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RANDOM DETERMINANTS, MIXED VOLUMES OF ELLIPSOIDS, AND ZEROS OF GAUSSIAN RANDOM FIELDS

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ABSTRACT. Consider a $d \times d$ matrix M whose rows are independent centered non-degenerate Gaussian vectors $\xi_1, \dots, \xi_d$ with covariance matrices $\Sigma_1, \dots, \Sigma_d$. Denote by $\mathcal{E}_i$ the location-dispersion ellipsoid of ξ_i : $\mathcal{E}_i = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^\top \Sigma_i^{-1} \mathbf{x} \leqslant 1\}$. We show that

$$\mathbb{E}\,|\det M| = \frac{d!}{(2\pi)^{d/2}} V_d(\mathcal{E}_1, \dots, \mathcal{E}_d),$$

where $V_d(\cdot, \dots, \cdot)$ denotes the *mixed volume*. We also generalize this result to the case of rectangular matrices. As a direct corollary we get an analytic expression for the mixed volume of d arbitrary ellipsoids in $\mathbb{R}^d$.

As another application, we consider a smooth centered non-degenerate Gaussian random field $X = (X_1, \dots, X_k)^\top : \mathbb{R}^d \to \mathbb{R}^k$. Using Kac-Rice formula, we obtain the geometric interpretation of the intensity of zeros of X in terms of the mixed volume of location-dispersion ellipsoids of the gradients of $X_i/\sqrt{\mathbf{Var} X_i}$. This relates zero sets of equations to mixed volumes in a way which is reminiscent of the well-known Bernstein theorem about the number of solutions of the typical system of algebraic equations.

1. Main results

1.1. **Random determinant and mixed volume of ellipsoids.** Consider independent centered non-degenerate Gaussian random vectors $\xi_1, \dots, \xi_k \in \mathbb{R}^d, k \leqslant d$, with covariance matrices $\Sigma_1, \dots, \Sigma_k$. Denote by $\mathcal{E}_i$ the location-dispersion ellipsoid of ξ_i:

$$\mathcal{E}_i = \{\mathbf{x} = (x_1, \dots, x_d)^\top \in \mathbb{R}^d : \mathbf{x}^\top \Sigma_i^{-1} \mathbf{x} \leqslant 1\}, \quad i = 1, \dots, k. \tag{1.1}$$

Denote by M a $k \times d$ matrix whose rows are $\xi_1, \dots, \xi_k$.

Theorem 1.1. *It holds that*

$$\mathbb{E} \sqrt{\det(MM^\top)} = \frac{(d)_k}{(2\pi)^{k/2} \kappa_{d-k}} V_d(\mathcal{E}_1, \dots, \mathcal{E}_k, B, \dots, B), \tag{1.2}$$

where $V_d(\cdot, \dots, \cdot)$ denotes the mixed volume of d convex bodies in $\mathbb{R}^d$ (see Sect. 2 for details), B is the unit ball in $\mathbb{R}^d$, $(d)_k = d(d-1) \cdot \dots \cdot (d-k+1)$ is the Pochhammer symbol, and $\kappa_n = \pi^{n/2}/\Gamma(1+n/2)$ denotes the volume of the unit ball in $\mathbb{R}^n$.

The left-hand side of (1.2) can be interpreted as the average k-dimensional volume of a Gaussian random parallelotope.

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Corollary 1.2. In the case $k = d$ it holds that

$$\mathbb{E}\,|\det M| = \frac{d!}{(2\pi)^{d/2}} V_d(\mathcal{E}_1, \dots, \mathcal{E}_d).$$

As another direct corollary we can calculate the mixed volume of d arbitrary ellipsoids in $\mathbb{R}^d$.

Corollary 1.3. If $\mathcal{E}_1, \dots, \mathcal{E}_d$ are arbitrary ellipsoids defined by the symmetric positive-definite matrices $\Sigma_1, \dots, \Sigma_d$ as in (1.1), then

$$\begin{aligned} V_d(\mathcal{E}_1, \dots, \mathcal{E}_d) = \frac{1}{d!} \prod_{i=1}^{d} (\det \Sigma_i)^{-1/2} \\ \times \int_{\mathbb{R}^{d^2}} |\det(x_{ij})| \prod_{i=1}^{d} \exp\left(-\frac{1}{2}\mathbf{x}_i^\top \Sigma_i^{-1} \mathbf{x}_i\right) dx_{11} \dots dx_{dd}, \end{aligned}$$

where

$$\mathbf{x}_i = (x_{i1}, \dots, x_{id})^\top.$$

The only estimate of the mixed volume of ellipsoids that we know is due to Barvinok [2]. He showed that

$$\frac{\kappa_d}{3^{(d-1)/2}} \sqrt{D_d(\Sigma_1, \dots, \Sigma_d)} \leqslant V_d(\mathcal{E}_1, \dots, \mathcal{E}_d) \leqslant \kappa_d \sqrt{D_d(\Sigma_1, \dots, \Sigma_d)},$$

where $D_d(\cdot, \dots, \cdot)$ denotes the mixed discriminant of d symmetric $d \times d$ matrices:

$$D_d(A_1, \dots, A_d) = \frac{1}{d!} \frac{\partial^d}{\partial \lambda_1 \dots \partial \lambda_d} \det(\lambda_1 A_1 + \dots + \lambda_d A_d)\Big|_{\lambda_1 = \dots = \lambda_d = 0}.$$

If $\xi_1, \dots, \xi_k$ are independent standard Gaussian vectors, then $MM^\top$ is a Wishart matrix, and (1.2) turns to (see [10],[5])

$$\mathbb{E}\sqrt{\det(MM^\top)} = \frac{(d)_k \kappa_d}{(2\pi)^{k/2} \kappa_{d-k}}.$$

1.2. **Intrinsic volumes.** If $\xi_1, \xi_2, \dots, \xi_k \in \mathbb{R}^d, k \leqslant d$, are identically distributed with the common covariance matrix Σ and the location-dispersion ellipsoid $\mathcal{E}$, then (1.2) turns to

$$\mathbb{E}\sqrt{\det(MM^\top)} = \frac{k!}{(2\pi)^{k/2}} V_k(\mathcal{E}), \tag{1.3}$$

where $V_k(\cdot)$ denotes the k-th *intrinsic volume* of a convex body in $\mathbb{R}^d$:

$$V_k(K) = \frac{\binom{d}{k}}{\kappa_{d-k}} V_d(\underbrace{K, \dots, K}_{k \text{ times}}, B, \dots, B).$$

The normalization is chosen so that $V_k(K)$ depends only on K and not on the dimension of the surrounding space, that is, if $\dim K < d$, then the computation of $V_k(K)$ in $\mathbb{R}^d$ leads to the same result as the computation in the affine span of K. In particular, if $\dim K = k$, then $V_k(K) = \mathrm{Vol}_k(K)$, the k-dimensional volume of K.

It is known that $V_1(K)$ is proportional to the mean width of K:

$$V_1(K) = \frac{d\kappa_d}{2\kappa_{d-1}} w(K).$$

Taking $k=1$ in (1.3), we obtain that for any centered Gaussian vector ξ with the location-dispersion ellipsoid $\mathcal{E}$ it holds that

$$\mathbb{E}\|\xi\| = \frac{1}{\sqrt{2\pi}} V_1(\mathcal{E}). \tag{1.4}$$

It was pointed out by Mikhail Lifshits that (1.4) is a special case of the following remarkable result of Sudakov.

1.3. **Connection with Sudakov's result.** For our purposes the following finite-dimensional version of Sudakov's theorem suffices. The result in full generality can be found in [9, Proposition 14].

Proposition 1.4. For an arbitrary subset $A \subset \mathbb{R}^d$ we have

$$\mathbb{E} \sup_{\mathbf{x}\in A} \langle \mathbf{x}, \eta\rangle = \frac{1}{\sqrt{2\pi}} V_1(\mathrm{conv}(A)), \tag{1.5}$$

where η is a standard Gaussian vector in $\mathbb{R}^d$ and $\mathrm{conv}(A)$ is the convex hull of A.

Let us deduce (1.4) from (1.5). Consider a matrix U such that $\Sigma = U^{-1}(U^{-1})^\top$ and $U\xi$ is a standard Gaussian vector. Using (1.5) with $A=\mathcal{E}$ and $\eta = U\xi$, we get

$$\begin{aligned}\mathbb{E}\|\xi\| &= \mathbb{E} \sup_{\|\mathbf{x}\|\leqslant 1} \langle \mathbf{x}, \xi\rangle = \mathbb{E} \sup_{\|\mathbf{x}\|\leqslant 1} \langle (U^{-1})^\top \mathbf{x}, U\xi\rangle \\ &= \mathbb{E} \sup_{\|U^\top \mathbf{x}\|\leqslant 1} \langle \mathbf{x}, U\xi\rangle = \mathbb{E}\sup_{\mathbf{x}\in\mathcal{E}} \langle \mathbf{x}, U\xi\rangle = \frac{1}{\sqrt{2\pi}} V_1(\mathcal{E}).\end{aligned}$$

Open problem: to obtain a formula generalizing (1.3) and (1.5).

1.4. **Zeros of Gaussian random fields.** Let $X(\mathbf{t}) = (X_1(\mathbf{t}),\dots,X_k(\mathbf{t}))^\top : \mathbb{R}^d \to \mathbb{R}^k$, $k \leqslant d$, be a random field. Following Azaïs and Wschebor [1], we always assume that the following conditions hold:

(a) X is Gaussian;
(b) almost surely, the function $X(\cdot)$ is of class $\mathcal{C}^1$;
(c) for all $\mathbf{t}\in\mathbb{R}^d$, $X(\mathbf{t})$ has a non-degenerate distribution;
(d) almost surely, if $X(\mathbf{t}) = 0$, then $X'(\mathbf{t})$, the Jacobian matrix of $X(\mathbf{t})$, has the full rank.

Then, almost surely, the level set $X^{-1}(0)$ is a $\mathcal{C}^1$-manifold of dimension $d-k$, and for any Borel set F the Lebesgue measure $\mathrm{Vol}_{d-k}(X^{-1}(0)\cap F)$ is well-defined ($\mathrm{Vol}_0(\cdot)$ denotes the counting measure).

It was shown in [1, p. 177] that

$$\mathbb{E}\,\mathrm{Vol}_{d-k}(X^{-1}(0)\cap F) = \int_F \mathbb{E}\left(\sqrt{\det\left(X'(\mathbf{t})X'(\mathbf{t})^\top\right)}\,\middle|\, X(\mathbf{t})=0\right) p_{X(\mathbf{t})}(0)\,d\mathbf{t}, \tag{1.6}$$

where $p_{X(\mathbf{t})}(\cdot)$ is a density of $X(\mathbf{t})$. Thus, the integrand in (1.6) can be interpreted as the intensity of zeros of X.

In this paper we consider the special case when X is centered and its coordinates $X_1,\dots,X_k$ are independent. Denote by $\mathcal{E}_i(\mathbf{t})$ the location-dispersion ellipsoid of $\nabla[X_i(\mathbf{t})/\sqrt{\mathrm{Var}\,X_i(\mathbf{t})}]$.

Theorem 1.5. *Let X be a centered random field with independent coordinates defined as above and satisfying conditions (a)-(d). Then*

$$\mathbb{E}\,\mathrm{Vol}_{d-k}(X^{-1}(0)\cap F) = \frac{(d)_k}{(2\pi)^k \kappa_{d-k}} \int_F V_d(\mathcal{E}_1(\mathbf{t}),\dots,\mathcal{E}_k(\mathbf{t}),B,\dots,B)\,d\mathbf{t}. \tag{1.7}$$

Formula (1.7) relates zero sets of random equations to mixed volumes. In the case $k = d$ it is therefore reminiscent of the well-known fact from the algebraic geometry which we formulate in the next subsection.

1.5. Bernstein's theorem.

Consider a complex polynomial in d variables

$$f(z_1, \dots, z_d) = \sum c_{j_1,\dots,j_d} z_1^{j_1} \dots z_d^{j_d}.$$

The Newton polytope of f is a subset of $\mathbb{R}^d$ defined as

$$\mathrm{Nw}(f) = \mathrm{conv}\{(j_1, \dots, j_d) \in \mathbb{Z}^d \,:\, c_{j_1,\dots,j_d} \neq 0\}.$$

Let $K_1, \dots, K_d$ be compact convex polytopes in $\mathbb{R}^d$ with vertexes in $\mathbb{Z}^d$. Consider a system of algebraic equations

$$\begin{cases} f_1(z_1, \dots, z_d) = 0, \\ \qquad \dots \\ f_d(z_1, \dots, z_d) = 0, \end{cases}$$

such that $\mathrm{Nw}(f_i) = K_i$. Bernstein showed [3] that for almost all such systems (with respect to the Lebesgue measure in the space of the coefficients of the polynomials) the number of nonzero solutions is equal to

$$\mathrm{Vol}_0(f_1^{-1}(0) \cap \dots \cap f_d^{-1}(0) \setminus \{\mathbf{0}\}) = d! V_d(K_1, \dots, K_d).$$

2. Some essential tools from geometry

For the basic facts from integral and convex geometry we refer the reader to [4] and [8].

2.1. Mixed volumes.

Consider arbitrary convex bodies $K_1, \dots, K_d \subset \mathbb{R}^d$. Minkowski showed [7] that $\mathrm{Vol}_d(\lambda_1 K_1 + \dots + \lambda_d K_d)$, where $\lambda_1, \dots, \lambda_d \geq 0$, is a homogeneous polynomial of degree d with nonnegative coefficients:

$$\mathrm{Vol}_d(\lambda_1 K_1 + \dots + \lambda_d K_d) = \sum_{i_1=1}^{d} \dots \sum_{i_d=1}^{d} \lambda_{i_1} \dots \lambda_{i_d} V_d(K_{i_1}, \dots, K_{i_d}). \tag{2.1}$$

The coefficients $V_d(K_{i_1}, \dots, K_{i_d})$ are uniquely determined by the assumption that they are symmetric with respect to permutations of $K_{i_1}, \dots, K_{i_d}$. The coefficient $V_d(K_1, \dots, K_d)$ is called the mixed volume of $K_1, \dots, K_d$. Differentiating (2.1), we get an alternative definition of the mixed volume:

$$V_d(K_1, \dots, K_d) = \frac{1}{d!} \frac{\partial^d}{\partial \lambda_1 \dots \partial \lambda_d} \mathrm{Vol}_d(\lambda_1 K_1 + \dots + \lambda_d K_d)\big|_{\lambda_1 = \dots = \lambda_d = 0}.$$

For any affine transformation L it holds

$$V_d(LK_1, \dots, LK_d) = |\det L| \cdot V_d(K_1, \dots, K_d). \tag{2.2}$$

The following relation can also be stated:

$$\int_{\mathbb{S}^{d-1}} V_{d-1}(P_{\mathbf{u}} K_1, \dots, P_{\mathbf{u}} K_{d-1}) \, d\mathbf{u} = \frac{\kappa_{d-1}}{\kappa_d} V_d(K_1, \dots, K_{d-1}, B), \tag{2.3}$$

where $d\mathbf{u}$ is the surface measure on $\mathbb{S}^{d-1}$ normalized to have total mass 1, and $P_{\mathbf{u}}$ denotes the orthogonal projection to the linear hyperplane $\mathbf{u}^{\perp}$.

2.2. **Volumes of parallelotopes.** For any $A \subset \mathbb{R}^d$ and $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^d$ denote by $P_{\mathbf{x}_1,\dots,\mathbf{x}_k} A$ the orthogonal projection of A to $\operatorname{span}^\perp\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ (the orthogonal complement of the linear span of $\mathbf{x}_1, \dots, \mathbf{x}_k$). Denote by $H_{\mathbf{x}_1,\dots,\mathbf{x}_k}$ the parallelotope generated by the vectors $\mathbf{x}_1, \dots, \mathbf{x}_k$. It is known that

$$\operatorname{Vol}_k(H_{\mathbf{x}_1,\dots,\mathbf{x}_k}) = \sqrt{\det(AA^\top)}, \tag{2.4}$$

where A is a matrix whose rows are $\mathbf{x}_1, \dots, \mathbf{x}_k$.

For any $\mathbf{x}_1, \dots, \mathbf{x}_d \in \mathbb{R}^d$ and $k = 1, \dots, d-1$ it holds that

$$\operatorname{Vol}_d(H_{\mathbf{x}_1,\dots,\mathbf{x}_d}) = \operatorname{Vol}_k(H_{\mathbf{x}_1,\dots,\mathbf{x}_k}) \operatorname{Vol}_{d-k}(P_{\mathbf{x}_1,\dots,\mathbf{x}_k} H_{\mathbf{x}_{k+1},\dots,\mathbf{x}_d}). \tag{2.5}$$

2.3. **Ellipsoids.** There is a bijection $A \mapsto \mathcal{E}$ between $d \times d$ symmetric positive-definite matrices and d-dimensional non-degenerate ellipsoids centered on the origin (see [6] for details):

$$\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^\top A^{-1} \mathbf{x} \leqslant 1\}.$$

Any non-degenerate linear coordinate transformation of the form $\mathbf{x} \mapsto L\mathbf{x}$ is reflected by a change of the corresponding representing matrix A to a matrix A_L given by

$$A_L = LAL^\top. \tag{2.6}$$

Let $\mathcal{E}'$ be an orthogonal projection of $\mathcal{E}$ onto an k-dimensional subspace with some orthonormal basis $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^d$. Denote by A' a $k \times k$ matrix representing the ellipsoid $\mathcal{E}'$ in this basis. If C is a $k \times d$ matrix whose rows are $\mathbf{x}_1, \dots, \mathbf{x}_k$, then

$$A' = CAC^\top. \tag{2.7}$$

3. Proofs

3.1. **Proof of Theorem 1.1. Case $k = d$.** We proceed by induction on d. First let us assume that ξ_d is a standard Gaussian vector. Denote by χ_d a random variable having the chi distribution with d degrees of freedom and independent from $\xi_1, \dots, \xi_{d-1}$. Using (2.4) and (2.5) with $k = 1$ we get

$$\begin{aligned}
\mathbb{E}\,|\det M| = \mathbb{E}\operatorname{Vol}_d(H_{\xi_1,\dots,\xi_d}) &= \int_{\mathbb{S}^{d-1}} \mathbb{E}\operatorname{Vol}_d(H_{\xi_1,\dots,\xi_{d-1},\chi_d \mathbf{u}})\,d\mathbf{u} \\
&= \mathbb{E}\chi_d \int_{\mathbb{S}^{d-1}} \mathbb{E}\operatorname{Vol}_{d-1}(P_{\mathbf{u}} H_{\xi_1,\dots,\xi_{d-1}})\,d\mathbf{u} \\
&= \frac{d\kappa_d}{\sqrt{2\pi}\kappa_{d-1}} \int_{\mathbb{S}^{d-1}} \mathbb{E}\operatorname{Vol}_{d-1}(H_{P_{\mathbf{u}}\xi_1,\dots,P_{\mathbf{u}}\xi_{d-1}})\,d\mathbf{u}.
\end{aligned}$$

It follows from (2.7) that $P_{\mathbf{u}}\xi_i$ has a location-dispersion ellipsoid $P_{\mathbf{u}}\mathcal{E}_i$. By the induction assumption,

$$\mathbb{E}\operatorname{Vol}_{d-1}(H_{P_{\mathbf{u}}\xi_1,\dots,P_{\mathbf{u}}\xi_{d-1}}) = \frac{(d-1)!}{(2\pi)^{(d-1)/2}} V_{d-1}(P_{\mathbf{u}}\mathcal{E}_1, \dots, P_{\mathbf{u}}\mathcal{E}_{d-1}).$$

Combining the latter two relations with (2.3), we obtain

$$\mathbb{E}\,|\det M| = \frac{d!}{(2\pi)^{d/2}} V_d(\mathcal{E}_1, \dots, \mathcal{E}_{d-1}, B). \tag{3.1}$$

If ξ_d is an arbitrary non-degenerate Gaussian vector, then there exists a linear transformation L such that $L\xi_d$ is a standard Gaussian vector. It follows from (2.6) that $L\mathcal{E}_i$ is the location-dispersion ellipsoid of $L\xi_i$, and in particular $L\mathcal{E}_d = B$. Applying (3.1) to the matrix $LM^\top$ and using (2.2), we get

$$\begin{aligned}\mathbb{E}\,|\det M| = |\det L|^{-1}\,\mathbb{E}\,|\det LM^\top| &= \frac{d!}{(2\pi)^{d/2}}|\det L|^{-1}V_d(L\mathcal{E}_1,\dots,L\mathcal{E}_{d-1},B)\\ &= \frac{d!}{(2\pi)^{d/2}}V_d(\mathcal{E}_1,\dots,\mathcal{E}_{d-1},\mathcal{E}_d).\end{aligned}$$

3.2. **Proof of Theorem 1.1. Case $k<d$.** Consider a $d\times d$ matrix M' whose first k rows form the matrix M and the last $d-k$ rows are independent standard Gaussian vectors $\xi_{k+1},\dots,\xi_d$ (independent from M). By the previous case,

$$\mathbb{E}\,|\det M'| = \frac{d!}{(2\pi)^{d/2}}V_d(\mathcal{E}_1,\dots,\mathcal{E}_k,B,\dots,B).$$

On the other hand, by (2.5),

$$\begin{aligned}\mathbb{E}\,|\det M'| = \mathbb{E}\ \mathrm{Vol}_d(H_{\xi_1,\dots,\xi_d}) &= \mathbb{E}\ \mathrm{Vol}_k(H_{\xi_1,\dots,\xi_k})\,\mathrm{Vol}_{d-k}(P_{\xi_1,\dots,\xi_k}H_{\xi_{k+1},\dots,\xi_d})\\ &= \mathbb{E}\sqrt{\det(MM^\top)}\,\mathbb{E}\ \mathrm{Vol}_{d-k}(H_{\eta_1,\dots,\eta_{d-k}}),\end{aligned}$$

where $\eta_1,\dots,\eta_{d-k}$ are independent standard Gaussian vectors in $\mathbb{R}^{d-k}$. By the previous case,

$$\mathbb{E}\ \mathrm{Vol}_{d-k}(H_{\eta_1,\dots,\eta_{d-k}}) = \frac{(d-k)!}{(2\pi)^{(d-k)/2}}\kappa_{d-k}.$$

Combining the latter three relations completes the proof.

3.3. **Proof of Theorem 1.5.** First we suppose that X_j has a unit variance: $\mathrm{Var}\,X_j(\mathbf{t})\equiv 1$ for all $j=1,\dots,k$. Differentiating the relation $\mathbb{E}X_j(\mathbf{t})X_j(\mathbf{t})=1$ with respect to t_i, we obtain

$$\mathbb{E}\frac{\partial X_j}{\partial t_i}(\mathbf{t})X_j(\mathbf{t}) = 0,$$

which, together with the independence of the coordinates of X, implies that $X'(\mathbf{t})$ and $X(\mathbf{t})$ are independent. This means that conditioning on $X(\mathbf{t})=0$ in (1.6) may be dropped. To complete the proof of the theorem in the case $\mathrm{Var}\,X_j(\mathbf{t})\equiv 1$ it remains to combine (1.6) with (1.2).

To cover the general case, it suffices to note that $X_j/\sqrt{\mathrm{Var}\,X_j}$ has the same zero set as X_j.

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