

TOPOLOGICAL COMPLEXITY OF MOTION PLANNING IN PROJECTIVE PRODUCT SPACES

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ABSTRACT. We study Farber’s topological complexity (TC) of Davis’ projective product spaces (PPS’s). We show that, in many non-trivial instances, the TC of PPS’s coming from at least two sphere factors is (much) lower than the dimension of the manifold. This is in high contrast with the known situation for (usual) real projective spaces for which, in fact, the Euclidean immersion dimension and TC are two facets of the same problem. Low TC-values have been observed for infinite families of non-simply connected spaces only for H-spaces, for finite complexes whose fundamental group has cohomological dimension not exceeding 2, and now in this work for infinite families of PPS’s. We discuss general bounds for the TC (and the Lusternik-Schnirelmann category) of PPS’s, and compute these invariants for specific families of such manifolds. Some of our methods involve the use of an equivariant version of TC. We also give a characterization of the Euclidean immersion dimension of PPS’s through generalized concepts of axial maps and, alternatively, non-singular maps. This gives an explicit explanation of the known relationship between the generalized vector field problem and the Euclidean immersion problem for PPS’s.

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1. INTRODUCTION AND NOTATION

As shown in [9], the topological complexity (TC) and the Euclidean immersion dimension (Imm) of the n -dimensional real projective space P^n are related by

$$\text{TC}(P^n) = \text{Imm}(P^n) - \epsilon(n) = 2n - \delta(n) \quad (1)$$

where

$$\epsilon(n) = \begin{cases} 1, & n = 1, 3, 7; \\ 0, & \text{otherwise,} \end{cases}$$

$\delta(n) = O(\alpha(n))$, and $\alpha(n)$ denotes the number of ones in the binary expansion of n . It is natural to ask whether the nice phenomenon in the first equality in (1) is part of a general property of manifolds. Not only does this question have a negative answer, but even close relatives of real projective spaces fail to satisfy the first equality in (1). For instance, in view of [3] and [13], the failure holds for lens spaces whose fundamental group has torsion of the form 2^e for $e > 1$. The same answer is observed in a forthcoming paper by two of the authors in which they study flag manifolds whose fundamental group is an elementary 2-group of rank greater than 1. This paper now shows that the list of counterexamples extends to Davis' projective product spaces, a family of manifolds giving a rather natural generalization of real projective spaces, and which, in particular, have \mathbb{Z}_2 as their fundamental groups (in the 'generic' case). Indeed, Theorem 3.8 in this paper shows that, in contrast to the second equality in (1), the topological complexity of a projective product space coming from at least two sphere factors can be much lower than the dimension of the manifold. Thus, in those cases, more than half the homotopy obstructions in the motion planning problem for $P_{\bar{n}}$ are trivial (cf. Remark 3.6 and the considerations after Theorem 3.8). Up to the authors knowledge, this gives the first infinite family of non-simply connected closed manifolds which are not H-spaces and whose TC is lower than their dimension (cf. [10, 22]; the upper bound in [6, Theorem 3] should be noted, too).

In the rest of this introductory section we set up notation and recall needed preliminary results. We use the reduced form of the Schwarz genus (also called sectional category, and denoted by secat) of a fibration, i.e. a trivial fibration has zero genus. In particular, we consider the reduced form of the Lusternik-Schnirelmann category (cat) and that of Farber's topological complexity (TC) of a space X —the latter being the reduced Schwarz genus of the double evaluation map $X^{[0,1]} \rightarrow X \times X$ sending a path $\gamma: [0, 1] \rightarrow X$ to the pair $(\gamma(0), \gamma(1))$. Thus, $\text{cat}(X) = \text{TC}(X) = 0$ for a contractible space X . We will also assume the reader is familiar with [7], and we next briefly recall the required results from that paper.

We let \bar{n} stand for an r -tuple (n_1, \dots, n_r) of positive integers with $n_1 \leq \dots \leq n_r$. We consider the diagonal action of \mathbb{Z}_2 on $S_{\bar{n}} := S^{n_1} \times \dots \times S^{n_r}$, and let $P_{\bar{n}}$ denote the resulting orbit space (so $P_{(n_1)}$ is the usual real projective

space P^{n_1}). We set $|\bar{n}| := \dim(P_{\bar{n}}) = \dim(S_{\bar{n}}) = \sum n_i$ and $\ell(\bar{n}) = r$. The real line bundle associated to the obvious covering $S_{\bar{n}} \rightarrow P_{\bar{n}}$, denoted by $\xi_{\bar{n}}$ and called the canonical line bundle over $P_{\bar{n}}$, can be used to identify the stable class of the tangent bundle $\tau_{P_{\bar{n}}}$ since

$$\tau_{P_{\bar{n}}} \oplus r\varepsilon \approx (|\bar{n}| + r)\xi_{\bar{n}}. \quad (2)$$

Here ε stands for a trivial line bundle.

The total space of the k -fold iterated Whitney sum of $\xi_{\bar{n}}$ is given by the Borel construction $k\xi_{\bar{n}} = S_{\bar{n}} \times_{\mathbb{Z}_2} \mathbb{R}^k$. In particular, the projectivization of $k\xi_{\bar{n}}$ is given by

$$P(k\xi_{\bar{n}}) = P_{\bar{n}} \times P^{k-1}. \quad (3)$$

The diagonal inclusion $S^{n_1} \hookrightarrow S_{\bar{n}}$ and the projection onto the first factor $S_{\bar{n}} \rightarrow S^{n_1}$ induce corresponding maps $j: P^{n_1} \hookrightarrow P_{\bar{n}}$ and $p: P_{\bar{n}} \rightarrow P^{n_1}$ satisfying

$$j^*(\xi_{\bar{n}}) \approx \xi_{n_1}, \quad p^*(\xi_{n_1}) \approx \xi_{\bar{n}}, \quad \text{and} \quad p \circ j = \text{Id}. \quad (4)$$

For $2 \leq i \leq r$ there are mod 2 cohomology classes x_i in $P_{\bar{n}}$ with $\dim(x_i) = n_i$ such that the mod 2 cohomology ring of $P_{\bar{n}}$ is given by

$$H^*(P_{\bar{n}}; \mathbb{Z}_2) = H^*(P^{n_1}; \mathbb{Z}_2) \otimes \Lambda[x_2, \dots, x_r] \quad (5)$$

(where Λ denotes an exterior algebra) with the only exception that, if n_1 is even, then $x_i^2 = x^{n_1}x_i$ whenever $n_i = n_1$. Here $x \in H^1(P^{n_1}; \mathbb{Z}_2)$ satisfies

$$x = w_1(\xi_{\bar{n}}), \quad \text{but all classes } x_i \text{ restrict trivially under the inclusion } j. \quad (6)$$

We also need the concept of ‘‘generalized axial map’’ as defined in [3]: For a real vector bundle α over a space X , we let $S(\alpha)$ and $P(\alpha)$ stand, respectively, for the sphere and projectivized bundles associated to α . Let h_α denote the Hopf line bundle over $P(\alpha)$ splitting off $\pi^*(\alpha)$, where $\pi: P(\alpha) \rightarrow X$ is the projection. A Hopf-type map¹ for α is any continuous map $P(\alpha) \rightarrow P^N$ for which the composite $P(\alpha) \rightarrow P^N \hookrightarrow P^\infty$ classifies h_α . In particular, (3) allows us to talk about Hopf-type maps defined on products of the form $P_{\bar{n}} \times P^s$.

2. IMMERSION DIMENSION

2.1. Axial maps. Consider a pair of sequences $\bar{n} = (n_1, \dots, n_r)$ and $\bar{m} = (m_1, \dots, m_s)$.

Definition 2.1. *A continuous map $\alpha: P_{\bar{n}} \times P_{\bar{m}} \rightarrow P^\infty$ is said to be axial if its restriction to each of the axes classifies the corresponding canonical bundle. By (6) this means that α corresponds to the class $x \otimes 1 + 1 \otimes x$. A continuous map $P_{\bar{n}} \times P_{\bar{m}} \rightarrow P^L$ is called axial if the composite $P_{\bar{n}} \times P_{\bar{m}} \rightarrow P^L \hookrightarrow P^\infty$ is axial.*

¹This is called an ‘axial map’ in [3], but we have to modify the name in view of Definition 2.1 in the next section.

Remark 2.2. *By (4), the existence of an axial map $P_{\bar{n}} \times P_{\bar{m}} \rightarrow P^L$ depends only on n_1 and m_1 . In particular, according to [9], if $n_1 = m_1$, $\text{TC}(P^{n_1})$ is the minimal integer L for which there is an axial map $P_{\bar{n}} \times P_{\bar{m}} \rightarrow P^L$. In any case, an axial map $P_{\bar{n}} \times P_{\bar{m}} \rightarrow P^L$ can exist only if $L \geq \max\{n_1, m_1\}$.*

A slightly weaker concept of axiality arises by requiring that the restriction of $\alpha: P_{\bar{n}} \times P_{\bar{m}} \rightarrow P^\infty$ to $j \times j: P^{n_1} \times P^{m_1} \hookrightarrow P_{\bar{n}} \times P_{\bar{m}}$ is axial in the usual sense. Yet, nothing is lost with respect to the more restrictive Definition 2.1 if we only care (as we will in this subsection) about the *existence* of such maps. Indeed, in view of (5), the only potential problem arises when $n_2 = 1$ or $m_2 = 1$. To fix ideas, assume $n_2 = \dots = n_\ell = 1 < n_{\ell+1}$ ($\ell \leq r$). Then, the restriction of α to its first axis might conceivably correspond to a class of the form $x + \sum_{i=2}^\ell \epsilon_i x_i$. Although such a situation is perfectly attainable, it can be easily fixed. Indeed, [7, Theorem 2.20] asserts that, under the present conditions, $P_{\bar{n}}$ is homeomorphic to $(S^1)^{\ell-1} \times P_{\bar{q}}$ where $\bar{q} = (1, n_{\ell+1}, \dots, n_r)$. Thus, unless $m_2 = 1$ (in which case the following adjustment would have to be made on the second axis, too), the required axial map is given by the composite $P_{\bar{n}} \times P_{\bar{m}} \rightarrow P_{\bar{n}} \times P_{\bar{m}} \xrightarrow{\alpha} P^\infty$ where the first map is $\gamma \times 1$, and γ is the projection $P_{\bar{n}} \rightarrow P_{\bar{q}}$ followed by the inclusion $P_{\bar{q}} \hookrightarrow P_{\bar{n}}$.

As a consequence of Remark 2.2, the nice relationship between TC and the existence of suitable axial maps between (usual) real projective spaces cannot hold for a $P_{\bar{n}}$ with $\ell(\bar{n}) > 1$. Yet, the axial map approach can be used to characterize the immersion dimension of $P_{\bar{n}}$ in a suitable range of dimensions. Indeed, the following are standard consequences of [7] and [25]:

- (I) The existence of a smooth immersion $P_{\bar{n}} \looparrowright \mathbb{R}^M$ implies the existence of an axial map $P^{n_1} \times P^{|\bar{n}|+r-1} \rightarrow P^{M+r-1}$.
- (II) The converse of (I) holds provided $P_{\bar{n}}$ is not stably parallelizable and $n_1 < 2(M - |\bar{n}|)$.

We will now elaborate on the previous facts from a purely ‘projective-product’ viewpoint—without relying on the connection through the generalized vector field problem.

Proposition 2.3. *The existence of an immersion $P_{\bar{n}} \looparrowright \mathbb{R}^M$ implies the existence of a Hopf-type map $P_{\bar{n}} \times P^{|\bar{n}|+r-1} \rightarrow P^{M+r-1}$.*

Proof. Let ε be the trivial line bundle over $P_{\bar{n}}$ and ν the normal bundle of the given immersion. From (2) we have the composite

$$(|\bar{n}| + r)\xi_{\bar{n}} \hookrightarrow (|\bar{n}| + r)\xi_{\bar{n}} \oplus \nu = (\tau_{P_{\bar{n}}} \oplus r\varepsilon) \oplus \nu = (M + r)\varepsilon.$$

The required Hopf-type map is given by the composite

$$P_{\bar{n}} \times P^{|\bar{n}|+r-1} = P((|\bar{n}| + r)\xi_{\bar{n}}) \hookrightarrow P((M + r)\varepsilon) = P_{\bar{n}} \times P^{M+r-1} \xrightarrow{\text{proj}} P^{M+r-1}$$

(cf. [3, Section 2]). □

Remark 2.4. *The converse of Proposition 2.3 can be proved (under an additional hypothesis) in terms of a standard application of Haefliger-Hirsch homotopy approximation of monomorphisms by skew-maps in the metastable range ([17], compare with [2] or [3, Corollary 2.8]). Indeed, the axial map in the conclusion of Proposition 2.3 is double covered by a \mathbb{Z}_2 -equivariant map*

$$S((|\bar{n}| + r)\xi_{\bar{n}}) = S_{\bar{n}} \times_{\mathbb{Z}_2} S^{|\bar{n}|+r-1} \rightarrow S^{M+r-1}.$$

This and the projection $(|\bar{n}| + r)\xi_{\bar{n}} \rightarrow P_{\bar{n}}$ determine a map $S((|\bar{n}| + r)\xi_{\bar{n}}) \rightarrow P_{\bar{n}} \times S^{M+r-1}$ which, after radial extension, yields a skew map $(|\bar{n}| + r)\xi_{\bar{n}} \rightarrow (M + r)\varepsilon$ over $P_{\bar{n}}$. Theorem 1.2 in [17] claims that the latter map can be skew-deformed to a bundle monomorphism $\phi: (|\bar{n}| + r)\xi_{\bar{n}} \hookrightarrow (M + r)\varepsilon$ provided

$$3|\bar{n}| < 2M. \quad (7)$$

Coker(ϕ) is then an $(M - |\bar{n}|)$ -dimensional bundle which, after taking into account (2) and cancelling r trivial sections, yields an isomorphism $\tau_{P_{\bar{n}}} \oplus \text{Coker}(\phi) = M\varepsilon$. Thus [18] asserts that Coker(ϕ) is the normal bundle of an immersion, as required.

Of course, the hypothesis (7) is much stronger than the arithmetical condition in (II), a hypothesis where n_1 plays a more relevant role (and which is in accordance to Remark 2.2).

Proposition 2.5. *There is a Hopf-type map $P_{\bar{n}} \times P^{|\bar{n}|+r-1} \rightarrow P^{M+r-1}$ if and only if there is an axial map $P^{n_1} \times P^{|\bar{n}|+r-1} \rightarrow P^{M+r-1}$.*

Proof. In view of (6), it suffices to check that the map $P_{\bar{n}} \times P^{|\bar{n}|+r-1} \rightarrow P^\infty$ that classifies the Hopf line bundle $h_{(|\bar{n}|+r)\xi_{\bar{n}}}$ corresponds to $x \otimes 1 + 1 \otimes x$. For this purpose, we may assume without loss of generality that the given Hopf-type map arises from an immersion as in Proposition 2.3 (say for a large enough M —this is irrelevant for the intended goal). Then, with the notation of that result, we see from (4) that, by restricting the isomorphism $(|\bar{n}| + r)\xi_{\bar{n}} \oplus \nu = (M + r)\varepsilon$ under the inclusion $j: P^{n_1} \hookrightarrow P_{\bar{n}}$, we get a Hopf-type map

$$P^{n_1} \times P^{|\bar{n}|+r-1} = P((|\bar{n}|+r)\xi_{n_1}) \hookrightarrow P((|\bar{n}|+r)\xi_{\bar{n}}) = P_{\bar{n}} \times P^{|\bar{n}|+r-1} \rightarrow P^{M+r-1}$$

which, as proved in [2], must also be an axial map. Thus, (6) implies that $h_{(|\bar{n}|+r)\xi_{\bar{n}}}$ corresponds, under the identification $P((|\bar{n}|+r)\xi_{\bar{n}}) = P_{\bar{n}} \times P^{|\bar{n}|+r-1}$ in (3), to a class of the form

$$1 \otimes x + (x + \sum \mu_i x_i) \otimes 1$$

where the summation runs over indexes i with $n_i = 1$, and each μ_i is either 0 or 1. But the first isomorphism in (4) and the naturality of the construction of Hopf line bundles imply $\mu_i = 0$ for all relevant i . \square

Example 2.6. *The arithmetical hypothesis in (II) is superfluous when $\ell(\bar{n}) = 1$, but it is needed if $\ell(\bar{n}) > 1$. From our perspective, such a phenomenon is due to the fact that, although the immersion dimension of any standard*

real projective space holds within Haefliger’s metastable range ([2]), as noted in [7], a projective product space $P_{\bar{n}}$ with $\ell(\bar{n}) > 1$ usually admits (very) low-codimension Euclidean immersions—compare to Remark 2.8 below. For instance², the non-parallelizable $P_{(12,14)}$ does not immerse in \mathbb{R}^{30} in view of [7, Theorem 3.4], [25, Lemma 2.2], and [21] (in that order), but the existence of the corresponding axial map in (I) is obtained in [21] through a Postnikov tower argument.

Despite Example 2.6, the method of proof of the main result in [2] yields:

Proposition 2.7. *If $\text{gd}(-(|\bar{n}| + r)\xi_{n_1}) > \lceil (n_1 + 1)/2 \rceil$, then the arithmetical hypothesis in (II) is superfluous.*

Proof. Assume for a contradiction that, for some M , there is an axial map $P^{n_1} \times P^{|\bar{n}|+r-1} \rightarrow P^{M+r-1}$ but that the non-stably parallelizable $P_{\bar{n}}$ does not immerse in \mathbb{R}^M . Without loss of generality we can assume $M = \text{imm}(P_{\bar{n}}) - 1 > |\bar{n}|$. Then, [7, Theorem 3.4] gives

$$M - |\bar{n}| = \text{imm}(P_{\bar{n}}) - |\bar{n}| - 1 = \text{gd}(-(|\bar{n}| + r)\xi_{n_1}) - 1 \geq \left\lceil \frac{n_1 + 1}{2} \right\rceil,$$

which amounts to having the arithmetical hypothesis in (II). \square

Remark 2.8. *In the same line of reasoning as in Example 2.6, it follows from [1] that, for any large n_1 , there are instances of spaces $P_{(n_1, \dots, n_r)}$ for which the hypothesis in Proposition 2.7 fails.*

It is worth mentioning that, for $n_1 \leq 9$, the arithmetical hypothesis in (II) above is superfluous³. As in the proof of Proposition 2.7, such an assertion can be verified by checking that, in the indicated range, there is no axial map $P^{n_1} \times P^{|\bar{n}|+r-1} \rightarrow P^{M+r-1}$ with $M = \text{imm}(P_{\bar{n}}) - 1 > |\bar{n}|$. Indeed, under the current hypothesis, such an axial map is prevented by the relation

$$(x + y)^{M+r} \neq 0 \tag{8}$$

where x and y denote respectively the generators of the mod 2 cohomology groups $H^1(P^{n_1}; \mathbb{Z}_2)$ and $H^1(P^{|\bar{n}|+r-1}; \mathbb{Z}_2)$. Explicitly, the basis element $x^g y^{|\bar{n}|+r-1} \in H^*(P^{n_1} \times P^{|\bar{n}|+r-1}; \mathbb{Z}_2)$ appears in the expansion of (8) with coefficient

$$\binom{|\bar{n}| + r + g - 1}{g} \tag{9}$$

where $g = \text{gd}(-(|\bar{n}| + r)\xi_{n_1})$. But, under the current hypothesis, a direct verification using [20] (or, alternatively, Table 4.4 and Proposition 4.5 in [7]) shows that (9) is odd. For instance, consider the case $n_1 = 6$, where the assumption that $P_{\bar{n}}$ is not stably parallelizable means $|\bar{n}| + r \neq$

²We thank Kee Lam for kindly pointing out this example.

³Kee Lam has brought the author’s attention that the smallest case where the arithmetical hypothesis in (II) is actually needed takes place when $n_1 = 10$.

0 mod 8. Then, [7, Table 4.4] gives $g = (6, 6, 5, 4, 3, 2, 1)$ for $|\bar{n}| + r \equiv (1, 2, 3, 4, 5, 6, 7) \pmod{8}$. So

$$\binom{|\bar{n}| + r + g - 1}{g} \equiv \left(\binom{6}{6}, \binom{7}{6}, \binom{7}{5}, \binom{7}{4}, \binom{7}{3}, \binom{7}{2}, \binom{7}{1} \right) \equiv 1 \pmod{2}.$$

We close this subsection by remarking that, just as the situation in Example 2.6 for the condition $n_1 < 2(M - |\bar{n}|)$, the hypothesis that $\mathbb{P}_{\bar{n}}$ is not stably parallelizable is also needed in (II). Yet, the full TC-axial picture is well understood in the stably parallelizable case. In fact, the situation is entirely similar to that in the classical case with $\ell(\bar{n}) = 1$, where there are well-known axial maps $\mathbb{P}^n \times \mathbb{P}^n \rightarrow \mathbb{P}^n$ for $n = 1, 3, 7$, but of course no immersion $\mathbb{P}^n \looparrowright \mathbb{R}^n$. Namely, since the immersion dimension of a stably parallelizable $\mathbb{P}_{\bar{n}}$ is $|\bar{n}| + 1$, there is an axial map $\mathbb{P}_{\bar{n}} \times \mathbb{P}^{|\bar{n}|+r-1} \rightarrow \mathbb{P}^{|\bar{n}|+r}$. But there is a finer (and optimal) axial map

$$\mathbb{P}_{\bar{n}} \times \mathbb{P}^{|\bar{n}|+r-1} \rightarrow \mathbb{P}^{|\bar{n}|+r-1} \quad (10)$$

(which cannot come from an immersion). Indeed, as shown in [7], the stable parallelizability of $\mathbb{P}_{\bar{n}}$ means that the exponent in the highest 2-power dividing $|\bar{n}| + r$ is no less than $\phi(n_1)$ —the number of positive integers less than or equal to n_1 and which are congruent to 0, 1, 2, or 4 mod 8. Therefore, classical work of Hurwitz, Radon, and Eckmann on the so-called Hurwitz-Radon matrix equations gives in fact a non-singular bilinear map $\mathbb{R}^{n_1+1} \times \mathbb{R}^{|\bar{n}|+r} \rightarrow \mathbb{R}^{|\bar{n}|+r}$ and, in view of Remark 2.2, an axial map of the form (10). An intriguing possibility is that explicit ‘linear’ formulæ leading to an axial map (10) could be deduced from a refinement of the Clifford-algebra input in the Hurwitz-Radon number—without relying on Remark 2.2.

2.2. Non-singular maps. The existence of axial maps can be translated into the existence of certain non-singular maps. Not only is such a fact a straightforward generalization of the corresponding well-known property for usual projective spaces, but the language of non-singular maps turns out to be irrelevant for the purposes of the paper, since they fail to provide local motion planners as in the classical case. Consequently, these ideas are loosely treated in this subsection, mentioned only for completeness purposes.

There are two closely related notions of non-singular maps associated to an axial map between projective product spaces. In the first one, for an ℓ -tuple $\bar{q} = (q_1, \dots, q_\ell)$, we consider the cone $Q_{\bar{q}}$ in $\mathbb{R}^{q_1+1} \times \dots \times \mathbb{R}^{q_\ell+1}$ consisting of tuples $\bar{x} = (x_1, \dots, x_\ell)$ with $|x_1| = \dots = |x_\ell|$. Thus, $\mathbb{P}_{\bar{q}}$ is the projectivization of $Q_{\bar{q}}$, i.e. $\mathbb{P}_{\bar{q}}$ is the subspace of $\mathbb{P}^{|\bar{q}|+\ell-1}$ consisting of the lines contained in $Q_{\bar{q}}$. Then, a continuous map $f : Q_{\bar{n}} \times Q_{\bar{m}} \rightarrow \mathbb{R}^{k+1}$ is said to be non-singular if $f(\lambda\bar{x}, \mu\bar{y}) = \lambda\mu f(\bar{x}, \bar{y})$ for $\lambda, \mu \in \mathbb{R}$, and if the equality $f(\bar{x}, \bar{y}) = 0$ holds only with $\bar{x} = 0$ or $\bar{y} = 0$. With this definition, *there is a one-to-one correspondence between the set of non-singular maps $f : Q_{\bar{n}} \times Q_{\bar{m}} \rightarrow \mathbb{R}^{k+1}$ (taken up to multiplication by a non-zero scalar) and*

the set of axial maps $g : P_{\bar{n}} \times P_{\bar{m}} \rightarrow P^k$. Such a corresponding pair (f, g) fits in a commutative diagram

$$\begin{array}{ccccccc} Q_{\bar{n}} \times Q_{\bar{m}} & \longleftarrow & S_{\bar{n}} \times S_{\bar{m}} & \longrightarrow & S_{\bar{n}} \times_{\mathbb{Z}_2} S_{\bar{m}} & \longrightarrow & P_{\bar{n}} \times P_{\bar{m}} \\ f \downarrow & & f' \downarrow & & h \downarrow & & g \downarrow \\ \mathbb{R}^{k+1} & \longleftarrow & \mathbb{R}^{k+1} - \{0\} & \xrightarrow{\rho} & S^k & \longrightarrow & P^k. \end{array}$$

Here the unlabelled horizontal maps facing east are the obvious two fold coverings, ρ is the normalization map $\rho(u) = u/|u|$, f' is the restriction $f|_{S_{\bar{n}} \times S_{\bar{m}}}$, and the right hand square is a pullback (hence h is \mathbb{Z}_2 -equivariant). Explicitly, given f , $g([\bar{x}], [\bar{y}])$ is the line in \mathbb{R}^{k+1} that goes through the origin and $f(\bar{x}, \bar{y})$. Conversely, given g , pick h as in the diagram above and precompose it with the double covering $S_{\bar{n}} \times S_{\bar{m}} \rightarrow S_{\bar{n}} \times_{\mathbb{Z}_2} S_{\bar{m}}$ to get a \mathbb{Z}_2 -biequivariant map $\tilde{g} : S_{\bar{n}} \times S_{\bar{m}} \rightarrow S^k$. Then f is the ‘‘bi-radial’’ extension of \tilde{g} given by

$$f(\bar{x}, \bar{y}) = \begin{cases} \frac{|\bar{x}|}{\sqrt{r}} \frac{|\bar{y}|}{\sqrt{s}} \tilde{g} \left(\frac{\sqrt{r}}{|\bar{x}|} \bar{x}, \frac{\sqrt{s}}{|\bar{y}|} \bar{y} \right), & \text{if } \bar{x} \neq 0 \text{ and } \bar{y} \neq 0; \\ 0, & \text{if } \bar{x} = 0 \text{ or } \bar{y} = 0. \end{cases} \quad (11)$$

Note that if $f : \mathbb{R}^{n_1+1} \times \mathbb{R}^{m_1+1} \rightarrow \mathbb{R}^{k+1}$ is a non-singular map (in the usual sense), then for any $\bar{n} = (n_1, n_2, \dots, n_r)$ and $\bar{m} = (m_1, m_2, \dots, m_s)$ a non-singular map $Q_{\bar{n}} \times Q_{\bar{m}} \rightarrow \mathbb{R}^{k+1}$ can be defined by $(\bar{x}, \bar{y}) \mapsto f(x_1, y_1)$. Of course, this fact is compatible with Remark 2.2.

A slight variation of the notion of non-singular maps goes as follows: Set $V_{\bar{t}} = \mathbb{R}^{t_1+1} \times \dots \times \mathbb{R}^{t_\ell+1}$. A map $f : V_{\bar{n}} \times V_{\bar{m}} \rightarrow \mathbb{R}^{k+1}$ is said to be non-singular if $f(\lambda \bar{x}, \mu \bar{y}) = \lambda \mu f(\bar{x}, \bar{y})$ for $\lambda, \mu \in \mathbb{R}$, and if the equality $f(\bar{x}, \bar{y}) = 0$ holds only when a coordinate x_i of \bar{x} or a coordinate y_j of \bar{y} vanishes. Then the above considerations apply basically without change, except that (11) takes the slightly more elaborated form

$$f(\bar{x}, \bar{y}) = \begin{cases} N(\bar{x}, \bar{y}) \tilde{g} \left(\frac{x_1}{|x_1|}, \dots, \frac{x_r}{|x_r|}, \frac{y_1}{|y_1|}, \dots, \frac{y_s}{|y_s|} \right), & \text{if no } x_i \text{ nor } y_j \text{ is zero;} \\ 0, & \text{otherwise,} \end{cases}$$

where $N(\bar{x}, \bar{y}) = (|x_1| \cdots |x_r|)^{\frac{1}{r}} (|y_1| \cdots |y_s|)^{\frac{1}{s}}$.

3. TOPOLOGICAL COMPLEXITY

In this section we give several general estimates for $\text{TC}(P_{\bar{n}})$. We find that $\text{TC}(P_{\bar{n}}) < \dim(P_{\bar{n}})$ in certain cases, indicating that a simple relation to immersion dimension such as (1) does not hold for these manifolds. We also compute the exact value of $\text{TC}(P_{\bar{n}})$ in many cases (Proposition 3.7), and give evidence toward the appealing possibility that $\text{TC}(P_{\bar{n}})$ would depend mostly on $\text{TC}(P^{n_1})$ and $\ell(\bar{n})$.

Let ∞ stand for the r -tuple (∞, \dots, ∞) , and let P_{∞} denote the quotient of $\prod_r S^\infty$ by the diagonal action of \mathbb{Z}_2 (with the antipodal action on each factor). Note that P_{∞} is an Eilenberg-MacLane space $K(\mathbb{Z}_2, 1)$ containing $P_{\bar{n}}$.

Lemma 3.1. *There is a CW decomposition for P_∞ whose n_1 -skeleton is contained in $P_{\bar{n}}$.*

Proof. Let $e_+^0 \cup e_-^0 \cup \dots \cup e_+^m \cup e_-^m$ be the usual \mathbb{Z}_2 -equivariant cell structure on a sphere S^m , and consider the resulting product structure

$$S_{\bar{n}} = \bigcup e_{\pm}^{i_1} \times \dots \times e_{\pm}^{i_r}. \quad (12)$$

If τ stands for the generator of \mathbb{Z}_2 , then a cell structure on $P_{\bar{n}}$ can be formed by identifying a cell $e_{\pm}^{i_1} \times \dots \times e_{\pm}^{i_r}$ in (12) with the corresponding cell $\tau \cdot (e_{\pm}^{i_1} \times \dots \times e_{\pm}^{i_r})$. If $\ell(\bar{m}) = \ell(\bar{n})$ and $n_i \leq m_i$, the inclusion $P_{\bar{n}} \hookrightarrow P_{\bar{m}}$ contains the n_1 -skeleton of $P_{\bar{m}}$. Thus the required cell structure in P_∞ is the inductive one under the above inclusions. \square

We are indebted to Sergey Melikhov for pointing out (in [23]) the proof of the following fact:

Proposition 3.2. *Let M^m and N^n be closed smooth manifolds, and let $C^\infty(M, N)$ denote the space of smooth maps in the Whitney C^∞ -topology. Then for $f: M \rightarrow N$ in a dense subset of $C^\infty(M, N)$, the fibers $f^{-1}(y)$ with $y \in N$ are all polyhedra of dimension $\leq \min(m - n, 0)$.*

Proof. First we note that the set of triangulable maps is dense in $C^\infty(M, N)$. Recall that a smooth map $f: M \rightarrow N$ is *triangulable* if there exists a PL map $g: K \rightarrow L$ between PL manifolds, and homeomorphisms $h: M \rightarrow K$ and $h': N \rightarrow L$ such that $g \circ h = h' \circ f$. By Verona's proof of Thom's triangulation conjecture [26], we know that all proper, topologically stable maps $f: M \rightarrow N$ are triangulable. By the Thom-Mather theorem (a full proof of which appears in [11]), such maps form an open dense subset of $C^\infty(M, N)$.

Next, we note that the fibers $f^{-1}(y)$ of a triangulable map $f: M \rightarrow N$ are all polyhedra (they are homeomorphic to simplicial complexes). For given $y \in N$, we may choose a triangulation $h': N \rightarrow L$ as above with $h'(y)$ a vertex of L . Then $f^{-1}(y)$ is homeomorphic with $g^{-1}(h'(y))$, a subcomplex of K .

Finally, we claim that for $f: M \rightarrow N$ in an open dense subset of the space $C^\infty(M, N)$, the fibers $f^{-1}(y)$ all have covering dimension $\leq \min(m - n, 0)$. Intersecting this set with the set of proper, topologically stable maps, we find an open dense set of maps whose fibres are all polyhedra of covering dimension $\leq \min(m - n, 0)$. Since covering dimension is a topological property, this proves the Proposition.

The proof of the final claim follows from the multi-jet transversality theorem [12], which implies that for an open dense set of mappings $f: M \rightarrow N$, the fibers $f^{-1}(y)$ all have the structure of a smooth submanifold of M of dimension $\min(m - n, 0)$ away from at most finitely many isolated singular points. \square

Theorem 3.3. $\text{TC}(P_{\bar{n}}) \leq 2|\bar{n}| - n_1 + 1$ for $\ell(\bar{n}) > 1$. On the other hand, the following numbers are equal, giving a lower bound for $\text{TC}(P_{\bar{n}})$:

- The Schwartz genus of the obvious double cover $S_{\bar{n}} \times_{\mathbb{Z}_2} S_{\bar{n}} \rightarrow P_{\bar{n}} \times P_{\bar{n}}$.
- The smallest integer L for which $(L + 1)\xi_{\bar{n}} \otimes \xi_{\bar{n}}$ admits a nowhere zero section.
- The smallest integer L for which there is an axial map $P_{\bar{n}} \times P_{\bar{n}} \rightarrow P^L$.
- $\text{TC}(P^{n_1})$

Proof. It follows from Remark 2.2 and the first two conditions in (4) that the number described in each of the first three items does not change if \bar{n} is replaced by n_1 (for the first item we use the fact that the indicated double cover is the sphere bundle associated to $\xi_{\bar{n}} \otimes \xi_{\bar{n}}$). Therefore, the equality of the four listed numbers follows from [9, Theorem 6.1]. The fact that they give a lower bound for $\text{TC}(P_{\bar{n}})$ follows from the third condition in (4) and the behavior of TC under retracts.

We use the argument in [15, Corollary 4.5] (which is inspired in turn by [24]) to prove the upper bound in this theorem. Set $L = 2|\bar{n}| - n_1 + 1$. By (1) and Remark 2.2, we can choose an axial map $q: P_{\bar{n}} \times P_{\bar{n}} \rightarrow P^L$. Since the axial condition is homotopical, we can assume first that q is smooth and then, by Proposition 3.2, that for each $z \in P^L$ the inverse image $q^{-1}(z)$ is homeomorphic to a CW complex of dimension at most $n_1 - 1$. Then, the axiomaticity of q implies that the image of the class x in (6) under the composite

$$q^{-1}(z) \hookrightarrow P_{\bar{n}} \times P_{\bar{n}} \xrightarrow{\pi_i} P_{\bar{n}} \quad (13)$$

is independent of the projection $\pi_i: P_{\bar{n}} \times P_{\bar{n}} \rightarrow P_{\bar{n}}$ ($i = 1, 2$) used. In fact, Lemma 3.1 and the dimensionality assumption on $q^{-1}(z)$ imply that the actual homotopy type of (13) is independent of i . The result then follows from Lemma 2.5 and Theorem 4.3 in [15]. \square

Of course, part of the argument for the lower bound in Theorem 3.3 actually yields $\text{TC}(P^{n_1}) \leq \text{TC}(P_{(n_1, n_2)}) \leq \cdots \leq \text{TC}(P_{(n_1, \dots, n_{r-1})}) \leq \text{TC}(P_{\bar{n}})$. On the other hand, the argument proving the upper bound uses and corrects the proof of [15, Corollary 4.5] which, instead of using Proposition 3.2, is based on an assertion about approximating axial maps by submersions. But such a claim is false in general, as illustrated next.

Example 3.4. Since $P^2 \looparrowright \mathbb{R}^3$, there exists an axial map $q: P^2 \times P^2 \rightarrow P^3$. Note that $2 < 3 < 2 \cdot 2 = 4$. However, q is not homotopic to a submersion. In fact, there does not exist any submersion $P^2 \times P^2 \rightarrow P^3$, by the following easy argument involving Stiefel-Whitney classes: Suppose $g: P^2 \times P^2 \rightarrow P^3$ is a submersion. Then we obtain the short exact sequence of vector bundles over $P^2 \times P^2$

$$0 \rightarrow E \rightarrow T(P^2 \times P^2) \xrightarrow{dg} g^*T(P^3) \rightarrow 0$$

where the kernel E is a real line bundle. It then follows that

$$w(P^2 \times P^2) = w(E) g^*w(P^3) = w(E)$$

(the latter equality since P^3 is parallelizable). But this is impossible since, for example, $w_2(P^2 \times P^2) \neq 0$.

Remark 3.5. *It is possible to prove the upper bound in Theorem 3.3 by applying [10, Theorem 3] to an axial map $\mathbb{P}_{\bar{n}} \times \mathbb{P}_{\bar{n}} \rightarrow \mathbb{P}^{2|\bar{n}|-n_1+1}$, and noticing that the canonical inclusion $\mathbb{P}_{\bar{n}} \hookrightarrow \mathbb{P}^{2|\bar{n}|-n_1+1}$ is an n_1 -equivalence. We have chosen the approach in [15] due to the intrinsic interest of Proposition 3.2.*

Remark 3.6. *The standard upper bound $\text{TC} \leq 2 \dim$ means that, in general, there are up to twice $\dim(X)$ classical homotopy obstructions to consider when bounding $\text{TC}(X)$ from above. For instance, the first top two are central in [6], with the very top one being critical for Costa-Farber’s applications—the next-to-the-top one comes for free from [4]. Thus, the upper bound in Theorem 3.3 is already taking care of the first $n_1 - 1$ of these obstructions for $X = \mathbb{P}_{\bar{n}}$.*

The lower bound in Theorem 3.3 is rather crude, as it ignores information coming from $S^{n_2} \times \cdots \times S^{n_r}$. For instance, [9, Theorem 4.5], (5), and ‘zero-divisors’ cup-length (zcl) considerations (as defined in [8]) easily yield

$$\text{TC}(\mathbb{P}_{\bar{n}}) \geq 2^{e+1} + \ell(\bar{n}) - 2 \text{ provided } n_1 \geq 2^e, \quad (14)$$

which improves by arbitrarily large amount the lower bound in Theorem 3.3 when $\ell(\bar{n}) \gg 0$. On the other hand, the general philosophy behind (1) implies that the lower bound in Theorem 3.3 can be much stronger than that in (14) if $\ell(\bar{n}) = 2$. For instance [19] gives

$$\text{TC}(\mathbb{P}_{(2^e-1, 2^e-1)}) \geq \text{TC}(\mathbb{P}^{2^e-1}) \geq 2^{e+1} - 2e - (2, 1, 1, 3) \quad (15)$$

provided $e \equiv (0, 1, 2, 3) \pmod{4}$, a bound which is almost twice that in (14). Of course, further results of this sort can be deduced from our current knowledge of the immersion dimension of (usual) real projective spaces. In view of [14, Theorems 2.1 and 2.4], it should be possible to use zcl-considerations based on *generalized* cohomology theories in order to insert the nice $\ell(\bar{n})$ -feature of (14) into the lower bound in Theorem 3.3, thus merging the corresponding strengths of (14) and (15) into a single lower bound (we hope to explore such a possibility elsewhere).

More interesting is the fact that $\text{TC}(\mathbb{P}_{\bar{n}})$ can be arbitrarily smaller than the dimension of $\mathbb{P}_{\bar{n}}$. The simplest of such situations originates from the subadditivity of TC ([8]), as $\text{TC}(\mathbb{P}_{\bar{n}}) - \text{TC}(\mathbb{P}_{\bar{m}}) \leq 2$ whenever $\text{TC}(\mathbb{P}_{\bar{n}}) \approx \mathbb{P}_{\bar{m}} \times S^{n_i}$ (the latter decomposition is characterized arithmetically in [7, Theorem 2.20]). As an extreme situation consider the following partial analogue of [7, (2.21)]:

Proposition 3.7. *Let $\phi(n_1)$ be the number of positive integers equal to or less than n which are congruent to 0, 1, 2, or 4 (mod 8). If $\nu(n_i + 1) \geq \phi(n_1)$ for all $i > 1$, then*

$$\text{zcl}_{\mathbb{Z}_2}(\mathbb{P}^{n_1}) + \ell(\bar{n}) - 1 \leq \text{TC}(\mathbb{P}_{\bar{n}}) \leq \text{TC}(\mathbb{P}^{n_1}) + \ell(\bar{n}) - 1.$$

Further, both inequalities above become equalities precisely for n_1 a 2-power.

Proof. The first inequality is (14); the second inequality follows from [7, Theorem 2.20]. The final assertion follows from the standard fact that $\mathrm{TC}(\mathbb{P}^{n_1}) = \mathrm{zcl}_{\mathbb{Z}_2}(\mathbb{P}^{n_1})$ precisely for n_1 a 2-power. \square

Proposition 3.7 suggests the possibility that $\mathrm{TC}(\mathbb{P}_{\bar{n}})$ can be estimated for any \bar{n} in terms of $\mathrm{TC}(\mathbb{P}^{n_1})$ and $\ell(\bar{n})$ alone. Theorem 3.8 below (whose proof is postponed to the next section) fits into such a general philosophy, and shows that the low TC-phenomenon in Proposition 3.7 holds even if there are no spheres factoring out $\mathbb{P}_{\bar{n}}$.

Theorem 3.8. *If k denotes the number of spheres S^{n_i} with n_i even and $i > 1$, then $\mathrm{TC}(\mathbb{P}_{\bar{n}}) < (\mathrm{TC}(\mathbb{P}^{n_1}) + 1)(\ell(\bar{n}) + k)$.*

The upper bound in Theorem 3.8 will be much lower than the dimension of $\mathbb{P}_{\bar{n}}$ provided the sum $n_2 + \cdots + n_r$ is large enough—which can hold even if there are no spheres S^{n_i} factoring out $\mathbb{P}_{\bar{n}}$. Thus, in such cases, most of the homotopy obstructions in the motion planning problem for $\mathbb{P}_{\bar{n}}$ already vanish. It is worth noticing that $\mathrm{TC}(\mathbb{P}_{\bar{n}})$ is not always less than $\dim(\mathbb{P}_{\bar{n}})$: if 1^r stands for the r -tuple $(1, \dots, 1)$, then $\mathrm{TC}(\mathbb{P}_{1^r}) = \dim(\mathbb{P}_{1^r})$, in view of Proposition 3.7. On the other hand, the upper bound in Theorem 3.8 not always improves that in Theorem 3.3. For instance, in the case of $\mathbb{P}_{(2^e, 2^e)}$, the former bound is $6 \cdot 2^e$ while the latter one is only $3 \cdot 2^e + 1$.

4. EQUIVARIANT TOPOLOGICAL COMPLEXITY

In a recent paper [5] Hellen Colman and the second author explore an equivariant generalization of topological complexity, in the setting of compact group actions. Here we give additional examples and results which will be useful in applying their results to the estimation of topological complexity of projective product spaces.

Let G be a compact Hausdorff topological group (in our present applications G will be the cyclic group \mathbb{Z}_2). If $p: E \rightarrow B$ is a G -map, the *equivariant sectional category* of p , denoted $\mathrm{secat}_G(p)$, is defined in [5, Section 5] to be the least integer k such that B may be covered by k invariant open sets U_1, \dots, U_k on each of which there exists a G -homotopy section, that is a G -map $s: U_i \rightarrow E$ such that $p \circ s$ is G -homotopic to the inclusion $i_{U_i}: U_i \hookrightarrow B$. If p is a G -fibration, then this is equivalent to requiring the existence of a G -section $s: U_i \rightarrow E$ such that $p \circ s = i_{U_i}$.

In particular, for any G -space X the *equivariant topological complexity* of X is defined in [5, Section 6] to be the equivariant sectional category of the double evaluation map $X^{[0,1]} \rightarrow X \times X$. Here G acts diagonally on the product and by composition on the path space of X .

In keeping with the conventions in place in this paper, we will define the equivariant topological complexity to be one less than the number of sets in the open cover; thus

$$\mathrm{TC}_G(X) = \mathrm{secat}_G(X^{[0,1]} \rightarrow X \times X) - 1.$$

Lemma 4.1. *Let $G = \mathbb{Z}_2$ act antipodally on the sphere S^n , where $n \geq 1$. Then*

$$\mathrm{TC}_G(S^n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. We argue that the usual motion planning rules on the spheres can be made equivariant with respect to the antipodal action, by choosing vector fields which are equivariant.

Suppose n is odd. Then the projective space P^n has zero Euler characteristic and so admits a nowhere-vanishing vector field. Using the double cover immersion $S^n \rightarrow P^n$, this pulls back to a nowhere-vanishing vector field v on S^n which is equivariant in the sense that $dg(v(A)) = v(gA)$ for $g \in G$ and $A \in S^n$. We consider the open sets $U_0 = \{(A, B) \in S^n \times S^n \mid A \neq -B\}$ and $U_1 = \{(A, B) \in S^n \times S^n \mid A \neq B\}$. We define s_0 on U_0 by choosing the shortest geodesic path from A to B (traveled at constant velocity). We define s_1 on U_1 in two stages: first travel from A to $-A$ along the great circle in the direction determined by $v(A)$; second travel from $-A$ to B along the shortest geodesic path. It is easy to check that these sets and motion planning rules are G -invariant.

When n is even, removing a point $[C]$ from P^n gives an open manifold homotopy equivalent to P^{n-1} , which therefore admits a nowhere-vanishing vector field. Again we pull this back to obtain a nowhere-vanishing equivariant vector field v' on $S^n - \{-C, C\}$. We let U_0 and s_0 be as before. We let $U'_1 = \{(A, B) \in S^n \times S^n \mid A \neq B, C, -C\}$ and define s'_1 using v' similarly to s_1 . Finally we let $U'_2 = \{(A, -A) \mid A \in W \cup -W\}$, where W is a small open disk neighbourhood centred on C . The path $s'_2(A, -A)$ for $A \in W$ travels first along the geodesic segment to the centre C of W ; then along some fixed path γ from C to $-C$; then along the geodesic segment in $-W$ to $-A$. For $A \in -W$ the path $s'_2(A, -A)$ travels first along the geodesic segment in $-W$ to $-C$; then along $-\gamma$ from $-C$ to C ; then along the geodesic segment in W to $-A$.

The lower bounds are given by the obvious inequality $\mathrm{TC}(X) \leq \mathrm{TC}_G(X)$, which holds for any G -space X . \square

Theorem 4.2. *Let G be a compact Lie group, and let X and Y be smooth G -manifolds. Then*

$$\mathrm{TC}_G(X \times Y) \leq \mathrm{TC}_G(X) + \mathrm{TC}_G(Y)$$

where $X \times Y$ is given the diagonal G -action.

Proof. Let $\mathrm{TC}_G(X) = n$ and $\mathrm{TC}_G(Y) = m$. Suppose that $X \times X = U_0 \cup \dots \cup U_n$ where the U_i are open invariant sets with G -sections $s_i: U_i \rightarrow X^{[0,1]}$. Suppose further that $Y \times Y = V_0 \cup \dots \cup V_m$ where the V_j are open invariant sets with G -sections $\sigma_j: V_j \rightarrow Y^{[0,1]}$. We can find a G -invariant partition of unity $\{f_i\}$ on $X \times X$ subordinate to $\{U_i\}$ (see [16, Corollary B.33]). Likewise let $\{g_j\}$ be a G -invariant partition of unity on $Y \times Y$ subordinate to $\{V_j\}$.

The rest of the proof proceeds by direct analogy with the proof in the non-equivariant case given in [8, Theorem 11], hence is omitted. \square

Remark 4.3. *Theorem 4.2 is certainly not the most general setting in which the product inequality holds. For instance, we believe it holds whenever X and Y are G -ENRs.*

Corollary 4.4. *Consider the diagonal action of \mathbb{Z}_2 on $S_{\bar{n}} = S^{n_1} \times \cdots \times S^{n_r}$. If k denotes the number of spheres with n_i even, then*

$$\mathrm{TC}_G(S_{\bar{n}}) = \ell(\bar{n}) + k.$$

The main result we will apply from [5] gives an upper bound for the (non-equivariant) topological complexity of a Borel fibration in terms of the topological complexity of the base and the equivariant topological complexity of the fibre.

Theorem 4.5 ([5, Theorem 6.21]). *Let X be a G -space, and let $E \rightarrow B = E/G$ be a numerable principal G -bundle. Then*

$$\mathrm{TC}(X_G) < (\mathrm{TC}_G(X) + 1)(\mathrm{TC}(B) + 1),$$

where $X_G = E \times_G X$ is the corresponding Borel space of X .

Proof of Theorem 3.8. Let $\bar{m} = (n_2, \dots, n_r)$. Note that $P_{\bar{n}}$ can be thought of as the Borel space $S^{n_1} \times_{\mathbb{Z}_2} S_{\bar{m}}$. The result then follows from Theorem 4.5. \square

The argument in the proof of Theorem 3.8 can be used to give low upper bounds for the LS-category of projective product spaces (extending the phenomenon noted in [7, (2.21)] when $P_{\bar{n}}$ has a full set of factoring spheres). Namely

$$\mathrm{cat}(P_{\bar{n}}) < (n_1 + 1)\ell(\bar{n}). \quad (16)$$

Since $\mathrm{TC} \leq 2\mathrm{cat}$, we get in particular

$$\mathrm{TC}(P_{\bar{n}}) < 2(n_1 + 1)\ell(\bar{n}) - 1 \quad (17)$$

which improves on Theorem 3.8 only when $P_{\bar{n}}$ comes from a product having ‘enough’ even dimensional spheres, i.e. $k \geq C_{n_1}\ell(\bar{n})$ where

$$C_{n_1} = \frac{2n_1 + 1 - \mathrm{TC}(P^{n_1})}{\mathrm{TC}(P^{n_1}) + 1}.$$

Note that, although $k \leq \ell(\bar{n})$, $C_{n_1} \ll 1$ for any ‘generic’ n_1 .

Just as (17) and Theorem 3.8 may fail to improve the upper bound in Theorem 3.3, the bound in (16) is not always useful (for instance if all the n_i are equal). For such cases it is worth keeping in mind that, in view of [4, Theorem 3.5], the standard estimate $\mathrm{cat} \leq \dim$ is improved by the inequality $\mathrm{cat}(P_{\bar{n}}) < \dim(P_{\bar{n}})$ provided $n_1 > 1$ and $\ell(\bar{n}) > 1$.

REFERENCES

- [1] J. F. Adams. Geometric dimension of bundles over RP^n . In *Proceedings of the International Conference on Prospects in Mathematics (Taniguchi Internat. Sympos., Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1973)*, pages 1–17. Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1974.
- [2] J. Adem, S. Gitler, and I. M. James. On axial maps of a certain type. *Bol. Soc. Mat. Mexicana (2)*, 17:59–62, 1972.
- [3] Luis Astey, Donald M. Davis, and Jesús González. Generalized axial maps and Euclidean immersions of lens spaces. *Bol. Soc. Mat. Mexicana (3)*, 9(1):151–163, 2003.
- [4] Israel Bernstein. On the Lusternik-Schnirelmann category of Grassmannians. *Math. Proc. Cambridge Philos. Soc.*, 79(1):129–134, 1976.
- [5] H. Colman and M. Grant. Equivariant topological complexity. arXiv:1205.0166v1 [math.AT].
- [6] Armando Costa and Michael Farber. Motion planning in spaces with small fundamental groups. *Commun. Contemp. Math.*, 12(1):107–119, 2010.
- [7] Donald M. Davis. Projective product spaces. *J. Topol.*, 3(2):265–279, 2010.
- [8] Michael Farber. Topological complexity of motion planning. *Discrete Comput. Geom.*, 29(2):211–221, 2003.
- [9] Michael Farber, Serge Tabachnikov, and Sergey Yuzvinsky. Topological robotics: motion planning in projective spaces. *Int. Math. Res. Not.*, (34):1853–1870, 2003.
- [10] José M. García-Calines and Lucile Vandembroucq. Topological complexity and the homotopy cofibre of the diagonal map. ArXiv:1106.4943v2 [math.AT].
- [11] Christopher G. Gibson, Klaus Wirthmüller, Andrew A. du Plessis, and Eduard J. N. Looijenga. *Topological stability of smooth mappings*. Lecture Notes in Mathematics, Vol. 552. Springer-Verlag, Berlin-New York, 1976.
- [12] M. Golubitsky and V. Guillemin. *Stable mappings and their singularities*. Graduate Texts in Mathematics, Vol. 14. Springer-Verlag, New York-Heidelberg, 1973.
- [13] Jesús González. Topological robotics in lens spaces. *Math. Proc. Cambridge Philos. Soc.*, 139(3):469–485, 2005.
- [14] Jesús González, Enrique Torres-Giese, and Maurilio Velasco-Fuentes. In preparation.
- [15] Mark Grant. Topological complexity, fibrations and symmetry. *Topology Appl.*, 159(1):88–97, 2012.
- [16] Victor Guillemin, Viktor Ginzburg, and Yael Karshon. *Moment maps, cobordisms, and Hamiltonian group actions*, volume 98 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Appendix J by Maxim Braverman.
- [17] André Haefliger and Morris W. Hirsch. Immersions in the stable range. *Ann. of Math. (2)*, 75:231–241, 1962.
- [18] Morris W. Hirsch. Immersions of manifolds. *Trans. Amer. Math. Soc.*, 93:242–276, 1959.
- [19] I. M. James. On the immersion problem for real projective spaces. *Bull. Amer. Math. Soc.*, 69:231–238, 1963.
- [20] Kee Yuen Lam. Sectioning vector bundles over real projective spaces. *Quart. J. Math. Oxford Ser. (2)*, 23:97–106, 1972.
- [21] Kee Yuen Lam and Duane Randall. Geometric dimension of bundles on real projective spaces. In *Homotopy theory and its applications (Cocoyoc, 1993)*, volume 188 of *Contemp. Math.*, pages 137–160. Amer. Math. Soc., Providence, RI, 1995.
- [22] Gregory Lupton and Jérôme Scherer. Topological complexity of H-spaces. *To appear in Proc. Amer. Math. Soc.*
- [23] S. Melikhov. Singular fibers of generic smooth maps of negative codimension. <http://mathoverflow.net/questions/94404#95597>.

- [24] J. Oprea and J. Walsh. Quotient maps, group actions and Lusternik-Schnirelmann category. *Topology Appl.*, 117(3):285–305, 2002.
- [25] B. J. Sanderson. A non-immersion theorem for real projective space. *Topology*, 2:209–211, 1963.
- [26] Andrei Verona. *Stratified mappings—structure and triangulability*. Lecture Notes in Mathematics, Vol. 1102. Springer-Verlag, Berlin, 1984.

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