array}{ccc}
\mathbf{C}^q & \xrightarrow{b_i^{n,k}} & \mathbf{C}^p \\
U^n_{k-1} \downarrow & & \downarrow U^n_{n-k} \\
\wedge^{k-1}\mathbf{C}^n & & \wedge^{n-k}\mathbf{C}^n \\
C^{n,k}_{e_i} \downarrow & & \downarrow W^n_k \\
\wedge^k\mathbf{C}^n & \xrightarrow{V^n_k} & \wedge^{n-k}\mathbf{C}^n
\end{array}
$$

- $U^n_{k-1}(e_I) = e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}$, where $I = \{i_1 < \cdots < i_{k-1}\}$. $U^n_{n-k}$ is similar.
- $V^n_k(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$, where $\{j_1 < \cdots < j_{n-k}\}$ is the complement of $\{i_1 < \cdots < i_k\}$.
- $W^n_k(e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}) = \epsilon(i, I)\epsilon(I, i, J)e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$ for any $i$ and $I$ such that $I \cap J = \emptyset$ and $(I \cup J)^c = \{i\}$ (which is independent of the choice of $i$ or $I$).

The following properties of creation operators are used in the proof of Theorem 9.10.

- $C^{n,k*}_h C^{n,k}_h(h_1 \wedge \cdots \wedge h_{k-1}) =$
  $(h|h)h_1 \wedge \cdots \wedge h_{k-1} - (h_1|h)h \wedge h_2 \wedge \cdots \wedge h_{k-1}$
  $$+ \cdots \pm (h_{k-1}|h)h \wedge h_1 \wedge \cdots \wedge h_{k-2}. \tag{7}$$
- $C^{n,k}_h C^{n,k*}_h(h_1 \wedge \cdots \wedge h_k) =$
  $\sum_{j=1}^k (h_j|h)h_1 \wedge \cdots \wedge h'_j \wedge \cdots \wedge h_k \quad (h'_j = h)$.

  In particular, $C^{n,1}_h C^{n,1*}_h = h \otimes \overline{h}$, for $h \in \mathbf{C}^n$.
- $\mathrm{tr}(C^{n,k*}_h C^{n,k}_h) = \binom{n-1}{k-1}\|h\|^2$.

  In particular, $C^{n,1*}_h C^{n,1}_h = \|h\|^2$.
- The eigenvalues of $\sum_{i=1}^m C^{n,k}_{h_i} C^{n,k*}_{h_i}$ are sums of $k$ eigenvalues of $\sum_{i=1}^m C^{n,1}_{h_i} C^{n,1*}_{h_i}$.

## 9.3 Modern operator spaces

The following problems were considered in [143] and leads to our notion of modern operator space. This subsection is a description of the results of [143].

- Classify all infinite dimensional rank 1 JC*-triples up to complete isometry (**algebraic**)
  ANSWER: $\Phi$, $H^{m,R}_\infty$, $H^{m,L}_\infty$, $H^{m,R}_\infty \cap H^{m,L}_\infty$
- Give a suitable "classification" of all Hilbertian operator spaces which are contractively complemented in a C*-algebra or normally contractively complemented in a W*-algebra (**analytic**)
  ANSWER: $\Phi$, $\mathbf{C}$, $\mathbf{R}$, $\mathbf{C} \cap \mathbf{R}$

It is known that $\mathbf{R}$ and $\mathbf{C}$ are the only **completely** contractively complemented Hilbertian operator spaces [168].

### 9.3.1 Rank one JC*-triples

In order to attack the structure theory of rank 1 JC*-triples, let us first review some of the basic concepts about them, as well as those concerning ternary rings of operators (TROs).

A $JC^*$-triple is a norm closed complex linear subspace of $B(H,K)$ (equivalently, of a $C^*$-algebra) which is closed under the operation $a \mapsto aa^*a$. $JC^*$-triples were defined and studied (using the name $J^*$-algebra) as a generalization of $C^*$-algebras by Harris [86] in connection with function theory on infinite dimensional bounded symmetric domains. By a polarization identity (involving $\sqrt{-1}$), any $JC^*$-triple is closed under the triple product

$$(a,b,c) \mapsto \{abc\} := \frac{1}{2}(ab^*c + cb^*a), \tag{8}$$

under which it becomes a Jordan triple system. A linear map which preserves the triple product (8) will be called a *triple homomorphism.* Cartan factors are examples of $JC^*$-triples, as are $C^*$-algebras, and Jordan $C^*$-algebras. We shall only make use of Cartan factors of type 1, that is, spaces of the form $B(H,K)$ where $H$ and $K$ are complex Hilbert spaces.

A special case of a $JC^*$-triple is a *ternary algebra*, that is, a subspace of $B(H,K)$ closed under the *ternary product* $(a,b,c) \mapsto ab^*c$. A *ternary homomorphism* is a linear map $\phi$ satisfying $\phi(ab^*c) = \phi(a)\phi(b)^*\phi(c)$. These spaces are also called ternary rings of operators and abbreviated TRO. TROs have come to play a key role in operator space theory, serving as the algebraic model in the category. The algebraic models for the categories of order-unit spaces, operator systems, and Banach spaces, are respectively Jordan $C^*$-algebras, $C^*$-algebras, and $JB^*$-triples. For TROs, a ternary isomorphism is the same as a complete isometry.

Every $JW^*$-triple of rank one is isometric to a Hilbert space and every maximal collinear family of partial isometries corresponds to an orthonormal basis. Conversely, every Hilbert space with the abstract triple product $\{xyz\} := ((x|y)z + (z|y)x)/2$ can be realized as a $JC^*$-triple of rank one in which every orthonormal basis forms a maximal family of mutually collinear minimal partial isometries. Collinear means:

$$vv^*w + wv^*v = w \text{ and } ww^*v + vw^*w = v$$

### 9.3.2 Operator space structure of Hilbertian JC*-triples

We shall now outline the proof from [143] of the classification of infinite dimensional Hilbertian operator spaces up to complete isometry .

- The general setting: $Y$ is a $JC^*$-subtriple of $B(H)$ which is Hilbertian in the operator space structure arising from $B(H)$, and $\{u_i : i \in \Omega\}$ is an orthonormal basis consisting of a maximal family of mutually collinear partial isometries of $Y$.

- We let $T$ and $A$ denote the TRO and the $C^*$-algebra respectively generated by $Y$. For any subset $G \subset \Omega$, $(uu^*)_G := \prod_{i\in G} u_i u_i^*$ and $(u^*u)_G := \prod_{i\in G} u_i^* u_i$. The elements $(uu^*)_G$ and $(u^*u)_G$ lie in the weak closure of $A$ and more generally in the left and right linking von Neumann algebras of $T$.

  Fix $m \geq 0$. To construct $H_\infty^{m,R}$ we make a temporary assumption on the ternary envelope of $Y$.

- **Assume** $(u^*u)_G \neq 0$ for $|G| \leq m+1$ and $(u^*u)_G = 0$ for $|G| \geq m+2$.

- Define elements which are indexed by an arbitrary pair of subsets $I, J$ of $\Omega$ satisfying

  $$|\Omega - I| = m+1,\ |J| = m, \tag{9}$$

  as follows:

  $u_{IJ} = (uu^*)_{I-J} u_{c_1} u_{d_1}^* u_{c_2} u_{d_2}^* \cdots u_{c_s} u_{d_s}^* u_{c_{s+1}} (u^*u)_{J-I}$, where

  $I \cap J = \{d_1, \ldots, d_s\}$ and $(I \cup J)^c = \{c_1, \ldots, c_{s+1}\}$.

- In the special case where $I \cap J = \emptyset$, $u_{I,J}$ has the form

  $$u_{I,J} = (uu^*)_I u_c (u^*u)_J,$$

  where $I \cup J \cup \{c\} = \Omega$ is a partition of $\Omega$. We call such an element a "one", and denote it also by $u_{I,c,J}$.

- **Lemma 9.12** *Fix $m \geq 0$. Assume $(u^*u)_G \neq 0$ for $|G| \leq m+1$ and $(u^*u)_G = 0$ for $|G| \geq m+2$. For any $c \in \Omega$,*

$$u_c = \sum_{I,J} u_{I,J} = \sum_{I,J} u_{I,c,J} \tag{10}$$

*where the sum is taken over all disjoint $I, J$ satisfying (9) and not containing $c$, and converges weakly in the weak closure of $T$.*

- The family $\{\epsilon(IJ)u_{I,J}\}$ forms a rectangular grid which satisfies the extra property

$$\epsilon(IJ)u_{IJ}[\epsilon(IJ')u_{IJ'}]^*\epsilon(I'J')u_{I'J'} = \epsilon(I'J)u_{I'J}. \tag{11}$$

- The map $\epsilon(IJ)u_{IJ} \to E_{JI}$ is a ternary isomorphism (and hence complete isometry) from the norm closure of $\mathrm{sp}_C\, u_{IJ}$ to the norm closure of $\mathrm{sp}_C\,\{E_{JI}\}$, where $E_{JI}$ denotes an elementary matrix, whose rows and columns are indexed by the sets $J$ and $I$, with a 1 in the $(J, I)$-position.

- This map can be extended to a ternary isomorphism from the w*-closure of $\mathrm{sp}_C\, u_{IJ}$ onto the Cartan factor of type I consisting of all $\aleph_0$ by $\aleph_0$ complex matrices which act as bounded operators on $\ell_2$. By restriction to $Y$ and (10), $Y$ is completely isometric to a subtriple $\tilde{Y}$, of this Cartan factor of type 1.

- **Definition 9.13** *We shall denote the space $\tilde{Y}$ above by $H_\infty^{m,R}$.*

  Explicitly, $H_\infty^{m,R} = \overline{\mathrm{sp}}_C\{b_i^m : i \in \mathbf{N}\}$, where

$$b_i^m = \sum_{I\cap J=\emptyset,(I\cup J)^c=\{i\},|J|=m} \epsilon(I,i,J)e_{J,I}.$$

- An entirely symmetric argument (with $J$ infinite and $I$ finite) under an entirely symmetric assumption on $Y$ defines the space $H_\infty^{m,L}$.

- Explicitly, $H_\infty^{m,L} = \overline{\mathrm{sp}}_C\{\tilde{b}_i^m : i \in \mathbf{N}\}$, where

$$\tilde{b}_i^m = \sum_{I\cap J=\emptyset,(I\cup J)^c=\{i\},|I|=m} \epsilon(I,i,J)e_{J,I},$$

  with $\epsilon(I, i, J)$ defined in the obvious analogous way with $I$ finite instead of $J$.

- Having constructed the spaces $H_\infty^{m,R}$ and $H_\infty^{m,L}$, we now assume that $Y$ is an arbitrary rank-one JC*-subtriple of B(H)

  Our analysis will consider the following three mutually exhaustive and mutually exclusive possibilities (in each case, the set $F$ is allowed to be empty):

  **Case 1** $(uu^*)_{\Omega-F} \neq 0$ for some finite set $F \subset \Omega$;

  **Case 2** $(u^*u)_{\Omega-F} \neq 0$ for some finite set $F \subset \Omega$;

  **Case 3** $(uu^*)_{\Omega-F} = (u^*u)_{\Omega-F} = 0$ for all finite subsets $F$ of $\Omega$.

- In case 1,

  $Y$ is completely isometric to an intersection $Y_1 \cap Y_2$ such that $Y_1$ is completely isometric to a space $H_\infty^{m,R}$, and $Y_2$ is a Hilbertian $JC^*$-triple.

- In case 2,

  $Y$ is completely isometric to an intersection $Y_1 \cap Y_2$ such that $Y_1$ is completely isometric to a space $H_\infty^{m,L}$, and $Y_2$ is an Hilbertian $JC^*$-triple.

- In case 3,

  $Y$ is completely isometric to $\Phi$.

**Theorem 9.14** *Let $Y$ be a $JC^*$-subtriple of $B(H)$ which is a separable infinite dimensional Hilbertian operator space. Then $Y$ is completely isometric to one of the following spaces:*

$$\Phi, \quad H_\infty^{m,R}, \quad H_\infty^{m,L}, \quad H_\infty^{m,R} \cap H_\infty^{n,L}.$$

*Proof.* Let $\equiv$ denote "completely isometric to."

1. $Y \equiv \Phi$, or $Y = Y_1 \cap Z$, where

   $Y_1 \equiv H_\infty^{m,R}$ or $H_\infty^{k,L}$.

2. $Y_1 \equiv H_\infty^{m_1,R} \Rightarrow Y = \mathcal{R} \cap Z$, where $\mathcal{R} = \cap H_\infty^{m_j,R}$, $m_j \uparrow$ (finite or infinite).

3. $Z \equiv \Phi$, or $Z = Z_1 \cap W$, where $Z_1 \equiv H_\infty^{k,L}$.

4. $Z = \mathcal{L} \cap W$, where $\mathcal{L} = \cap_j H_\infty^{k_j,L}$, $k_j \uparrow$. Hence $Y = \mathcal{R} \cap \mathcal{L} \cap \Phi$.

5. $H_\infty^{m',L} \to H_\infty^{m,L}$ is a complete contraction when $m' > m$ (and similarly for $H_\infty^{k,R}$).

6. $\Phi$ is completely isometric to $\cap_{m=0}^\infty H_\infty^{m,L}$ and to $\cap_{m=1}^\infty H_\infty^{m,R}$ and to $\cap_{j=1}^\infty H_\infty^{n_j,L}$ for any sequence $n_j \to \infty$.

### 9.3.3 Further properties of $H_\infty^{m,R}$ and $H_\infty^{m,L}$

We now describe, from [143] the relation of these spaces with Fock space and compute some completely bounded Banach-Mazur distances.

If $H$ is a separable Hilbert space, $l_m(h) : H^{\wedge m} \to H^{\wedge m+1}$ is the creation operator $l_m(h)x = h \wedge x$. Creation operators form a linear space $\mathcal{C}^m = \overline{\text{sp}}\{l_m(e_i)\}$, where $\{e_i\}$ is an orthonormal basis, and inherit an operator space structure from $B(H^{\wedge m}, H^{\wedge m+1})$. Annihilation operators $\mathcal{A}^m$ consists of the adjoints of the creation operators on $H^{\wedge m-1}$.

**Lemma 9.15** *$H_\infty^{m,R}$ is completely isometric to $\mathcal{A}^{m+1}$ and $H_\infty^{m,L}$ is completely isometric to $\mathcal{C}^m$.*

**Remark 9.16** *Every finite or infinite dimensional separable Hilbertian $JC^*$-subtriple $Y$ is completely isometric to a finite or infinite intersection of spaces of creation and annihilation operators, as follows*

**(a)** *If $Y$ is infinite dimensional, then it is completely isometric to one of $\mathcal{A}^m$, $\mathcal{C}^m$, $\mathcal{A}^m \cap \mathcal{C}^k$, $\cap_{k=1}^\infty \mathcal{C}^k$*

**(b)** *If $Y$ is of dimension $n$, then $Y$ is completely isometric to $\cap_{j=1}^m \mathcal{C}^{k_j}$, where $n \geq k_1 > \cdots > k_m \geq 1$*

**Theorem 9.17** *For $m$, $k \geq 1$,*

**(a)** $d_{cb}(H_\infty^{m,R}, H_\infty^{k,R}) = d_{cb}(H_\infty^{m,L}, H_\infty^{k,L}) = \sqrt{\frac{m+1}{k+1}}$ *when $m \geq k$*

**(b)** $d_{cb}(H_\infty^{m,R}, H_\infty^{k,L}) = \infty$

**(c)** $d_{cb}(H_\infty^{m,R}, \Phi) = d_{cb}(H_\infty^{m,L}, \Phi) = \infty$

### 9.3.4 Contractively complemented Hilbertian operator spaces

**Theorem 9.18** *Suppose $Y$ is a separable infinite dimensional Hilbertian operator space which is contractively complemented (resp. normally contractively complemented) in a $C^*$-algebra $A$ (resp. $W^*$-algebra $A$) by a projection $P$.*

*Then,*

**(a)** *$\{Y, A^{**}, P^{**}\}$ (resp. $\{Y, A, P\}$) is an expansion of its support $\{H, A^{**}, Q\}$ (resp. $\{H, A^{**}, Q\}$, which is essential)*

**(b)** *$H$ is contractively complemented in $A^{**}$ (resp. $A$) by $Q$ and is completely isometric to either $R$, $C$, $R \cap C$, or $\Phi$.*

**Theorem 9.19 (Converse)** *The operator spaces $R, C, R\cap C$, and $\Phi$ are each essentially normally contractively complemented in a von Neumann algebra.*

# 10 Quantum operator algebras

## 10.1 Operator space characterization of TROs

In the category of operator spaces, that is, subspaces of the bounded linear operators $B(H)$ on a complex Hilbert space $H$ together with the induced matricial operator norm structure, objects are equivalent if they are completely isometric, *i.e.* if there is a linear isomorphism between the spaces which preserves this matricial norm structure. Since operator algebras, that is, subalgebras of $B(H)$, are motivating examples for much of operator space theory, it is natural to ask if one can characterize which operator spaces are operator algebras. One satisfying answer was given by Blecher, Ruan and Sinclair in [27], where it was shown that among operator spaces $A$ with a (unital but not necessarily associative) Banach algebra product, those which are completely isometric to operator algebras are precisely the ones whose multiplication is completely contractive with respect to the Haagerup norm on $A \otimes A$ (For a completely bounded version of this result, see [23]).

A natural object to characterize in this context are the so called *ternary rings of operators* (TRO's). These are subspaces of $B(H)$ which are closed under the ternary product $xy^*z$. This class includes C*-algebras. TRO's, like C*-algebras, carry a natural operator space structure. In fact, every TRO is (completely) isometric to a corner $pA(1-p)$ of a C*-algebra $A$. TRO's are important because, as shown by Ruan [171], the injectives in the category of operator spaces are TRO's (corners of injective C*-algebras) and not, in general, operator algebras (For the dual version of this result see [66]). Injective envelopes of operator systems and of operator spaces ([83],[171]) have proven to be important tools, see for example [26]. The characterization of TRO's among operator spaces is the subject of this subsection.

Closely related to TRO's are the so called JC*-triples, norm closed subspaces of $B(H)$ which are closed under the triple product $(xy^*z + zy^*x)/2$. These generalize the class of TRO's and have the property, as shown by Harris in [86], that isometries coincide with algebraic isomorphisms. It is not hard to see this implies that the algebraic isomorphisms in the class of TRO's are complete isometries, since for each TRO $A$, $M_n(A)$ is a JC*-triple (For the converse of this, see [84, Proposition 2.1]). As a consequence, if an operator space $X$ is completely isometric to a TRO, then the induced ternary product on $X$ is unique, *i.e.*, independent of the TRO.

Building on the pioneering work of Arveson ([13],[15]) on noncommutative analogs of the Choquet and Shilov boundaries, Hamana (see [84]) proved that every operator space $A$ has a unique enveloping TRO $\mathcal{T}(A)$ which is an invariant of complete isometry and has the property that for any TRO $B$ generated by a realization of $A$, there exists a homomorphism of $B$ onto $\mathcal{T}(A)$. The space $\mathcal{T}(A)$ is also called the *Hilbert C*-envelope* of $A$. The work in [24] suggests that the Hilbert C*-envelope is an appropriate noncommutative generalization to operator spaces of the classical theory of Shilov boundary of function spaces.

It is also true that a commutative TRO ($xy^*z = zy^*x$) is an associative JC*-triple and hence by [71, Theorem 2], is isometric (actually completely isometric) to a complex $C_{hom}$-space, that is, the space of weak*-continuous functions on the set of extreme points of the unit ball of the dual of a Banach space which are homogeneous with respect to the natural action of the circle group, see [71]. Hence, if one views operator spaces as noncommutative Banach spaces, and C*-algebras as noncommutative $C(\Omega)$'s, then TRO's and JC*-triples can be viewed as noncommutative $C_{hom}$-spaces.

As noted above, injective operator spaces, *i.e.*, those which are the range of a *completely* contractive projection on some $B(H)$, are completely isometrically TRO's; the so called *mixed injective* operator spaces, those which are the range of a contrative projection on some $B(H)$, are isometrically JC*-triples. The operator space classification of mixed injectives was begun in [145] and is described in the preceding subsection.

Relevant to this subsection is another property shared by all JC*-triples (and hence all TRO's). For any Banach space $X$, we denote by $X_0$ its open unit ball: $\{x \in X : \|x\| < 1\}$. The open unit ball of every JC*-triple is a *bounded symmetric domain*. This is equivalent to saying that it has a transitive group of biholomorphic automorphisms. It was shown by Koecher in finite dimensions (see [133]) and Kaup [118] in the general case that this is a defining property for the slightly larger class of JB*-triples. The only illustrative basic examples of JB*-triples which are not JC*-triples are the space $H_3(\mathcal{O})$ of 3 x 3 hermitian matrices over the octonians and a certain subtriple of $H_3(\mathcal{O})$. These are called *exceptional* triples, and they cannot be represented as a JC*-triple. This holomorphic characterization has been useful as it gives an elegant proof, due to Kaup [119], that the range of a contractive projection on a JB*-triple is isometric to another JB*-triple. The same statement holds for JC*-triples, as proven earlier by Friedman and Russo in [73]. Youngson proved in [197] that the range of a completely contractive projection on a C*-algebra is completely isometric to a TRO. These results, as well as those of [10] and [67], are rooted in the fundamental result of Choi-Effros [41] for completely positive projections on C*-algebras and the classical result ([129],[70, Theorem 5]) that the range of a contractive projection on $C(\Omega)$ is isometric to a $C_\sigma$-space, hence a $C_{hom}$-space.

Motivated by this characterization for JB*-triples, a holomorphic characterization of TRO's up to complete isometry is given in [146] and stated below. As a consequence, a holomorphic operator space characterization of C*-algebras is obtained as well. It should be mentioned that Upmeier (for the category of Banach spaces) in [192] and El Amin-Campoy-Palacios (for the category of Banach algebras) in [2], gave different but still holomorphic characterizations of C*-algebras up to isometry. We note in passing that injective operator spaces satisfy the hypothesis of Theorem 10.1, so it follows that they are (completely isometrically) TRO's based on deep results about JB*-triples rather than the deep result of Choi-Effros.

**Theorem 10.1** *Let $A \subset B(H)$ be an operator space and suppose that $M_n(A)_0$ is a bounded symmetric domain for all $n \geq 2$, then $A$ is ternary isomorphic and completely isometric to a TRO.*

**Theorem 10.2** *Let $A \subset B(H)$ be an operator space and suppose that $M_n(A)_0$ is a bounded symmetric domain for all $n \geq 2$ and $A_0$ is of tube type. Then $A$ is completely isometric to a C*-algebra.*

## 10.2 Holomorphic characterization of operator algebras

If an operator space $A$ (i.e., a closed linear subspace of $B(H)$) is also a unital (not necessarily associative) Banach algebra with respect to a product which is completely contractive, then according to [27], it is completely isometric and algebraically isomorphic to an operator algebra (i.e., an associative subalgebra of some $B(K)$). The main result of [149] drops the algebra assumption on $A$ in favor of a holomorphic assumption. Using only natural conditions on holomorphic vector fields on Banach spaces, an algebra product is constructed on $A$ which is completely contractive and unital, so that the result of [27] can be applied.

We state this result as Theorem 10.3. In this theorem, for any element $v$ in the symmetric part (recalled below) of a Banach space $X$, $h_v$ denotes the corresponding complete holomorphic vector field on the open unit ball of $X$.

**Theorem 10.3** *An operator space $A$ is completely isometric to a unital operator algebra if and only there exists an element $v$ in the completely symmetric part of $A$ such that:*

1. $h_v(x+v) - h_v(x) - h_v(v) + v = -2x$ *for all* $x \in A$
2. *For all* $X \in M_n(A)$, $\|V - h_V(X)\| \leq \|X\|^2$.

This result is thus an instance where the consideration of a ternary product, called the partial triple product, which arises from the holomorphic structure via the symmetric part of the Banach space, leads to results for binary products. Examples of this phenomenon occurred in [8], [12] where this technique is used to describe the algebraic properties of isometries of certain operator algebras. The technique was also used in [120] to show that Banach spaces with holomorphically equivalent unit balls are linearly isometric (see [7] for an exposition of [120]).

Suppose that $X$ is a TRO (i.e., a closed subspace of $B(H)$ closed under the ternary product $ab^*c$) which contains an element $v$ satisfying $xv^*v = vv^*x = x$ for all $x \in X$. Then it is trivial that $X$ becomes a unital C*-algebra for the product $xv^*y$, involution $vx^*v$, and unit $v$. By comparison, the above result starts only with an operator space $X$ containing a distinguished element $v$ in the completely symmetric part of $X$

(defined below) having a unit-like property. This is to be expected since the result of [27] fails in the absence of a unit element.

The main technique in the proof of this result is to use a variety of elementary isometries on $n$ by $n$ matrices over $A$ (most of the time, $n = 2$) and to exploit the fact that isometries of arbitrary Banach spaces preserve the partial triple product. The first occurrence of this technique appears in the construction, for each $n$, of a contractive projection $P_n$ on $K\overline{\otimes}A$ ($K$= compact operators on separable infinite dimensional Hilbert space) with range $M_n(A)$ as a convex combination of isometries. The completely symmetric part of $A$ is defined to be the intersection of $A$ (embedded in $K\overline{\otimes}A$) and the symmetric part of $K\overline{\otimes}A$. It is the image under $P_1$ of the symmetric part of $K\overline{\otimes}A$. It follows from [146] that the completely symmetric part of $A$ is a TRO, which is a crucial tool in this work.

The completely symmetric part of an arbitrary operator space $A$ is defined below. The binary product $x \cdot y$ on $A$ is constructed using properties of isometries on 2 by 2 matrices over $A$ and it is shown that the symmetrized product can be expressed in terms of the partial Jordan triple product on $A$.

According to [28], "The one-sided multipliers of an operator space $X$ are a key to the 'latent operator algebraic structure' in $X$." The unified approach through multiplier operator algebras developed in [28] leads to simplifications of known results and applications to quantum $M$-ideal theory. They also state "With the extra structure consisting of the additional matrix norms on an operator algebra, one might expect to not have to rely as heavily on other structure, such as the product." Our result is certainly in the spirit of this statement.

Another approach to operator algebras is [117], in which the set of operator algebra products on an operator space is shown to be in bijective correspondence with the space of norm one quasi-multipliers on the operator space.

### 10.2.1 Symmetric part of a Banach space

We review the construction and properties of the partial Jordan triple product in an arbitrary Banach space. Let $X$ be a complex Banach space with open unit ball $X_0$. Every holomorphic function $h : X_0 \to X$, also called a holomorphic vector field, is locally integrable, that is, the initial value problem

$$\frac{\partial}{\partial t}\varphi(t, z) = h(\varphi(t, z)) \ , \ \varphi(0, z) = z,$$

has a unique solution for every $z \in X_0$ for $t$ in a maximal open interval $J_z$ containing 0. A **complete holomorphic vector field** is one for which $J_z = (-\infty, \infty)$ for every $z \in X_0$.

It is a fact that every complete holomorphic vector field is the sum of the restriction of a skew-Hermitian bounded linear operator $A$ on $X$ and a function $h_a$ of the form $h_a(z) = a - Q_a(z)$, where $Q_a$ is a quadratic homogeneous polynomial on $X$.

The **symmetric part** of $X$ is the orbit of 0 under the set of complete holomorphic vector fields, and is denoted by $S(X)$. It is a closed subspace of $X$ and is equal to $X$ precisely when $X$ has the structure of a $JB^*$-triple (by [118]).

If $a \in S(X)$, we can obtain a symmetric bilinear form on $X$, also denoted by $Q_a$ via the polarization formula

$$Q_a(x, y) = \frac{1}{2}\left(Q_a(x + y) - Q_a(x) - Q_a(y)\right)$$

and then the partial Jordan triple product $\{\cdot, \cdot, \cdot\} : X \times S(X) \times X \to X$ is defined by $\{x, a, z\} = Q_a(x, z)$. The space $S(X)$ becomes a $JB^*$-triple in this triple product.

It is also true that the "main identity" (4) holds whenever $a, y, b \in S(X)$ and $x, z \in X$. The following lemma is an elementary consequence of the definitions.

**Lemma 10.4** *If $\psi$ is a linear isometry of a Banach space $X$ onto itself, then*

**(a)** *For every complete holomorphic vector field $h$ on $X_0$, $\psi \circ h \circ \psi^{-1}$ is a complete holomorphic vector field. In particular, for $a \in S(X)$, $\psi \circ h_a \circ \psi^{-1} = h_{\psi(a)}$.*

**(b)** *$\psi(S(X)) = S(X)$ and $\psi$ preserves the partial Jordan triple product:*

$$\psi\{x, a, y\} = \{\psi(x), \psi(a), \psi(y)\} \text{ for } a \in S(X),\ x, y \in X.$$

The symmetric part of a Banach space behaves well under contractive projections (see [7, 5.2,5.3]).

**Theorem 10.5 (Stacho [186])** *If $P$ is a contractive projection on a Banach space $X$ and $h$ is a complete holomorphic vector field on $X_0$, then $P \circ h|_{P(X)_0}$ is a complete holomorphic vector field on $P(X)_0$. In addition $P(S(X)) \subset S(X)$ and the partial triple product on $P(S(X))$ is given by $\{x, y, z\} = P\{x, y, z\}$ for $x, z \in P(X)$ and $y \in P(S(X))$.*

### 10.2.2 Completely symmetric part of an operator space

Let $A \subset B(H)$ be an operator space. We let $K$ denote the compact operators on a separable infinite dimensional Hilbert space, say $\ell_2$. Then $K = \overline{\cup_{n=1}^{\infty} M_n(\mathbf{C})}$ and thus

$$K\overline{\otimes}A = \overline{\cup_{n=1}^{\infty} M_n \otimes A} = \overline{\cup_{n=1}^{\infty} M_n(A)}$$

By an abuse of notation, we shall use $K \otimes A$ to denote $\cup_{n=1}^{\infty} M_n(A)$. We tacitly assume the embeddings $M_n(A) \subset M_{n+1}(A) \subset K\overline{\otimes}A$ induced by adding zeros.

The *completely symmetric part* of $A$ is defined by $CS(A) = A \cap S(K\overline{\otimes}A)$. More precisely, if $\psi : A \to M_1(A)$ denotes the complete isometry identification, then $CS(A) = \psi^{-1}(\psi(A) \cap S(K\overline{\otimes}A))$.

For $1 \leq m < N$ let $\psi_{1,m}^N : M_N(A) \to M_N(A)$ and $\psi_{2,m}^N : M_N(A) \to M_N(A)$ be the isometries of order two defined by

$$\psi_{j,m}^N : \begin{bmatrix} M_m(A) & M_{m,N-m}(A) \\ M_{N-m,m}(A) & M_{N-m}(A) \end{bmatrix} \to \begin{bmatrix} M_m(A) & M_{m,N-m}(A) \\ M_{N-m,m}(A) & M_{N-m}(A) \end{bmatrix}$$

and

$$\psi_{1,m}^N : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

and

$$\psi_{2,m}^N : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}.$$

These two isometries give rise in an obvious way to two isometries $\tilde{\psi}_{1,m}$ and $\tilde{\psi}_{2,m}$ on $K \otimes A$, which extend to isometries $\psi_{1,m}, \psi_{2,m}$ of $K\overline{\otimes}A$ onto itself, of order 2 and fixing elementwise $M_m(A)$. The same analysis applies to the isometries defined by, for example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & b \\ -c & -d \end{bmatrix}, \begin{bmatrix} -a & -b \\ c & d \end{bmatrix}, \begin{bmatrix} -a & b \\ c & -d \end{bmatrix}.$$

We then can define a projection $\tilde{P}_m$ on $K \otimes A$ with range $M_m(A)$ via

$$\tilde{P}_m x = \frac{\tilde{\psi}_{2,m}\left(\frac{\tilde{\psi}_{1,m}(x)+x}{2}\right) + \frac{\tilde{\psi}_{1,m}(x)+x}{2}}{2}.$$

The projection $\tilde{P}_m$ on $K \otimes A$ extends to a projection $P_m$ on $K\overline{\otimes}A$, with range $M_m(A)$ given by

$$P_m x = \frac{\psi_{2,m}\left(\frac{\psi_{1,m}(x)+x}{2}\right) + \frac{\psi_{1,m}(x)+x}{2}}{2},$$

or

$$P_m = \frac{1}{4}(\psi_{2,m}\psi_{1,m} + \psi_{2,m} + \psi_{1,m} + \mathrm{Id}).$$

**Proposition 10.6** *With the above notation,*

**(a)** $P_n(S(K\overline{\otimes}A)) = M_n(CS(A))$

**(b)** [30] *$M_n(CS(A))$ is a JB*-subtriple of $S(K\overline{\otimes}A)$, that is,*

$$\{M_n(CS(A)), M_n(CS(A)), M_n(CS(A))\} \subset M_n(CS(A));$$

*Moreover,*

$$\{M_n(A), M_n(CS(A)), M_n(A)\} \subset M_n(A).$$

**(c)** *$CS(A)$ is completely isometric to a TRO.*

**Corollary 10.7** *$CS(A) = M_1(CS(A)) = P_1(S(K\overline{\otimes}A))$ and $CS(A) \subset S(A)$.*

**Definition 10.8** *Let us now define a product $y \cdot x$ by*

$$\begin{bmatrix} y\cdot x & 0 \\ 0 & 0 \end{bmatrix} = 2\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\}$$

*and denote the corresponding matrix product by $X \cdot Y$. That is, if $X = [x_{ij}]$ and $Y = [y_{ij}]$, then $X \cdot Y = [z_{ij}]$ where*

$$z_{ij} = \sum_k x_{ik} \cdot y_{kj}.$$

It follows that

$$\{xvy\} = \frac{1}{2}(y \cdot x + x \cdot y).$$

With these tools at hand, the proof of Theorem 10.3 can be completed.

# 11 Universal Enveloping TROs and C*-algebras

## 11.1 Operator space structure of $JC^*$-triples and TROs

The paper [35], together with its sequel [36], initiates a systematic investigation of the operator space structure of $JC^*$-triples via a study of the TROs they generate. The approach is through the introduction and development of a variety of universal objects (TROs and $C^*$-algebras). Explicit descriptions of operator space structures of Cartan factors of arbitrary dimension are obtained as a consequence. Much of this subsection is taken literally from the introduction to [35].

It should be noted that Theorem 11.1 and its corollary, as well as the computation of the universal enveloping TROs of all Cartan factors of finite dimension, were also obtained roughly around the same time in [30].

It is necessary to widen the definition of $JC^*$-triple to include any complex Banach space with a surjective isometry onto a subspace of a TRO, which is closed under the (Jordan) triple product $\{a, b, c\} = (ab^*c + cb^*a)/2$. A counteractively complemented subspace of $B(H)$ is an example of such a $JC^*$-triple, not necessarily a $JC^*$-subtriple of $B(H)$. In the reflexive case, operator space structure of $JC^*$-triples of this kind has been investigated in [145], [147], and [143].

The aim of [35] is to explore the operator space structures of an arbitrary $JC^*$-triple, and, quoting the authors of [35], "to place the ground-breaking results of [145], [147], [143] in the general setting of a full theory." This new device (universal TRO of a $JC^*$-triple) facilitates a general inquiry into the operator space structure of $JC^*$-triples.

**Theorem 11.1 (Theorem 3.1 of [35])** *Let $E$ be a $JC^*$-triple. Then there is a unique pair $(C^*(E), \alpha_E)$ where $C^*(E)$ is a $C^*$-algebra and $\alpha_E : E \to C^*(E)$ is an injective triple homomorphism, with the following properties:*

*(a) $\alpha_E(E)$ generates $C^*(E)$ as a $C^*$-algebra;*

*(b) for each triple homomorphism $\pi : E \to A$, where $A$ is a $C^*$-algebra, there is a unique $*$-homomorphism $\tilde{\pi} : C^*(E) \to A$ with $\tilde{\pi} \circ \alpha_E = \pi$.*

[30] note that in the first displayed formula of (b), the triple product is the one on the JB*-triple $M_n(CS(A))$, namely, $\{xyz\}_{M_n(CS(A))} = P_n(\{xyz\}_{S(K\overline{\otimes}A)})$, which, it turns out, is actually the restriction of the triple product of $S(K\overline{\otimes}A)$: whereas in the second displayed formula, the triple product is the partial triple product on $K\overline{\otimes}A$

**Corollary 11.2 (Corollary 3.2 of [35])** *Let $E$ be a $JC^*$-triple. Then there is a unique pair $(T^*(E), \alpha_E)$ where $T^*(E)$ is a TRO and $\alpha_E : E \to T^*(E)$ is an injective triple homomorphism, with the following properties:*

*(a) $\alpha_E(E)$ generates $T^*(E)$ as a TRO;*

*(b) for each triple homomorphism $\pi : E \to T$, where $T$ is aTRO, there is a unique TRO-homomorphism $\tilde{\pi} : T^*(E) \to T$ with $\tilde{\pi} \circ \alpha_E = \pi$.*

In a technical tour de force, $T^*(E)$ is explicitly calculated for the Cartan factors of any dimension: Hilbert space [35, Theorem 5.1], spin factor [35, Lemma 5.2], rectangular of rank 2 or more [35, Theorem 5.4], symmetric and antisymmetric [35, Theorem 5.5]. The corresponding references in [30] to these results, but only for the finite dimensional cases as noted above, are: Hilbert space [30, Theorem 3.15], spin factor [30, Theorem 3.5], rectangular of rank 2 or more [30, Theorem 3.13], symmetric and antisymmetric [30, Theorem 3.6]).

As a technical aid of possibly independent interest, the notion of a reversible $JC^*$-triple is also introduced in [35]. and the universal reversibility (or not) of all Cartan factors, is also determined. Building on the definition of reversibility for Jordan algebras leads to the following definition.

**Definition 11.3 (Definition 4.1 of [35])** *A $JC^*$-subtriple $E$ of a TRO $T$ is said to be reversible in $T$ if*

$$a_1 a_2^* a_3 \cdots a_{2n}^* a_{2n+1} + a_{2n+1} a_{2n}^* \cdots a_2^* a_1 \in E$$

*whenever $a_1, \ldots a_{2n+1} \in E$. A $JC^*$-triple $E$ is universally reversible if $\alpha_E(E)$ is reversible in $T^*(E)$.*

**Theorem 11.4 (Theorem 5.6 of [35])** *Spin factors of dimension greater than 4 and Hilbert spaces of dimension greater than 2 are not universally reversible. All other Cartan factors are universally reversible.*

In the next theorem, operator space structures of $JC^*$-triples arising from concrete triple embeddings in $C^*$-algebras are considered, making use of universal TROs and establishing links with injective envelopes and triple envelopes, and concentrating on Cartan factors. It is necessary first to refine the definition of operator space structure.

**Definition 11.5 (Definition 6.1 of [35])** *A $JC$-operator space structure on a $JC^*$-triple $E$ is an operator space structure determined by a linear isometry from $E$ onto a $JC^*$-subtriple of $B(H)$.*

Another important definition is that of an operator space ideal. This is a norm closed ideal $\mathcal{I}$ of $T^*(E)$ for which $\mathcal{I} \cap \alpha_E(E) = \{0\}$. For each operator space ideal $\mathcal{I}$, we have the $JC$-operator space structure $E_{\mathcal{I}}$ on $E$ determined by the isometric embedding $E \mapsto T^*(E)/\mathcal{I}$ given by $x \mapsto \alpha_E(x) + \mathcal{I}$.

**Theorem 11.6 (Theorem 6.8 of [35])** *Let $E$ be a Cartan factor which is not a Hilbert space. Then $\mathcal{I} \leftrightarrow E_{\mathcal{I}}$ is a bijective correspondence between the operator space ideals of $T^*(E)$ and the $JC$-operator space structures of $E$.*

The case of a Hilbert space, excluded from Theorem 11.6 is covered in [36, Theorem 2.7]. It follows from this that the triple envelope, in the sense of [84], of a $JC^*$-triple $E$ which is isometric to a Cartan factor is identified with the TRO generated by $E$. The following theorem shows that distinct $JC$-operator space structures on a Hilbert space cannot be completely isometric.

**Theorem 11.7 (Theorem 2.6 of [36])** *For a Hilbert space $E$ and operator space ideals $\mathcal{I}$ and $\mathcal{J}$ of $T^(E)$, the following are equivalent:*

*(a) $E_{\mathcal{I}} = E_{\mathcal{J}}$*

*(b) $\mathcal{I} = \mathcal{J}$*

*(c) $E_{\mathcal{I}}$ and $E_{\mathcal{J}}$ are completely isometric.*

It is also shown in [36, Theorem 3.4] how infinite-dimensional Hilbertian $JC$-operator spaces are determined explicitly by their finite-dimensional subspaces, and that, in turn, they impose very rigid constraints upon the operator space structure of their finite-dimensional subspaces. Finally, the operator space ideals of the universal TRO of a Hilbert space are identified [36, Theorem 3.7], as well as the corresponding injective envelopes [36, Theorem 4.4].

Finally we make four remarks about the paper [30]. First, the methods for computing the universal enveloping TROs of the Cartan factors (in finite dimensions only) are different from those of [35] in that they use grids and depend strongly on the results on grids in [145].

Second, because the paper [30] works in the slightly more general context of $JB^*$-triples, they are able to obtain a new proof of a version of the Gelfand-Naimark theorem for $JB^*$-triples ([75, Theorem 2].

**Corollary 11.8 (Corollary 2.6 of [30])** *Any $JB^*$-triple $Z$ contains a unique purely exceptional ideal $J$, such that $Z/J$ is $JB^*$-triple isomorphic to a $JC^*$-triple.*

Third, they define a radical for universally reversible $JC^*$-triples and use it to give the structure of the universal enveloping TRO of a universally reversible TRO in a $C^*$-algebra with a TRO antiautomorphism of order 2, generalizing [35, Corollary 4.5].

Finally, the results of [30] are used in [31] to show that the classification of the symmetric spaces, derived initially from Lie theory and then from Jordan theory, can be achieved by $K$-theoretic methods. This in spite of the fact that $JB^*$-triples do not, in general, behave well under the formation of tensor products.

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