f-EIKONAL HELIX SUBMANIFOLDS AND f-EIKONAL HELIX CURVES

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Abstract. Let $M \subset \mathbb{R}^n$ be a Riemannian helix submanifold with respect to the unit direction $d \in \mathbb{R}^n$ and $f : M \to \mathbb{R}$ be a eikonal function. We say that M is a f-eikonal helix submanifold if for each $q \in M$ the angle between ∇f and d is constant.Let $M \subset \mathbb{R}^n$ be a Riemannian submanifold and $\alpha : I \to M$ be a curve with unit tangent T. Let $f : M \to \mathbb{R}$ be a eikonal function. We say that α is a f-eikonal helix curve if the angle between ∇f and T is constant along the curve α. ∇f will be called as the axis of the f-eikonal helix curve.The aim of this article is to give that the relations between f-eikonal helix submanifolds and f-eikonal helix curves, and to investigate f-eikonal helix curves on Riemannian manifolds.

1. Introduction

In differential geometry of manifolds, an helix submanifold of $\mathbb{R}^n$ with respect to a fixed direction d in $\mathbb{R}^n$ is defined by the property that tangent space makes a constant angle with the fixed direction d (helix direction) in [3]. Di Scala and Ruiz-Hernández have introduced the concept of these manifolds in [3].

Recently, M. Ghomi worked out the shadow problem given by H.Wente. And, He mentioned the shadow boundary in [6]. Ruiz-Hernández investigated that shadow boundaries are related to helix submanifolds in [10].

Helix hypersurfaces have been worked in nonflat ambient spaces in [4,5]. Cermelli and Di Scala have also studied helix hypersurfaces in liquid cristals in [2].

The plan of this article is as follows. Section 2, we give some important definitions which will be used in other sections.In section 3, we define f-eikonal helix submanifolds and define f-eikonal helix curves. And also, we give an important property between f-eikonal helix submanifolds and f-eikonal helix curves, see Theorem 3.1. In Theorem 3.2, we show that when a curve on a manifold is f-eikonal helix curve. Besides,we give the important relation between geodesic curves and f-eikonal helix curves, see Theorem 3.3. Section 4, in 3-dimensional Riemannian manifold, we find out the axis of a f-eikonal helix curve and we give the relation between the curvatures of the curve in Theorem 4.1 and Theorem 4.2. Then, we give more important corollary relating to helix submanifolds. In section 5, we briefly specify the relatıon between ∇f and helix submanifolds, see Lemma 5.1 and Theorem 5.1.

2. Basic Definitions

Definition 2.1. *Given a submanifold $M \subset \mathbb{R}^n$ and an unitary vector d in $\mathbb{R}^n$, we say that M is a helix with respect to d if for each $q \in M$ the angle between d and T_qM is constant.*

Let us recall that a unitary vector d can be decomposed in its tangent and orthogonal components along the submanifold M, i.e. $d = \cos(\theta)T + \sin(\theta)\xi$ with $\|T\| = \|\xi\| = 1$, where $T \in TM$ and $\xi \in \vartheta(M)$.The angle between d and T_qM is constant if and only if the tangential component of d has constant length $\|\cos(\theta)T\| = \cos(\theta)$. We can assume that $0 < \theta < \frac{\pi}{2}$ and we can say that M is a helix of angle θ.

We will call T and ξ the tangent and normal directions of the helix submanifold M. We can call d the helix direction of M and we will assume d always to be unitary [3].

Definition 2.2. *Let $M \subset \mathbb{R}^n$ be a helix submanifold of angle $\theta \neq \frac{\pi}{2}$ w.r. to the direction $d \in \mathbb{R}^n$. We will call the integral curves of the tangent direction T of the helix M, the helix lines of M w.r.to d [3].*

Remark 2.1 *We say that ξ is parallel normal in the direction $X \in TM$ if $\nabla^{\perp}_X \xi = 0$. Here, $\nabla^{\perp}$ denotes the normal connection of M induced by the standard covariant derivative of the Euclidean ambient. And, we denote by D the standard covariant derivative in $\mathbb{R}^n$ and denote by $\overline{D}$ the induced covariant derivative in M. [3].*

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Definition 2.3. *Let M be a submanifold of the Riemannian manifold $\mathbb{R}^n$ and let D be the Riemannian connexion on $\mathbb{R}^n$. For C^∞ fields X and Y with domain A on M (and tangent to M), define $\overline{D}_X Y$ and $V(X,Y)$ on A by decomposing $D_X Y$ into unique tangential and normal components, respectively; thus,*

$$D_X Y = \overline{D}_X Y + V(X,Y).$$

Then, $\overline{D}$ is the Riemannian connexion on M and V is a symmetric vector-valued 2-covariant C^∞ tensor called the second fundamental tensor. The above composition equation is called the Gauss equation [7].

Remark 2.2 *Let us observe that for any helix euclidean submanifold M, the following system holds for every $X \in TM$, where the helix direction $d = \cos(\theta)T + \sin(\theta)\xi$.*

$$\cos(\theta)\nabla_X T - \sin(\theta)A^{\xi}(X) = 0 \tag{2.1}$$

$$\cos(\theta)V(X,T) + \sin(\theta)\nabla_X^{\perp}\xi = 0 \tag{2.2}$$

[3].

Definition 2.4. *Let (M,g) be a Riemannian manifold, where g is the metric. Let $f : M \to \mathbb{R}$ be a function and let ∇f be its gradient, i.e., $df(X) = g(\nabla f\ , X)$. We say that f is eikonal if it satisfies:*

$$\|\nabla f\| = constant.$$

[3].

Definition 2.5. *Let $\alpha = \alpha(t) : I \subset \mathbb{R} \to M$ be an immersed curve in 3-dimensional Riemannian manifold M . The unit tangent vector field of α will be denoted by T. Also, $\kappa > 0$ and τ will denote the curvature and torsion of α, respectively.Therefore if $\{T, N, B\}$ is the Frenet frame of α and $\overline{D}$ is the Levi-Civita connection of M, then one can write the Frenet equations of α as*

$$\overline{D}_T T = \kappa N$$

$$\overline{D}_T N = -\kappa T + \tau B$$

$$\overline{D}_T B = -\tau N$$

[1].

Throughout all section, the submanifolds $M \subset \mathbb{R}^n$ have the induced metric by $\mathbb{R}^n$.

3. f-EIKONAL HELIX CURVES

Definition 3.1. *Let $M \subset \mathbb{R}^n$ be a Riemannian helix submanifold with respect to the unit direction $d \in \mathbb{R}^n$ and $f : M \to \mathbb{R}$ be a eikonal function. We say that M is a f-eikonal helix submanifold if for each $q \in M$ the angle between ∇f and d is constant.*

For definition 3.1, $\langle \nabla f, d\rangle =$ constant since $\|\nabla f\|$ and d are constant.

Example 3.1. *Let $M \subset \mathbb{R}^n$ be a Riemannian helix submanifold with respect to the unit direction $d \in \mathbb{R}^n$. Let us assume that the tangent component of d equals ∇f for a eikonal function $f : M \to \mathbb{R}$. Because of the definition helix submanifold, we have $\langle \nabla f, d\rangle = constant$. That is, M is a f-eikonal helix submanifold.*

Definition 3.2. *Let $M \subset \mathbb{R}^n$ be a Riemannian submanifold and $\alpha : I \to M$ be a curve with unit tangent T. Let $f : M \to \mathbb{R}$ be a eikonal function. We say that α is a f-eikonal helix curve if the angle between ∇f and T is constant along the curve α. ∇f will be called as the axis of the f-eikonal helix curve.*

Example 3.2. *Let $M \subset \mathbb{R}^n$ be a Riemannian submanifold and $\alpha : I \to M$ be a curve with unit tangent T. Let $f : M \to \mathbb{R}$ be a eikonal function. If ∇f equals T, then $\langle \nabla f\ , \nabla f\ \rangle =$constant. That is, α is a f-eikonal helix curve.*

Theorem 3.1. *Let $M \subset \mathbb{R}^n$ be a f-eikonal helix submanifold .Then, the helix lines of M are f-eikonal helix curves.*

Proof. Recall that $d = \cos(\theta)T + \sin(\theta)\xi$ is the decomposition of d in its tangent and normal components.Let α be the helix line of M with unit speed. That is, $\frac{d\alpha}{ds} = T$. Hence, doing the dot product with ∇f in each part of d along the helix lines of M, we obtain:

$$\langle \nabla f, d\rangle = \cos(\theta)\left\langle \nabla f, \frac{d\alpha}{ds}\right\rangle + \sin(\theta)\langle \nabla f, \xi\rangle$$

Due to the fact that M is a f-eikonal helix submanifold, $\langle \nabla f, d \rangle =$ constant along the helix lines of M. On the other hand, $\langle \nabla f, \xi \rangle = 0$ since $\nabla f \in TM$. So, $\left\langle \nabla f, \frac{d\alpha}{ds} \right\rangle$ is constant along the helix lines of M. It follows that the helix lines of M are f-eikonal helix curves. □

Theorem 3.2. *Let $i : M \to \mathbb{R}^n$ be a submanifold and let $f : M \to \mathbb{R}$ be a eikonal function, where M has the induced metric by $\mathbb{R}^n$. Let us assume that $\alpha : I \subset IR \to M$ is a unit speed (parametrized by arc length function s) curve on M with unit tangent T . Then,α is a f-eikonal helix curve if and only if*

$$\beta(s) = \phi(\alpha(s)) = (i(\alpha(s)), f(\alpha(s))) \subset \mathbb{R}^n \times \mathbb{R}$$

is a general helix with the axis $d = (0, 1)$. Here, $\phi : M \to \mathbb{R}^n \times \mathbb{R}$ is given by $\phi(p) = (i(p), f(p))$ and $i : M \to \mathbb{R}^n$ is given by $i(p) = p$, where $p \in M$.

Proof. We consider the curve $\beta(s) = (i(\alpha(s)), f(\alpha(s))) = (\alpha(s), f(\alpha(s)))$. Then, the tangent of β

$$\acute{\beta}(s) = \left(T, \frac{d(f \circ \alpha)}{ds}\right),$$

where T is the unit tangent of α.On the other hand, we know that $X[f] = \langle \nabla f, X \rangle$ for each $X \in TM$ (see definition 2.4). In particular, for $X = T$,

$$T[f] = \langle \nabla f, T \rangle$$
$$\frac{d\alpha}{ds}[f] = \langle \nabla f, T \rangle$$

and so, we have:

$$\frac{d(f \circ \alpha)}{ds} = \langle \nabla f, T \rangle .$$

Therefore, we obtain

$$\acute{\beta}(s) = (T, \langle \nabla f, T \rangle) . \tag{3.1}$$

Hence, doing the dot product with d in each part of (3.1) , we get:

$$\left\langle \acute{\beta}(s), d \right\rangle = \langle \nabla f, T \rangle . \tag{3.2}$$

From the equality (3.2), we can write

$$\left\| \acute{\beta}(s) \right\| . \cos(\theta) = \langle \nabla f, T \rangle ,$$

where θ is the angle between d and $\acute{\beta}(s)$. It follows that

$$\cos(\theta) = \frac{\langle \nabla f, T \rangle}{\sqrt{1 + \langle \nabla f, T \rangle^2}}. \tag{3.3}$$

If α is a f-eikonal helix curve,i.e. $\langle \nabla f, T \rangle =$ constant, it can be easily seen that $\cos(\theta)$ =constant by using (3.3). That is, β is a general helix with the axis $d = (0, 1)$.Conversely, we assume that β is a general helix, i.e. $\cos(\theta)$ =constant.Hence, by using (3.3), we can write

$$\langle \nabla f, T \rangle^2 = \frac{\cos^2(\theta)}{\sin^2(\theta)} = \text{constant } (\theta \neq 0). \tag{3.4}$$

And so, from (3.4), we deduce that $\langle \nabla f, T \rangle$ =constant. In other words, α is a f-eikonal helix curve. □

Theorem 3.3. *Let $M \subset \mathbb{R}^n$ be a complete connected smooth Riemannian submanifold without boundary and let M be isometric to a Riemannian product $N \times \mathbb{R}$. Let us assume that $f : M \to \mathbb{R}$ be a non-trivial affine function (see main theorem in [9]). Then, all geodesic curves on M are f-eikonal helix curves.*

Proof. Since $f : M \to \mathbb{R}$ is a affine function, for each unit geodesic $\alpha : (-\infty, \infty) \to M$ there are constants a and $b \in \mathbb{R}$ such that

$$f(\alpha(s)) = as + b.$$

for all $s \in (-\infty, \infty)$ (see [8] or see [9]). On the other hand, we know that

$$X[f] = \langle \nabla f, X \rangle$$

for each $X \in TM$. In particular, for $X = T$,

$$T[f] = \langle \nabla f, T \rangle$$
$$\frac{d\alpha}{ds}[f] = \langle \nabla f, T \rangle$$

and so, we have

$$\frac{d(f\circ\alpha)}{ds} = \langle \nabla f, T\rangle . \tag{3.5}$$

Moreover, since $f(\alpha(s)) = as + b$, $\dfrac{d(f\circ\alpha)}{ds}$ =constant. Hence, from (3.5), we obtain

$$\langle \nabla f, T\rangle = \text{constant}$$

along the curve α.On the other hand, from Lemma 2.3 (see [11]), $\|\nabla f\|$ =constant. Consequently, all geodesic curves on M are f-eikonal helix curves. □

4. THE AXIS OF f-EIKONAL HELIX CURVES

Theorem 4.1. *Let $M \subset \mathbb{R}^4$ be a 3-dimensional Riemannian manifold and let M be a complete connected smooth without boundary. Also, let M be isometric to a Riemannian product $N \times \mathbb{R}$. Let us assume that $f : M \to \mathbb{R}$ be a non-trivial affine function (see main theorem in [9]) and be $\alpha : I \to M$ a f-eikonal helix curve. Then, the following properties are hold:*

(1) The axis of α:

$$\nabla f = \|\nabla f\| \left(\cos(\theta)T + \sin(\theta)B\right).$$

(2) $\dfrac{\tau}{\kappa}$ =constant.

Proof. (1) Since α is f-eikonal helix curve, we can write

$$\langle \nabla f, T\rangle = \text{constant}. \tag{4.1}$$

If we take the derivative in each part of (4.1) in the direction T on M, we have

$$\left\langle \overline{D}_T \nabla f, T\right\rangle + \left\langle \nabla f, \overline{D}_T T\right\rangle = 0. \tag{4.2}$$

On the other hand, from Lemma 2.3 (see [11]), ∇f is parallel in M, i.e. $\overline{D}_X \nabla f = 0$ for arbitrary $X \in TM$. So, we get $\overline{D}_T \nabla f = 0$.Then, by using (4.2) and Frenet formulas, we obtain

$$\kappa \langle \nabla f, N\rangle = 0. \tag{4.3}$$

Since κ is assumed to be positive, (4.3) implies that $\langle \nabla f, N\rangle = 0$. Hence, we can write the axis of α as

$$\nabla f = \lambda_1 T + \lambda_2 B. \tag{4.4}$$

Doing the dot product with T in each part of (4.4), we get

$$\langle \nabla f, T\rangle = \lambda_1 = \|\nabla f\| \cos(\theta), \tag{4.5}$$

where θ is the angle between ∇f and T. And, since $\|\nabla f\|^2 = \lambda_1^2 + \lambda_2^2$, we also have

$$\lambda_2 = \|\nabla f\| \sin(\theta)$$

by using (4.5).Finally, the axis of α

$$\nabla f = \|\nabla f\| \left(\cos(\theta)T + \sin(\theta)B\right).$$

(2) From the proof of (1), we can write

$$\langle \nabla f, N\rangle = 0. \tag{4.6}$$

If we take the derivative in each part of (4.6) in the direction T on M, we have

$$\left\langle \overline{D}_T \nabla f, N\right\rangle + \left\langle \nabla f, \overline{D}_T N\right\rangle = 0. \tag{4.7}$$

And, from the proof of (1), $\overline{D}_T \nabla f = 0$. Hence, from (4.7),

$$\left\langle \nabla f, \overline{D}_T N\right\rangle = 0. \tag{4.8}$$

By using Frenet formulas, from (4.8) we obtain

$$-\kappa \langle \nabla f, T\rangle + \tau \langle \nabla f, B\rangle = 0. \tag{4.9}$$

On the other hand, by using (4.4), we can write as $\langle \nabla f, T\rangle = \lambda_1$ and $\langle \nabla f, B\rangle = \lambda_2$.Since $\lambda_1 = \|\nabla f\| \cos(\theta)$ and $\lambda_2 = \|\nabla f\| \sin(\theta)$ from the proof of (1), we obtain

$$\langle \nabla f, T\rangle = \|\nabla f\| \cos(\theta) \text{ and } \langle \nabla f, B\rangle = \|\nabla f\| \sin(\theta). \tag{4.10}$$

So, by using (4.9) and the equalities (4.10), we have

$$\frac{\tau}{\kappa} = \cot(\theta)\text{=constant.}$$

This completes the proof of the Theorem. $\square$

Theorem 4.2. *Let $M \subset \mathbb{R}^4$ be a 3-dimensional Riemannian manifold and let M be a complete connected smooth without boundary. Also, let M be isometric to a Riemannian product $N \times \mathbb{R}$ .Let us assume that $f : M \to \mathbb{R}$ be a non-trivial affine function (see main theorem in [9]) and be $\alpha : I \to M$ a curve with a unit tangent T. If $\dfrac{\tau}{\kappa}$ =constant, then the curve α is a f-eikonal helix curve (with the axis $\nabla f = \|\nabla f\| (\cos(\theta)T + \sin(\theta)B)$).*

Proof. We consider the vector field

$$\nabla f = \|\nabla f\| (\cos(\theta)T + \sin(\theta)B) \tag{4.11}$$

If we take the derivative in each part of (4.11) in the direction T on M, we have

$$\overline{D}_T \nabla f = \|\nabla f\| \cos(\theta)\overline{D}_T T + \|\nabla f\| \sin(\theta)\overline{D}_T B \tag{4.12}$$

And, from Theorem 4.1, we know that $\overline{D}_T \nabla f = 0$. So, by using Frenet formulas, from (4.12), we can write

$$0 = \|\nabla f\| (\kappa \cos(\theta) - \tau \sin(\theta)) N.$$

It follows that $\frac{\tau}{\kappa} = \cot(\theta)$. On the other hand, since $\dfrac{\tau}{\kappa}$ =constant, we deduce that θ is constant. Hence, from (4.11), we obtain

$$\langle \nabla f, T \rangle = \|\nabla f\| . \cos(\theta). \tag{4.13}$$

On the other hand, from Lemma 2.3 (see [11]), $\|\nabla f\|$ =constant and so, from (4.13), we get $\langle \nabla f, T \rangle$ is constant. Consequently, the curve α is a f-eikonal helix curve (with the axis $\nabla f = \|\nabla f\| (\cos(\theta)T + \sin(\theta)B)$). $\square$

The latter Theorem 4.1 and Theorem 4.2 have the following corollaries.

Corollary 4.1. *Let $M \subset \mathbb{R}^4$ be a 3-dimensional Riemannian manifold and let M be a complete connected smooth without boundary. Also, let M be isometric to a Riemannian product $N \times \mathbb{R}$. Let us assume that $f : M \to \mathbb{R}$ be a non-trivial affine function (see main theorem in [9]) and be $\alpha : I \to M$ a curve with a unit tangent T. The curve α is a f-eikonal helix curve with the axis $\nabla f = \|\nabla f\| (\cos(\theta)T + \sin(\theta)B)$ if and only if $\dfrac{\tau}{\kappa}$ =constant.*

Example 4.1. *In corollary 4.1, all f-eikonal helix curves in M are also LC-helix curves (see [12]).*

Corollary 4.2. *Let $M \subset \mathbb{R}^4$ be a 3-dimensional Riemannian helix submanifold and let M be a complete connected smooth without boundary. Also, let M be isometric to a Riemannian product $N \times \mathbb{R}$. Let us assume that $f : M \to \mathbb{R}$ be a non-trivial affine function (see main theorem in [9]).Then, $\frac{\tau}{\kappa}$ is constant along the helix lines of M.*

Proof. From Theorem 3.1, we know that the helix lines of M are f-eikonal helix curves.And, by using corollary 4.1, this concludes the proof. $\square$

5. THE RELATION HELIX SUBMANIFOLDS AND ∇f

Lemma 5.1. *Let $M \subset \mathbb{R}^n$ be a Riemannian helix submanifold with respect to the unit direction $d \in \mathbb{R}^n$ and $f : M \to \mathbb{R}$ be a function . Let D be Riemannian connexion (standard covariant derivative) on $\mathbb{R}^n$ and $\overline{D}$ be Riemannian connexion on M. Let us assume that $\alpha : I \subset IR \to M$ is a unit speed (parametrized by arc length function s) curve on M with unit tangent T . Then,the normal component ξ of d is parallel normal in the direction T if and only if $(\nabla f)' \in TM$ along the curve α, where ∇f is the unit tangent component of the direction d.*

Proof. We assume that the normal component ξ of d is parallel normal in the direction T. Since T and $\nabla f \in TM$, from the Gauss equation in Definition 2.3,

$$D_T \nabla f = \overline{D}_T \nabla f + V(T, \nabla f) \tag{5.1}$$

According to this Lemma, since the normal component ξ of d is parallel normal in the direction T, i.e.$\nabla_T^{\perp} \xi = 0$ (see Remark 2.1), from (2.2) in Remark 2.2 ($0 < \theta < \frac{\pi}{2}$)

$$V(T, \nabla f) = 0 \tag{5.2}$$

So, by using (5.1),(5.2) and Frenet formulas, we have:

$$D_T \nabla f = \frac{d\nabla f}{ds} = (\nabla f)^{'} = \overline{D}_T \nabla f.$$

That is, the vector field $(\nabla f)^{'} \in TM$ along the curve α, where TM is the tangent space of M.

Conversely, let us assume that $(\nabla f)^{'} \in TM$ along the curve α. Then, from Gauss equation, $V(T, \nabla f) = 0$. Hence, from (2.2) in Remark 2.2 $(0 < \theta < \frac{\pi}{2})$, $\nabla_T^{\perp} \xi = 0$. That is, the normal component ξ of d is parallel normal in the direction T. This completes the proof. □

Theorem 5.1. *Let $M \subset \mathbb{R}^n$ be a Riemannian helix submanifold with respect to the unit direction $d \in \mathbb{R}^n$ and $f : M \to \mathbb{R}$ be a function . Let D be Riemannian connexion (standard covariant derivative) on $\mathbb{R}^n$ and $\overline{D}$ be Riemannian connexion on M. Let us assume that $\alpha : I \subset IR \to M$ is a unit speed (parametrized by arc length function s) curve on M with unit tangent T . Then, if the normal component ξ of d is parallel normal in the direction T and if ∇f parallel in M , then the tangent component of d is euclidean parallel along the curve α, where ∇f is the unit tangent component of the direction d.*

Proof. Since T and $\nabla f \in TM$, from the Gauss equation in Definition 2.3,

$$D_T \nabla f = \overline{D}_T \nabla f + V(T, \nabla f) \tag{5.3}$$

Since ∇f parallel in M, i.e. $\overline{D}_X \nabla f = 0$ for arbitrary $X \in TM$, $\overline{D}_T \nabla f = 0$. On the other hand, according to the Lemma 5.1, $(\nabla f)^{'} \in TM$ due to the fact that the normal component ξ of d is parallel normal in the direction T. Therefore, from Gauss equation, $V(T, \nabla f) = 0$. Hence, from (5.3), we have:

$$D_T \nabla f = \frac{d\nabla f}{ds} = (\nabla f)^{'} = 0$$

along the curve α. That is, the tangent component of d is euclidean parallel along the curve α. This completes the proof. □

Here, we emphasize an important point. In Theorem 5.1, if M is the n-dimensional Euclidean space and if the function f is affine, then grad f is parallel on M (see [9]).

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