NULLSPACES OF CONFORMALLY INVARIANT OPERATORS. APPLICATIONS TO Q_k-CURVATURE

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Abstract. We study conformal invariants that arise from functions in the nullspace of conformally covariant differential operators. The invariants include nodal sets and the topology of nodal domains of eigenfunctions in the kernel of GJMS operators. We establish that on any manifold of dimension $n \geq 3$, there exist many metrics for which our invariants are nontrivial. We discuss new applications to curvature prescription problems.

Introduction

In this note, we announce various results about conformal invariants arising from nodal sets and nullspaces of conformally invariant operators, including some applications to Q_k-curvatures. The invariants include nodal sets and the topology of nodal domains of eigenfunctions in the kernel of GJMS operators (see Section 2).

We also establish that, on any manifold of dimension $n \geq 3$, there exist many metrics for which our invariants are nontrivial (see Theorem 3.1). In addition, we discuss new applications to curvature prescription problems (see Section 5).

This note is organized as follows. In Section 1, we review the main definitions and properties about the GJMS operators $P_{k,g}$. In Section 2, we describe various conformally invariant quantities arising from the nullspaces of the GJMS operators. In Section 3, we present several results related to their negative values. In Section 4, we report on a careful analysis of the spectral properties of the Yamabe and Paneitz operators on compact Heisenberg manifolds. In Section 5, we study Q_k-curvature problems.

Full details on the results mentioned in this note will appear in [5, 6].

1. Conformal Powers of the Laplacian and Q_k-curvatures

Let (M^n, g) be a Riemannian manifold ($n \geq 3$). We are interested in the conformal powers of the Laplacian as constructed in [11]; these are called the GJMS operators.

For any positive integer k if n is odd, or for any positive integer $k \leq \frac{n}{2}$ if n is even, there is a covariant differential operator $P_{k,g} : C^\infty(M) \to C^\infty(M)$ such that

(i) P_k has with same leading part as Δ_g^k, where Δ_g is the Laplacian on M.

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(ii) P_k is conformally invariant in the sense that

$$P_{k,e^{2\Upsilon}g} = e^{-(\frac{n}{2}+k)\Upsilon} P_{k,g} e^{(\frac{n}{2}-k)\Upsilon} \qquad \forall \Upsilon \in C^\infty(M,\mathbb{R}). \tag{1}$$

For $k = 1$ we get the Yamabe operator $P_{1,g} := \Delta_g + \frac{n-2}{4(n-1)} R_g$, where R_g is the scalar curvature. For $k = 2$ we recover the Paneitz operator. In any case, the operator P_k is formally self-adjoint (see, e.g., [8]).

For $w \in \mathbb{R}$ denote by $\mathcal{E}[w]$ the space of (smooth) conformal densities of weight w on M. A conformal density of weight w can be defined as a section of the line bundle of conformal densities of weight w (see, e.g., [14]). It can be equivalently seen as a family of $(u_{\hat{g}})_{\hat{g}\in[g]} \subset C^\infty(M)$ parametrized by the conformal class $[g]$, in such way that

$$u_{e^{2\Upsilon}g}(x) = e^{w\Upsilon(x)} u_g(x) \qquad \forall \Upsilon \in C^\infty(M,\mathbb{R}).$$

Then (1) exactly means that $P_{k,g}$ uniquely extends to a differential operator,

$$P_{k,g} : \mathcal{E}\left[-\frac{n}{2}+k\right] \longrightarrow \mathcal{E}\left[-\frac{n}{2}-k\right], \tag{2}$$

which is independent of the representative metric in the conformal class $[g]$.

It follows from this that null eigenvectors of $P_{k,g}$ can be interpreted as conformal densities of weight $-\frac{n}{2}+k$. Incidentally, the dimension of the nullspace of $P_{k,g}$ is an invariant of the conformal class $[g]$.

In addition, the operator $P_{k,g}$ is intimately related to the Q_k-curvature $Q_{k,g}$, which, for $k \neq \frac{n}{2}$, is defined by

$$Q_{k,g} := \frac{2}{n-2k} P_{k,g}(1).$$

For $k = \frac{n}{2}$ the $Q_{\frac{n}{2}}$-curvature is often defined by a limit procedure by letting $k \to \frac{n}{2}$ (see [2, 12]). It is often referred to as Branson's Q-curvature.

2. Conformal Invariants from the Nullspace of $P_{k,g}$

Let $k \in \mathbb{N}$ and further assume $k \leq \frac{n}{2}$ if n is even. If $u = (u_{\hat{g}})_{\hat{g}\in[g]}$ is a conformal density of weight w, then the zero locus $u_{\hat{g}}^{-1}(0)$ is independent of the metric $\hat{g}$, and hence is an invariant of the conformal class $[g]$.

In view of (2) we can regard null-eigenvectors of $P_{k,g}$ as conformal densities of weight $-\frac{n}{2}+k$. Therefore, we obtain

Proposition 2.1.

(1) *If* $\dim\ker P_{k,g} \geq 1$*, then the nodal set and nodal domain of any nonzero null-eigenvector of* $P_{k,g}$ *give rise to invariants of the conformal class* $[g]$.
(2) *If* $\dim\ker P_{k,g} \geq 2$*, then (non-empty) intersections of nodal sets of null-eigenvectors of* $P_{k,g}$ *and their complements are invariants of the conformal class* $[g]$.

For a conformal density of weight 0, all its level sets, not just its zero-locus, give rise to conformal invariants. Observe that if n is even, then for $k = \frac{n}{2}$ the nullspace of $P_{\frac{n}{2}}$ gives rise to a subspace of $\mathcal{E}[0]$. It can be shown that the constant functions are contained in the nullspace of $P_{\frac{n}{2}}$. We thus obtain

Proposition 2.2. *Assume* n *is even. If* $\dim\ker P_{\frac{n}{2},g} \geq 2$*, then the level sets of any non-constant null-eigenvector of* $P_{\frac{n}{2},g}$ *are invariants of the conformal class* $[g]$.

Let $u_{1,g}, \cdots, u_{m,g}$ be a basis of $\ker P_{k,g}$ (where the $u_{j,g}$ are conformal densities of weight $k-\frac{n}{2}$). In addition, set $\mathcal{N} := \bigcap_{1\leq j\leq m} u_{j,g}^{-1}(0)$ and define $\Phi : M\setminus\mathcal{N} \to \mathbb{RP}^{m-1}$ by

$$\Phi(x) := (u_{1,g}(x) : \cdots : u_{m,g}(x)) \qquad \forall x \in M \setminus \mathcal{N}.$$

Observe that if $x \in M\setminus\mathcal{N}$, then the m-uple $(u_{1,g}(x), \cdots u_{m,g}(x))$ depends on the representative metric g only up to positive scaling. That is, the projective vector $(u_{1,g}(x) : \cdots : u_{m,g}(x)) \in \mathbb{RP}^{m-1}$ is independent of g. Therefore, we have

Proposition 2.3. *The map Φ above is an invariant of the conformal class $[g]$.*

For $k = \frac{n}{2}$, the nullspace of $P_{\frac{n}{2}}$ always contains the constant functions, so we may assume that $u_{1,g}(x) = 1$. Moreover, as the $u_{j,g}(x)$ are conformal densities of weight 0, for any $x \in M$, the $(m-1)$-uple $(u_{2,g}(x), \cdots, u_{m,g}(x))$ is actually *independent* of the representative metric g. Therefore, defining $\Psi : M \to \mathbb{R}^{m-1}$ by

$$\Psi(x) := (u_{2,g}(x), \cdots, u_{m,g}(x)) \qquad \forall x \in M,$$

we obtain

Proposition 2.4. *The map Ψ above is an invariant of the conformal class $[g]$.*

Finally, assume M compact and denote by $dV_g(x)$ the Riemannian measure defined by g. If u_g is a conformal density of weight $w < 0$ and we set $p = \frac{n}{|w|}$, then a simple exercise shows that the value of the integral $\int_M |u(x)|^p dV_g(x)$ is independent of the representative metric g. Applying this result to null-eigenvectors of $P_{k,g}$ we arrive at the following statement.

Proposition 2.5. *Assume M compact and $k < \frac{n}{2}$. Let u_g be a null-eigenvector of $P_{k,g}$ and regard it as a conformal density of weight $-\frac{n}{2}+k$. Then the integral $\displaystyle\int_M |u_g(x)|^{\frac{2n}{n-2k}} dV_g(x)$ is an invariant of the conformal class $[g]$.*

3. Negative Eigenvalues of the $P_{k,g}$

In this section, we assume M compact. Let $k \in \mathbb{N}$ (and further assume $k \leq \frac{n}{2}$ if n is even). We are interested in metrics for which $P_{k,g}$ has negative eigenvalues.

For $m \in \mathbb{N}_0$, denote by $\mathcal{G}_{k,m}$ the set of metrics g on M such that $P_{k,g}$ has at least m negative eigenvalues, where the eigenvalues are counted with multiplicity. Using the continuity of the eigenvalues of the $P_{k,g}$ with respect to the metric (cf. [4]), it is not hard to show that

- $\mathcal{G}_{k,m}$ is an open subset of the space of metrics equipped with the C^{2k}-topology.
- If $g \in \mathcal{G}_{k,m}$, then the whole conformal class $[g]$ is contained in $\mathcal{G}_{k,m}$.

It follows from this that the number of negative eigenvalues defines a partition of the set of conformal classes.

The family $\{\mathcal{G}_{k,m}\}_{m\geq 0}$ is a non-increasing sequence of open subsets. The question that naturally arises is whether $\mathcal{G}_{k,m}$ can be empty for large m.

In the case of the Yamabe operator, using results of Lokhamp [13] we get

Theorem 3.1. *Assume M compact. Then*

1. *For any $m \in \mathbb{N}$, there is a metric g on M for which the Yamabe operator $P_{1,g}$ has at least m negative eigenvalues.*

(2) *There are infinitely many metrics on M for which the nullspace of $P_{1,g}$ has dimension ≥ 1.*

In particular, we see that there are infinitely many metrics that fall into the scope of application of Proposition 2.1 and Proposition 2.3.

It would be interesting to obtain similar results for higher powers of the Laplacian. A first result in this direction is the following.

Theorem 3.2. *Assume $M = \Sigma \times \Sigma$, where Σ is a compact surface of genus ≥ 2. Then, for any $m \in \mathbb{N}$, there is a metric g on M for which the Paneitz operator $P_{2,g}$ has at least m negative eigenvalues.*

There is a similar result on compact Heisenberg manifolds (see next section).

In addition, as an application of Courant's nodal domain theorem, we obtain

Theorem 3.3. *Let g be a metric such that the Yamabe operator $P_{1,g}$ has exactly m negative eigenvalues. Then any (non-zero) null-eigenvector has at most $m+1$ nodal domains.*

It would be interesting to extend this result to higher conformal powers of the Laplacian.

Remark 3.4. In contrast to the conformal invariance of $\dim \ker P_{k,g}$, the non-zero eigenvalues of $P_{k,g}$ exhibit a very different behavior. As shown by Canzani in [4], arbitrary small pertubations within the conformal class can produce simple non-zero eigenvalues. In fact, if we equip the set of metrics with the C^{2k}-topology, then $P_{k,g}$ has simple non-zero eigenvalues for all metrics but that of a meager set (see [4]).

4. The Yamabe and Paneitz Operators on Compact Heisenberg Manifolds

In this section, we consider the example of a Heisenberg manifold $M := \Gamma\backslash\mathbb{H}_d$, $d \in \mathbb{N}$. Here $\mathbb{H}_d$ is the $(2d+1)$-dimensional Heisenberg group consisting of unipotent matrices,

$$A = \begin{pmatrix} 1 & x & t \\ 0 & 1 & y^T \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y \in \mathbb{R}^d, \quad t \in \mathbb{R},$$

and Γ is its lattice subgroup of unipotent matrices with integer-entries.

In the sequel we shall use coordinates $x = (x_1, \cdots x_d)$, $y = (y_1, \cdots y_n)$ and t as above to represent an element of $\mathbb{H}_d$. A left-invariant frame of vector fields on $\mathbb{H}_d$ is then given by the left-invariant vector fields,

$$T = \frac{\partial}{\partial t}, \qquad X_j = \frac{\partial}{\partial x_j}, \qquad Y_j = \frac{\partial}{\partial y_j} + x_j \frac{\partial}{\partial t}, \quad j = 1, \cdots, d.$$

As these vector fields are left-invariant, they descend to vector fields on M.

Following [10], for $s > 0$, we endow $\mathbb{H}_d$ with the left-invariant metric,

$$g_s := \sum_{1 \leq j \leq d} dx_j \otimes dx_j + \sum_{1 \leq j \leq d} s^{-2} dy_j \otimes dy_j + s^{2d} \theta \otimes \theta, \tag{3}$$

where $\theta := dt - \sum_{1 \leq j \leq d} x_j dy_j$ is the standard contact form of $\mathbb{H}_d$. This metric descends to a metric on M.

For the metric g_s the Laplace operator is $\Delta_{g_s} = -\sum_{1\leq j\leq d}(X_j^2+s^2Y_j^2)-s^{-2d}T^2$ and the scalar curvature is $R_{g_s} = -\frac{d}{2}s^{2d+2}$. Therefore, the Yamabe operator is given by

$$P_{1,g_s} = -\sum_{1\leq j\leq d}(X_j^2+s^2Y_j^2)-s^{-2d}T^2-\frac{2d-1}{16}s^{2d+2}.$$

The Paneitz operator too can be computed. We find

$$P_{2,g_s} = \Delta_{g_s}^2 + \frac{12-(2d-1)^2}{8(2d-1)}s^{2d+2}\Delta_{g_s} + 2\frac{d+1}{2d-1}s^2T^2 + (2d-3)\frac{(2d+1)(2d-1)^2-4(22d+1)}{256(2d-1)^2}s^{4d+4}.$$

Using the representation theory of the Heisenberg group enables us to give explicit spectral resolutions of the Yamabe and Paneitz operators. In this note, we shall only present some applications of these spectral resolutions. A fully detailed account is given in [5].

Proposition 4.1. *There is a constant $c(d)$ depending only on d such that, for all $s>0$ large enough, the number of negative eigenvalues of the Yamabe operator P_{1,g_s} is $\geq c(d)s^{2d+1}$.*

This result provides us with a quantitative version of Theorem 3.1 on Heisenberg manifolds.

In addition, the nullspace of P_{1,g_s} can be expressed in terms of Jacobi's theta function,

$$\vartheta(z,\tau) := \sum_{k\in\mathbb{Z}} e^{i\pi k^2\tau}e^{2i\pi kz}, \qquad z,\tau\in\mathbb{C},\ \Im\tau>0.$$

Proposition 4.2. *Suppose that $s^{d+2}=\frac{8\pi}{2d-1}\left(2d+\sqrt{4d^2+2d-1}\right)$. Then*

(1) *The nullspace of the Yamabe operator P_{1,g_s} is spanned by the functions,*

$$u_\pm(x,y,t) := e^{\pm 2i\pi t}e^{\mp\pi s|x|^2}\prod_{1\leq j\leq d}\vartheta(y_j\pm isx_j, is).$$

(2) *The nodal set of the function $u_\pm(x,y,t)$ is equal to the join,*

$$\bigcup_{1\leq j\leq d}\left\{(x,y,t)\in M; x_j=y_j=\frac{1}{2}\right\}.$$

Finally, for the Paneitz operator in low dimension we obtain

Proposition 4.3. *For $d=1,2,3$ the number of negative eigenvalues of the Paneitz operator P_{2,g_s} goes to ∞ as $s\to\infty$.*

5. Q_k-Curvature Prescription Problems

The Q_k-curvature prescription problem has been an important focus of interest in conformal geometry, especially for Branson's Q-curvature $Q_{\frac{n}{2}}$ (n even) (see, e.g., [1, 3, 7]). In this section, we look at the Q_k-curvature prescription problem for $k\neq\frac{n}{2}$ (further assuming $k<\frac{n}{2}$ when n is even).

In the sequel, we denote by $\mathcal{R}(Q_k)$ the set of functions $v\in C^\infty(M)$ for which there is a metric $\hat{g}$ in the conformal class $[g]$ such that $v=Q_{k,\hat{g}}$.

Let $\hat{g}$ be a metric in the conformal class $[g]$. The definition of $Q_{k,\hat{g}}$ implies that $Q_{k,\hat{g}} = 0$ if and only if $P_{k,\hat{g}}(1) = 0$. Furthermore, it follows from the transformation law (1) that $P_{k,\hat{g}}(1) = 0$ if and only if the nullspace of $P_{k,g}$ contains a positive function. Therefore, we obtain

Theorem 5.1. *$\mathcal{R}(Q_k)$ contains the zero function if and only if the nullspace of $P_{k,g}$ contains a nowhere-vanishing function.*

For the Yamabe operator $P_{1,g}$ we actually get a finer result.

Theorem 5.2. *Let $u_g \in \ker P_{1,g} \setminus \{0\}$ and let Ω be a nodal domain of u_g. Then, for any metric $\hat{g}$ in the conformal class $[g]$,*

1. $\int_\Omega |u_{\hat{g}}(x)|\, R_{\hat{g}}(x) dV_{\hat{g}}(x) < 0$, *where $R_{\hat{g}}$ is the scalar curvature.*
2. *The scalar curvature $R_{\hat{g}}$ cannot be nonnegative on Ω.*

Finally, assume M is compact. In [9] constraints on $\mathcal{R}(Q_{\frac{n}{2}})$ are described; these arise from $\ker P_{\frac{n}{2}}$. In Appendix to [5], Gover-Malchiodi prove the following.

Theorem 5.3. *Let $v \in \mathcal{R}(Q_k)$. Then, for all $u \in \ker P_{k,g} \setminus \{0\}$, there is a metric $\hat{g}$ in the conformal class $[g]$ such that*

$$\int_M u(x)v(x)dV_{\hat{g}}(x) = 0.$$

When $\ker P_{k,g}$ is non-trivial this results yields infinitely many constraints on $\mathcal{R}(Q_k)$. In particular, there is an infinite-dimensional (non-linear) subspace of $C^\infty(M)$ that is disjoint from $\mathcal{R}(Q_k)$.

An immediate consequence of Theorem 5.3 is the following.

Theorem 5.4. *Let $v \in C^\infty(M)$ and assume there is $u \in \ker P_{k,g} \setminus \{0\}$ such that uv is non-negative everywhere and positive at a point. Then v cannot be contained in $\mathcal{R}(Q_k)$.*

Gover-Malchiodi apply the above result to two special cases:

- $u \in \ker P_{k,g} \setminus \{0\}$ and $v = s_u$ where s_u is any function with the same strict sign as u.
- $u \in \ker P_{k,g} \setminus \{0\}$ is non-negative (resp., non-positive) and v is a positive (resp., negative) constant function.

As a result, they obtain the following:

Theorem 5.5. *Assume $\ker P_{k,g}$ is non-trivial.*

1. *For $u \in \ker P_{k,g} \setminus \{0\}$ no function with the same strict sign as u can be contained in $\mathcal{R}(Q_k)$. In particular $u \notin \mathcal{R}(Q_k)$.*
2. *If $\ker P_{k,g} \setminus \{0\}$ contains a non-negative function, then there is no metric in the conformal class $[g]$ with non-zero constant Q_k-curvature.*

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