A NOTE ON COMMUTING FOLIATIONS

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Abstract. The aim of this note is to extend the notion of commutativity of vector fields to the category of singular foliations, using Nambu structures, i.e. integrable multi-vector fields. We will show some basic results about commuting Nambu structures.

1. Introduction

Foliations are often viewed as generalizations of dynamical systems. A lot of natural notions can be extended from the world of dynamical systems to the world of foliations. For example, many authors studied the entropy of foliations (see, e.g., [9] and references therein, and also [11]). However, to my knowledge, the general notions of commutativity and integrability, which are very important for dynamical systems, have been given very little attention so far in the world of foliations. The only explicit mentions of the words "(non)commuting foliations" that I found in the literature are a paper of Movshev [7] and some papers of Katok and his collaborators (see, e.g., [3]). The aim of this note is to attract attention to commuting foliations, and to give some preliminary results about them, which are relatively simple but nevertheless interesting in our view.

So what are commuting foliations? As V.I. Arnold said, a good definition is five good examples. Instead of giving a formal definition, let us list here some examples which could be considered as (singular) commuting foliations. These examples are different but related to each other, and they all generalize the notion of commuting vector fields:

- Commuting actions of Lie groups and Lie algebras. In particular, foliations generated by commuting vector fields in an integrable dynamical system.

- A pair of transverse foliations $\mathcal{F}_1, \mathcal{F}_2$ such that the holonomy of $\mathcal{F}_1$ acts trivially on $\mathcal{F}_2$ and vice versa (Movshev [7]).

- Almost direct products, i.e. constructions of the type $(F_1 \times F_2)/G$, where F_1 and F_2 are two manifolds, and G is a discrete group which acts on the product $F_1 \times F_2$ freely and diagonally, i.e. G acts on both F_1 and F_2, and the action of G on $F_1 \times F_2$ is composed of these two actions of G on F_1 and F_2. The two foliations on $(F_1 \times F_2)/G$ are the "horizontal" foliation with

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leaves $(F_1 \times \{pt\})/G$ and the "vertical" foliation with leaves $(\{pt\} \times F_2)/G$ respectively.

- Foliations generated by compatible Poisson structures, symplectic structures, Dirac structures, etc.
- Parallelizable webs (see, e.g., [1]).
- Commuting Nambu structures (see below).

There are many different ways to generate (singular) foliations. One of the most general and convenient ways is via the so called Nambu structures, i.e. multi-vector fields which are integrable à la Frobenius. In this note we will also be mainly concerned with Nambu structures, We refer the reader to Chapter 6 of [2] and also to [12] for basic notions about Nambu structures which will be used in this paper. We will use the following definition of Nambu structures [12] which is a bit different from the original definition of Takhtajan [8]: A **Nambu structure** (or tensor) of order q on a manifold M is a q-vector field Λ on M which satisfies the following condition: for any point $p \in M$ such that $\Pi(p) \neq 0$, there is a local coordinate system $(x_1, \ldots, x_n)$ in a neighborhood of p such that

$$\Lambda = \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_q} \tag{1.1}$$

in that neighborhood. Geometrically, a Nambu structure of order q is nothing but a singular q-dimensional foliation together with a contravariant volume form (i.e. q-vector field) on its leaves: in the local normal form $\Lambda = \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_q}$ the foliation is generated by the commuting vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_q}$. A Nambu structure is called *regular* if it does not vanish anywhere. In that case its foliation is also a regular foliation.

An advantage of multi-vector fields in the study of foliations is that we can use the Schouten bracket in the calculus of multi-vector fields. In particular, it would be natural to say that when two Nambu stuctures commute then their Schouten bracket vanish. This definition, which was already mentioned in [2], works well in the case when the sum of the orders of the two Nambu structures does not exceed the dimension of the manifold (see Definition 2.1 and Proposition 2.2 below). However, if Λ_1 and Λ_2 are two Nambu structures of orders q_1 and q_2 respectively on a n-dimensional manifold, such that $q_1 + q_2 > n$, then the condition $[\Lambda_1, \Lambda_2] = 0$ does not mean much. (It means nothing at all when $q_1 + q_2 \geq n + 2$, because the order of $[\Lambda_1, \Lambda_2]$ is $q_1 + q_2 - 1 > n$ in that case so $[\Lambda_1, \Lambda_2]$ is automatically zero). In this note, we will give a meaningful definition of commutativity of two Nambu structures Λ_1 and Λ_2 in the case when $q_1 + q_2 > n$, by replacing the equation $[\Lambda_1, \Lambda_2] = 0$ by an appropriate stronger condition (see Definition 2.6 and Proposition 2.7).

2. Commuting Nambu tensors

2.1. The case $q_1 + q_2 \leq n$.

Definition 2.1. *Let Λ_1 and Λ_2 be Nambu tensors of orders q_1 and q_2 respectively on a manifold M of dimension n, such that $q_1+q_2 \leq n$ and $\Lambda_1 \wedge \Lambda_2 \neq 0$ almost everywhere. Then we will say that Λ_1 and Λ_2* **commute** *with each other if their Schouten bracket vanishes:*

$$[\Lambda_1, \Lambda_2] = 0. \tag{2.1}$$

Proposition 2.2 ([2])**.** *Let Λ_1 and Λ_2 be two commuting Nambu tensors of orders q_1 and q_2 respectively on a manifold M of dimension n, with $q_1+q_2 \leq n$. Suppose that $\Lambda_1(O) \wedge \Lambda_2(O) \neq 0$ at a point $O \in M$. Then Λ_1 and Λ_2 can be put into the following simultaneous normal form with respect to a local coordinate system $(x_1, \dots, x_n)$ in a neighborhod of O:*

$$\begin{array}{l} \Lambda_1 = \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_{q_1}}, \\ \Lambda_2 = \frac{\partial}{\partial x_{q_1+1}} \wedge \dots \wedge \frac{\partial}{\partial x_{q_1+q_2}}. \end{array} \tag{2.2}$$

Proof. The above proposition was mentioned without proof in [2], so for the completeness of exposition let us present here a proof.

Let us first consider the case $n = q_1 + q_2$. In this case, we have two local transverse foliations near O, generated by Λ_1 and Λ_2 respectively. We can find a coordinate system $(y_1, \dots, y_{q_1}, z_1, \dots, z_{q_2})$ near O such that the foliation generated by Λ_1 is $\{z_1 = const., \dots, z_{q_2} = const.\}$, and the foliation generated by Λ_1 is $\{y_1 = const., \dots, y_{q_1} = const.\}$. In other words, $\Lambda_1 = f_1 \frac{\partial}{\partial y_1} \wedge \dots \wedge \frac{\partial}{\partial y_{q_1}}$ and $\Lambda_2 = f_2 \frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_{q_2}}$, where f_1 and f_2 are some functions. The equality $[\Lambda_1, \Lambda_2] = 0$ implies that f_1 (resp. f_2) does not depend on the variables $z_1, \dots, z_{q_2}$ (resp. $y_1, \dots, y_{q_1}$). One can then find functions $x_1, \dots, x_{q_1}$ (resp. $x_{q_1+1}, \dots, x_n$) which depend only on the coordinates $y_1, \dots, y_{q_1}$ (resp. $z_1, \dots, z_{q_2}$), such that Λ_1 and Λ_2 have the canonical form (2.2) in the new coordinate system $(x_1, \dots, x_n)$.

The case when $n > q_1 + q_2$ can be reduced to a parametrized version of the case with the dimension equal to $q_1 + q_2$. The main point is to prove that $\Lambda_1 \wedge \Lambda_2$ is a Nambu structure. Locally we can write

$$\begin{array}{l} \Lambda_1 = X_1 \wedge \dots \wedge X_{q_1}, \\ \Lambda_2 = Y_1 \wedge \dots \wedge Y_{q_2}, \end{array} \tag{2.3}$$

where the vector fields $X_1, \dots, X_{q_1}, Y_1, \dots, Y_{q_2}$ are linearly independent can can be competed by vector fields $Z_1, \dots, Z_{q_3}$, where $q_3 = n - q_1 - q_2$, to become a basis field for the tangent bundle of M near O. The fact that Λ_1 is a Nambu structure means that $X_1, \dots, X_{q_1}$ satisfy the Frobenius integrability condition, i.e. $[X_i, X_j](x)$ lies in the linear span of $X_1(x), \dots, X_{q_1}(x)$ for any x near O. The same holds for the vector fields $Y_1, \dots, Y_{q_2}$. The equality

$$\begin{aligned} 0 = [\Lambda_1, \Lambda_2] = \sum_{i,j} ((-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \dots \wedge X_{i-1} \wedge X_{i+1} \dots \wedge X_{q_1} \\ \wedge Y_1 \wedge \dots \wedge Y_{j-1} \wedge Y_{j+1} \wedge \dots \wedge Y_{q_2}) \end{aligned} \tag{2.4}$$

implies that $[X_i, Y_j]$ must also lie in the span of $X_1, \dots, X_{q_1}, Y_1, \dots, Y_{q_2}$, because it cannot contain a component of the type Z_k in its decomposition in the basis (X_i, Y_j, Z_k). It means that $X_1, \dots, X_{q_1}, Y_1, \dots, Y_{q_2}$ span an integrable distribution, which is tangent to a $(q_1 + q_2)$-dimensional foliation, and $\Lambda_1 \wedge \Lambda_2$ is a contravariant volume form on the leaves of this foliation,. Thus $\Lambda_1 \wedge \Lambda_2$ is a Nambu structure. □

Proposition 2.3. *Let $\Lambda_1, \dots, \Lambda_s$ be pairwise commuting Nambu tensors of orders $q_1, \dots, q_s$ respectively on a manifold M of dimension n, with $q_1 + \dots + q_s \leq n$. Suppose that $\Lambda_1(O) \wedge \dots \wedge \Lambda_s(O) \neq 0$ at a point $O \in M$. Then $\Lambda_1, \dots, \Lambda_s$ can be put into the following simultaneous normal form with respect to a local coordinate system $(x_1, \dots, x_n)$ in a neighborhod of O:*

$$\begin{aligned} \Lambda_1 &= \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_{q_1}}, \\ \Lambda_2 &= \frac{\partial}{\partial x_{q_1+1}} \wedge \dots \wedge \frac{\partial}{\partial x_{q_1+q_2}}, \\ &\dots \\ \Lambda_s &= \frac{\partial}{\partial x_{q_1+\dots+q_{s-1}+1}} \wedge \dots \wedge \frac{\partial}{\partial x_{q_1+\dots+q_s}}. \end{aligned} \tag{2.5}$$

Proof. By induction. Apply Proposition 2.2 to Λ_1 and $\Pi_2 = \Lambda_2 \wedge \dots \wedge \Lambda_s$, we get a coordinate system in which Λ_1 and Π_2 are normalized. The problem is then reduced to (a parametrized version of) the problem of normalization of the $(s-1)$-tuple of Nambu structures $\Lambda_2, \dots, \Lambda_s$. □

Example 2.4. Consider a non-identity linear automorphism ϕ from a torus $\mathbb{T}^n$ to itself, and denote by M its suspension: M is a torus fibration over the circle $\mathbb{S}^1$, with a "horizontal" vector field X which is a lifting of the standard constant vector field on $\mathbb{S}^1$ such that the Poincaré map of X on a fiber of M is isomorphic to ϕ. Denote by Λ the standard contravariant volume form on the torus fibers of M. Then Λ is a Nambu structure on M which is preserved by X. We can also view X as a Nambu structure of order 1 on M. Then Λ and X are two transverse commuting Nambu structures on M. Notice that the holonomy of the foliation generated by X near the closed orbits of X are not trivial on the tori (i.e. the leaves of Λ), i.e. we have here two transverse foliations which are generated by two commuting Nambu structures, but which do not satisfy Movshev's holonomy condition [7]. Nevertheless, this example is of almost direct product type, and it is reasonable to consider almost direct products of manifolds as examples of commuting foliations.

2.2. **Reduction of Nambu structures.** Before treating the case $q_1 + q_2 > n$, let us make a digression and discuss briefly about the reduction of Nambu structure, because we will reduce the case with $q_1 + q_2 > n$, to the case with $q_1 + q_2 = n$.

A vector field X is called a **Nambu vector field** with respect to a Nambu structure Λ if X preserves Λ, i.e. $\mathcal{L}_X \Lambda = [X, \Lambda] = 0$, where $\mathcal{L}$ denotes the Lie derivation, and the bracket is the Schouten bracket. X is called a **Hamiltonian vector field** with respect to Λ if there are $q - 1$

functions $f_1, \ldots, f_{q-1}$ such that

$$X = (df_1 \wedge \ldots \wedge df_{q-1}) \lrcorner \Lambda. \quad (2.6)$$

Any Hamiltonian vector field is a Nambu vector field which is tangent to the foliation generated by Λ. Conversely, a Nambu vector field which is tangent to the foliation of the Nambu tensor Λ is a locally Hamiltonian vector field near each non-singular point of Λ. More generally, if $f_1, \ldots, f_{q-k}$ are $q-k$ functions, where $1 \leq k < q$, and Λ is a Nambu tensor of order q, then the k-vector field

$$\Pi_{f_1,\ldots,f_{q-k}} = (df_1 \wedge \ldots \wedge df_k) \lrcorner \Lambda \quad (2.7)$$

is a Nambu structure of order k which will be called a **Hamiltonian Nambu structure** with respect to Λ : the foliation of $\Pi_{f_1,\ldots,f_{q-k}}$ is tangent to the foliation of Λ, and $\Pi_{f_1,\ldots,f_{q-k}}$ "preserves" Λ in the sense that

$$[\Pi_{f_1,\ldots,f_{q-k}}, \Lambda] = 0. \quad (2.8)$$

Assume Λ is a given Nambu structure of order q, and Π is a regular Nambu structure of order k $(1 \geq k < q)$ on a manifold M of dimension n, with the following properties:
i) The leaf space $M/\mathcal{F}^{\Pi}$ of the regular foliation $\mathcal{F}^{\Pi}$ of Π in M is a Hausdorff $(n-k)$-dimensional manifold.
ii) The foliation of Π is tangent to the foliation of Λ, and $[\Pi, \Lambda] = 0$. Then there is a unique Nambu structure Θ of order $q-k$ on the quotient manifold $M/\mathcal{F}^{\Pi}$ (the leaf space of $\mathcal{F}^{\Pi}$), which is the **reduction of** Λ **by** Π in the following sense: locally near each point $z \in M$ there is a coordinate system $(x_1, \ldots, x_n)$ in which

$$\Pi = \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_k}, \quad (2.9)$$

the variables $(x_{k+1}, \ldots, x_n)$ are local coordinate system on the quotient manifold $M/\mathcal{F}^{\Pi}$,

$$\Lambda = \Pi \wedge \Theta \quad (2.10)$$

and the expression of Θ involves only the variables $(x_{k+1}, \ldots, x_n)$. The above reduction process is an imitation of the reduction of Poisson structures. It can be done locally, i.e. we can talk about the **local reduction** of Λ by Π near any given regular point of Π (without the need of the assumption that Π is globally regular).

2.3. **The case $q_1 + q_2 > n$.** When $q_1 + q_2 \geq n + 2$ then we always have $[\Lambda_1, \Lambda_2] = 0$ for any q_1-vector field Λ_1 and q_2-vector field Λ_2, and we have to change the definition of commutativity in this case in order for it to be meaningful. When $q_1 + q_2 = n + 1$ then the condition is non-trivial, but not sufficient to imply that Λ_1 and Λ_2 can be put into a constant form simultaneously near a non-singular point. The best that we can have when $q_1 + q_2 = n + 1$ under the condition $[\Lambda_1, \Lambda_2] = 0$ is the following:

Proposition 2.5. *Let Λ_1 and Λ_2 be two Nambu tensors of orders q_1 and q_2 respectively on a manifold M of dimension n, such that $q_1 + q_2 = n + 1$ and $[\Lambda_1, \Lambda_2] = 0$. $\Lambda_1(O) \wedge \Lambda_2(O) \neq 0$ at a point $O \in M$. Then in a neighborhood of O there exists a vector field X which is locally Hamiltonian with respect to both Λ_1 and Λ_2. Conversely, if Λ_1 and Λ_2 are two Nambu tensors of orders q_1 and q_2 respectively on a manifold M of dimension $n = q_1 + q_2 - 1$, which admit a common Hamiltonian vector field X, then $[\Lambda_1, \Lambda_2] = 0$.*

Proof. Denote by $\mathcal{F}_1$ and $\mathcal{F}_2$ the foliations of Λ_1 and Λ_2 respectively. Then the intersection of $\mathcal{F}_1$ with $\mathcal{F}_2$ near O is a regular 1-dimensional foliation. Let $\tilde{X}$ be a local vector field which is tangent to this intersection foliation, i.e. $\tilde{X}$ is tangent to both Λ_1 and Λ_2 : $\tilde{X} \wedge \Lambda_1 = \tilde{X} \wedge \Lambda_2 = 0$. Since $\tilde{X}$ is tangent to Λ_1, we have $[\tilde{X}, \Lambda_1] = a\Lambda_1$ for some function a. By putting $X = f\tilde{X}$, where f is a local solution of the ordinary differential equation $\tilde{X}(f) = af$, we get $[X, \Lambda] = 0$, i.e. X is a locally Hamiltonian vector field of Λ_1. Locally near O we can write $\Lambda_1 = X \wedge \Pi_1$, and also $\Lambda_2 = X \wedge \Pi_2$, where Π_1 and Π_2 are Nambu structures and Π_1 is invariant with respect to X. The equality $0 = [\Lambda_1, \Lambda_2] = \pm[X, \Lambda_2] \wedge \Pi_1$ implies that $[X, \Lambda_2] = 0$, i.e. X is also a local Hamiltonian vector field with respect to Λ_2. □

Remark that, in the above proposition, even though we can choose X, Π_1 and Π_2 such that $\Lambda_1 = X \wedge \Pi_1$, $\Lambda_2 = X \wedge \Pi_2$, and Π_1 and Π_2 are two Nambu tensors invariant with respect to X, we cannot arrange so that $[\Pi_1, \Pi_2] = 0$ in general. A way to define commutativity of two Nambu structures whose total rank is greater than the dimension of the manifold is as follows:

Definition 2.6. *Let Λ_1 and Λ_2 be two Nambu structures of orders q_1 and q_2 respectively on a manifold M of dimension n, such that $q_1 + q_2 - n = k > 0$. Then we say that Λ_1 commutes with Λ_2 if in a neighborhood of any point $O \in M$ such that the foliations generated by Λ_1 and Λ_2 are transverse to each other near O, there is a local coordinate system $(x_1, \ldots, x_n)$ such that*

$$\begin{aligned} \Lambda_1 &= \tfrac{\partial}{\partial x_1} \wedge \ldots \wedge \tfrac{\partial}{\partial x_{q_1}}, \\ \Lambda_1 &= \tfrac{\partial}{\partial x_1} \wedge \ldots \wedge \tfrac{\partial}{\partial x_k} \wedge \tfrac{\partial}{\partial x_{q_1+1}} \wedge \ldots \wedge \tfrac{\partial}{\partial x_n}. \end{aligned} \tag{2.11}$$

Another equivalent definition of commutativity of Nambu structures in the case $q_1 + q_2 > n$ is given by the following proposition and by induction on $q_1 + q_2 - n$:

Proposition 2.7. *Λ_1 commutes with Λ_2 if and only if near any point O such that Λ_1 and Λ_2 are transverse at O, there is a vector field X such that $X(O) \neq 0$, X is Hamiltonian with respect to both Λ_1 and Λ_2, and the local reductions of Λ_1 and Λ_2 with respect to X commute with each other.*

The proof is straightforward and by induction on $q_1 + q_2 - n$.

In Definition 2.1 for the case with $q_1+q_2 \leq n$, we assumed that $\Lambda_1 \wedge \Lambda_2 \neq 0$. The case when $\Lambda_1 \wedge \Lambda_2$ is identically zero is a degenerate case, and in that case the definition of commutativity has to be changed as follows to make sense:

Definition 2.8. *Let Λ_1 and Λ_2 be two Nambu structures of order q_1 and q_2 respectively on a manifold M, and $k > 0$ is a constant, such that $\dim(T_z\mathcal{F}_1 \cap T_z\mathcal{F}_2) = k$ for almost every $z \in M$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the foliations of Λ_1 and Λ_2 respectively. Then we will say that Λ_1 commutes with Λ_2 if near any point $z \in M$ such that $\dim(T_z\mathcal{F}_1 \cap T_z\mathcal{F}_2) = k$ there is a local coordinate system $(x_1, \ldots, x_n)$ such that*

$$\begin{aligned} \Lambda_1 &= \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_{q_1}}, \\ \Lambda_1 &= \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_{q_1+1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{q_1+q_2-k}}. \end{aligned} \tag{2.12}$$

Example 2.9. Given an action of a Lie algebra $\mathfrak{g}$ on a manifold M, i.e. a Lie morphism $\mathfrak{g} \to \mathcal{X}(M)$ from $\mathfrak{g}$ to the Lie algebras of vector fields on M, such that its general orbits have dimension equal to q, we can construct a family of Nambu structure as follows: for each element $\xi \in \wedge^q \mathfrak{g}$, denote by Λ_ξ the image of ξ via the natural extension $\wedge^q \mathfrak{g} \to \wedge^q \mathcal{X}(M)$ of the map $\mathfrak{g} \to \mathcal{X}(M)$. Then Λ_ξ is a Nambu structure of order q on M for any ξ, and is ξ is chosen well enough then almost all the regular orbits of the action of $\mathfrak{g}$ on M are also the regular leaves of the foliation of Λ_ξ. Now if there are two commuting actions of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on M, and two elements $\xi_1 \in \wedge^{q_1} \mathfrak{g}_1$ and $\xi_2 \in \wedge^{q_2} \mathfrak{g}_2$, then the two associated Nambu structures Λ_{ξ_1} and Λ_{ξ_2} will commute with each other. We will leave the verification of this fact as a simple exercise to the reader.

3. Almost direct products

The almost direct product example mentioned in the introduction of this note is in fact a general consruction of foliations which are transverse to each other and have complementary dimensions. More precisely, we have the following simple proposition:

Proposition 3.1. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two foliations on a connected compact manifold M, such that $T_xM = T_x\mathcal{F}_1 \oplus T_x\mathcal{F}_2$ for any $x \in M$. Then the triple $(M, \mathcal{F}_1, \mathcal{F}_2)$ is isomorphic to an almost direct product model*

$$(F_1 \times F_2)/G, \tag{3.1}$$

where F_1 and F_2 are two connected manifolds (which are not necessarily compact), G is a discrete group which acts on the product $F_1 \times F_2$ freely and diagonally.

The about result seems to be folkloric, but unfortunately I don't have an exact reference for it. So for the sake of completeness, let us give here a proof of it.

Proof. First notice that, due to the fact that $\mathcal{F}_1$ and $\mathcal{F}_2$ are transverse and have complementary dimensions, $\mathcal{F}_1$ creates a locally flat parallel transport among the leaves of $\mathcal{F}_2$ and vice versa: given any two paths γ_1 tangent to $\mathcal{F}_1$ and γ_2 tangent to $\mathcal{F}_2$ such that $\gamma_1(0) = \gamma_2(0)$, there is a unique natural way to transport γ_1 along γ_2 such that γ_1 remains always tangent to $\mathcal{F}_1$. This locally flat parallel transport exists locally even if the manifold M is not compact. The theorem still holds if we replace the compactness condition by the following weaker *completeness condition*: the parallel transport exists not only locally, but also globally, i.e. given any two paths γ_1 tangent to $\mathcal{F}_1$ and γ_2 tangent to $\mathcal{F}_2$ such that $\gamma_1(0) = \gamma_2(0)$, there is a unique natural way to transport γ_1 (resp. γ_2) along γ_2 (resp. γ_1) such that γ_1 (resp. γ_2) remains always tangent to $\mathcal{F}_1$ (resp. $\mathcal{F}_2$).

Take a point $z \in M$. Denote by $\mathcal{F}_1(z)$ (resp. $\mathcal{F}_2(z)$) the leaf of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) passing through z. Let γ be any loop in M starting at z. We can approximate γ by a zig-zag piecewise horizontal-vertical loop (also starting at z_0) which is homotopic to γ. Using the parallel transport to commute vertical pieces with horizontal pieces, one sees easily that γ is homotopic to the concatenation $\gamma_1 + \gamma_2$ where $\gamma_1 : [0, 1/2] \to \mathcal{F}_1(z)$ and $\gamma_2 : [1/2, 1] \to \mathcal{F}_2(z)$ are two paths lying in $\mathcal{F}_1(z)$ and $\mathcal{F}_2(z)$ respectively such that $\gamma_1(0) = \gamma_2(1)$ and the end point of γ_1 is the starting point of γ_2. We will denote this point by $[\gamma].z = \gamma_1(1/2) = \gamma_2(1/2)$. Observe that $[\gamma].z \in \mathcal{F}_1(z) \cap \mathcal{F}_2(z)$, and it depends only on the homotopy class $[\gamma]$ of γ in the fundamental group $\pi_1(M, z)$.

Fix a point $z_0 \in M$. Denote by $\Gamma \in \pi_1(M, z_0)$ the set of elements α in the fundamental group $\pi_1(M, z_0)$ such that for any $z \in M$, any path μ from z_0 to z we have $\pi_\mu(\alpha).z = z$, where $\pi_\mu(\alpha)$ denotes the image of α in $\pi_1(M, z)$ via the natural isomorphism ϕ_μ from $\pi_1(M, z_0)$ to $\pi_1(M, z)$ generated by the path μ. One verifies easily that Γ is a normal subgroup of $\pi_1(M, z_0)$, i.e. the quotient $G = \pi_1(M, z_0)/\Gamma$ is a group. The group Γ also satisfies the following remarkable property: if $[\gamma_1 + \gamma_2] \in \Gamma$, where γ_1 is a loop tangent to $\mathcal{F}_1$ and γ_2 is a loop tangent to $\mathcal{F}_2$, then $[\gamma_1], [\gamma_2] \in \Gamma$.

Denote by $\tilde{M}$ the normal covering of M associated to Γ, i.e. $\pi_1(\tilde{M}) \cong \Gamma$, $G = \pi_1(M, z_0)/\Gamma$ acts freely on $\tilde{M}$, and $\tilde{M}/G = M$. The foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ can be lifted naturally to $\tilde{M}$. We will denote the lifted foliations on $\tilde{M}$ by $\tilde{F}_1$ and $\tilde{F}_2$ respectively. Since $(M, \mathcal{F}_1, \mathcal{F}_2)$ satisfies the completeness condition, $(\tilde{M}, \tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ also satifies this condition. We want to show that $(\tilde{M}, \tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ has direct product type. Due to the completeness condition, it is enough to verify that if F_1 is a leaf of $\tilde{\mathcal{F}}_1$ and F_2 is a leaf of $\tilde{\mathcal{F}}_2$, then F_1 intesects with F_2 at exactly one point.

Assume to the contrary that $y_0, y_1 \in F_1 \cap F_2$, $y_0 \neq y_1$. Then there is a loop $\gamma = \gamma_1 + \gamma_2$ such that γ_1 (resp. γ_2) starts at y_0 (resp. y_1), ends at y_1 (resp. y_0) and is tangent to $\tilde{F}_1$ (resp. $\tilde{F}_2$). Denote the projection map from $\tilde{M}$ to M by the hat, e.g. $\hat{y}_0 \in M$ is the image of y_0, $\hat{\gamma}_1$ is the image of γ_1 by the projection $\tilde{M} \to M$. By construction, $[\hat{\gamma}] = [\hat{\gamma}_1 + \hat{\gamma}_2] \in \Gamma$, which implies that $\hat{y}_1 = \hat{y}_0$, which in turns implies that $[\hat{\gamma}_1] \in \Gamma$. But since γ_1 is a lifting of γ, the fact that $\hat{\gamma}_1 \in \Gamma$ implies that $y_1 = y_0$ by construction of $\tilde{M}$, which is a contradiction. Thus any leaf of $\tilde{\mathcal{F}}_1$ intersects with any leaf of $\tilde{\mathcal{F}}_2$ at exactly one point, and $(\tilde{M}, \tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ has direct product type. The rest of the proof is straightforward. □

In Proposition 3.1, a-priori there are no Nambu structures. But of course, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are generated by two commuting Nambu structures, then we will have volume forms on the manifolds F_1, F_2 in the almost direct product model, and the action of G on F_1 and F_2 will be volume-preserving. Proposition 3.1 can be naturally extended to the case of k foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$, where $k > 2$. But in the case $k > 2$, the transversality condition $T_xM = \bigoplus_{i=1}^k T_x\mathcal{F}_i$ is far from being sufficient for the decomposition of the picture in to a semi-direct product $(F_1 \times \ldots \times F_k)/G$, and we really need some more meaningful commutativity condition. Fortunately, the commutativity of k regular Nambu structures $\Lambda_1, \ldots, \Lambda_k$ which generate $\mathcal{F}_1, \ldots, \mathcal{F}_k$ will do the job.

Proposition 3.2. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_2$ be k foliations on a connected compact manifold M, generated by k regular pairwise commuting commuting Nambu structures $\Lambda_1, \ldots, \Lambda_k$ respectively, such that $T_xM = \bigoplus_{i=1}^k T_x\mathcal{F}_i$ for any $x \in M$. Then the multi-foliation $(M, \mathcal{F}_1, \ldots, \mathcal{F}_k)$ is isomorphic to an almost direct product model*

$$(F_1 \times \ldots \times F_k)/G, \tag{3.2}$$

where $F_1, \ldots, F_2$ are connected manifolds with (contravariant) volume forms, G is a discrete group which acts on the product $F_1 \times F_2$ freely and diagonally, and the action of G on each F_i is volume-preserving.

The proof of Proposition 3.2 is absolutely similar to Proposition 3.1, and it can also be deduced from the proof of Proposition 3.1 by induction on k. (Notice that, for example, $\Lambda_1 \wedge \Lambda_2$ is again a regular Numbu structure which commutes with the other Λ_i).

Proposition 3.2 is reminiscent of other almost direct product theorems, in particular the clasical theorem about the almost direct product decomposition of reductive algebraic groups (see e.g. [6]), and also the topological decomposition theorem for nondegenerate singularities of integrable Hamiltonian systems [10]. A particular case of the above theorem is when the foliations $\mathcal{F}_i$ are one-dimensional, i.e. the Nambu tensors Λ_1 are commuting vector fields. In this case the manifold M is a k-dimensional torus, G is (in the generic case) isomorphic to $\mathbb{Z}^k$, and one recovers the classical Liouville's theorem about quasi-periodicity of motion of integrable systems [4]. Inspired by this, one can extend the notion of integrability of dynamical systems to the case of foliations as follows:

Definition 3.3. *A foliation $\mathcal{F}$ of dimension q on a manifold M will be called* **integrable** *if it can be represented by a Nambu structure $\Lambda = \Lambda_1$ or order q, and there are Nambu structures $\Lambda_2, \ldots, \Lambda_s$ and functions $F_1, \ldots, F_r$ such that: $q_1 + \ldots + q_s + r = n$, F_i are first integrals of Λ_j, the Λ_i commute pairwise, and $\Lambda_1 \wedge \ldots \wedge \Lambda_s \neq 0$ almost everywhere.*

Proposition 3.2, or rather a parametrized version of it which involves also first integrals, can then be viewed as a generalization of the classical Lioville's theorem to the case of integrable foliations.

4. Some final remarks and questions

In this note, I considered only the regular case. But what about the singular case, when, for example, two foliations are regular but together they have have singularities, or at least one of the two foliations is singular ? One should be able to develop a normal form theory for such singular commuting foliations, at least in the case when the singularities are nondegenerate or generic in some sense.

What about differential forms which are invariants ? Apparently, those forms must be analogous to basic differential forms of fibrations. (In particular, the contraction of the form with any vector field tangent to the foliation must vanish, i.e. they are transverse forms, or more generally, one can talk about invariant transverse structures). Will they play a role in a generalized theory of integrability of foliations . And what about a Galoisian theory of obstructions to the integrability of singular foliations ?

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