ON THE ALGEBRAIC K-THEORY OF TRUNCATED POLYNOMIAL ALGEBRAS IN SEVERAL VARIABLES

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Abstract. We consider the algebraic K-theory of a truncated polynomial algebra in several commuting variables, $K(k[x_1,\dots,x_n]/(x_1^{a_1},\dots,x_n^{a_n}))$. This naturally leads to a new generalization of the big Witt vectors. If k is a perfect field of positive characteristic we describe the K-theory computation in terms of a cube of these Witt vectors on $\mathbb{N}^n$. If the characteristic of k does not divide any of the a_i we compute the K-groups explicitly. We also compute the K-groups modulo torsion for $k=\mathbb{Z}$.

To understand this K-theory spectrum we use the cyclotomic trace map to topological cyclic homology, and write $\mathrm{TC}(k[x_1,\dots,x_n]/(x_1^{a_1},\dots,x_n^{a_n}))$ as the iterated homotopy cofiber of an n-cube of spectra, each of which is easier to understand.

1. Introduction

About 15 years ago, Hesselholt and Madsen [9] computed the relative algebraic K-theory groups of $k[x]/(x^a)$ for a perfect field k of positive characteristic. In this paper we extend their result to a truncated polynomial ring in multiple commuting variables. We study

$$K(k[x_1,\dots,x_n]/(x_1^{a_1},\dots,x_n^{a_n}),(x_1),\dots,(x_n)),$$

the appropriate multi-relative version of $K(k[x_1,\dots,x_n]/(x_1^{a_1},\dots,x_n^{a_n}))$. For ease of exposition, let $A=k[x_1,\dots,x_n]/(x_1^{a_1},\dots,x_n^{a_n})$, and let $\widetilde{K}(A)$ denote the multi-relative version of $K(A)$.

Hesselholt and Madsen (loc. cit.) expressed $K_*(k[x]/(x^a),(x))$ in terms of big Witt vectors. Recall that given a truncation set $S\subset\mathbb{N}$, one can define the Witt ring $\mathbb{W}_S(k)$. With $S=\{1,2,\dots,m\}$ this gives the length m big Witt vectors, and they proved that for a perfect field k of positive characteristic

$$K_{2q-1}(k[x]/(x^a),(x))\cong\mathbb{W}_{aq}(k)/V_a(\mathbb{W}_q(k))$$

and

$$K_{2q}(k[x]/(x^a),(x))=0.$$

Here V_a is the a^{th} Verschiebung, one of the structure maps between Witt vectors.

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When $k = \mathbb{F}_p$, for example, it is not much harder to write down the answer explicitly without refering to Witt vectors. But to do that one has to consider the cases $p \mid a$ and $p \nmid a$ separately, and the answer looks somewhat less elegant.

To express the answer in the n-variable case, we again take advantage of Witt vectors to organize our calculation. Our computations lead us naturally to an n-dimensional version of the big Witt vector construction. We will define these Witt vectors on $\mathbb{N}^n$ carefully in the next section. As far as we have been able to determine this construction is new; it is not equivalent to the definition given by Dress and Siebeneicher [4].

We say a set $S \subset \mathbb{N}^n$ is a *truncation set in* $\mathbb{N}^n$ if $(ds_1, \ldots, ds_n) \in S$ implies $(s_1, \ldots, s_n) \in S$, for $d \in \mathbb{N}$. Given such an S, we define the Witt vectors on $\mathbb{N}^n$, $\mathbb{W}_S(k)$, in a way that generalizes the construction for $n = 1$. We then prove the following. Let $\mathbb{T}^{n-1}$ denote an $(n-1)$-dimensional torus.

Theorem 1.1. *Let k be any commutative ring and let A be as above. Then*

$$\widetilde{K}_*(A) \cong \widehat{TC}_*(A) \otimes H_*(\mathbb{T}^{n-1})$$

for a spectrum $\widehat{\mathrm{TC}}(A)$ which is the iterated homotopy pushout over an n-cube $\mathrm{TC}(\mathrm{THH}(k) \wedge \widehat{\mathcal{X}})$ of spectra.

For each vertex $I \subset \{1, \ldots, n\}$ in the n-cube we give an explicit description of the spectrum $\mathrm{TC}(\mathrm{THH}(k) \wedge \widehat{X}_I)$, at least up to profinite completion, as well as all the maps in the cube. This description is given in terms of fixed points of the topological Hochschild homology of k. If k is a perfect field of positive characteristic we can identify the homotopy groups of $\mathrm{TC}(\mathrm{THH}(k) \wedge \widehat{X}_I)$ with certain Witt vectors on $\mathbb{N}^n$ of k.

Theorem 1.2. *Suppose k is a perfect field of positive characteristic. Then for each $I \subset \{1, \ldots, n\}$ the homotopy groups of $\mathrm{TC}(\mathrm{THH}(k) \wedge \widehat{X}_I)$ are concentrated in odd degrees with*

$$\mathrm{TC}_{2q-1}(\mathrm{THH}(k) \wedge \widehat{X}_I) \cong \mathbb{W}_{S_q(I)}(k).$$

Here $S_q(I) \subset \mathbb{N}^n$ is the truncation set in $\mathbb{N}^n$:

$$S_q(I) = \{(s_1, \ldots, s_n) \quad \mid \quad \sum_{i=1}^{n} \lfloor \frac{s_i - 1}{a_i^{\epsilon_i}} \rfloor \leq q - 1\},$$

where $\epsilon_i = 0$ if $i \notin I$ and $\epsilon_i = 1$ if $i \in I$. The maps in the n-cube $\mathrm{TC}(\mathrm{THH}(k) \wedge \widehat{\mathcal{X}})$ are given by the Verschiebung maps $V^1_{a_1}, \ldots, V^n_{a_n}$.

In the 1-dimensional case the Verschiebung map $V_a : \mathbb{W}_m(k) \to \mathbb{W}_{am}(k)$ is always injective. This is no longer true in the n-dimensional case unless we impose further conditions.

Theorem 1.3. *Suppose k is a perfect field of characteristic $p > 0$ and suppose p does not divide any of the a_i. Then all the maps in $\mathrm{TC}(\mathrm{THH}(k) \wedge \widehat{\mathcal{X}})$ are split injective and $\widehat{\mathrm{TC}}_*(A)$ is concentrated in odd degrees with*

$$\widehat{\mathrm{TC}}_{2q-1}(A) \cong \mathbb{W}_{S_q(\{1,\ldots,n\})}(k) / \left(\sum_{i=1}^{n} V^i_{a_i} \left(\mathbb{W}_{S_q(\{1,\ldots,\hat{i},\ldots,n\})}(k) \right) \right),$$

where $\hat{i}$ denotes skipping i.

Along with Theorem 1.1, this completely determines the algebraic K-theory groups in this case. We can be even more explicit. Given a truncation set in $\mathbb{N}^n$, S, and $(s_1, \ldots, s_n) \in S$ we can extract a 1-dimensional "p-typical" truncation set $p^{\mathbb{N}_0}(s_1, \ldots, s_n) \cap S$ consisting of those p-power multiples of $(s_1, \ldots, s_n)$ which lie in S. Then the group $\widehat{\mathrm{TC}}_{2q-1}(A)$ in Theorem 1.3 above is isomorphic to

$$\bigoplus \mathbb{W}_{p^{\mathbb{N}_0}(s_1,\ldots,s_n)\cap S}(k;p),$$

where the sum is over all $s_1, \ldots, s_n \geq 1$ with $p \nmid \gcd(s_i)$ and $a_i \nmid s_i$.

We also compute the ranks of the multi-relative K-theory groups of truncated polynomials over $\mathbb{Z}$. To state this result, we first recall the definition of the Poincare series of a sequence of groups.

Definition 1.4. *The Poincare series of a sequence of groups G_i is the power series $\sum b_i t^i$ such that* $\mathrm{rank}(G_i) = b_i$.

We prove the following theorem:

Theorem 1.5. *The Poincare series of*

$$K_*(\mathbb{Z}[x_1, \ldots, x_n]/(x_1^{a_1} \ldots x_n^{a_n}), (x_1), \ldots, (x_n))$$

is given by

$$\sum_{j \geq 1} t^{2j-1}(1+t)^{n-1} \binom{n+j-2}{n-1} \prod_{i=1}^{n} (a_i - 1).$$

This generalizes results of Soulé [13] and Staffeldt [14] in the single variable case.

1.1. **Organization.** In §2 we give a precise definition of the Witt vectors on $\mathbb{N}^n$. This section might be of independent interest. In §3 we define cyclotomic spectra and give some relevant examples. In §4 we review the cyclotomic trace map, before we prove the main theorems in §5 and §6.

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2. Witt vectors

At the heart of our computation is building an n-cube of S^1-equivariant spectra, analogous to a cofiber sequence of S^1-spectra that Hesselholt and Madsen used in the one variable case. The various vertices of this hypercube are underlain by n-fold smash powers of S^1, and the maps in the diagram arise from the Hesselholt-Madsen maps restricted to one of the factors. This suggests that a useful language will again be Witt vectors, but our Witt vectors will be modeled on truncation sets in $\mathbb{N}^n$, rather than those modeled on $\mathbb{N}$.

2.1. **Classical Case.** For the classical construction, recall that given a truncation set $S \subset \mathbb{N} = \{1, 2, \ldots\}$ and a commutative ring R we can define a commutative ring $\mathbb{W}_S(R)$. As a set, $\mathbb{W}_S(R) = R^S$, and we define addition and multiplication in such a way that the ghost map

$$w : \mathbb{W}_S(R) \to R^S$$

that takes a vector $(a_s|s \in S)$ to the vector $(w_s|s \in S)$ with

$$w_s = \sum_{dt=s} t a_t^d$$

is a ring map. One can show that there is exactly one functorial way to do this.

We will be particularly interested in three families of truncation sets. If $S = \{1, 2, \dots, m\}$ we denote $\mathbb{W}_S(R)$ by $\mathbb{W}_m(R)$ and call it the big Witt vectors of length m. If $S = \{1, p, \dots, p^{m-1}\}$ we denote $\mathbb{W}_S(R)$ by $\mathbb{W}_m(R; p)$ and call it the p-typical Witt vectors of length m. If R is a $\mathbb{Z}_{(p)}$-algebra there is a splitting

$$\mathbb{W}_m(R) \cong \prod_{p \nmid s} \mathbb{W}_{\lfloor \log_p(m/s)+1 \rfloor}(R; p).$$

Finally, we might consider the set $\langle m \rangle = \{d \in \mathbb{N} | \quad d \mid m\}$ and the associated Witt vectors $\mathbb{W}_{\langle m \rangle}(R)$.

Note that associating Witt vectors to a truncation set is a special case of a more general construction. Consider $\mathbb{N}$ as a set with an action of the multiplicative monoid $\mathbb{N}$. A truncation set is then just a subposet of $\mathbb{N}$ which is closed under divisibility. The $\mathbb{N}$-action endows this with further structure, though: for any two elements r and s such that $r \leq s$, we have a "weight": $s/r \in \mathbb{N}$. It is possible to build Witt vectors on other weighted posets of this form. For instance, Dress-Siebeneicher's construction of Witt vectors relative to a profinite group [4] is built on the poset of finite index subgroups of a profinite group. Our new construction described below is built on $\mathbb{N}^n$.

2.2. **Witt vectors on $\mathbb{N}^n$.** Now we wish to replace $\mathbb{N}$ by $\mathbb{N}^n$. Coordinate multiplication gives an action by $\mathbb{N}$ and hence a weighted poset: $(t_1, \dots, t_n)$ divides $(s_1, \dots, s_n)$ if there is some $d \in \mathbb{N}$ such that $(dt_1, \dots, dt_n) = (s_1, \dots, s_n)$, and the number d is the weight. We say that $S \subset \mathbb{N}^n$ is a truncation set in $\mathbb{N}^n$ if it is a closed subposet: if $(s_1, \dots, s_n) \in S$ and $(t_1, \dots, t_n)$ divides $(s_1, \dots, s_n)$, then $(t_1, \dots, t_n) \in S$. From now on we will use vector notation for n-tuples of natural numbers.

As a weighted poset, $\mathbb{N}^n$ splits into a disjoint union of countably infinitely many copies of $\mathbb{N}$, indexed by those sequences $\overrightarrow{s}$ with $\gcd(\overrightarrow{s}) = 1$:

$$\mathbb{N}^n = \coprod_{\gcd(\overrightarrow{s})=1} \mathbb{N} \cdot \overrightarrow{s}.$$

Thus our truncation sets in $\mathbb{N}^n$ are simply disjoint unions of ordinary truncation sets. Given $\overrightarrow{s} \in S$, we use the notation $\mathbb{N}\overrightarrow{s} \cap S$ to denote all multiples of $\overrightarrow{s}$ in S. This can be thought of as a 1-dimensional truncation set by identifying $\overrightarrow{s} \in S$ with $1 \in \mathbb{N}$.

Now with these, we can copy the classical construction. Given a truncation set in $\mathbb{N}^n$, S, we construct the Witt vectors $\mathbb{W}_S(R)$. As a set, $\mathbb{W}_S(R) = R^S$. We define a ghost map

$$w : \mathbb{W}_S(R) \to R^S$$

as the map that takes a vector $(a_{\overrightarrow{s}} | \overrightarrow{s} \in S)$ to the vector $(w_{\overrightarrow{s}} | \overrightarrow{s} \in S)$ with

$$w_{\overrightarrow{s}} = \sum_{d\overrightarrow{t}=\overrightarrow{s}} \gcd(\overrightarrow{t})(a_{\overrightarrow{t}})^d.$$

Lemma 2.1. *There is a unique functorial way to put a ring structure on $\mathbb{W}_S(R)$ in such a way that the ghost map is a ring map.*

Moreover, there is a canonical splitting

$$\mathbb{W}_S(R) \cong \prod_{\gcd(\overrightarrow{s})=1} \mathbb{W}_{\mathbb{N}\overrightarrow{s}\cap S}(R).$$

Proof. Since our truncation set splits as a disjoint union of classical truncation sets, this lemma is simply a restatement of two classical facts:

(1) There is a unique functorial ring structure for a classical truncation set, and
(2) the functor $S \mapsto R^S$ takes disjoint unions to cartesian products.

□

The classical Witt construction had a great many structure maps linking the Witt vectors for various S. Here we have all of the classical ones and more. The real power of this construction is that we have various structure maps which mix the disjoint factors in $\mathbb{N}^n$!

The restriction map is the easiest to define. Given $S' \subset S$ we get a restriction map

$$R^S_{S'} : \mathbb{W}_S(R) \to \mathbb{W}_{S'}(R)$$

in the obvious way.

Next, we have Frobenius maps in each of the n directions. Fix $i \in \{1, \ldots, n\}$ and $r \geq 2$. Then we have a truncation set

$$S/(1, \ldots, r \ldots, 1) = \{(t_1, \ldots, t_n) | (t_1, \ldots, rt_i, \ldots, t_n) \in S\}.$$

We can define a Frobenius map

$$F^i_r : \mathbb{W}_S(R) \to \mathbb{W}_{S/(1,\ldots,r,\ldots,1)}(R)$$

by requiring that the diagram

$$\begin{array}{ccc} \mathbb{W}_S(R) & \xrightarrow{\ w\ } & R^S \\ \Big\downarrow{\scriptstyle F^i_r} & & \Big\downarrow{\scriptstyle (F^i_r)^w} \\ \mathbb{W}_{S/(1,\ldots,r,\ldots,1)}(R) & \xrightarrow{\ w\ } & R^{S/(1,\ldots,r\ldots,1)} \end{array}$$

commutes. Here $(F^i_r)^w$ is defined by

$$(F^i_r)^w(x_{\overrightarrow{s}} | \overrightarrow{s} \in S)_{(t_1,\ldots,t_n)} = x_{(t_1,\ldots,rt_i,\ldots,t_n)}.$$

We note that F^i_r and F^j_s commute if $i \neq j$.

Associated to these Frobenius maps, we also have Verschiebung maps in each of the n directions. Once again fix $i \in \{1, \ldots, n\}$ and $r \geq 2$. Then we can define a Verschiebung map

$$V^i_r : \mathbb{W}_{S/(1,\ldots,r,\ldots,1)}(R) \to \mathbb{W}_S(R)$$

by

$$V^i_r((x_{\overrightarrow{s}}))_{(t_1,\ldots,t_n)} = \begin{cases} x_{(t_1,\ldots,t_i/r,\ldots,t_n)} & \text{if } t_i/r \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

Once again we note that V^i_r and V^j_s commute if $i \neq j$.

These maps satisfy the usual compatibility relations, deducible from the standard definitions in the Burnside ring, such as $F^i_r V^i_r = r$.

We need some basic computations with these Witt vectors on $\mathbb{N}^n$. In all of these propositions, fix a truncation set S, let $\overrightarrow{s} \in S$ have $\gcd(\overrightarrow{s}) = 1$, and let $S' = \mathbb{N} \cdot \overrightarrow{s} \cap S$. The truncation set S' can be considered as a truncation set in $\mathbb{N}^n$ or a classical one, and we will not distinguish between the two. As such, the canonical splitting of $\mathbb{N}^n$ into its disjoint factors passes to the Frobenii and Verschiebungs:

Proposition 2.2. *The map V_r^i splits as a cartesian product of maps*

$$V_r^i : \mathbb{W}_{S'/(1,\ldots,r,\ldots,1)}(R) \to \mathbb{W}_{S'}(R),$$

and similarly for F_r^i.

This lets us focus attention on the simple factors. Fix an i, and let $d_i = \gcd(s_i, r)$ and let $e_i = r/d_i$.

Proposition 2.3. *We have an immediate isomorphism between $S'/(1,\ldots,r,\ldots,1)$ considered as a truncation set in $\mathbb{N}^n$ and S'/e_i considered as a 1-dimensional truncation set.*

Since this truncation set is just as in the classical case, the classical results tell us the value of the Frobenius and Verschiebung.

Proposition 2.4. *If we identify $\mathbb{W}_{S'/(1,\ldots,r,\ldots,1)}(R)$ with $\mathbb{W}_{S'/e}(R)$ then V_r^i is given by the composite*

$$\mathbb{W}_{S'/e}(R) \xrightarrow{V_{e_i}} \mathbb{W}_{S'}(R) \xrightarrow{\cdot d_i} \mathbb{W}_{S'}(R),$$

where V_{e_i} is the classical Verschiebung and $\cdot d_i$ denotes multiplication by d_i.

Similarly, the Frobenius map F_r^i is given by

$$\mathbb{W}_{S'/e}(k) \xrightarrow{F_{e_i}} \mathbb{W}_{S'}(k).$$

3. Cyclotomic spectra

Certain S^1-spectra, called *cyclotomic spectra*, are particularly important to computations of algebraic K-theory. We first recall the definition of a cyclotomic spectrum [10]. Let G be a compact Lie group. Given a genuine G-spectrum X indexed on a complete G-universe $\mathcal{U}$, and a normal subgroup $H \lhd G$, there are two notions of H-fixed points [11]. Both notions of fixed points yield a G/H-(pre)spectrum. The first, denoted X^H, has V^{th} space

$$X^H(V) = X(V)^H$$

for each $V \subset \mathcal{U}^H$. The second notion, that of geometric fixed points, is defined as follows. Let $\mathcal{F}_H$ denote the family of subgroups of G not containing H. Let $E\mathcal{F}_H$ denote the universal space of this family, and let $\widetilde{E}\mathcal{F}_H$ denote the cofiber of the map $E\mathcal{F}_{H+} \to S^0$ given by projection onto the non-basepoint. Then the geometric fixed point spectrum X^{gH} is defined as

$$X^{gH} = (X \wedge \widetilde{E}\mathcal{F}_H)^H.$$

The fixed points and geometric fixed points are related as follows. Starting with the cofiber sequence

$$E\mathcal{F}_{H+} \to S^0 \to \widetilde{E}\mathcal{F}_H,$$

then smashing with X and taking H-fixed points yields a cofiber sequence, the isotropy separation sequence

$$(X \wedge E\mathcal{F}_{H+})^H \to X^H \to (X \wedge \widetilde{E}\mathcal{F}_H)^H = X^{gH}.$$

Thus there is always a map $X^H \to X^{gH}$ from the H-fixed points of X to the H-geometric fixed points of X. If X is a structured ring spectrum, this is a map of structured ring spectra.

We are now ready to recall the definition of cyclotomic spectra. Let $C_n \subset S^1$ denote the cyclic subgroup of order n, and let $\rho_n : S^1 \to S^1/C_n$ denote the isomorphism given by the nth root. Using this isomorphism, an S^1/C_n-spectrum E indexed on $\mathcal{U}^{C_n}$ determines an S^1-spectrum indexed on $\mathcal{U}$. We write this spectrum as $\rho_n^* E$.

Definition 3.1. *A cyclotomic spectrum is a genuine S^1-equivariant spectrum X together with equivalences of S^1-spectra*

$$r_n : \rho_n^*(X^{gC_n}) \xrightarrow{\simeq} X$$

for all $C_n \subset S^1$, such that for any $m, n > 0$ the following diagram commutes

$$\begin{array}{ccc}
\rho_n^*((\rho_m^*(X^{gC_m}))^{gC_n}) & = & \rho_{nm}^*(X^{gC_{nm}}) \\
\downarrow{\scriptstyle \rho_n^*((r_m)^{gC_n})} & & \downarrow{\scriptstyle r_{mn}} \\
\rho_n^*(X^{gC_n}) & \xrightarrow{\quad r_n \quad} & X
\end{array}$$

We will also need the notion of a cyclotomic map between cyclotomic spectra.

Definition 3.2. *A cyclotomic map $f : X \to Y$ between cyclotomic spectra is a map of genuine S^1-equivariant spectra such that the diagram*

$$\begin{array}{ccc}
\rho_n^*(X^{gC_n}) & \xrightarrow{r_n} & X \\
\downarrow & & \downarrow \\
\rho_n^*(Y^{gC_n}) & \xrightarrow{r_n} & Y
\end{array}$$

commutes for all n.

It is clear that cyclotomic spectra together with cyclotomic maps form a category. Before proceeding we give a couple of examples of cyclotomic spectra.

Example 3.3. *For a space X, the free loop space LX has an S^1-action given by rotation of loops. There is an equivariant homeomorphism*

$$r : LX \to \rho_n^*((LX)^{C_n})$$

given by $r(\lambda)(z) = \lambda(z^n)$. The S^1-suspension spectrum of this space, $\Sigma_{S^1}^\infty LX_+$, is a cyclotomic spectrum.

Example 3.4. *For a ring A, the topological Hochschild homology S^1-spectrum* $\mathrm{THH}(A)$ *is a cyclotomic spectrum* [10].

The importance of cyclotomic spectra stems from the fact that for any cyclotomic spectrum X, we have a notion of the topological cyclic homology of X, $\mathrm{TC}(X)$. This is defined as follows. For any S^1-spectrum X, there are maps $F_n : X^{C_{mn}} \to X^{C_m}$ for all $n, m \geq 1$, given by inclusion of fixed points. These maps are called Frobenius maps. If X is cyclotomic, there is a second class of maps: $R_n : X^{C_{mn}} \to X^{C_m}$, for all $n, m \geq 1$. These maps, called restriction maps, are given as composites

$$X^{C_{mn}} = (X^{C_n})^{C_m} \to (X^{gC_n})^{C_m} \xrightarrow{r_n} X^{C_m}$$

where the first map is the map from fixed points to geometric fixed points described earlier in the section. The topological cyclic homology of X is then defined as

$$\mathrm{TC}(X) = \mathrm{holim}_{R,F} X^{C_n}.$$

As discussed in Example 3.4, for a ring A, the S^1-spectrum $\mathrm{THH}(A)$ is cyclotomic. Hence we can take the topological cyclic homology of this spectrum. The resulting spectrum $\mathrm{TC}(\mathrm{THH}(A))$ is usually denoted $\mathrm{TC}(A)$, and we will use this convention as well. This is the topological cyclic homology of the ring A. For our application to algebraic K-theory, three sources of cyclotomic spectra will be particularly important:

(1) Topological Hochschild homology of rings
(2) Homotopy fibers of cyclotomic maps
(3) Suspension spectra of cyclotomic spaces

The first source of cyclotomic spectra, from THH, has been used extensively throughout algebraic K-theory computations. The consideration of homotopy fibers of cyclotomic maps as cyclotomic spectra is used implicitly in work of Blumberg and Mandell [2]. The third approach can be thought of as building cyclotomic spectra by hand. Below, we describe in detail the latter two approaches to generating cyclotomic spectra.

3.1. **Cyclotomic spectra as homotopy fibers.** For a cyclotomic map $f : X \to Y$, it follows immediately that the induced maps $X^{C_n} \to Y^{C_n}$ commute with R and F. Hence we get an induced map $\mathrm{TC}(X) \to \mathrm{TC}(Y)$.

Proposition 3.5. *Suppose $f : X \to Y$ is a cyclotomic map. Then*

(1) *The homotopy fiber* $\mathrm{hofib}(f)$ *is cyclotomic.*
(2) $\mathrm{TC}(\mathrm{hofib}(f)) \simeq \mathrm{hofib}(\mathrm{TC}(X) \to \mathrm{TC}(Y))$

Proof. The homotopy fiber is taken in the category of S^1-spectra, so it is an S^1-spectrum. Taking fixed points or geometric fixed points preserves (co)fiber sequences, so we get a diagram

$$\begin{array}{ccccc} \rho_n^*(\mathrm{hofib}(f)^{gC_n}) & \longrightarrow & \rho_n^*(X^{gC_n}) & \longrightarrow & \rho_n^*(Y^{gC_n}) \\ \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ \mathrm{hofib}(f) & \longrightarrow & X & \longrightarrow & Y \end{array}$$

where the rows are (co)fiber sequences. It follows that we have an equivalence $\rho_n^*(\mathrm{hofib}(f)^{gC_n}) \to \mathrm{hofib}(f)$ of S^1-spectra. For the second claim, note that both TC and hofib are homotopy limits, so they commute. Hence

$$\begin{aligned} \mathrm{TC}(\mathrm{hofib}(f)) &= \mathrm{holim}_{R,F}\mathrm{hofib}(X \to Y)^{C_n} \simeq \mathrm{holim}_{R,F}\mathrm{hofib}(X^{C_n} \to Y^{C_n}) \\ &\simeq \mathrm{hofib}(\mathrm{holim}_{R,F} X^{C_n} \to \mathrm{holim}_{R,F} Y^{C_n}) = \mathrm{hofib}(\mathrm{TC}(X) \to \mathrm{TC}(Y)). \end{aligned}$$

□

Example 3.6. *Suppose we have a map $f : A \to B$ of rings. Then the induced map* $\mathrm{THH}(A) \to \mathrm{THH}(B)$ *is cyclotomic. The homotopy fiber is sometimes denoted* $\mathrm{THH}(f)$. *If our map is the quotient $A \to A/I$ the homotopy fiber is usually denoted* $\mathrm{THH}(A, I)$. *It follows that the two possible definitions of* $\mathrm{TC}(f)$, *as* $\mathrm{TC}(\mathrm{THH}(f))$ *or as* $\mathrm{hofib}(\mathrm{TC}(A) \to \mathrm{TC}(B))$, *agree.*

3.2. **Cyclotomic spectra as suspension spectra of cyclotomic spaces.** The category of cyclotomic spectra is tensored over a corresponding category of spaces.

Definition 3.7. *A cyclotomic space A is an S^1-equivariant space together with compatible equivalences*

$$\rho_n^*(A^{C_n}) \to A$$

of S^1-spaces for all n.

A map of cyclotomic spaces is defined in the analogous way to that of a cyclotomic spectrum. The S^1-equivariant suspension spectrum of a cyclotomic space is a cyclotomic spectrum. Indeed, $\Sigma^\infty_{S^1}(-_+)$ is a functor from unbased cyclotomic spaces to cyclotomic spectra, and $\Sigma^\infty_{S^1}(-)$ is a functor from based cyclotomic spaces to cyclotomic spectra. Below we give several important examples of cyclotomic spaces. A number of these will naturally arise in our algebraic K-theory computations.

Example 3.8. *A free loop space LX is a cyclotomic space, as described in Example 3.3. Suppose we have a map $X \to Y$ of spaces. Then the induced map $LX \to LY$ is a map of cyclotomic spaces, and by taking suspension spectra we get a map* $\Sigma^\infty_{S^1} LX_+ \to \Sigma^\infty_{S^1} LY_+$ *of cyclotomic spectra.*

Example 3.9. *If Π is a pointed monoid, the cyclic bar construction $B^{cy}(\Pi)$ is a based cyclotomic space. This is particularly relevant to our computation because of the splitting:*

$$\mathrm{THH}(k(\Pi)) \simeq \mathrm{THH}(k) \wedge B^{cy}(\Pi).$$

If we can describe how $B^{cy}(\Pi)$ is built out of cyclotomic spaces then we get a corresponding description of $\mathrm{THH}(k(\Pi))$, *and hence* $\mathrm{TC}(k(\Pi))$.

Example 3.10. *Suppose $\{A(s)\}_{s\geq 1}$ is a family of S^1-equivariant based spaces with compatible homeomorphisms*

$$\rho_n^*(A(s)^{C_n}) \approx \begin{cases} A(s/n) & \text{if } n|s \\ * & \text{if } n \nmid s \end{cases}$$

Then

$$A = \bigvee_{s\geq 1} A(s)$$

is a based cyclotomic space.

Example 3.11. *As a concrete example of a family as described in Example 3.10, suppose we have real S^1-representations $\lambda(s)$ for $s \geq 1$ with the property that $\rho_n^*(\lambda(s)^{C_n}) \cong \lambda(s/n)$ whenever $n \mid s$. Then $\{S^{\lambda(s)}\}_{s\geq 1}$ is such a family. Further, if we let $A(s) = S^1/C_{s+} \wedge S^{\lambda(s)}$, then $\{A(s)\}_{s\geq 1}$ is also such a family. Therefore*

$$A = \bigvee_{s\geq 1} S^1/C_{s+} \wedge S^{\lambda(s)}$$

is a based cyclotomic space. This example will arise in our calculations.

4. Algebraic K-theory Calculations

For a ring A, there is a map relating the algebraic K-theory of A and the topological cyclic homology of A:

$$trc : K(A) \to \mathrm{TC}(A).$$

This map, called the cyclotomic trace, is due to Bökstedt, Hsiang, and Madsen [3]. This map is often close to an equivalence [12, 10, 7]. Indeed, in the case of $A = k[x_1, \ldots, x_n]/(x_1^{a_n}, \ldots, x_n^{a_n})$, the cyclotomic trace map on the multi-relative K-theory group $\widetilde{K}(A)$:

$$trc : \widetilde{K}(A) \to \widetilde{\mathrm{TC}}(A)$$

is an equivalence. It follows from [5] that this map is an equivalence after p-completion. The proof that this can be extended to an integral statement can be found in the preprint [6].

Thus, we aim to compute the multi-relative topological cyclic homology. As noted above, $\mathrm{TC}(A) = \mathrm{TC}(\mathrm{THH}(A))$, so the first step is understanding the topological Hochschild homology of A. For any pointed monoid algebra $k(\Pi)$ there is splitting [10]:

$$\mathrm{THH}(k(\Pi)) \simeq \mathrm{THH}(k) \wedge B^{cy}(\Pi).$$

This splitting will be essential to our calculations. Note that in the single variable case, $k[x]/(x^a)$ is a pointed monoid algebra $k(\Pi_a)$ where Π_a is the monoid $\Pi_a = \{0, 1, x, \ldots, x^{a-1}\}, x^a = 0$. Thus the topological Hochschild homology of $k[x]/(x^a)$ splits:

$$\mathrm{THH}(k[x]/(x^a)) \simeq \mathrm{THH}(k) \wedge B^{cy}(\Pi_a).$$

To proceed with the K-theory computation, one needs to compute the fixed points of topological Hochschild homology, $\mathrm{THH}(k[x]/(x^a))^{C_n}$. In order to do this, one needs to understand the S^1-equivariant homotopy type of the cyclic bar construction $B^{cy}(\Pi_a)$. In general, computing the S^1-equivariant homotopy type of $B^{cy}(\Pi)$ for a pointed monoid Π is very difficult, and indeed it has only been done in a small number of cases [9, 8]. In the case of the pointed monoid Π_a, Hesselholt and Madsen specified the equivariant homotopy type [9]. Let $\widetilde{B}^{cy}(\Pi_a)$ denote the piece of the cyclic bar construction where the degree of x is required to be positive. This yields the relative topological Hochschild homology (and hence the relative algebraic K-theory). Hesselholt and Madsen wrote $\widetilde{B}^{cy}(\Pi_a)$ as the cofiber of a map of S^1-spaces:

$$\bigvee_{s\geq 1} S^1/C_{s+} \wedge S^{\lambda_{s-1}} \to \bigvee_{s\geq 1} S^1/C_{s+} \wedge S^{\lambda_{\lfloor \frac{s-1}{a} \rfloor}}. \tag{4.1}$$

Here λ_s denotes the S^1-representation $\mathbb{C}(1) \oplus \ldots \oplus \mathbb{C}(s)$, where $\mathbb{C}(i)$ is $\mathbb{C}$ with S^1 acting by $z \cdot w = z^i w$. The wedge summand indexed by s maps by a multiplication by a map to the wedge summand indexed by as.

Remark 4.2. *Following Example 3.11, both of the spaces in Equation 4.1 are cyclotomic spaces.*

Lemma 4.3. *The map in Equation 4.1 is a map of cyclotomic spaces.*

Proof. If $r|s$, we have the following commutative diagram:

$$\begin{array}{ccc} \rho_r^*\big(\big(S^1/C_{s+}\wedge S^{\lambda_{s-1}}\big)^{C_r}\big) & \xrightarrow{a} & \rho_r^*\big(\big(S^1/C_{as+}\wedge S^{\lambda_{s-1}}\big)^{C_r}\big) \\ \downarrow\approx & & \downarrow\approx \\ S^1/C_{s/r+}\wedge S^{\lambda_{s/r-1}} & \xrightarrow{a} & S^1/C_{as/r+}\wedge S^{\lambda_{s/r-1}} \end{array}$$

It follows that the map in Equation 4.1 is a map of cyclotomic spaces. □

To proceed with the calculation, we establish the following notation: Let

$$X_\varnothing = \bigvee_{s\geq 1} S^1/C_{s+}\wedge S^{\lambda_{s-1}}$$

and let

$$X_{\{1\}} = \bigvee_{s\geq 1} S^1/C_{s+}\wedge S^{\lambda_{\lfloor \frac{s-1}{a}\rfloor}}.$$

Then the relative cyclic bar construction fits into a cofiber sequence.

$$X_\varnothing \longrightarrow X_{\{1\}} \longrightarrow \widetilde{B}^{cy}(\Pi_a)$$

To recover the relative topological Hochschild homology $\mathrm{THH}(k[x]/(x^a),(x))$, we smash the cofiber sequence with $T = \mathrm{THH}(k)$:

$$T\wedge X_\varnothing \longrightarrow T\wedge X_{\{1\}} \longrightarrow \mathrm{THH}(k[x]/(x^a),(x))$$

To recover the relative topological cyclic homology group, we take TC of the cyclotomic spectrum $\mathrm{THH}(k[x]/(x^a),(x))$:

$$\mathrm{TC}(k[x]/(x^a),(x)) = \mathrm{TC}(\mathrm{hocofib}(T\wedge X_\varnothing \to T\wedge X_{\{1\}})).$$

By Proposition 3.5, if $T\wedge X_\varnothing \to T\wedge X_{\{1\}}$ is a cyclotomic map of cyclotomic spectra, then

$$\mathrm{TC}(\mathrm{hocofib}(T\wedge X_\varnothing \to T\wedge X_{\{1\}})) \simeq \mathrm{hocofib}(\mathrm{TC}(T\wedge X_\varnothing)\to \mathrm{TC}(T\wedge X_{\{1\}})).$$

Indeed, from Remark 4.2 each of $X_\varnothing$ and $X_{\{1\}}$ are cyclotomic. The spectrum $T = \mathrm{THH}(k)$ is also cyclotomic, and hence by Definition 3.1 each of $X_\varnothing\wedge T$ and $X_{\{1\}}\wedge T$ are cyclotomic. Further, it follows from Lemma 4.3 that the map between them is also cyclotomic. Hence

$$\mathrm{TC}(k[x]/(x^a),(x)) = \mathrm{hocofib}(\mathrm{TC}(T\wedge X_\varnothing)\to \mathrm{TC}(T\wedge X_{\{1\}})).$$

We can now generalize this approach to studying the multi-relative topological cyclic homology of truncated polynomials in multiple variables and get a formula for

$$\widetilde{\mathrm{TC}}(A) = \mathrm{TC}(k[x_1,\dots,x_n]/(x_1^{a_n},\dots,x_n^{a_n}),(x_1),\dots,(x_n)).$$

Note that $A = k[x_1,\dots,x_n]/(x_1^{a_n},\dots,x_n^{a_n})$ is a pointed monoid algebra $k(\Pi_{a_1}\wedge \dots\wedge\Pi_{a_n})$ where Π_i is defined as above. Thus we can write

$$\mathrm{THH}(A)\simeq \mathrm{THH}(k)\wedge B^{cy}(\Pi_{a_1}\wedge\dots\wedge\Pi_{a_n}),$$

and the multi-relative THH-spectrum, $\widetilde{\mathrm{THH}}(A)$ splits as

$$\widetilde{\mathrm{THH}}(A)\simeq \mathrm{THH}(k)\wedge \widetilde{B}^{cy}(\Pi_{a_1}\wedge\dots\wedge\Pi_{a_n}).$$

Here $\widetilde{B}^{cy}(\Pi_{a_1}\wedge\dots\wedge\Pi_{a_n})$ denotes the piece of the cycle bar construction where the degree of each x_i is required to be positive. In the one variable case $\widetilde{B}^{cy}(\Pi_a)$

was written as the homotopy cofiber of a map of S^1-spaces. In the n-variable case, we write $\widetilde{B}^{cy}(\Pi_{a_1} \wedge \ldots \wedge \Pi_{a_n})$ as the iterated homotopy cofiber of an n-cube of S^1-spaces. This n cube is defined as follows.

Given $I \subset \{1, \ldots, n\}$, let

$$\lambda_I(s_1, \ldots, s_n) = \bigoplus_{i=1}^{n} \begin{cases} \lambda_{s_i - 1} & \text{if } i \notin I \\ \lambda_{\lfloor \frac{s_i - 1}{a_i} \rfloor} & \text{if } i \in I \end{cases}$$

and let

$$X_I = \bigvee_{s_1, \ldots, s_n \geq 1} S^1/C_{s_1 +} \wedge \ldots \wedge S^1/C_{s_n +} \wedge S^{\lambda_I(s_1, \ldots, s_n)}.$$

The maps in this n-cube are constructed from identity maps and the maps of Hesselholt and Madsen in the 1-variable case. Then $\widetilde{B}^{cy}(\Pi_{a_1} \wedge \ldots \wedge \Pi_{a_n})$ is the iterated homotopy cofiber of the cube $\mathcal{X} = \{X_I\}_{I \subset \{1, \ldots, n\}}$. For example, in the two variable case, $\widetilde{B}^{cy}(\Pi_{a_1} \wedge \Pi_{a_2})$ is the iterated homotopy cofiber of the square:

$$\begin{array}{ccc} X_{\varnothing} & \longrightarrow & X_{\{1\}} \\ \downarrow & & \downarrow \\ X_{\{2\}} & \longrightarrow & X_{\{1,2\}} \end{array}$$

Smashing the n-cube with $T = \mathrm{THH}(k)$ we find that $\widetilde{\mathrm{THH}}(A)$ is the iterated homotopy cofiber of the cube $T \wedge \mathcal{X} = \{T \wedge X_I\}_{I \subset \{1, \ldots, n\}}$. We then have the following result.

Theorem 4.4. *The multi-relative topological cyclic homology of A, $\widetilde{\mathrm{TC}}(A)$, is given as the iterated homotopy cofiber of the cube* $\mathrm{TC}(T \wedge \mathcal{X}) = \{\mathrm{TC}(T \wedge X_I)\}_{I \subset \{1, \ldots, n\}}$.

Proof. The spectrum $\widetilde{\mathrm{THH}}(A)$ is cyclotomic, and hence we can take its topological cyclic homology, $\widetilde{\mathrm{TC}}(A) = \mathrm{TC}(\widetilde{\mathrm{THH}}(A))$. From above, the multi-relative topological Hochschild homology of A is an iterated homotopy cofiber:

$$\widetilde{\mathrm{THH}}(A) = \mathrm{hocofib}(\{T \wedge X_I\}_{I \subset \{1, \ldots, n\}}).$$

Hence

$$\widetilde{\mathrm{TC}}(A) = \mathrm{TC}(\mathrm{hocofib}(\{T \wedge X_I\}_{I \subset \{1, \ldots, n\}})).$$

It follows from Example 3.11 and Lemma 4.3 that the spectra $T \wedge X_I$ and maps in the cube $T \wedge \mathcal{X}$ are cyclotomic. Thus by Proposition 3.5 the multi-relative topological cyclic homology is given as an iterated homotopy cofiber

$$\widetilde{\mathrm{TC}}(A) = \mathrm{hocofib}(\{\mathrm{TC}(T \wedge X_I)\}_{I \subset \{1, \ldots, n\}}).$$

□

In order to proceed with the computation, we need to understand $\mathrm{TC}(T \wedge X_I)$ for each $I \subset \{1, \ldots n\}$, as well as the maps between them. Topological cyclic homology can be computed in two stages. First, we take the homotopy limit of $\mathrm{THH}(T \wedge X_I)^{C_n}$ over the Frobenius maps to get the spectrum $\mathrm{TF}(T \wedge X_I)$. Then $\mathrm{TC}(T \wedge X_I)$ is the homotopy equalizer of the identity and the restriction maps on $\mathrm{TF}(T \wedge X_I)$.

First we need a better understanding of X_I as an S^1-space.

Lemma 4.5. *There is an S^1-equivariant homeomorphism*

$$S^1/C_{s_1} \times \ldots \times S^1/C_{s_n} \to S^1/C_{\gcd(\overrightarrow{s})} \times \mathbb{T}^{n-1},$$

where $\mathbb{T}^{n-1}$ denotes an $(n-1)$-torus with trivial S^1-action.

Proof. It suffices by induction to consider the case of $n = 2$. By considering the natural pull-back used above, we may assume that $d = \gcd(s_1, s_2) = 1$ (as otherwise, we use the d^{th} root map to identify S^1 and S^1/C_d).

The homeomorphism in this case will be

$$S^1 \times S^1/C_{s_1 s_2} \to S^1/C_{s_1} \times S^1/C_{s_2},$$

where the domain has the left action of S^1 on only the left factor and the range has the diagonal action. Choose natural numbers n and m such that $ms_2 - ns_1 = 1$. Then if we denote the coset $a \cdot C_j \in S^1/C_j$ by a, our map is given by

$$(a, x) \mapsto (ax^{ms_2}, ax^{ns_1}),$$

the inverse to which is

$$(a, b) \mapsto (b^{ms_2} a^{-ns_1}, ab^{-1}),$$

where in both cases, we use the natural group structures. □

This implies that we can write X_I as

$$X_I = \bigvee_{\overrightarrow{s}} \mathbb{T}^{n-1}_+ \wedge S^1/C_{\gcd(\overrightarrow{s})+} \wedge S^{\lambda_I(\overrightarrow{s})}.$$

Now we define

$$\widehat{X}_I = \bigvee_{\overrightarrow{s}} S^1/C_{\gcd(\overrightarrow{s})_+} \wedge S^{\lambda_I(\overrightarrow{s})},$$

i.e., X_I without the $(n-1)$-torus. Now we can prove Theorem 1.1.

Proof of Theorem 1.1. It follows from Example 3.11 and Lemma 4.3 that $T \wedge \widehat{X}_I$ is also cyclotomic and that

$$\mathrm{TC}(T \wedge X_I) \simeq \mathrm{TC}(T \wedge \widehat{X}_I) \wedge \mathbb{T}^{n-1}_+.$$

Let $\widehat{\mathrm{TC}}(A)$ denote the iterated homotopy cofiber of the cube $\{\mathrm{TC}(T \wedge \widehat{X}_I)\}_{I \subset \{1,\ldots,n\}}$. The result then follows. □

To understand $\mathrm{TC}(T \wedge \widehat{X}_I)$ we use the following result:

Lemma 4.6. *Up to profinite completion we have*

$$\mathrm{TF}(T \wedge \widehat{X}_I) = \bigvee_{\overrightarrow{s}} \Sigma[T \wedge S^{\lambda_I(\overrightarrow{s})}]^{C_{\gcd(\overrightarrow{s})}}.$$

Proof. It is clear that $\mathrm{TF}(T \wedge \widehat{X}_I)$ splits as a wedge over $\overrightarrow{s}$, and the result is essentially [10, Lemma 8.2]. □

Now the restriction map R^d maps the wedge summand $d\overrightarrow{s}$ to the wedge summand $\overrightarrow{s}$, so it follows that

$$\mathrm{TC}(T \wedge \widehat{X}_I) = \bigvee_{\gcd(\overrightarrow{s})=1} \mathrm{holim}_R \Sigma[T \wedge S^{\lambda_I(d\overrightarrow{s})}]^{C_d},$$

again up to profinite completion.

We pause here and observe that the indexing set for our wedge is the same as the indexing set that identifies $\mathbb{N}^n$ as a disjoint union of copies of $\mathbb{N}$. This plays an essential role in our computation.

Now let $\mathrm{TC}(T \wedge \widehat{X}_I; \overrightarrow{s})$ denote the summand corresponding to $\overrightarrow{s}$ in the afore-described wedge sum.

5. Calculations for perfect fields

For any ring R, there is a close connection between $\mathrm{THH}(R)$ and Witt vectors. In fact we have

$$\pi_0 \mathrm{THH}(R)^{C_m} \cong \mathbb{W}_{\langle m \rangle}(R),$$

the Witt vectors defined by the truncation set $\langle m \rangle$ [10, Addendum 3.3].

Now let k be a perfect field of characteristic $p > 0$. Recall that in this case

$$\mathrm{THH}_*(k) \cong k[\mu_0]$$

with $|\mu_0| = 2$, and that

$$\pi_*(\mathrm{THH}(k)^{C_{p^{m-1}}}) = \mathrm{TR}^m_*(k;p) \cong \mathbb{W}_m(k;p)[\mu_0].$$

It follows that

$$\pi_*(\mathrm{THH}(k)^{C_m}) \cong \mathbb{W}_{\langle m \rangle}(k)[\mu_0]$$

for all m.

From this we can recover the big Witt vectors by taking the limit

$$\lim_{R, m \leq n} \pi_*(\mathrm{THH}(k)^{C_m}) \cong \mathbb{W}_n(k)[\mu_0].$$

Now we prove the following:

Theorem 5.1. *Let k be a perfect field of characteristic $p > 0$. Then $\mathrm{TF}_*(T \wedge \widehat{X}_I; \overrightarrow{s})$ is concentrated in odd degrees and*

$$\mathrm{TF}_{2q-1}(T \wedge \widehat{X}_I; \overrightarrow{s}) \cong \mathbb{W}_{\{\overrightarrow{t} \mid \overrightarrow{s}, \overrightarrow{t} \in S_q(I)\}}(k).$$

Proof. By Lemma 4.6, up to profinite completion we have

$$\mathrm{TF}(T \wedge \widehat{X}_I) = \bigvee_{\overrightarrow{s}} \Sigma [T \wedge S^{\lambda_I(\overrightarrow{s})}]^{C_{\gcd(\overrightarrow{s})}}.$$

So we would like to understand

$$\pi_{2q-2}((T \wedge S^{\lambda_I(\overrightarrow{s})})^{C_{\gcd(\overrightarrow{s})}}).$$

We consider the more general situation where λ is any complex S^1-representation and we will compute

$$\pi_n((T \wedge S^{\lambda})^{C_{p^r d}}),$$

where $(p, d) = 1$. If k is a $\mathbb{Z}_{(p)}$-algebra, by [9, Proof of Proposition 4.2.5] there is a splitting

$$(T \wedge S^{\lambda})^{C_{p^r d}} \xrightarrow{\simeq} \prod_{e \mid d} (T(k) \wedge S^{\lambda^{C_{d/e}}})^{C_{p^r}},$$

where the map on the e^{th} factor is $R_{d/e} \circ F_e$. By [10, Proposition 9.1], the homotopy groups are concentrated in even degrees and

$$\pi_{2q}((T(k) \wedge S^{\lambda^{C_{d/e}}})^{C_{p^r}}) \cong \begin{cases} \mathbb{W}_s(k;p) & \dim_{\mathbb{C}}(\lambda^{C_{(d/e)(p^{r-s+1})}}) \leq q < \dim_{\mathbb{C}}(\lambda^{C_{(d/e)(p^{r-s})}}) \\ \mathbb{W}_{r+1}(k;p) & q \geq \dim_{\mathbb{C}}(\lambda^{C_{d/e}}) \end{cases}$$

Recall that if k is a $\mathbb{Z}_{(p)}$-algebra, then for any truncation set S there is a splitting

$$\mathbb{W}_S(k) \cong \prod_{p \nmid e} \mathbb{W}_{p^{\mathbb{N}_0} \cap S/e}(k).$$

On the e^{th} factor this is given by $R_{d/e} \circ F_e$. Hence, $\mathbb{W}_S(k)$ splits as a product of p-typical Witt vectors. Let S be the truncation set of all divisors m of $p^r d$ such that $\dim_{\mathbb{C}}(\lambda^{C_{p^r d/m}}) \leq q$. Then

$$p^{\mathbb{N}_0} \cap S/e = \{1, p, p^2, \ldots, p^{s-1}\}$$

where s is the unique integer such that $\dim_{\mathbb{C}}(\lambda^{C_{(d/e)(p^{r-s+1})}}) \leq q < \dim_{\mathbb{C}}(\lambda^{C_{(d/e)(p^{r-s})}})$, and $s = r+1$ if $q \geq \dim_{\mathbb{C}}(\lambda^{C_{d/e}})$. It follows that

$$\pi_{2q}((T \wedge S^{\lambda})^{C_{p^r d}}) \cong \mathbb{W}_S(k),$$

for S as defined above.

We now specialize to our case of interest. We would like to compute

$$\mathrm{TF}_{2q-1}(T \wedge \widehat{X}_I; \overrightarrow{s}) = \pi_{2q-2}([T \wedge S^{\lambda_I(\overrightarrow{s})}]^{C_{\gcd(\overrightarrow{s})}})$$

From above,

$$\mathrm{TF}_{2q-1}(T \wedge \widehat{X}_I; \overrightarrow{s}) \cong \mathbb{W}_S(k)$$

where S is the truncation set

$$S = \{m | \gcd(\overrightarrow{s}) \ \text{ and } \ \dim_{\mathbb{C}}(\lambda_I(\overrightarrow{s})^{C_{\gcd(\overrightarrow{s})/m}}) \leq q-1\}$$

Associating to each $m | \gcd(\overrightarrow{s})$ a $\overrightarrow{t} | \overrightarrow{s}$ via

$$\overrightarrow{t} = \frac{m\overrightarrow{s}}{\gcd(\overrightarrow{s})}$$

we see that this truncation set S is isomorphic to the truncation set

$$\{\overrightarrow{t} | \overrightarrow{s}, \overrightarrow{t} \in S_q(I)\}$$

where $S_q(I)$ is as defined in the Introduction. The result follows. □

This means, in particular, that $\mathrm{TF}_{2q-1}(T \wedge \widehat{X}_I; \overrightarrow{s})$ is the colimit

$$\mathrm{TF}_{2q-1}(T \wedge \widehat{X}_I; \overrightarrow{s}) = \mathrm{colim}_{d | \gcd(\overrightarrow{s})} \mathbb{W}_{\langle d \rangle}(k).$$

When we take the equalizer for R and Id to recover topological cyclic homology we then get

Theorem 5.2. *Let k be a perfect field of characteristic $p > 0$. Then $\mathrm{TC}_*(T \wedge \widehat{X}_I)$ is concentrated in odd degrees and*

$$\mathrm{TC}_{2q-1}(T \wedge \widehat{X}_I) \cong \mathbb{W}_{S_q(I)}(k).$$

Proof. From Theorem 5.1 it follows that if we take the limit over R of $\mathrm{TF}_{2q-1}(T \wedge \widehat{X}_I; \overrightarrow{s})$ over all $\overrightarrow{s} \in S_q(I)$ we get $\mathbb{W}_{S_q(I)}(k)$. There is an isomorphism between the limit over R of this system and the equalizer of R and Id over all $\overrightarrow{s}$, so the result follows. □

We have now computed the homotopy groups at each vertex of our cube. The maps in the n-cube are constructed out of the same maps that yielded the ordinary Verscheibung in the 1-variable case of Hesselholt and Madsen [9]. By the same argument, and the definition of the n-dimensional Verschiebung maps, it follows that the maps in our n-cube are indeed $V^i_{a_i}$, for $i = 1, \dots n$. This completes the proof of Theorem 1.2. In the case where p does not divide any of the truncations a_i, it follows from the computations of the Verschiebung in §2 above that the maps $V^i_{a_i}$ are injective. Theorem 1.3 follows.

6. Calculations for $\mathbb{Z}$

In the single variable case, the algebraic K-theory groups $K_q(\mathbb{Z}[x]/x^n, (x))$ have been studied in work of Angeltveit, Gerhardt, and Hesselholt [1]. They compute these groups completely for q odd and up to extensions for q even. Here we consider the n-variable case.

In this section we prove Theorem 1.5, stated in the Introduction. To prove the theorem we compute the Poincare series of $\mathrm{TC}(T \wedge X_I)$ at each vertex of our n-cube $\mathrm{TC}(T \wedge \mathcal{X})$.

Proposition 6.1. *For $k = \mathbb{Z}$ the Poincare series of $\mathrm{TC}(T \wedge \widehat{X}_I)$ is given by*

$$\sum_{j \geq 1} t^{2j-1} \binom{n+j-2}{n-1} \prod_{i \in I} a_i.$$

Proof. We use the formula from Section 4,

$$\mathrm{TF}(T \wedge \widehat{X}_I) = \bigvee_{\overrightarrow{s}} \Sigma (T \wedge S^{\lambda_I(\overrightarrow{s})})^{C_{\gcd(\overrightarrow{s})}}$$

and the corresponding formulae for $\mathrm{TC}(T \wedge X_I)$. Using the computations of $RO(S^1)$-graded TR-groups of $\mathbb{Z}$ found in [1] we conclude that

$$\Sigma(\mathrm{THH}(\mathbb{Z}) \wedge S^{\lambda_I(\overrightarrow{s})})^{C_{\gcd(s_1, \dots, s_n)}},$$

contributes a $\mathbb{Z}$ in degree $2 \dim_{\mathbb{C}} (S^{\lambda_I(\overrightarrow{s})})^{C_d} + 1$ for each $d \mid \gcd(\overrightarrow{s})$. Therefore, to compute the rank in degree $2j-1$ we need to count the number of vectors $\overrightarrow{s}$ such that $2 \dim_{\mathbb{C}} (S^{\lambda_I(\overrightarrow{s})}) + 1 = 2j - 1$. Hence we want to count the number of n-tuples such that $\dim_{\mathbb{C}}(S^{\lambda_I(\overrightarrow{s})}) = j - 1$. There are $\binom{n+j-2}{n-1}$ n-tuples $(b_1, \dots, b_n)$ with $b_1 + \ldots + b_n = j - 1$, and there are

$$\begin{cases} 1 & \text{if } i \notin I \\ a_i & \text{if } i \in I \end{cases}$$

s_i's with

$$\dim_{\mathbb{C}} \left(\begin{cases} \lambda_{s_i - 1} & \text{if } i \notin I \\ \lambda_{\lfloor \frac{s_i - 1}{a_i} \rfloor} & \text{if } i \in I \end{cases} \right) = b_i.$$

The result follows. □

Theorem 1.5 follows directly from Proposition 6.1 and the computation of Verschiebung maps in [1].

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