APERIODIC SEQUENCES AND APERIODIC GEODESICS

VIKTOR SCHROEDER AND STEFFEN WEIL

ABSTRACT. We introduce a quantitative condition on orbits of dynamical systems which measures their aperiodicity. We show the existence of sequences in the Bernoulli-shift and geodesics on closed hyperbolic manifolds which are as aperiodic as possible with respect to this condition.

1. MAIN RESULTS.

In this section we state our main results in the case of sequences in a finite alphabet and of geodesics in hyperbolic manifolds. Denote by $\mathbb{N}_0$ the natural numbers including 0 and let $\mathbb{N} = \mathbb{N} \setminus \{0\}$. Given a finite set $\mathcal{A}$ with $k \geq 2$ elements, let $\Sigma = \mathcal{A}^{\mathbb{Z}}$ be the set of biinfinite sequences in the alphabet $\mathcal{A}$, which we call *words*. With $[w(i) \dots w(i+l)]$ denote the subword of $w \in \Sigma$ starting at *time* $i \in \mathbb{Z}$ and of *length* $l \in \mathbb{N}_0$. For a word $w \in \Sigma$ define the *recurrence time* $R_w^i : \mathbb{N}_0 \to \mathbb{N} \cup \{\infty\}$ at time $i \in \mathbb{Z}$ by

$$R_w^i(l) = \min\{s \geq 1 : [w(i+s) \dots w(i+s+l)] = [w(i) \dots w(i+l)]\},$$

(i.e. the first instant when the sub word $[w(i) \dots w(i+l)]$ of w is seen again), and by

$$R_w(l) := \min\{R_w^i(l) : i \in \mathbb{Z}\}.$$

For a periodic word $w \in \Sigma$ with period $p \in \mathbb{N}$, i.e. $w(i) = w(i+p)$ for all $i \in \mathbb{Z}$, we have $R_w(l) \leq p$ for all $l \in \mathbb{N}_0$. Thus, if R_w is unbounded, then w is aperiodic and we view the growth rate of R_w as a measure for the aperiodicity of the word w. Note that R_w is nondecreasing and by a trivial counting argument we have $R_w(l) \leq k^{l+1}$ for every word w, in particular

$$\lim_{l \to \infty} \frac{1}{l} \ln R_w(l) \leq \ln(k).$$

One of our main results is the existence of words w such that the growth rate is as near as possible to this bound.

Theorem 1.1. *Let $\varphi : \mathbb{N}_0 \to [0, \infty)$ be a non-decreasing function such that*

$$\lim_{l \to \infty} \frac{1}{l} \ln(\varphi(l)) \leq \delta \ln(k) \tag{1.1}$$

for some $0 < \delta < 1$. Then there exist $l_0 = l_0(\varphi, k, \delta) \in \mathbb{N}_0$ and a word $w \in \Sigma$ such that, for every $l_0 \leq l \in \mathbb{N}_0$, we have $R_w(l) \geq \varphi(l)$.

Now let M be a closed n-dimensional hyperbolic manifold, where $n \geq 2$. Let $i_M > 0$ denote the injectivity radius of M and let d be the Riemannian distance function on M. For

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a unit speed geodesic $\gamma : \mathbb{R} \to M$ we define the *recurrence time* $R_\gamma^{t_0} : [0, \infty) \to [i_M/2, \infty]$ at time $t_0 \in \mathbb{R}$ by

$$R_\gamma^{t_0}(l) = \inf\{s > i_M/2 : d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \frac{i_M}{2} \text{ for all } 0 \leq t \leq l\}.$$

and

$$R_\gamma(l) := \inf\{R_\gamma^{t_0}(l) : t_0 \in \mathbb{R}\}.$$

If γ is a periodic geodesic, then R_γ is bounded and again one can view the growth rate of R_γ as a measure for the aperiodicity of γ.

Theorem 1.2. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a non-decreasing function such that*

$$\lim_{l \to \infty} \frac{1}{l} \ln(\varphi(l)) \leq \delta(n - 1) \tag{1.2}$$

for some $0 < \delta < 1$. If $i_M > 2\ln(2)$ then there exist $l_0 = l_0(\varphi, \delta, n, i_M) \geq 0$ and a unit speed geodesic $\gamma : \mathbb{R} \to M$ such that for all $l \geq l_0$, we have $R_\gamma(l) \geq \varphi(l)$.

The theorems will be shown in greater generality.

Remark. The bounds $\ln(k)$ and $n - 1$ equal the topological entropies of the respective dynamical systems. Moreover, we believe that the assumption on the injectivity radius in Theorem 1.2 is not necessary. A version of this theorem is also true if M is of strictly negative curvature. However, for the sake of clarity of the paper we restrict to these assumptions.

Organization of the paper. In Section 2 we will introduce the measure of aperiodcitiy for general dynamical systems and deduce immediate properties. In Section 3 and 4 we examine two examples and state the main results, namely of the Bernoulli-shift and the geodesic flow on a closed hyperbolic manifold. These will be proven in Section 5.

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2. F-Aperiodic Points.

Let (X, d) be a compact metric space and let $T : X \to X$ be a given continuous transformation. For $n \in \mathbb{N}_0$ let T^n be the n-times composition of T (where $T^0 = id_X$) and for a point $x \in X$ let $T^n x$ be the point in the orbit $\mathcal{T}(x) := \{T^n x\}_{n \in \mathbb{N}_0}$ of x at *time* n. Let moreover μ be a finite Borel-measure on the Borel-σ-algebra $\mathcal{B}$ of (X, d) such that T is measure-preserving; see [5].

A point $x \in X$ is called *periodic* (with respect to T) if there exists an integer $p \in \mathbb{N}$, called a *period* of x, such that $T^p x = x$. Denote by $\mathcal{P}_T$ the T-invariant set of T-periodic points of X. A point is called *aperiodic*, if it is not periodic.

A point $x \in X$ is *recurrent* with respect to T, if for any $\varepsilon > 0$ there exists $s = s(x, \varepsilon) \in \mathbb{N}$ such that $d(T^s x, x) < \varepsilon$. Periodic points are obviously recurrent. The set $\mathcal{R}_T$ of recurrent points is nonempty (see [6]) and T-invariant. However $s(T^i x, \varepsilon)$ can differ from $s(x, \varepsilon)$ in general, unless T is an isometry on its orbit $\mathcal{T}(x)$; that is, $d(T^{i+s}x, T^i x) = d(T^s x, x)$ for all i and $s \in \mathbb{N}_0$. We recall that by the Poincaré-recurrence theorem, μ-almost every point is recurrent.

In this paper we give a quantitative version of recurrence and aperiodicity. Given a point $x \in X$ and a *time* $i \in \mathbb{N}_0$, we ask for a lower bound on the *shift* s such that $T^{i+s}x$ is allowed to be ε-close to $T^i x$:

Definition 2.1. For a non-increasing function $F : (0,\infty) \to [0,\infty)$ a point $x \in X$ is called F*-aperiodic* at *time* $i \in \mathbb{N}_0$ if for every $\varepsilon > 0$, whenever

$$d(T^i x, T^{i+s}x) < \varepsilon$$

for some $s \in \mathbb{N}$, then $s > F(\varepsilon)$. If x is F-aperiodic at every time $i \in \mathbb{N}_0$ then it is called F*-aperiodic*.

We emphasize that although we called the condition "F-aperiodic", a periodic point x is F-aperiodic for a suitable bounded function F. However, if the function F is unbounded, an F-aperiodic point must be aperiodic. Moreover, if x is not recurrent, then it is easy to find an unbounded function F such that x is F-aperiodic at least at time 0.

Let $F : (0,\infty) \to [0,\infty)$ be a given non-increasing function. Clearly, if a non-increasing function $\bar{F}$ satisfies $\bar{F}(s) \le F(s)$ for all $s \in (0,\infty)$ then an F-aperiodic point is also $\bar{F}$-aperiodic. On the other hand, using the upper box dimension $\dim_B(X)$ for metric spaces, we obtain an upper bound on the growth rate (as ε tends to 0) of functions F such that an F-aperiodic point might exist. For $\varepsilon > 0$ let $N(X,\varepsilon)$ denote the largest number of disjoint metric balls of radius ε. Then the upper box dimension ([16]) is given by

$$\dim_B(X) = \limsup_{\varepsilon\to 0} \frac{\ln(N(X,\varepsilon))}{-\ln(\varepsilon)}.$$

Lemma 2.2. *Let x be an F-aperiodic point. Then* $\limsup_{\varepsilon\to 0} \frac{\ln(F(\varepsilon))}{\ln(2/\varepsilon)} \le \dim_B(X)$.

Proof. Let $\varepsilon > 0$. If $B(T^{s_1}x, \varepsilon/2) \cap B(T^{s_2}x, \varepsilon/2) \neq \emptyset$ for some $0 \le s_1 < s_2 \le F(\varepsilon)$, we have $d(T^{s_1}x, T^{s_2}x) < \varepsilon_0$ which is impossible since $s_2 - s_1 \le F(\varepsilon_0)$. Therefore the metric balls $B(T^s x, \varepsilon/2)$ must be disjoint for $s \le F(\varepsilon)$. Hence we have $F(\varepsilon) \le N(X, \varepsilon/2)$. □

Moreover, since F is independent of the time $i \in \mathbb{N}_0$, the set $\mathcal{F}_T \subset X$ of F-aperiodic points is T-invariant. In the case when $(X, \mathcal{B}, \mu, T)$ is ergodic, $\mathcal{F}_T$ is either of full or of zero μ-measure. When $\mathcal{P}_T$ is nonempty, this question is related to the distribution of periodic orbits. In fact, let $x_0 \in \mathcal{P}_T$ be of minimal period p_0 and assume that $F(\varepsilon) \ge p_0$ for some $\varepsilon_{p_0} > 0$. In the case when F is continuous, we may choose $\varepsilon_{p_0} := \sup\{\varepsilon > 0 : F(\varepsilon) \ge p_0\}$. Define the *critical neighborhood* of x_0 with respect to F and p_0 by

$$\mathcal{N}_{x_0} := B(x_0, \varepsilon_{p_0}/2) \cap T^{-p_0}(B(x_0, \varepsilon_{p_0}/2)). \tag{2.1}$$

Whenever $x \in \mathcal{N}_{x_0}$ we have by the triangle inequality that $d(x, T^{p_0}x) < \varepsilon_{p_0}$, but $p_0 \le F(\varepsilon_{p_0})$. Thus, no point in $\mathcal{N}_{x_0}$ can be F-aperiodic and we see that the orbit of an F-aperiodic point must avoid the critical neighborhoods of periodic points. If in addition $\mu(\mathcal{N}_{x_0}) > 0$ then the set of F-aperiodic points cannot be of full and must therefore be of zero μ-measure. Thus, we showed the following criterion.

Lemma 2.3. *Assume $\mathcal{P}_T \neq \emptyset$ and let x_0 be a periodic point of period p_0 and $F(\varepsilon) \ge p_0$ for some $\varepsilon > 0$. If μ is ergodic and positive on $\mathcal{N}_{x_0}$ then the set $\mathcal{F}_T$ has μ-measure 0.*

In particular, this result is interesting for the *systolic point* $x_0 \in \mathcal{P}_T$ of *systolic period* $p_0 \in \mathbb{N}$, that is, x_0 has minimal period p_0 and for every periodic point in X of period p we have $p \ge p_0$.

Lemma 2.4. *F-aperiodicity is a closed condition.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of F-aperiodic points in X converging to $x \in X$. Let i and $s \in \mathbb{N}$ be fixed. For $\varepsilon > 0$ such that $d(T^i x, T^{i+s}x) < \varepsilon$ let $d := \frac{1}{2}(\varepsilon - d(T^i x, T^{i+s}x))$. Since T is continuous, there exists $N = N(i, s, d) \in \mathbb{N}_0$ such that for all $n \geq N$ we have $d(T^i x, T^i x_n) < d$ and $d(T^{i+s}x, T^{i+s}x_n) < d$. From the triangle inequality we obtain

$$d(T^i x_n, T^{i+s}x_n) \leq d(T^i x_n, T^i x) + d(T^i x, T^{i+s}x) + d(T^{i+s}x, T^{i+s}x_n) < \varepsilon$$

for $n \geq N$ so that $s > F(\varepsilon)$ since x_n is F-aperiodic. Hence, x is also F-aperiodic. □

Finally, note that if T acts as an isometry on the orbit $\mathcal{T}(x)$ of a point $x \in X$, then x is F-aperiodic as soon as it is F-aperiodic at a given time. For instance, we consider the rotation on the circle as a motivating example:

Example 1. Let $\mathbb{Z}$ act on $\mathbb{R}$ by translations and let $X = \mathbb{R}/\mathbb{Z}$ be the compact quotient space with the induced metric d obtained from the Euclidean metric. Given an irrational number $0 < \alpha \in \mathbb{R} \setminus \mathbb{Q}$, we let $T = T_\alpha : X \to X$ be the automorphism induced by the translation $\tilde{T} : \mathbb{R} \to \mathbb{R}$, $\tilde{T}(x) := x + \alpha$. For $c > 0$ we let $F_c : (0, \infty) \to [0, \infty)$, $F_c(t) = ct^{-1}$. In fact, since $\dim_B(X) = 1$, -1 is the optimal exponent due to Lemma 2.2. The point $[0]$ is F_c-aperiodic if and only if every point $[x]$ is F_c-aperiodic and hence $\mathcal{F}_T$ is either empty or X itself. Moreover, since T is an isometry, $[0]$ is F_c-aperiodic as soon as it is F_c-aperiodic at time 0. The question for which c and α there exist F_c-aperiodic points can be answered by classical Diophantine approximation; see for instance [1] for the following well-known results: Let μ be the Lebesgue measure on $\mathbb{R}$. For μ-almost every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we have $c_0(\alpha) = 0$, where

$$c_0(\alpha) = \inf\{c > 0 : \text{there exist infinitely many } p \in \mathbb{Z}, q \in \mathbb{N} \text{ such that } |\alpha - \frac{p}{q}| < \frac{c}{q^2}\}.$$

However, there exists a set of Hausdorff-dimension one such that $c_0(\alpha)$ is positive. Such an α is called badly approximable. The supremum $\sup_{\alpha\in\mathbb{R}\setminus\mathbb{Q}} c_0(\alpha)$ of this set, called the Hurwitz-constant, is equal to $1/\sqrt{5}$ and attained at the golden ratio.

First, let α such that $c_0(\alpha) = 0$. Then for $c > 0$ we have for infinitely many $p \in \mathbb{Z}$, $q \in \mathbb{N}$,

$$|\tilde{T}^q 0 - p| = |q\alpha - p| = q|\alpha - \frac{p}{q}| < cq^{-1}, \tag{2.2}$$

hence $q \leq F_c(cq^{-1})$ and we see that $[0]$ is not F_c-aperiodic for any $c > 0$. Thus, $\mathcal{F}_T$ is empty. In particular, this shows that for $c > 1/\sqrt{5}$ the set $\mathcal{F}_T$ is empty for every $T = T_\alpha$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ irrational. However, for α a badly approximable number we have $c_0(\alpha) > 0$ and for $c < c_0(\alpha)$ there are only finitely many p, q as in (2.2). Hence we can choose some $0 < \bar{c} \leq c_0(\alpha)$ such that $[0]$ is $F_{\bar{c}}$-aperiodic and therefore $\mathcal{F}_T = X$.
If we conversely assume that $[0]$ is F_c-aperiodic, then whenever $|\tilde{T}^q 0 - p| < \varepsilon$ for some $\varepsilon > 0$ we have $q > F_c(\varepsilon) = c/\varepsilon > \frac{c}{q|\alpha - p/q|}$. Thus, $|\alpha - \frac{p}{q}| > \frac{c}{q^2}$ for every $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and α is necessarily a badly approximable number.

In the following we are concerned with the examples of the Bernoulli-shift and the geodesic flow on a closed hyperbolic manifold where the question of existence of F-aperiodic points is more delicate.

Remark. A somewhat orthogonal problem has been studied by many authors. For instance, [2] showed that the rate of recurrence can be quantified in the case when X has finite Hausdorff-dimension. More precisely, assume that the α-dimensional Hausdorff-measure

H_α is σ-finite for some $\alpha > 0$, then for μ-almost every point $x \in X$ there exists a finite constant $c(x) \geq 0$ such that

$$\liminf_{n\to\infty} n^{1/\alpha} d(x, T^n(x)) \leq c(x).$$

Assume that there exists a point $x \in X$ which is F-aperiodic at time 0 for the function $F(\varepsilon) = c \cdot \varepsilon^{-\alpha}$ for some $c > 0$ (compare with Lemma 2.2). Then it is not hard to show that for every $n > 0$,

$$n^{1/\alpha} d(x, T^n x) \geq c^{1/\alpha}.$$

The main point in our paper is that we study the recurrence for every point of the orbit and not only for the initial one.

3. Sequences.

Let $\mathcal{A}$ be a finite set of $k \geq 2$ elements which we call *alphabet*. Let $\Sigma^+ = \{w : \mathbb{N} \to \mathcal{A}\}$ and $\Sigma = \{w : \mathbb{Z} \to \mathcal{A}\}$ be the set two-sided sequences in symbols from $\mathcal{A}$. The elements of Σ are called *words*. Given words w and $\bar{w}$ in Σ we let $a(w, \bar{w}) = \max\{i \geq 0 : w(i) = \bar{w}(i)$ for $|j| \leq i\}$ for $w \neq \bar{w}$ and define $\bar{d}(w, \bar{w}) := 2^{-a(w,\bar{w})}$, and $\bar{d}(w, w) := 0$ otherwise. Let T denote the shift operator acting on Σ, with $T(w) = \bar{w}$ where $\bar{w}(i) = w(i+1)$. Then, $(\Sigma, \bar{d})$ is a compact metric space such that T is a homeomorphism. Moreover, let $\mathcal{B}$ denote the product σ-algebra of the power set $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ which equals the Borel-σ-algebra of $(\Sigma, \bar{d})$. Let (the probability measure) $\mu = \prod_{\mathbb{Z}} \mu_{\mathcal{A}}$ be the infinite product measure of $\mathcal{B}$ where $\mu_{\mathcal{A}}$ is a probability measure on $(\mathcal{A}, \mathcal{P}(\mathcal{A}))$. Then the *Bernoulli-shift* $(\Sigma, \mathcal{B}, \mu, T)$ is ergodic. For details we refer to [5].

Note that by definition of $\bar{d}$, two words are close if and only if the length of their subwords around position 0 on which they agree is large. In particular, if $w \in \mathcal{R}_T$ then, by recurrence applied to the word $T^i w$, for every length $l \in \mathbb{N}_0$ we can find an $s = s(i, l) \in \mathbb{N}$ such that $[w(i) \dots w(i + l)] = [w(i + s) \dots w(i + s + l)]$. In the case of sequences it is suitable to reformulate F-aperiodicity as follows (see Proposition 3.2).

Definition 3.1. For a non-decreasing function $\varphi : \mathbb{N}_0 \to [0, \infty)$ a word $w \in \Sigma$ is called *φ-aperiodic* at *time* $i \in \mathbb{Z}$, if for every *length* $l \in \mathbb{N}_0$, whenever

$$[w(i) \dots w(i + l)] = [w(i + s) \dots w(i + s + l)] \tag{3.1}$$

for some *shift* $s \in \mathbb{N}$, then $s > \varphi(l)$. If w is φ-aperiodic at every time $i \in \mathbb{Z}$ it is called φ-aperiodic.

A φ-aperiodic word $w \in \Sigma$ is F-aperiodic for the following function F.

Proposition 3.2. *A φ-aperiodic word $w \in \Sigma$ is F-aperiodic for $F(\varepsilon) = \varphi(-2\lceil\log_2(\varepsilon)\rceil)$. Conversely, an F-aperiodic word w is φ-aperiodic for $\varphi(l) = F(2^{-(l/2-1)})$.*

Proof. Let $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. For every $l \in \mathbb{N}_0$ such that $\bar{d}(T^i w, T^{i+s} w) \leq 2^{-l}$ we have $[w(i - l) \dots w(i + l)] = [w(i - l + s) \dots w(i + s + l)]$. Thus, for $2^{-l} < \varepsilon \leq 2^{-(l-1)}$,

$$s > \varphi(2l) = \varphi(-2\lceil\log_2(\varepsilon)\rceil) = F(\varepsilon).$$

Since $F(\bar{\varepsilon}) \leq F(\varepsilon)$ for $\bar{\varepsilon} \geq \varepsilon$, the first implication follows.

Conversely, if w is F-aperiodic, assume that $[w(i) \dots w(i + l)] = [w(i + s) \dots w(i + s + l)]$ for $s \in \mathbb{N}$, $l \in \mathbb{N}_0$ and let $\bar{l} = l/2$ if l is even and $\bar{l} = (l - 1)/2$ if l is odd. Hence, $\bar{d}(T^{i+\bar{l}} w, T^{i+\bar{l}+s} w) \leq 2^{-\bar{l}}$ and for every $2^{-\bar{l}} < \varepsilon \leq 2^{-(\bar{l}-1)}$ we have

$$s > F(\varepsilon) \geq F(2^{-(\bar{l}-1)}) \geq F(2^{-(l-3)/2}) = \varphi(l).$$

This finishes the proof. □

If a φ-aperiodic word contains a periodic subword of infinite length then the function φ is bounded, whereas if a word is φ-aperiodic for an unbounded function, the word must be aperiodic. We want to give some examples in order to make the definition more familiar, among them the prominent Morse-Thue-sequence:

Example 2. First, let $a,\ b \in \mathcal{A}$. One checks that the (non-recurrent) words $w_1 = \dots bbbaaa\dots$ and $w_2 = ..abaabaaabaaaab\dots$ are φ-aperiodic only for a function φ such that $1 = s > \varphi(l)$ for all $l \in \mathbb{N}_0$. Both, the orbits of w_1 and w_2, come closer and closer to the periodic word $\dots aaa\dots$ with respect to the metric $\bar{d}$. This is not the case for φ-aperiodic words when φ is unbounded; see Proposition 3.4.

Consider the Morse-Thue recurrent sequence $w \in \{0,1\}^{\mathbb{Z}}$ which is determined as follows: Let $a_0 = 0$, $b_0 = 1$. Then for $n \in \mathbb{N}_0$, let $a_{n+1} = a_n b_n$ and $b_{n+1} = b_n a_n$ be finite words of length $2^{n+1} - 1$. Then w is defined such that it satisfies $[w(0)\dots w(2^n - 2)] = a_n$ and $[w(-n)] = [w(n-1)]$ for every $n \in \mathbb{N}$. In particular, w contains the sub words $a_{n+2} = a_n b_n b_n a_n$. Hence for every length $l = 2^n - 1$, w contains subwords of the form WW where W has length l. A function φ such that w is φ-aperiodic must therefore be bounded by $\varphi(2^n - 1) \leq 2^n - 1$ for every $n \in \mathbb{N}$. On the other hand there are no sub words of the form WWa where a is the first letter of a sub word W (see [11]). In other words, w is overlap-free (which means that there are no sub words of the form $aWaWa$ for a finite sub word W and a letter a), from which follows that there are even no sub words of the form $wWwWw$ for w and W finite subwords. Hence we may choose $\varphi(l) \geq l$. We conclude that w is at least φ-aperiodic for the function $\varphi(l) = l$, $l \in \mathbb{N}_0$.

The example shows that the set of φ-aperiodic words $\mathcal{F}_T = \mathcal{F}_T(\varphi)$ is nonempty for the unbounded function $\varphi(l) = l$ and moreover, the Morse-Thue sequence gives an explicit example of such a word. However, let $a \in \mathcal{A}$ such that $\mu_{\mathcal{A}}(\{a\}) > 0$ and let $w = \dots aaa\dots$ be a periodic word which is of systolic period 1. Moreover, μ is positive on the critical neighborhood of w and hence by Lemma 2.3, $\mathcal{F}_T$ is of zero μ-measure unless φ is strictly bounded by 1.

Our main result for sequences is the following. It will be proved in Section 5.

Theorem 3.3. *Let $\varphi : \mathbb{N}_0 \to [0,\infty)$ be a non-decreasing unbounded function such that there exists $c \in (1,k)$ satisfying*

$$k - \lfloor\varphi(0)\rfloor - \sum_{l=1}^{\infty} \frac{\lfloor\varphi(l)\rfloor - \lfloor\varphi(l-1)\rfloor}{c^l} \geq c, \tag{3.2}$$

where $\lfloor\cdot\rfloor$ denotes the integer part. Then there exists a φ-aperiodic word in Σ.

Remark. The condition is satisfied for the following set of parameters:
(1) $k \geq 4$, then $\varphi(l) = l$ satisfies (3.2) for $c = 2$,
(2) $k \geq 5$, then $\varphi(l) = 2^l$ satisfies (3.2) for $c = 3$,
(3) $k \geq 2$, $0 < \delta < 1$ and $k^\delta < c < k$, then there exists $l_0 = l_0(k,\delta,c) \in \mathbb{N}_0$ such that

$$\varphi(l) = \begin{cases} 0, & \text{for } l \leq l_0 \\ k^{\delta l}, & \text{for } l > l_0 \end{cases} \tag{3.3}$$

satisfies (3.2).

Note that if a word w is φ-aperiodic then $R_w(l) > \varphi(l)$ for every $l \in \mathbb{N}_0$ where R_w is the recurrence time introduced in Paragraph 1. Theorem 1.1 is hence a corollary of Theorem 3.3.

Proof of Theorem 1.1. By condition (1.1), for every $\varepsilon_0 > 0$ there exists $l_1 = l_1(\varepsilon_0) \in \mathbb{N}$ such that for all $l \geq l_1$,

$$\frac{1}{l}\ln(\varphi(l)) \leq \delta \ln(k)(1+\varepsilon_0).$$

Since $\delta < 1$ we let $\varepsilon_0 > 0$ such that $\tilde{\delta} = (1+\varepsilon_0)\delta < 1$. Then, $\varphi(l) \leq k^{\tilde{\delta}l}$ for $l \geq l_1$. If we take $c := (k - k^{\tilde{\delta}})/2$ then by (3.3) there exists $l_2 = l_2(k, \tilde{\delta})$ such that condition (3.2) is satisfied for the function $\bar{\varphi}(l) := k^{\tilde{\delta}l}$ for $l > l_2$ and $\bar{\varphi}(l) = 0$ for $l \leq l_2$, $l \in \mathbb{N}_0$. Theorem 3.3 implies the existence of a $\bar{\varphi}$-aperiodic word $w \in \Sigma$. Thus, setting $l_0 := \max\{l_1, l_2\}+1$, we have that $\bar{\varphi}(l) \geq \varphi(l)$ for all $l \geq l_0$ and the claim follows. □

Remark. The critical function φ for which φ-aperiodic words cannot exist is the function $\varphi(l) = k^{l+1}$. The critical exponent $\ln(k)$ equals the topological entropy of the system $(\Sigma, \bar{d}, T)$ (see [20]) and is optimal. To see that there exists no $w \in \Sigma$ which is φ-aperiodic for a function φ such that $\varphi(l) \geq k^{l+1}-1$ for some $l \in \mathbb{N}_0$, fix a subword $[w(1)\dots w(1+l)]$ of any $w \in \Sigma$. Inductively one shows that at each step $1 \leq s \leq \varphi(l)$ one has at most $k^{l+1} - s$ possibilities to choose a sub word $[w(1+s)\dots w(1+s+l)]$ such that w stays φ-aperiodic. Then, at step $s = k^{l+1}$, there is no choice left such that w is φ-aperiodic.

Remark. Let $\Sigma^+(m) = \{w : \{1,\dots,m\} \to \mathcal{A}\}$ be the set of words of length m in $\mathcal{A}$ and $\mathcal{W}^g(m) \subset \Sigma^+(m)$ be the set of *good* words of length m which satisfy (3.1) for all $i, s \in \mathbb{N}$ and $l \in \mathbb{N}_0$ such that $i+s+l \leq m$. If φ satisfies (3.2) with respect to the parameter $c > 1$ we will see in the proof of Theorem 3.3 (see Lemma 5.6) that the good words $\mathcal{W}^g(m)$ increase in m by the factor c. Thus, $|\mathcal{W}^g(m)| \geq c^m$ which is a lower bound on the asymptotic growth of $|\mathcal{W}^g(m)|$, where $|\cdot|$ denotes its cardinality.

We may reformulate the critical neighborhood of a periodic point given in (2.1) to the setting of φ-aperiodicity. Moreover, since $\mathcal{P}_T$ is dense in Σ we can also give a sufficient condition on φ-aperiodicity in terms of periodic words. Therefore, for a non-decreasing unbounded function $\varphi : \mathbb{N}_0 \to [0,\infty)$, we define a discrete form of a right-inverse for φ by $\ell : \mathbb{N} \to \mathbb{N}_0$,

$$\ell(s) = \min\{j \in \mathbb{N}_0 : \varphi(j) \geq s\}, \tag{3.4}$$

which is also non-decreasing and unbounded.

Proposition 3.4. *Let $\varphi : \mathbb{N}_0 \to [0,\infty)$ be a non-decreasing unbounded function. If $w \in \Sigma$ is φ-aperiodic, then for every periodic word $\bar{w} \in \Sigma$ of period s and for all $i \in \mathbb{Z}$ we have*

$$\bar{d}(T^i w, \bar{w}) > 2^{-(s+\ell(s))/2}.$$

Conversely, if $\bar{d}(T^i w, \bar{w}) > 2^{-(s+\ell(s)-1)/2}$ for every periodic word $\bar{w}$ of period s and all $i \in \mathbb{Z}$, then w is φ-aperiodic.

Proof. If w is φ-aperiodic, w is aperiodic and there exists $m \in \mathbb{N}_0$ such that $\bar{d}(T^i w, \bar{w}) = 2^{-m}$ where we assume $2m \geq s$ (otherwise the first statement follows). Hence, $[w(i-m)\dots w(i+m)] = [\bar{w}(-m)\dots \bar{w}(m)]$ and we see that $[w(i-m)\dots w(i-m+s+(2m-s)] = [w(i-m+s)\dots w(i+m)]$. Thus, $s > \varphi(2m-s)$ and $m < (s+\ell(s))/2$ from (5.1).

Conversely, assume that $[w(i)\dots w(i+l)] = [w(i+s)\dots w(i+s+l)]$ for $s \in \mathbb{N}$, $l \in \mathbb{N}_0$ and let $\bar{l} = (s+l)/2$ if $s+l$ even and $\bar{l} = (s+l-1)/2$ if $s+l$ is odd. Moreover, let $\bar{w}$ be the periodic word of period s such that $[\bar{w}(i)\dots \bar{w}(i+s-1)] = [w(i)\dots w(i+s-1)]$. Thus, $2^{-\bar{l}} \geq d(T^{i+\bar{l}}w, T^{i+\bar{l}}\bar{w}) > 2^{-(s+\ell(s)-1)/2}$ and we see that $s + \ell(s) - 1 > 2\bar{l} \geq s + l - 1$. Hence, $l < \ell(s)$ and from (5.1) we have $s > \varphi(l)$. □

Remark. Consider the overlap-free recurrence time $\tilde{R}^0_w : \mathbb{N}_0 \to \mathbb{N}$ of the initial sub word,

$$\tilde{R}^0_w(l) = \min\{s > l : [w(s) \dots w(s+l)] = [w(0) \dots w(l)]\}.$$

Clearly, $R_w(l) \leq R^0_w(l) \leq \tilde{R}^0_w(l)$ for $l \in \mathbb{N}_0$. Then it follows from [12] that, since the Bernoulli-shift is ergodic, for μ-almost all $w \in \Sigma$ the limit

$$\lim_{l\to\infty} \frac{\ln \tilde{R}^0_w(l)}{l}$$

exists and equals the measure-entropy $h_\mu(T)$.

4. Geodesic flow on hyperbolic manifolds

Let M be a closed n-dimensional hyperbolic manifold, that is a compact connected Riemannian manifold without boundary of constant negative curvature -1, where $n \geq 2$. We denote by d the distance function on M and by $i_M > 0$ the injectivity radius.

Let SM be the unit tangent bundle of M and d^S the Sasaki-distance function on SM. For $v \in SM$ let $\gamma_v : \mathbb{R} \to M$ be the unit speed geodesic such that $\gamma_v'(0) = v$. The geodesic flow $\phi^t : SM \to SM$, $t \in \mathbb{R}$, acts on the compact metric space (SM, d^S) by diffeomorphisms, where $\phi^t v = \gamma_v'(t)$. For details and background we refer to [4].

A vector $v \in SM$ is *periodic*, if there exists a $t > 0$ such that $\phi^t v = v$ and v is *recurrent* if for every $\varepsilon > 0$ there exists $s > 0$ such that $d^S(\phi^s v, v) < \varepsilon$. Denote by $\mathcal{P}_\phi$ and $\mathcal{R}_\phi$ the flow-invariant sets of periodic respectively of recurrent vectors. Thus if $v \in \mathcal{R}_\phi$ then for a given $t \in \mathbb{R}$, $\varepsilon > 0$, there exists $s = s(t, \varepsilon)$ such that $d^S(\phi^{t+s} v, \phi^t v) < \varepsilon$.

We now adjust the definitions of F-aperiodic and φ-aperiodic points to the setting of the geodesic flow.

Definition 4.1. Let $F : (0, \infty) \to [0, \infty)$ be a non-increasing function and $s_0 > 0$ be a constant, called the *minimal shift*. A vector $v \in SM$ is called *F-aperiodic* (with minimal shift s_0) at $t_0 \in \mathbb{R}$ if for every $\varepsilon > 0$, whenever

$$d^S(\phi^{t_0} v, \phi^{t_0+s} v) < \varepsilon$$

for some shift $s > s_0$, then $s > F(\varepsilon)$. If v is F-aperiodic at every time t_0 then v is called *F-aperiodic* (with minimal shift s_0).

Note that in contrast to the discrete setting in Section 2 (where $s \in \mathbb{N}$, i.e. $s \geq 1$) we now have to specify the additional parameter s_0, since $d^S(\phi^{t_0} v, \phi^{t_0+s} v) = s$ for s small enough.

We also have to generalize the notion of φ-aperiodicity. All geodesics will be assumed to be unit speed. Note that as in the case of the Bernoulli-shift, two vectors in the Sasaki-distance are very close if and only if the trajectories of the corresponding geodesics are close (in the Riemannian distance) to each other for a long time. Thus we may reformulate φ-aperiodicity in terms of the *fellow traveller length*.

Herefore we introduce a second parameter, the *distance constant* $\varepsilon_0 > 0$.

Definition 4.2. Let $\varphi : [0, \infty) \to [0, \infty)$ be a non-decreasing function, let $0 < \varepsilon_0 < i_M$ and $s_0 \geq \varepsilon_0$. A geodesic $\gamma : \mathbb{R} \to M$ is called *φ-aperiodic* at *time* $t_0 \in \mathbb{R}$ if for every *length* $l > \varepsilon_0$, whenever

$$d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \varepsilon_0 \quad \text{for all } 0 \leq t \leq l$$

for some *shift* $s > s_0$, then $s > \varphi(l)$. If γ is φ-aperiodic at every time t_0, it is called *φ-aperiodic* (with parameters (s_0, ε_0)).

The geodesic flow on compact hyperbolic manifolds is ergodic with respect to the Liouville measure μ (on the Borel-σ-algebra of SM). A systole of M has length $2i_M$ which equals the systolic period. For a non-decreasing function φ let $\mathcal{F}_\phi$ be the set of φ-aperiodic geodesics (with respect to (s_0, ε_0)), which is invariant under the geodesic flow ϕ^t. Since μ is positive on open sets, one can show as in Lemma 2.3, that the set $\mathcal{F}_\phi$ is of zero μ-measure if and only if φ is not bounded by either s_0 or $2i_M - \varepsilon_0$.

The main result of this section is the following, which will be proved in the Section 5.

Theorem 4.3. *Assume that $i_M > \ln(2)$ and let $\varepsilon_0 > 0$ such that $\ln(2) + \varepsilon_0 < i_M$. Let*

$$\varphi_\delta(l) = e^{\delta(n-1)l},$$

where $0 < \delta < 1$. Then there exists a minimal length $l_0 = l_0(\delta, i_M, n, \varepsilon_0)$ and a geodesic $\gamma : \mathbb{R} \to M$ which satisfies for every $t_0 \in \mathbb{R}$ and all $l \geq l_0$, whenever

$$d(\gamma(t_0 + t), \gamma(t_0 + s + t) < \varepsilon_0 \quad \textit{for all } 0 \leq t \leq l \tag{4.1}$$

for some shift $s > \varepsilon_0$, then $s > \varphi_\delta(l)$.

Note that for $\varepsilon_0 = i_M/2$, if a geodesic $\gamma : \mathbb{R} \to M$ satisfies (4.1) then $R_\gamma(l) \geq \varphi_\delta(l)$ for all $l \geq l_0$, where R_γ is the recurrence time introduced in Paragraph 1. Theorem 1.2 is hence a corollary of Theorem 4.3.

Proof of Theorem 1.2. By (1.2) there exists for every $\tau > 0$ some $l_1 = l_1(\tau) \geq 0$ such that for all $l \geq l_1$ we have

$$\varphi(l) \leq e^{(1+\tau)(n-1)\delta l}.$$

Since $\delta < 1$ we let $\tau_0 > 0$ such that $\bar{\delta} := (1 + \tau_0)\delta < 1$. From Theorem 4.3 for $\varepsilon_0 = i_M/2$, there exists an $l_2 = l_2(\bar{\delta}, i_M, n)$ and a geodesic geodesic $\gamma : \mathbb{R} \to M$ such that for every $t_0 \in \mathbb{R}$ and $l \geq l_2$, whenever

$$d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \frac{i_M}{2} \quad \text{for all } \ 0 \leq t \leq l,$$

for some shift $s > i_M/2$, then $s > e^{\bar{\delta}(n-1)l}$. If we set $l_0 := \max\{l_1, l_2\}$ then $s > e^{\bar{\delta}(n-1)l} \geq \varphi(l)$ whenever $l \geq l_0$ and the proof is finished. □

In order to prove Theorem 4.3 we discretize our geodesics. Therefore we need a third parameter, the *discretization constant* $r_0 > 0$. To a geodesic $\gamma : \mathbb{R} \to M$ we consider the *discrete* geodesic

$$\bar{\gamma} : \mathbb{Z} \to M, \quad \bar{\gamma}(i) := \gamma(i \cdot r_0).$$

Definition 4.4. *(Discrete Definition)* Let $\bar{\varphi} : \mathbb{N}_0 \to [0, \infty)$ be a non-decreasing function and let the parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ be given where $\bar{s}_0 \in \mathbb{N}_0$, $0 < \bar{\varepsilon}_0 < i_M$ and $0 < r_0 < \bar{\varepsilon}_0$. A discrete geodesic $\bar{\gamma} : \mathbb{Z} \to M$ is called *$\bar{\varphi}$-aperiodic* at time $i \in \mathbb{Z}$ if for $l \in \mathbb{N}$, whenever

$$d(\bar{\gamma}(i + j), \bar{\gamma}(i + s + j)) < \bar{\varepsilon}_0 \quad \text{for all } j \in \{0, \dots, l\} \tag{4.2}$$

for some shift $s > \bar{s}_0$, then $s > \bar{\varphi}(l)$. $\bar{\gamma}$ is called *$\bar{\varphi}$-aperiodic* (with parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$) if it is $\bar{\varphi}$-aperiodic at every time $i \in \mathbb{Z}$.

Note that, given a $\bar{\varphi}$-aperiodic geodesic $\bar{\gamma} : \mathbb{Z} \to M$ (with the parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$), the corresponding geodesic $\gamma : \mathbb{R} \to M$ is continuously φ-aperiodic in the following way.

Lemma 4.5. *For a non-decreasing function $\bar{\varphi} : [0,\infty) \to [0,\infty)$ and the parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ let $\bar{\gamma} : \mathbb{Z} \to M$ be a $\bar{\varphi}|_{\mathbb{N}_0}$-aperiodic geodesic. For $r_0 \leq l \in \mathbb{R}$, define*

$$\varphi(l) := r_0 \cdot \bar{\varphi}(\frac{l - r_0}{r_0}) - r_0.$$

Then γ is φ-aperiodic with respect to the minimal shift $s_0 = (\bar{s}_0 + 1)r_0$ and the distance constant $\varepsilon_0 = \bar{\varepsilon}_0 - r_0 > 0$.

Conversely, if $\gamma : \mathbb{R} \to M$ is φ-aperiodic with parameters (s_0, ε_0) then for $r_0 < \varepsilon_0$, let

$$\bar{\varphi}(l) := \varphi(l \cdot r_0)/r_0.$$

Then $\bar{\gamma} : \mathbb{Z} \to M$ is $\bar{\varphi}$-aperiodic with parameters $(\lceil s_0/r_0 \rceil, \varepsilon_0, r_0)$.

Proof. For $t_0 \in \mathbb{R}$, $L \geq r_0$ and $s > (\bar{s}_0 + 1)r_0$ assume that $d(\gamma(t_0 + t), \gamma(t_0 + s + t)) < \varepsilon_0$ for all $0 \leq t \leq L$. If we set $i := \lceil \frac{t_0}{r_0} \rceil$ and $i + \bar{s} := \lceil \frac{t_0 + s}{r_0} \rceil$ whereas $l := \lfloor \frac{L}{r_0} \rfloor$, we have $i,\ l \geq 1$ and $\bar{s} > \bar{s}_0$. Then, since $\varepsilon_0 = \bar{\varepsilon}_0 - r_0 < i_M$ and the distance function is locally convex, one checks by the triangle inequality that $d(\bar{\gamma}(i), \bar{\gamma}(i + \bar{s})) < \bar{\varepsilon}_0$ and $d(\bar{\gamma}(i+l), \bar{\gamma}(i+\bar{s}+l)) < \bar{\varepsilon}_0$. In particular, $d(\bar{\gamma}(i+j), \bar{\gamma}(i+\bar{s}+j)) < \bar{\varepsilon}_0$ for all $0 \leq j \leq l$. Thus, $\bar{s} > \bar{\varphi}(l)$ so that

$$s \geq (\bar{s} - 1)r_0 > (\bar{\varphi}(l) - 1)r_0 \geq \big(\bar{\varphi}(\frac{L}{r_0} - 1) - 1\big)r_0 = \varphi(L)$$

since $(l + 1)r_0 \geq L$. This finishes the first part of the Lemma. The second part follows analogously. □

In terms of Lemma 4.5 we are left with stating the existence theorem for discrete $\bar{\varphi}$-aperiodic geodesics. Recall that for an unbounded function $\bar{\varphi}$ we defined its discrete right-inverse $\bar{\ell} : \mathbb{N} \to \mathbb{N}_0$ in (3.4) which is also non-decreasing and unbounded.

Theorem 4.6. *Let $\bar{\varphi} : \mathbb{N}_0 \to [0,\infty)$ be a non-decreasing, unbounded function. Assume that $\ln(2) < r_0 < \bar{\varepsilon}_0 < i_M$ and $\bar{s}_0 \in \mathbb{N}_0$ such that for all $l \geq \bar{s}_0$,*

$$\lfloor \bar{\varphi}(l) \rfloor > l, \quad \textit{and} \quad \bar{\ell}(\bar{s}_0) \geq 1, \tag{4.3}$$

and moreover, that there exists a constant $c \in (1, 2^{n-1})$ such that

$$2^{n-1} - \bar{c} \cdot \sum_{l=\bar{\ell}(\bar{s}_0)}^{\infty} \frac{\lfloor \bar{\varphi}(l) \rfloor - \lfloor \bar{\varphi}(l-1) \rfloor}{c^l} \geq c, \tag{4.4}$$

where $\bar{c}$ is an explicit constant depending only on n and i_M. Then there exist a $\bar{\varphi}$-aperiodic geodesic $\gamma : \mathbb{Z} \to M$ with the parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$.

Remark. Since $\bar{\ell}$ is unbounded, condition (4.4) depends again essentially on the convergence of the sum in (4.4). For instance, let $\delta \in (0,1)$ and define $\bar{\varphi}(l) = 2^{\delta(n-1)l}$ and let $c \in (2^{\delta(n-1)}, 2^{n-1})$. Then, since $\bar{\ell}(s) = \lceil \frac{1}{\delta(n-1)\ln(2)} \ln(s) \rceil$ for $s \geq 0$, there exists a minimal shift $\bar{s}_0 = \bar{s}_0(n, \delta, \bar{c}, c)$ such that (4.3) and (4.4) are satisfied.

The constant $\bar{c}$ of condition (4.4) can in fact be sharped to be also dependent on $\bar{s}_0$, in which case it is strictly decreasing in $\bar{s}_0$. It will be explicitly defined in the proof of claim 5.12. We may give a rough upper bound of $\bar{c}$ which is independent of $\bar{s}_0$ by

$$\bar{c} \leq \lceil \big(3\cosh(i_M)\sqrt{n+1}\big)^{n-1} \rceil \lceil \frac{\int_0^{5i_M + 4\ln(\sqrt{n+1}/2)} \sinh(t)^{n-1} dt}{\int_0^{i_M/2} \sinh(t)^{n-1} dt} \rceil. \tag{4.5}$$

The lower bound $\ln(2)$ on the injectivity radius is necessary for the proof. However we believe that the result should be valid without this bound. Moreover, a version of Theorem

4.6 remains true for M a closed n-dimensional Riemannian manifold of negative sectional curvature.

Remark. Again, the critical function φ such that φ-aperiodic geodesics might or might not exist seems to be the function $\varphi(s) = e^{(n-1)s}$ and the critical exponent $n-1$ equals the topological entropy of (SM, ϕ^t).

Lemma 2.2 gives an upper bound on the growth rate of non-increasing functions $F : (0,\infty) \to (0,\infty)$ for which F-aperiodic geodesics can exist. In fact, since SM is a $(2n-1)$-dimensional manifold, its box dimension is $2n-1$. Discretizing ϕ^t by the time t_0-map ϕ^{t_0} where $t_0 = t_0(i_M) > 0$ is sufficiently small, gives the upper bound

$$\limsup_{\varepsilon\to 0} \frac{\ln(F(\varepsilon))}{\ln(2/\varepsilon)} \leq 2n-1.$$

Remark. For a closed geodesic $\alpha : \mathbb{R} \to M$, let $\mathcal{N}_{\varepsilon_0}(\alpha)$ be the (closed) $\varepsilon_0/2$-neighborhood of α in M, where $\varepsilon_0 > 0$ sufficiently small. When a geodesic $\gamma : \mathbb{R} \to M$ enters $\mathcal{N}_{\varepsilon_0}(\alpha)$ at time t_0 let $\mathfrak{p}_\alpha(\gamma, t_0)$ be the *penetration length* of γ in α at time t_0, that is, the maximal length $L \in [0,\infty]$ of an interval I, $t_0 \in I$, such that $\gamma(t) \in \mathcal{N}_{\varepsilon_0}(\alpha)$ for all $t \in I$. Set $\mathfrak{p}_\alpha(\gamma, t_0) = 0$ if $\gamma(t_0) \notin \mathcal{N}_{\varepsilon_0}(\alpha)$. Then by [10], for μ-almost every $v \in SM$ the limit

$$\limsup_{t\to\infty} \frac{\mathfrak{p}(\gamma_v(t))}{\ln(t)} \tag{4.6}$$

exists and equals $1/(n-1)$.

Moreover, the penetration length reflects the *depth* in which γ enters the neighborhood $\mathcal{N}_{\varepsilon_0}(\alpha)$. The study of depths or penetration lengths in an adequate convex set of negatively curved manifolds, such as the ε-neighborhood of totally geodesic embedded submanifold or the cusp-neighborhood of a finite-volume hyperbolic manifold, leads to the theory of diophantine approximation in negatively curved manifolds; see for instance [7, 9, 10, 13, 14, 15, 17, 18] to give only a short and incomplete list. In general, a sequence of depths or penetration lengths and times of γ in these convex sets reflects "how well γ is approximated", where γ is called *badly approximable* if any such sequence is bounded; see [9, 10].

Now, let γ be a φ-aperiodic geodesic (φ unbounded) with respect to the parameters s_0 and ε_0 and let α be **any** closed geodesic in M. Then, it can be seen that the penetration lengths of γ in $\mathcal{N}_{\varepsilon_0}(\alpha)$ are bounded by a constant depending only on φ, ε_0 and the length of α (and s_0 respectively). Therefore, the notion of φ-aperiodicty is linked to bad approximation; recall also Example 1. In particular, the limit of (4.6) equals 0 for γ.

5. Proofs

Let $\varphi : \mathbb{N}_0 \to [0,\infty)$ be a non-decreasing unbounded function. Recall the definition of the function $\ell : \mathbb{N} \to \mathbb{N}_0$ given by

$$\ell(s) = \min\{j \in \mathbb{N}_0 : \varphi(j) \geq s\},$$

see (3.4). The following properties hold: ℓ is non-decrasing and for s and $l \in \mathbb{N}_0$, we have

$$\begin{aligned} \varphi(\ell(s)) &\geq s, \\ l < \ell(s) &\iff \varphi(l) < s, \\ l \geq \ell(s) &\iff \varphi(l) \geq s. \end{aligned} \tag{5.1}$$

Proof. For the first property, clearly $\varphi(\min\{j : \varphi(j) \geq s\}) \geq s$. Let $l < \ell(s)$ and assume $s \leq \varphi(l)$. Then $\ell(s) = \min\{j : \varphi(j) \geq s\} \leq l$; a contradiction. If $s > \varphi(l)$ then $\ell(s) = \min\{j : \varphi(j) \geq s\} > l$ and if $\varphi(l) \geq s$ then $\ell(j) = \min\{j : \varphi(j) \geq s\} \leq l$. Also, if $l \geq \ell(s)$ then $\varphi(l) \geq \varphi(\ell(s)) \geq s$. □

5.1. **Proof of Theorem 3.3.** Recall that $\Sigma^+(m) = \{w : \{1, \dots, m\} \to \mathcal{A}\}$ is the set of words of length $m-1$. We consider $\Sigma^+(m)$ to be a subset of $\Sigma^+ = \mathcal{A}^{\mathbb{N}}$ (for example, by extending an element $w \in \Sigma^+(m)$ to an element $\bar{w} \in \Sigma^+$ by setting $\bar{w}(i) = a$ for all $i > m$, where $a \in \mathcal{A}$ is fixed).

Definition 5.1. Let $m \in \mathbb{N}$. $w \in \Sigma^+(m)$ is called *φ-aperiodic* if for all $i, s \in \mathbb{N}$ and $l \in \mathbb{N}_0$ such that $i + s + l \leq m$ whenever

$$[w(i) \dots w(i+l)] = [w(i+s) \dots (w(i+s+l)]$$

we have $s > \varphi(l)$.

Let $l_0 := \min\{j \in \mathbb{N}_0 \cup \{-1\} : \varphi(j+1) \neq 0\}$ and note that $\ell(s) > l_0$ for all $s \in \mathbb{N}$. For $m \in \mathbb{N}$, define the *admissible set* by

$$A(m) := \{(i, s) \in \mathbb{N} \times \mathbb{N} : i + s + \ell(s) = m\},$$

if $m \geq m_0 := 2 + \ell(1) > 2 + l_0$ and let $A(m)$ be empty for $m < m_0$. Then, for $(i, s) \in A(m)$ where $m \geq m_0$, we define the sets

$$C_{is} := \{w \in \Sigma^+(m) : [w(i) \dots w(i + \ell(s))] \neq [w(i+s) \dots w(i+s+\ell(s))]\},$$

called *conditions*.

Remark. Note that $s > \varphi(\ell(s) - 1)$ for $\ell(s) > 0$ but $s \leq \varphi(\ell(s))$. Therefore $\ell(s)$ determines the critical length of a given shift s with respect to φ.

For $w \in \Sigma^+(m)$ and $1 \leq n \leq m$ let $w|_n := [w(1) \dots w(n)] \in \Sigma^+(n)$. This leads to the reformulation of φ-aperiodic words:

Lemma 5.2. *For $m < m_0$ every word $w \in \Sigma^+(m)$ is φ-aperiodic. For $m \geq m_0$, a word $w \in \Sigma^+(m)$ is φ-aperiodic if and only if for all $n \leq m$ and all $(i, s) \in A(n)$ we have $w|_n \in C_{is}$.*

Proof. First, let $m < m_0$. Then for every i,$s \in \mathbb{N}$, $l \in \mathbb{N}_0$ such that $i+s+l \leq m < 2+\ell(1)$ we have in particular $l < \ell(1)$. Equivalently, $\varphi(l) < 1$ so that $s > \varphi(l)$ and every word $[w(1) \dots w(m)]$ follows to be φ-aperiodic.

Now let $m \geq m_0$. Let w be φ-aperiodic and assume $w|_n \notin C_{is}$ for some i and s in $\mathbb{N}$ such that $i + s + \ell(j) = n \leq m$. Then

$$[w(i) \dots w(i + \ell(s))] = [w(i+s) \dots w(i+s+\ell(s))]$$

and by (3.1), we have $s > \varphi(\ell(s))$; a contradiction to $\varphi(\ell(s)) \geq s$.

Conversely, assume that w is not φ-aperiodic. Then there are i, $s \in \mathbb{N}$ and $l \in \mathbb{N}_0$ such that $i + s + l \leq m$ and

$$[w(i) \dots w(i+l)] = [w(i+s) \dots w(i+s+l)]$$

with $s \leq \varphi(l)$. This implies that $\ell(s) \leq l$ and in particular

$$[w(i) \dots w(i + \ell(s))] = [w(i+s) \dots w(i+s+\ell(s))].$$

Hence, it follows that $w|_n \notin C_{is}$ since $i + s + \ell(s) = n \leq m$ so that $(i, s) \in A(n)$. □

Note that by the same arguments as in the previous proof, a word $w \in \Sigma^+$ is φ-aperiodic if and only if for all $n \geq m_0$ and all $(i,s) \in A(n)$ we have $w|_n \in C_{is}$.

For $m \in \mathbb{N}$ such that $m \geq m_0$ the set of good words of length m is therefore given by

$$\mathcal{W}^g(m) = \{w \in \Sigma^+(m) : w|_n \in C_{is} \text{ for all } (i,s) \in A(n) \text{ where } n \leq m\},$$

and by $\mathcal{W}^g(m) = \Sigma^+(m)$ otherwise. Let

$$\mathcal{C}_m = \{C_{is} : (i,s) \in A(m)\}$$

be the set of conditions at place m which is empty if and only if $m < m_0$. Clearly, if $w \in \mathcal{W}^g(m)$ then $w|_n \in \mathcal{W}^g(n)$ for $n \leq m$.

Lemma 5.3. *For* $m \in \mathbb{N}$*,*

$$|\mathcal{W}^g(m+1)| \geq k \cdot |\mathcal{W}^g(m)| - \sum_{C_{is} \in \mathcal{C}_{m+1}} |\mathcal{W}^g(i+s-1)|$$

Proof. If $m+1 < m_0$ then $\mathcal{C}_{m+1}$ is empty and the claim follows. Hence let $m+1 \geq m_0$. Set $L = \{w \in \Sigma^+(m+1) : w|_m \in \mathcal{W}^g(m)\}$. Then

$$\mathcal{W}^g(m+1) = L \cap \Big(\bigcap_{C_{is} \in \mathcal{C}_{m+1}} C_{is} \Big) = L \setminus \Big(\bigcup_{C_{is} \in \mathcal{C}_{m+1}} (L \cap C_{is}^C) \Big),$$

where C_{is}^C denotes the complement of C_{is}. Fix some condition $C_{is} \in \mathcal{C}_{m+1}$. Since $|L| = k \cdot |\mathcal{W}^g(m)|$ the Lemma follows from the following claim. □

Claim 5.4. $|L \cap C_{is}^C| \leq |\mathcal{W}^g(i+s-1)|$.

Proof. If $Q := \{w|_{i+s-1} \in \Sigma^+(i+s-1) : w \in L\}$ then clearly $|Q| \leq |\mathcal{W}^g(i+s-1)|$. Decompose L into $L = \cup_{q \in Q} L_q$ where $L_q = \{w \in L : w|_{i+s-1} = q\}$. By definition, different elements in L_q have different subwords $[w(i+s) \ldots w(m+1)]$ and moreover

$$L \cap C_{is}^C = \{w \in L : [w(i) \ldots w(i+\ell(s)] = [w(i+s) \ldots w(m+1)]\}.$$

Hence, if $s > \ell(s)$ then an element w of L_q, which is also in C_{is}^C, is uniquely determined by q, that means, w is of the form $w|_{i+s-1} = q$ and

$$[w(i+s) \ldots w(m+1)] = [q(i) \ldots q(i+\ell(s))].$$

If $s \leq \ell(s)$ then one inductively checks that a word w in $L_q \cap C_{is}^C$ is of the form $w|_{i+s-1} = q$,

$$\begin{aligned} [w(i+js) \ldots w(i+(j+1)s-1)] &= [w(i+(j-1)j) \ldots w(i+js-1)] = \ldots = \\ &= [w(i) \ldots w(i+s-1)] = [q(i) \ldots q(i+s-1)] \end{aligned}$$

for $1 \leq j \leq j_0$ where j_0 is the maximal j such that $i+(j+1)s-1 \leq m+1$, and

$$[w(i+(j_0+1)j) \ldots w(m+1)] = [q(i) \ldots q(m+1-(i+(j_0+1)s))],$$

if $i+(j_0+1)s < m+1$. Again, w is uniquely determined by q. Hence in both cases, $|L_q \cap C_{is}^C| \leq 1$ and therefore

$$|L \cap C_{is}^C| \leq |Q| \leq |\mathcal{W}^g(i+s-1)|$$

which proves the claim. □

The above Lemma yields the following crucial estimate:

Lemma 5.5. *For* $m \in \mathbb{N}$*,*

$$|\mathcal{W}^g(m+1)| \geq \big(k - \lfloor\varphi(0)\rfloor\big)|\mathcal{W}^g(m)| - \sum_{j=1}^{m} \big(\lfloor\varphi(j)\rfloor - \lfloor\varphi(j-1)\rfloor\big)|\mathcal{W}^g(m-j)|. \quad (5.2)$$

Proof. For $0 \le j \le m$ let

$$H_j = \{C_{is} \in \mathcal{C}_{m+1} : i + s - 1 = m - j\}, \tag{5.3}$$

possibly empty. If $C_{is} \in H_j$ then $i + s + \ell(s) = m + 1$ and $i + s - 1 = m - j$; hence $\ell(s) = j$. Therefore, $|H_j| \le |\{s : \ell(s) = j\}|$. We have $\ell(s) \le j$ if and only if $s \le \varphi(j)$ and thus

$$|\{s : \ell(s) \le j\}| = |\{s : s \le \varphi(j)\}| = \lfloor \varphi(j) \rfloor.$$

For $j \ge 1$ this implies that

$$\begin{aligned} |H_j| &\le |\{s : \ell(s) = j\}| = |\{s : \ell(s) \le j\} \setminus \{s : \ell(s) \le j - 1\}| \\ &= \lfloor \varphi(j) \rfloor - \lfloor \varphi(j-1) \rfloor. \end{aligned}$$

Moreover,

$$|\{s : \ell(s) = 0\}| = |\{s \in \mathbb{N}_0 : \varphi(0) \ge s\}| = \lfloor \varphi(0) \rfloor.$$

Lemma 5.3 concludes the proof. □

Finally we show the existence of a φ-aperiodic word in Σ^+.

Lemma 5.6. *If condition* (3.2) *is satisfied, then* $|\mathcal{W}^g(m)| \ge c^m$. *In particular, there exists a* φ*-aperiodic word in* Σ^+.

Proof. For $m + 1 < m_0$ we have that $|\mathcal{W}^g(m+1)| = k^{m+1} \ge c^{m+1}$. For $m + 1 \ge m_0$ assume that $|\mathcal{W}^g(n)| \ge c \cdot |\mathcal{W}^g(n-1)|$ for all $n \le m$. Then, by the previous Lemma,

$$\begin{aligned} |\mathcal{W}^g(m+1)| &\ge (k - \lfloor \varphi(0) \rfloor)|\mathcal{W}^g(m)| - \textstyle\sum_{j=1}^{m} (\lfloor \varphi(j) \rfloor - \lfloor \varphi(j-1) \rfloor)|\mathcal{W}^g(m-j)| \\ &\ge (k - \lfloor \varphi(0) \rfloor)|\mathcal{W}^g(m)| - \textstyle\sum_{j=1}^{m} \frac{\lfloor \varphi(j) \rfloor - \lfloor \varphi(j-1) \rfloor}{c^j}|\mathcal{W}^g(m)| \\ &\ge \Big(k - \lfloor \varphi(0) \rfloor - \textstyle\sum_{j=1}^{\infty} \frac{\lfloor \varphi(j) \rfloor - \lfloor \varphi(j-1) \rfloor}{c^j}\Big)|\mathcal{W}^g(m)| \ge c \cdot |\mathcal{W}^g(m)|, \end{aligned} \tag{5.4}$$

where we used condition (3.2) in the last inequality. Now Lemma 5.2 implies the existence of a φ-aperiodic word in Σ^+. □

Given a φ-aperiodic word $w \in \Sigma^+$ and a letter $a \in \mathcal{A}$, extend w to a word $\dots aaaw =: \bar{w} \in \Sigma$ (in the obvious way). Consider the sequence $\{T^n \bar{w}\}_{n \in \mathbb{N}}$ in the compact space Σ and let w_0 be an accumulation point. Note that from the definition of the metric $\bar{d}$, a sequence w^n in Σ converges to a word $w_0 \in \Sigma$ if and only if for every $l \in \mathbb{N}_0$ there exists $N \in \mathbb{N}$ such that $[w^n(-l) \dots w^n(l)] = [w_0(-l) \dots w_0(l)]$ for every $n \ge N$. It therefore follows that φ-aperiodicity is a closed condition (as showed similarly in Lemma 2.4). Since every $T^n \bar{w}$ is φ-aperiodic starting at time $-(n-1)$, w_0 is a φ-aperiodic word in Σ. This proves Theorem 3.3.

5.2. **Proof of Theorem 4.6.** Recall that M is a closed hyperbolic manifold of dimension $n \ge 2$ and we have $\ln(2) < r_0 < \bar{\varepsilon}_0 < i_M$. Moreover $\bar{\varphi} : \mathbb{N}_0 \to [0, \infty)$ is a non-decreasing unbounded function for which conditions (4.3) and (4.4) are satisfied with respect to the given minimal shift $\bar{s}_0 \in \mathbb{N}_0$.

A reference for the following is given by [4, 19]. Let $\mathbb{H}^n$ be the n-dimensional hyperbolic upper half-space model where d denotes the hyperbolic distance function on $\mathbb{H}^n$. Let Γ be the discrete, torsion-free subgroup of the isometry group of $\mathbb{H}^n$ identified with the fundamental group $\pi_1(M)$ of M acting cocompactly on $\mathbb{H}^n$ such that the manifold $\Gamma \backslash \mathbb{H}^n$ with the induced smooth and metric structure is isometric to M. Let $\pi : \mathbb{H}^n \to \Gamma \backslash \mathbb{H}^n \cong M$ be the projection map. Assume all geodesic segments, rays or lines to be parametrized by arc length and identify their images with their point sets in $\mathbb{H}^n$. Let $\partial_\infty \mathbb{H}^n$ be the set of equivalence classes of asymptotic rays in $\mathbb{H}^n$ which we identify with the set $\mathbb{R}^{n-1} \cup \{\infty\}$, where $\bar{\mathbb{H}}^n - \{\infty\} = \mathbb{H}^n \cup \mathbb{R}^{n-1}$ is equipped with the induced Euclidean topology. If γ is a ray in $\mathbb{H}^n$ we will simply write $\gamma(\infty)$ for the corresponding point in $\partial_\infty \mathbb{H}^n$. For any two points p and q in $\bar{\mathbb{H}}^n$ denote by $[p, q]$ the geodesic segment, ray or line in $\mathbb{H}^n$ - depending on if $p, q \in \mathbb{H}^n$, $p \in \mathbb{H}^n$ and $q \in \partial_\infty \mathbb{H}^n$, or $p, q \in \partial_\infty \mathbb{H}^n$ respectively - connecting p and q.

For $t \in \mathbb{R}$ let $H_t := \mathbb{R}^{n-1} \times \{e^{-t}\} \subset \mathbb{H}^n$. This equals the horosphere based at ∞ through the point $\gamma(t)$ of the unit speed geodesic $\gamma(t) = (0, e^{-t})$. Let h_t be the induced length metric on H_t with respect to d . The geometry of horospheres in the hyperbolic space is well-known; see for instance [8] for the following facts. (H_t, h_t) is a complete and flat metric space, isometric to the $(n-1)$-dimensional Euclidean space. If $\gamma_i : \mathbb{R} \to \mathbb{H}^n$ with $\gamma_i(0) \in H_0$, $i = 1, 2$, are two geodesic lines in $\mathbb{H}^n$ with $\gamma_1(-\infty) = \gamma_2(-\infty) = \infty$ and $\gamma_1(0)$, $\gamma_2(0)$ in the same horosphere, let $\mu(t) := h_t(\gamma_1(t), \gamma_2(t))$. Then, for $t \geq 0$,

$$\mu(t) = e^t \mu(0). \tag{5.5}$$

Moreover, for two points p, q in the same horosphere H_t we have

$$h_t(p, q) = 2 \sinh(d(p, q)/2). \tag{5.6}$$

Now let $\tau > 0$ such that the discretization constant satisfies $r_0 = \ln 2 + \tau$. Let $R > 0$ be a fixed length, say $R = 1$. Define Q to be an isometric copy of a closed $(n-1)$-dimensional cube $[-R/2, R/2]^{n-1}$ of edge lengths R in the Euclidean space $\mathbb{E}^{n-1}$ and contained in the horosphere H_0. Starting with the cube Q as a reference, we inductively shed shadows in the horospheres H_{mr_0}, $m \in \mathbb{N}$, as follows:

Definition 5.7. Given two disjoint sets S and S' in $\bar{\mathbb{H}}^n$, the set $\mathcal{S}(S; S') := \{q \in S' : S \cap [\infty, q] \neq \emptyset\}$ is called the *shadow of S in S'* (with respect to ∞).

By (5.5), the shadow $\mathcal{S}(Q; H_{r_0})$ of Q is an isometric copy of a closed $(n-1)$-dimensional cube of edge lengths $e^{r_0} R = (2 + e^\tau) R$, contained in H_{r_0}. Hence, there exist 2^{n-1} disjoint isometric copies Q_j, $j \in \{1, \ldots, 2^{n-1}\}$, of Q in $\mathcal{S}(Q; H_{r_0})$; see Figure 5.2.

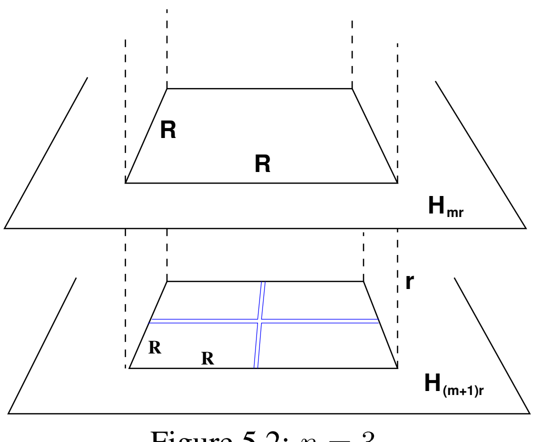

Figure 5.2: $n = 3$.

For $m \geq 1$, let the closed disjoint cubes $Q_{i_1 \ldots i_m}$ in H_{mr_0} be already defined. Fix a *cube* $Q_{i_1 \ldots i_m}$, then, as above, the shadow

$$\mathcal{S}(Q_{i_1 \ldots i_m}; H_{(m+1)r_0}) \subset H_{(m+1)r_0}$$

contains 2^{n-1} disjoint isometric copies $Q_{i_1 \ldots i_m j}$ of Q, $j \in \{1, \ldots, 2^{n-1}\}$. Hence, for an alphabet $\mathcal{A} = \{1, \ldots, 2^{n-1}\}$, we associate a finite word $[w(1) \ldots w(m+1)] \in \Sigma^+(m+1)$

to the cube $Q_{i_1\dots i_{m+1}}$ in $H_{(m+1)r_0}$ where $w(n) = i_n$ for all $n \in \{1, \dots, m+1\}$. In particular, we obtain a bijection of finite words $\Sigma^+(m)$ of length m with the set of cubes

$$\mathcal{Q}(m) := \{Q_{i_1\dots i_m} \subset H_{mr_0} : i_n \in \{1, \dots, 2^{n-1}\} \text{ for } 1 \le n \le m\}.$$

We denote the closed cubes $Q_{i_1\dots i_m}$ obtained in this way by $q(1)\dots q(m)$ where $q(n) \in \{1, \dots, 2^{n-1}\}$ for $n \in \{1, \dots, m\}$. Every sequence of cubes $\{q(1)q(2)\dots q(m)\}_{m\in\mathbb{N}}$, successively shadowed from the previous ones, determines a unique point

$$\eta := \bigcap_{m\in\mathbb{N}} \mathcal{S}(q(1)\dots q(m); \mathbb{R}^{n-1}) \in \mathbb{R}^{n-1},$$

since $\mathcal{S}(q(1)\dots q(m); \mathbb{R}^{n-1})$, $m \in \mathbb{N}$, is a sequence of closed nested subsets of $\mathbb{R}^{n-1}$ with diameters converging to 0. Define $\eta =: q(1)q(2)\dots$ in $\mathbb{R}^{n-1}$. By construction, the geodesic line $[\infty, \eta]$ runs through every cube $q(1)\dots q(m)$, $m \in \mathbb{N}$, of the particular sequence. Hence, we obtain a bijection of infinite sequences $q(1)q(2)\dots$ of cubes and words $w =: [w(1)w(2)\dots]$ in Σ^+.

Notation. Given a cube $q(1)\dots q(m)$ in $\mathcal{Q}(m)$ and an integer $n \le m$, let $q(1)\dots q(m)|_n \in \mathcal{Q}(n)$ be the unique cube such that $q(1)\dots q(m)$ lies in the shadow of $q(1)\dots q(m)|_n$. Moreover, for $\xi \in \mathbb{R}^n$ we denote the geodesic subsegment $[i, j](\xi)$ by

$$[i, j](\xi) := [\infty, \xi]|_{[ir_0, jr_0]} \colon [ir_0, jr_0] \to \mathbb{H}^n,$$

where we assume that $[\infty, \xi](0) \in H_0$ and that $i, j \in \mathbb{N}_0$ with $i \le j$, which connects the horospheres H_{ir_0} to H_{jr_0} and is orthogonal to both. If $i = j$, then we write $[i](\xi) := [i, i](\xi)$ which is the orthogonal projection of ξ on the horosphere H_{ir_0}.

We again define the *admissible set*

$$A(m) := \{(i, s) \in \mathbb{N} \times \mathbb{N} : i + s + \bar{\ell}(s) = m, s > \bar{s}_0\},$$

if $m \ge m_0 := 2 + \bar{s}_0 + \bar{\ell}(\bar{s}_0 + 1)$ and set $A(m)$ to be empty for $m < m_0$.

Definition 5.8. Let $\psi \in \Gamma$ be an isometry and let $i, s \in \mathbb{N}$, $l \in \mathbb{N}_0$. If $\xi \in \mathbb{R}^{n-1}$ such that $d(\psi([i](\xi)), [i + s](\xi)) < \bar{\varepsilon}_0$ and also $d(\psi([i + l](\xi)), [i + s + l](\xi)) < \bar{\varepsilon}_0$ we write

$$\psi\big([i, i + l](\xi)\big) \sim_{\bar{\varepsilon}_0} [i + s, i + s + l](\xi).$$

In particular, by convexity of the distance function, we have for all $j \in \{0, \dots, l\}$,

$$d(\psi\big([i, i + j](\xi)\big), [i + s, i + s + j](\xi)) < \bar{\varepsilon}_0. \tag{5.7}$$

We are now able to translate the proof of Theorem 3.3 for the existence of φ-aperiodic words into the existence of φ-aperiodic geodesics by counting good cubes:

Definition 5.9. Let $m \in \mathbb{N}$. A cube $q(1)\dots q(m)$ in $\mathcal{Q}(m)$ is called *good* if for every $\xi \in \mathcal{S}(q(1)\dots q(m); \mathbb{R}^{n-1})$, every $\psi \in \Gamma$ and every $i \in \mathbb{N}$, $l \in \mathbb{N}_0$, whenever

$$\psi\big([i, i + l](\xi)\big) \sim_{\bar{\varepsilon}_0} [i + s, i + s + l](\xi) \tag{5.8}$$

for some shift $s > \bar{s}_0$ such that $i + s + l \le m$, then $s > \bar{\varphi}(l)$. Otherwise $q(1)\dots q(m)$ is called *bad*.

If the cube $q(1)\dots q(m)$ is good, then, since $\bar{\varepsilon}_0 < i_M$, for every $x \in q(1)\dots q(m)$ the projection of the geodesic segment $[\infty, x]|_{[r_0, mr_0]}$ into M is $\bar{\varphi}$-aperiodic, up to length mr_0, with respect to condition (4.2) (see the proof Lemma 5.10 (2)).

Analogously to the proof of Theorem 3.3, for $(i,s) \in A(m)$ and $m \geq m_0$, define

$$C_{is} := \{q(1)\dots q(m) \in \mathcal{Q}(m) : \text{ for all } \xi \in \mathcal{S}(q(1)\dots q(m); \mathbb{R}^{n-1}) \text{ and } \psi \in \Gamma, \\ \psi\big([i, i+\bar{\ell}(s)](\xi)\big) \not\sim_{\bar{\varepsilon}_0} [i+s, m](\xi)\}$$

and let $\mathcal{C}_m$ be the set of all C_{ij} for $(i,j) \in A(m)$. Note that $\mathcal{C}_m$ is empty if $m < m_0$.

With respect to these definitions, the relationship between Definitions 4.4 and 5.9 respectively and the sets C_{is} is given by the following Lemma:

Lemma 5.10. (1) *For $m < m_0$ every cube $q(1)\dots q(m) \in \mathcal{Q}(m)$ is good. For $m \geq m_0$, the cube $q(1)\dots q(m) \in \mathcal{Q}(m)$ is good if $q(1)\dots q(m)|_n \in C_{is}$ for all $n \leq m$ and $(i,s) \in A(n)$.*

(2) *Let $q(1)q(2)\dots$ be an infinite sequence of cubes and let $\eta \in \mathbb{R}^{n-1}$ be the unique corresponding limit point. The discrete geodesic $\overline{\pi \circ [r_0, \infty)(\eta)}$ in M is $\bar{\varphi}$-aperiodic at every time $i \in \mathbb{N}$ if for all $m \in \mathbb{N}$ and $(i,s) \in A(m)$ the cube $q(1)\dots q(m)$ in $\mathcal{Q}(m)$ of the sequence $q(1)q(2)\dots$ belongs to C_{is}.*

Proof. For (1), let first $m < m_0$. Let i, $s \in \mathbb{N}$, $l \in \mathbb{N}_0$ such that $s > \bar{s}_0$ and $i+s+l \leq m < 2 + \bar{s}_0 + \bar{\ell}(\bar{s}_0+1)$. In particular, $l < \bar{\ell}(\bar{s}_0+1)$ so that $\varphi(l) < \bar{s}_0 + 1 \leq s$ and every cube $q(1)\dots q(m)$ follows to be good.

Now let $m \geq m_0$. Assume by absurd that $q(1)\dots q(m)$ is not good and let $\xi \in \mathcal{S}(q(1)\dots q(m); \mathbb{R}^{n-1})$ and $\psi \in \Gamma$ such that for some $i \in \mathbb{N}$, $l \in \mathbb{N}_0$, we have

$$\psi\big([i, i+l](\xi)\big) \sim_{\bar{\varepsilon}_0} [i+s, i+s+l](\xi),$$

where $s > \bar{s}_0$ with $i+s+l \leq m$ and $s \leq \bar{\varphi}(l)$. Hence, $\bar{\ell}(s) \leq l$ and for $n := i+s+\bar{\ell}(s)$ we have in particular by (5.7),

$$\psi\big([i, i+\bar{\ell}(s)](\xi)\big) \sim_{\bar{\varepsilon}_0} [i+s, n](\xi).$$

Hence, we see that $q(1)\dots q(m)|_n \notin C_{is}$ where $(i,s) \in A(n)$ for $n \leq m$; a contradiction.

For (2), assume that $\bar{\gamma} := \overline{\pi \circ [r_0, \infty)(\eta)}$ is not $\bar{\varphi}$-aperiodic at time $i \in \mathbb{N}$. Then there must be a shift $s \in \mathbb{N}$ with $s > \bar{s}_0$, and $l \in \mathbb{N}_0$ such that

$$d(\bar{\gamma}(i+j), \bar{\gamma}(i+s+j)) < \bar{\varepsilon}_0 \quad \text{for all } j \in \{0, \dots, l\},$$

where $s \leq \bar{\varphi}(l)$. Since $\bar{\varepsilon}_0 < i_M$ and the distance function is convex, we also have $d(\gamma((i+t)r_0), \gamma((i+s+t)r_0) < \bar{\varepsilon}_0$ for all $0 \leq t \leq l$ for the corresponding extended geodesic $\gamma : \mathbb{R} \to M$. By discreteness of Γ, there exist finitely many isometries $\psi_1,\dots,\psi_q \in \Gamma$ and a subdivision of the interval $[ir_0, (i+l)r_0]$ into $[l_0 r_0, l_1 r_0], [l_1 r_0, l_2 r_0], \dots, [l_{q-1} r_0, l_q r_0]$ where $l_0 = i$ and $l_q = i+l$ and $l_j \in \mathbb{R}$, such that (with analogous notation as above)

$$\psi_{j+1}\big([l_j, l_{j+1}](\eta)\big) \sim_{\bar{\varepsilon}_0} [s+l_j, s+l_{j+1}](\eta), \quad j = 0, \dots, q-1.$$

We thus have $d(\psi_{j+1}\big([l_{j+1}](\eta)\big), [s+l_{j+1}](\eta)) < \bar{\varepsilon}_0$ and $d(\psi_{j+2}\big([l_{j+1}](\eta)\big), [s+l_{j+1}](\eta)) < \bar{\varepsilon}_0$. Since $\bar{\varepsilon}_0 < i_M$ and every orbit of Γ is $2i_M$-separated (that is, for ψ, $\bar{\psi} \in \Gamma$ we have $d(\psi x, \bar{\psi} x) \geq 2i_M$ for any $x \in \mathbb{H}^n$) it follows from the triangle inequality that $\psi_{j+1}\big([l_{j+1}](\eta)\big) = \psi_{j+2}\big([l_{j+1}](\eta)\big)$; hence $\psi_{j+1} = \psi_{j+2}$ for all $j = 0, \dots, q-2$ since Γ acts freely. Therefore, we have an isometry $\psi \in \Gamma$ such that

$$\psi\big([i, i+l](\eta)\big) \sim_{\bar{\varepsilon}_0} [i+s, i+s+l](\eta)$$

where $s \leq \bar{\varphi}(l)$. The proof is now finished analogously to the case of (1). □

In view of Lemma 5.10, let for $m \geq m_0$,

$$\mathcal{Q}^g(m) = \{q(1)\dots q(m) \in \mathcal{Q}(m) : q(1)\dots q(m)|_n \in C_{is} \text{ for all } (i,s) \in A(n), n \leq m\},$$

and $Q^g(m) = \mathcal{Q}(m)$ for $m < m_0$, which is a subset of all good cubes at step m.

Lemma 5.11. *Assume that condition* (4.3) *is satisfied. Then, for* $m \in \mathbb{N}$,

$$|\mathcal{Q}^g(m+1)| \geq k|\mathcal{Q}^g(m)| - \bar{c} \cdot \sum_{C_{is} \in \mathcal{C}_{m+1}} |\mathcal{Q}^g(i+s-1)|, \tag{5.9}$$

where $\bar{c}$ *is a constant depending only on* n, i_M *and* $\bar{s}_0$, *and is strictly decreasing in* $\bar{s}_0$.

Proof. If $m+1 < m_0$ then $\mathcal{C}_{m+1}$ is empty and the claim follows. Hence assume $m+1 \geq m_0$. Let

$$L = \{q(1)\dots q(m+1) \in \mathcal{Q}(m+1) : q(1)\dots q(m+1)|_m \in \mathcal{Q}^g(m)\}$$

and note that $|L| = k|Q^g(m)|$. Then

$$\mathcal{Q}^g(m+1) = L \cap (\bigcap_{C_{is} \in \mathcal{C}_{m+1}} C_{is}) = L \setminus (\bigcup_{C_{is} \in \mathcal{C}_{m+1}} (L \cap C_{is}^C)),$$

where C_{is}^C is the complement of C_{is}. Fix some $C = C_{is} \in \mathcal{C}_{m+1}$. Define

$$Q = \{q(1)\dots q(m+1)|_{i+s-1} \in \mathcal{Q}(i+s-1) : q(1)\dots q(m+1) \in L\},$$

One checks that $|Q| \leq |\mathcal{Q}^g(i+s-1)|$. Let $L = \cup_{q \in Q} L_q$ where

$$L_q = \{q(1)\dots q(m+1) \in L : q(1)\dots q(m)|_{i+s-1} = q\}.$$

It remains to show that each $L_q \cap C^C$ contains at most $\bar{c}$ cubes; in this case,

$$|L \cap C^C| \leq \bar{c} \cdot |Q| \leq \bar{c} \cdot |\mathcal{Q}^g(i+s-1)|.$$

The following claim concludes the proof. □

Claim 5.12. $|L_q \cap C^C| \leq \bar{c} \cdot |\mathcal{Q}^g(i+s-1)|$.

For the proof of the claim note that if (4.4) is satisfied, then for all $l \geq \bar{s}_0$,

$$\lfloor \bar{\varphi}(l) \rfloor > l,$$

which implies that for all $s > \bar{s}_0$,

$$\bar{\ell}(s) < s. \tag{5.10}$$

To see this, assume $\bar{\ell}(s) \geq s$ for some $s > \bar{s}_0$. Then, by definition of $\bar{\ell}$, $\bar{\varphi}(j) < s$ for all $s > j \in \mathbb{N}_0$. In particular, for $\bar{s}_0 < s$ we have $\bar{\varphi}(\bar{s}_0) \geq \lfloor \bar{\varphi}(\bar{s}_0) \rfloor$; a contradiction to $\lfloor \bar{\varphi}(\bar{s}_0) \rfloor > \bar{s}_0$.

Proof of the Claim 5.12. L_q consists of cubes of the form $q \cdot q(i+s)\dots q(m+1) \in \mathcal{Q}(m+1)$. Hence, consider the point set W of all geodesic segments $[i, i+\bar{\ell}(s)](\xi)$ where $\xi \in \mathcal{S}(q, \mathbb{R}^{n-1})$; see Figure 5.2. Since $s > \bar{s}_0$ we have $\bar{\ell}(s) < s$ by (5.10), and therefore $s - 1 - \bar{\ell}(s) \geq 0$. Moreover, by definition, the cube q in $H_{(i+s-1)r_0}$ has h-edge lengths R. Thus from (5.5), the subset $H_{i+\bar{\ell}(s)} \cap W$ is isometric to an Euclidean cube with h-edge length

$$e^{-(i+s-1)r_0 + (i+\bar{\ell}(s))r_0} R = e^{-(s-1-\bar{\ell}(s))r_0} R \leq R.$$

Since an Euclidean cube in $\mathbb{E}^{n-1}$ of edge length L has diameter at most $\sqrt{n-1}L$, we obtain from (5.6) that the d-diameter of $H_{i+\bar{\ell}(s)} \cap W$ is bounded above by

$$2\,\text{arcsinh}(e^{-(s-1-\bar{\ell}(s))r_0}\sqrt{n-1}R/2). \tag{5.11}$$

In the same way, the h-edge length of $H_{ir_0} \cap W$ is given by

$$e^{-(s-1)r_0} R. \tag{5.12}$$

Now, by definition, for every $q \cdot q(i+s) \dots q(m+1) \in L_q \cap C^C$ there exists $\psi \in \Gamma$ such that $\psi\big([i, i+\bar{\ell}(s)](\xi)\big) \sim_{\bar{\varepsilon}_0} [i+s, m+1](\xi)$ for some $\xi \in \mathcal{S}(q, \mathbb{R}^{n-1})$. In particular, $x := [m+1](\xi)$ must belong to the $\bar{\varepsilon}_0$-neighborhood of $\psi(W \cap H_{i+s+\bar{\ell}(s)})$. Thus, we want to estimate the maximal number of cubes in $\mathcal{Q}(m+1)$ which intersect with the $\bar{\varepsilon}_0$-neighborhood of $\psi(W \cap H_{i+s+\bar{\ell}(s)})$. Let therefore also $y \in H_{(m+1)r_0}$ belong to the $\bar{\varepsilon}_0$-neighborhood of $\psi(W \cap H_{i+s+\bar{\ell}(s)})$. By the triangle inequality and by (5.11), we have

$$d(x, y) \leq 2\bar{\varepsilon}_0 + 2\,\mathrm{arcsinh}(e^{-(s-1-\bar{\ell}(s))r_0}\sqrt{n-1}R/2).$$

Therefore, again from (5.6), the h-diameter of the intersection of the $\bar{\varepsilon}_0$-neighborhood of $\psi(W \cap H_{i+s+\bar{\ell}(s)})$ with $H_{(m+1)r_0}$ is bounded above by

$$\bar{r}_1(s) := 2\sinh(\bar{\varepsilon}_0 + \mathrm{arcsinh}(e^{-(s-1-\bar{\ell}(s))r_0}\sqrt{n-1}R/2)).$$

On the other hand, the cubes $q \cdot q(i+s) \dots q(m+1) \in \mathcal{Q}(m+1)$ are disjoint and have Euclidean volume R^{n-1}. Therefore, we set

$$\bar{c}_1(s) := \lceil \frac{(\bar{r}_1(s) + \sqrt{n-1}R)^{n-1}}{R^{n-1}} \rceil.$$

Hence, the $\bar{\varepsilon}_0$-neighborhood of $\psi(W \cap H_{i+s+\bar{\ell}(s)})$ can intersect at most $\bar{c}_1(s)$ qubes in $\mathcal{Q}(m+1)$. Since $q(1) \dots q(m)$ is good for every $q(1) \dots q(m+1) \in L_q$, we conclude that, with respect to ψ, at most $\bar{c}_1(s)$ cubes can become bad in $L_q \cap C^C$.

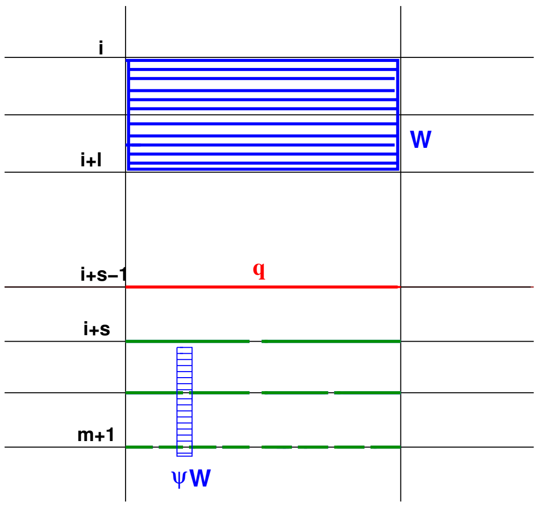

Figure 5.2: $n = 2$.

Now, let $\bar{y}$ be the center of $W \cap H_{ir_0}$, which is isometric to a cube in the Euclidean space of edge length $e^{-(s-1)r_0}R$ by (5.12) and contained in the cube $q|_i$. From (5.6), $W \cap H_{ir_0}$ must be contained in the hyperbolic ball $B_d(\bar{y}, \bar{r}_2(s))$, where

$$\bar{r}_2(s) = 2\,\mathrm{arcsinh}(e^{-(s-1)r_0}\sqrt{n-1}R/4).$$

Note that if there is some point $p \in W \cap H_{ir_0}$ and some $\psi \in \Gamma$ such that $d(\psi p, \bar{q}) < \bar{\varepsilon}_0$, where $\bar{q} := \mathcal{S}(q, H_{(i+s)r_0})$, then $d(\psi\bar{y}, \bar{q}) < \bar{\varepsilon}_0 + \bar{r}_2(s)$. In particular, for every cube $q \cdot q(i+s) \dots q(m+1) \in L_q \cap C^C$ there exists such an isometry ψ. But since the orbit $\Gamma\bar{y}$ is $2i_M$-separated, the open metric balls $B(\psi\bar{y}, i_M)$, $\psi \in \Gamma$, are disjoint and there can only be finitely many, say $\bar{c}_2(j)$, intersecting the $\max\{\bar{\varepsilon}_0 + \bar{r}_2(s) - i_M, 0\}$-neighborhood of $\bar{q}$.

In fact, from (5.5) and (5.6), the h-diameter of $\bar{q}$ is bounded above by $e^{r_0}\sqrt{n-1}R$ and $\bar{q}$ must be contained in a hyperbolic ball of radius $2\,\mathrm{arcsinh}(e^{r_0}\sqrt{n-1}R/4)$. Therefore, $\bar{c}_2(s)$ is bounded above by

$$\lceil \frac{\mathrm{vol}(B\big(2\,\mathrm{arcsinh}(e^{r_0}\sqrt{n-1}R/4)+2\,\mathrm{arcsinh}(e^{-(s-1)r_0}\sqrt{n-1}R/4)+\bar{\varepsilon}_0\big))}{\mathrm{vol}(B(i_M/2))}\rceil.$$

Since both, $\bar{c}_1(s)$ and $\bar{c}_2(s)$ are non-increasing in s, we conclude the claim by setting $\bar{c} := \bar{c}_1(\bar{s}_0+1)\bar{c}_2(\bar{s}_0+1)$. □

Analogously to the proof of Lemma 5.5, the previous Lemma yields the following.

Lemma 5.13. *Assume that condition* (5.10) *is satisfied. Then, for* $m \in \mathbb{N}$*,*

$$\begin{aligned} |\mathcal{Q}^g(m+1)| &\geq \big(k - \mathbf{1}_{\{\bar{\ell}(\bar{s}_0+1)=0\}}\bar{c}\lfloor\bar{\varphi}(0)\rfloor\big)|\mathcal{Q}^g(m)| \\ &- \bar{c}\cdot \textstyle\sum_{j=\max(\bar{\ell}(\bar{s}_0+1),1)}^{m}\big(\lfloor\bar{\varphi}(j)\rfloor - \lfloor\bar{\varphi}(j-1)\rfloor\big)|\mathcal{Q}^g(m-j)|. \end{aligned}$$

Proof. Recall the definition of the set $H_j = \{C_{is} \in \mathcal{C}_{m+1} : i+s-1 = m-j\}$ in (5.3). Since $\bar{\ell}$ is non-decreasing we have $j = m+1-(i+s) = \bar{\ell}(s) \geq \bar{\ell}(\bar{s}_0+1)$ if $s > \bar{s}_0$. □

Finally, if moreover condition (4.4) is satisfied, then the same inductive proof as in Lemma 5.6 shows that the number of good cubes in $Q^g(m+1)$ increases in $m+1$ by the factor $c > 1$; see (5.4). Lemma 5.10.(2) then shows the existence of a $\bar{\varphi}$-aperiodic geodesic $\bar{\gamma} : \mathbb{N} \to M$. Thus, we have shown the following.

Lemma 5.14. *Assume that conditions* (4.3) *and* (4.4) *are satisfied. Then, for* $m \in \mathbb{N}$*,* $|\mathcal{Q}^g(m)| \geq c^m$*. In particular, there exists a* $\bar{\varphi}$*-aperiodic geodesic* $\bar{\gamma} : \mathbb{N} \to M$ *with parameters* $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$*.*

Now, let $\bar{\gamma} : \mathbb{N} \to M$ be a $\bar{\varphi}$-aperiodic geodesic (with parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$ and let $\gamma : \mathbb{R} \to M$ be the corresponding extended geodesic. Consider the sequence $v^n := \phi^n\gamma'(r_0)$, $n \in \mathbb{N}$, in the compact space SM and let γ_0 be an accumulation point. The space of unit speed geodesics (identified with SM) is endowed with the topology of uniform convergence on bounded sets. Therefore note that a sequence v^n converges to v in SM if and only if for every $l \geq 0$ and every $\tau > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $d(\gamma_{v^n}(t), \gamma_v(t)) < \tau$ for every $t \in [-l, l]$. Therefore $\bar{\varphi}$-aperiodicity can be shown to be a closed condition (similarly as in Lemma 2.4). Since $\bar{\gamma}_{v^n}$ is $\bar{\varphi}$-aperiodic beginning at $t_n \geq -(n-1)$ (with parameters $(\bar{s}_0, \bar{\varepsilon}_0, r_0)$), it follows that $\bar{\gamma}_0 : \mathbb{Z} \to M$ is $\bar{\varphi}$-aperiodic. This completes the proof of Theorem 4.6.

5.3. **Proof of Theorem 4.3.** For $\delta \in (0,1)$ choose $\bar{\delta} \in [\delta, 1)$ such that for $r_0 = \ln(3-\bar{\delta})$ we have $\ln(3-\bar{\delta}) + \varepsilon_0 < i_M$. Note that $\tilde{\delta} = \bar{\delta}\ln(2)/\ln(3-\bar{\delta}) \to 1$ as $\bar{\delta} \to 1$ and assume therefore that $\tilde{\delta} > \delta$. For $l \geq 0$ let $\bar{\psi}(l) = 2^{\bar{\delta}(n-1)l}$ so that its right inverse $\lceil \frac{1}{\bar{\delta}(n-1)\ln(2)}\ln(s)\rceil$ is an unbounded function. Then, for $c = \frac{1}{2}(2^{n-1} + 2^{\bar{\delta}(n-1)})$, we have that for sufficiently large $\bar{s}_0 = \bar{s}_0(\bar{\delta}, n, i_M, \varepsilon_0) \in \mathbb{N}_0$ the conditions (4.3) and (4.4) are satisfied. Thus, from Theorem 4.6 there exists a discrete geodesic $\bar{\gamma} : \mathbb{Z} \to M$ which is $\bar{\psi}$-aperiodic with respect to $(\bar{s}_0, r_0+\varepsilon_0, r_0)$. From Lemma 4.5 we obtain that $\gamma : \mathbb{R} \to M$ is continuously ψ-aperiodic with parameters $s_0 = (\bar{s}_0+1)r_0$ and ε_0, where for $l \geq r_0$,

$$\begin{aligned} \psi(l) &= \ln(3-\bar{\delta})\cdot\bar{\psi}(\tfrac{l}{\ln(3-\bar{\delta})} - 1) - \ln(3-\bar{\delta}) \\ &= \tfrac{\ln(3-\bar{\delta})}{2^{\bar{\delta}(n-1)}} e^{\frac{\bar{\delta}\ln(2)}{\ln(3-\bar{\delta})}(n-1)l} - \ln(3-\bar{\delta}) \\ &= \big(\tfrac{\ln(3-\bar{\delta})}{2^{\bar{\delta}(n-1)}} - \tfrac{\ln(3-\bar{\delta})}{e^{\tilde{\delta}(n-1)l}}\big)e^{\tilde{\delta}(n-1)l} \\ &=: c(\tilde{\delta}, l)\cdot e^{\tilde{\delta}(n-1)l} = c(\tilde{\delta}, l)\varphi_{\tilde{\delta}}(l). \end{aligned}$$

Note that $c(\tilde{\delta}, l)$ is increasing in l and we restrict ψ to the interval $[l_1, \infty)$ for some $l_1 > \ln(3 - \bar{\delta})$ such that $c(\tilde{\delta}, l_1) > 0$.

We now translate the minimal shift s_0 into the minimal length l_0. Let to this end $N := \lceil \frac{s_0}{2i_M} \rceil$. Assume that for some t_0 we have $d(\gamma(t_0 + t), \gamma(t_0 + s + t) < \varepsilon_0$ for all $0 \le t \le l$ where $l \ge \max\{l_1, 3Ns_0 + 2i_M\} =: l_0$.

First, we assume that $s \le s_0$. Note that the function $t \mapsto d(\gamma(t_0 + t), \gamma(t_0 + s + t)$ is not only convex but decreases and increases exponentially (see [3]) so that we have $d(\gamma(t_0 + t), \gamma(t_0 + s + t) < \varepsilon_0/4$ for all $s' \le t \le l - s'$ where s' is sufficiently large, say $s' = 2i_M$. The closing lemma implies the existence of a closed geodesic nearby; in fact, we will prove the following Lemma.

Lemma 5.15. *In this setting, there exists a closed geodesic α of period $p \le s + \varepsilon_0/4$ such that (up to parametrization of α),*

$$d(\alpha(t), \gamma(t_0 + s' + t)) < \varepsilon_0/2 \quad \textit{for all } 0 \le t \le s + l - 2s' - \varepsilon_0.$$

Let $N' = \lceil s_0/p \rceil \in \mathbb{N}$ be the smallest integer such that $N'p > \bar{s}_0$ and note that $2Ns \ge N'p$. We then have by the triangle inequality,

$$\begin{aligned} & d(\gamma(t_0 + s' + t), \gamma(t_0 + s' + N'p + t)) \\ \le\ & d(\gamma(t_0 + s' + t), \alpha(t)) + d(\gamma(t_0 + s' + N'p + t), \alpha(t)) < \varepsilon_0 \end{aligned}$$

for all $0 \le t \le l - 2s' - N'p + s$ and in particular for all $0 \le t \le l - 2s' - 2Ns_0$. Thus,

$$2Ns \ge N'p > c(\tilde{\delta}, l_1)\varphi_{\tilde{\delta}}(l - 2s' - 2Ns_0)) = \frac{c(\tilde{\delta}, l_1)}{e^{\tilde{\delta}(n-1)(2s' + 2N\bar{s}_0)}}\varphi_{\tilde{\delta}}(l),$$

and we can find a positive constant $c_0 = c_0(\tilde{\delta}, i_M, n, \varepsilon_0)$ such that $s > c_0\varphi_{\tilde{\delta}}(l)$.

In the case when $s > s_0$, we have

$$s > c(\tilde{\delta}, l_1)\varphi_{\tilde{\delta}}(l) \ge\ c_0\varphi_{\tilde{\delta}}(l).$$

Finally, since $\delta < \tilde{\delta}$, we restrict if necessary to $\tilde{l}_0 \ge l_0$ such that $c_0\varphi_{\tilde{\delta}}(l) \ge \varphi_\delta(l)$ for all $l \ge \tilde{l}_0$. The proof of Theorem 4.3 is finished by the proof of Lemma 5.15.

Proof of Lemma 5.15. We consider the setting of the proof of Theorem 4.6. Let now d_M be the distance function on M and recall that we have $d_M(\gamma(t_0 + t), \gamma(t_0 + s + t) < \varepsilon_0/4$ for all $s' \le t \le l - s'$, where $s' = 2i_M > 2\ln(2)$. We denote a lift of the segment γ on $[t_0 + s', t_0 + l - s']$ by β and let the endpoints of β be x_1 and x_2. Since $\varepsilon_0 < i_M$, there exists an isometry $\psi \in \Gamma$ such that $d(\beta, \psi(\beta(t))) < \varepsilon_0/4$ for all $t \in [t_0 + s', t_0 + l - s']$ and in particular, $d(x_i, \psi x_i) < \varepsilon_0/4$ for $i = 1, 2$. Let $\tilde{\alpha}$ be the axis of ψ and denote by $d_1 = d(\tilde{\alpha}, x_1)$ and $d_2 = d(\tilde{\alpha}, x_2)$. We first show that d_1 is close to d_2 in the following sense. Namely, the displacement function $d_\psi(\cdot) = d(\psi\cdot, \cdot)$ grows at least linearly in the distance to $\tilde{\alpha}$. Since $s - \varepsilon_0/4 \le d_\psi(x_i) \le s + \varepsilon_0/4$ for $i = 1, 2$ we see that $|d_1 - d_2|$ is bounded by a constant depending only on ψ, s and ε_0.

Now, if we show that $d_i < \varepsilon_0/2$ for $i = 1, 2$, then the proof follows by convexity of the distance function. We show this for d_1. Since d_1 is close to d_2 and l is large, the distance function $t \mapsto d(\beta(t), \tilde{\alpha}(t))$ decreases exponentially on $[0, s']$, where $\tilde{\alpha}$ is parametrized such that $\tilde{\alpha}(0)$ equals the orthogonal projection $\bar{x}_1$ of x_1 on the convex set $\tilde{\alpha}$. Moreover, s' is large and thus $d(\tilde{\alpha}, \beta(s')) < d_1/2$. The orthogonal projection of $\psi(x_1)$ on $\tilde{\alpha}$ is given by $\psi(\bar{x}_1)$. Hence, $d(\psi(x_1), \tilde{\alpha}(s')) \ge d(\psi(x_1), \psi(\bar{x}_1)) = d_1$. On the other hand, we have by the triangle inequality $d(\psi(x_1), \tilde{\alpha}(s')) \le d(\psi(x_1), \beta(s')) + d(\beta(s'), \tilde{\alpha}(s')) < d_1/2 + \varepsilon_0/4$. Thus, $d_1 < d_1/2 + \varepsilon_0/4$ and the claim follows. ☐

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V. SCHROEDER, INSTITUT FÜR MATHEMATIK DER UNIVERSITÄT ZÜRICH,
E-mail address: `viktor.schroeder@math.uzh.ch`

S. WEIL, INSTITUT FÜR MATHEMATIK DER UNIVERSITÄT ZÜRICH,
E-mail address: `steffen.weil@math.uzh.ch`