ALGEBRAIC KASPAROV K-THEORY. II

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Abstract. A kind of motivic stable homotopy theory of algebras is developed. Explicit fibrant replacements for the S^1-spectrum and $(S^1, \mathbb{G})$-bispectrum of an algebra are constructed. As an application, unstable, Morita stable and stable universal bivariant theories are recovered. These are shown to be embedded by means of contravariant equivalences as full triangulated subcategories of compact generators of some compactly generated triangulated categories. Another application is to introduce and study the symmetric monoidal compactly generated triangulated category of K-motives. It is established that the triangulated category kk of Cortiñas–Thom [3] can be identified with K-motives of algebras. It is proved that the triangulated category of K-motives is a localization of the triangulated category of $(S^1, \mathbb{G})$-bispectra. Also, explicit fibrant $(S^1, \mathbb{G})$-bispectra representing stable algebraic Kasparov K-theory and algebraic homotopy K-theory are constructed.

Contents

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1. Introduction

$\mathbb{A}^1$-homotopy theory is a homotopy theory of motivic spaces, where each smooth algebraic variety $X \in Sm/F$ is regarded as the motivic space $\operatorname{Hom}_{Sm/F}(-, X)$ (see [22, 26]).

$k[t]$-homotopy theory is a homotopy theory of simplicial functors defined on algebras, where each algebra A is regarded as the space $rA = \operatorname{Hom}_{\mathrm{Alg}_k}(A, -)$, so that we can study algebras from a homotopy theoretic viewpoint (see [9, 11]). Many of basic ideas and techniques in this subject originate in $\mathbb{A}^1$-homotopy theory. Another source of ideas for $k[t]$-homotopy theory comes from Kasparov K-theory of C^*-algebras.

In [9] a kind of unstable motivic homotopy theory of algebras was developed. In order to develop stable motivic homotopy theory of algebras and – *what is most important* – make explicit computations presented in this paper, one first needs to introduce and study "unstable, Morita stable and stable Kasparov K-theory spectra" $\mathbb{K}^{unst}(A, B)$, $\mathbb{K}^{mor}(A, B)$ and $\mathbb{K}^{st}(A, B)$ respectively, where A, B are two algebras. We refer the reader to [11] for properties of the spectra. This paper is to develop stable motivic homotopy theory of algebras.

Throughout we work with a reasonable category of algebras $\Re$ and the category $U_\bullet\Re$ of some pointed simplicial functors on $\Re$. $U_\bullet\Re$ comes equipped with a motivic model structure. We write $Sp(\Re)$ to denote its category of S^1-spectra. $\mathbb{K}^{unst}(A, B)$, $\mathbb{K}^{mor}(A, B)$ and $\mathbb{K}^{st}(A, B)$ are examples of fibrant Ω-spectra in $Sp(\Re)$ (see [11]).

One of the main results of the paper computes a fibrant replacement of the suspension spectrum $\Sigma^\infty rA$ of an algebra A.

Theorem. *Given an algebra $A \in \Re$, there is a natural weak equivalence of spectra*

$$\Sigma^\infty rA \longrightarrow \mathbb{K}^{unst}(A, -)$$

in $Sp(\Re)$.

Let $SH_{S^1}(\Re)$ denote the homotopy category of $Sp(\Re)$. It is a compactly generated triangulated category with compact generators $\{\Sigma^\infty rA[n]\}_{A\in\Re, n\in\mathbb{Z}}$. One of the consequences of the theorem says that there is an isomorphism of abelian groups

$$SH_{S^1}(\Re)(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}_n^{unst}(A, B), \quad A, B \in \Re, n \in \mathbb{Z}.$$

Another consequence of the theorem states that the full subcategory $\mathcal{S}$ of $SH_{S^1}(\Re)$ spanned by the compact generators $\{\Sigma^\infty rA[n]\}_{A\in\Re, n\in\mathbb{Z}}$ is triangulated and there is a contravariant equivalence of triangulated categories

$$D(\Re, \mathfrak{F}) \xrightarrow{\sim} \mathcal{S},$$

where $j : \Re \to D(\Re, \mathfrak{F})$ is the universal unstable excisive homotopy invariant homology theory in the sense of [10]. Thus $D(\Re, \mathfrak{F})$ is recovered from $SH_{S^1}(\Re)$.

If we localize $SH_{S^1}(\Re)$ with respect to the family of compact objects

$$\{\operatorname{cone}(\Sigma^\infty r(M_n A) \to \Sigma^\infty rA)\}_{n>0}$$

we shall get a compactly generated triangulated category $SH_{S^1}^{mor}(\Re)$ with compact generators $\{\Sigma^\infty rA[n]\}_{A\in\Re, n\in\mathbb{Z}}$. It is in fact the homotopy category of a model category $Sp_{mor}(\Re)$. We construct in a similar way a compactly generated triangulated category

$SH^{\infty}_{S^1}(\Re)$, obtained from $SH_{S^1}(\Re)$ by localization with respect to the family of compact objects

$$\{\mathrm{cone}(\Sigma^\infty r(M_\infty A)\to\Sigma^\infty rA)\},$$

where $M_\infty A = \cup_n M_n A$. It is as well the homotopy category of a model category $Sp_\infty(\Re)$.

Theorem. *Given an algebra $A\in\Re$, there are a natural weak equivalences of spectra*

$$\Sigma^\infty rA \longrightarrow \mathbb{K}^{mor}(A,-)$$

and

$$\Sigma^\infty rA \longrightarrow \mathbb{K}^{st}(A,-)$$

in $Sp_{mor}(\Re)$ and $Sp_\infty(\Re)$ respectively.

As above the preceding theorem implies that for all $A,B\in\Re$ and $n\in\mathbb{Z}$ there are isomorphisms of abelian groups

$$SH^{mor}_{S^1}(\Re)(\Sigma^\infty rB[n],\Sigma^\infty rA)\cong\mathbb{K}^{mor}_n(A,B)$$

and

$$SH^{\infty}_{S^1}(\Re)(\Sigma^\infty rB[n],\Sigma^\infty rA)\cong\mathbb{K}^{st}_n(A,B)$$

respectively.

Another consequence of the theorem states that the full subcategories $\mathcal{S}_{mor}$ and $\mathcal{S}_\infty$ of $SH^{mor}_{S^1}(\Re)$ and $SH^{\infty}_{S^1}(\Re)$ spanned by the compact generators $\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}$ are triangulated and there are contravariant equivalence of triangulated categories

$$D_{mor}(\Re,\mathfrak{F})\xrightarrow{\sim}\mathcal{S}_{mor}$$

and

$$D_{st}(\Re,\mathfrak{F})\xrightarrow{\sim}\mathcal{S}_{\infty}.$$

Here $j:\Re\to D_{mor}(\Re,\mathfrak{F})$ (respectively $j:\Re\to D_{st}(\Re,\mathfrak{F})$) is the universal Morita stable (respectively stable) excisive homotopy invariant homology theory in the sense of [10]. Thus $D_{mor}(\Re,\mathfrak{F})$ and $D_{st}(\Re,\mathfrak{F})$ are recovered from $SH^{mor}_{S^1}(\Re)$ and $SH^{st}_{S^1}(\Re)$ respectively.

We next introduce the symmetric monoidal compactly generated triangulated category of K-motives $DK(\Re)$ together with a canonical contravariant functor

$$M_K:\Re\to DK(\Re).$$

The category $DK(\Re)$ is an analog of the triangulated category of K-motives for algebraic varieties introduced in [12].

For any algebra $A\in\Re$ its K-motive is, by definition, the object $M_K(A)$ of $DK(\Re)$. We have that

$$M_K(A)\otimes M_K(B)=M_K(A\otimes B)$$

for all $A,B\in\Re$. We also have the following

Theorem. *For any two algebras $A,B\in\Re$ and any integer n one has a natural isomorphism*

$$DK(\Re)(M_K(B)[n],M_K(A))\cong\mathbb{K}^{st}_n(A,B).$$

If $\mathcal{T}$ is the full subcategory of $DK(\Re)$ spanned by K-motives of algebras $\{M_K(A)\}_{A\in\Re}$ then $\mathcal{T}$ is triangulated and there is an equivalence of triangulated categories

$$D_{st}(\Re,\mathfrak{F})\to\mathcal{T}^{\mathrm{op}}$$

sending an algebra $A\in\Re$ to its K-motive $M_K(A)$.

We also prove that for any algebra $A \in \Re$ and any integer n one has a natural isomorphism

$$DK(\Re)(M_K(A)[n], M_K(k)) \cong KH_n(A),$$

where the right hand side is the n-th homotopy K-theory group in the sense of Weibel [28]. This result is a reminiscence of a similar result for K-motives of algebraic varieties in the sense of [12] identifying the K-motive of the point with algebraic K-theory.

In [3] Cortiñas–Thom constructed a universal excisive homotopy invariant and M_∞-invariant homology theory on all k-algebras

$$j : \mathrm{Alg}_k \to kk.$$

The triangulated category kk is an analog of Cuntz's triangulated category kk^{lca} whose objects are the locally convex algebras [4, 5, 6].

If we denote by $kk(\Re)$ the full subcategory of kk spanned by the objects from $\Re$ and assume that the cone ring Γk in the sense of [20] is in $\Re$ then we show that there is a natural triangulated equivalence

$$kk(\Re) \xrightarrow{\sim} \mathcal{T}^{\mathrm{op}}$$

sending $A \in kk(\Re)$ to its K-motive $M_K(A)$. Thus we can identify $kk(\Re)$ with K-motives of algebras.

One of the equivalent approaches to the motivic stable homotopy theory in the sense of Morel–Voevodsky [22] is the theory of $(S^1, \mathbb{G}_m)$-bispectra. The role of $\mathbb{G}_m$ in our context plays the representable functor $\mathbb{G} := r(\sigma)$, where $\sigma = (t-1)k[t^{\pm 1}]$. We develop the motivic theory of $(S^1, \mathbb{G})$-bispectra. As usual they form a model category which we denote by $Sp_{\mathbb{G}}(\Re)$. We construct an explicit fibrant $(S^1, \mathbb{G})$-bispectrum $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A, -)$, obtained from fibrant S^1-spectra $\mathbb{K}^{unst}(\sigma^n A, -)$, $n \geqslant 0$, by stabilization in σ-direction.

The main computational result for bispectra says that $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A, -)$ is a fibrant replacement of the suspension bispectrum associated with an algebra A.

Theorem. *Let A be an algebra in $\Re$; then there is a natural weak equivalence of bispectra in $Sp_{\mathbb{G}}(\Re)$*

$$\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA \to \Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A, -),$$

where $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA$ is the suspension bispectrum of rA.

Let k be the field of complex numbers $\mathbb{C}$ and let $\mathcal{K}^\sigma(A, -)$ be the (0,0)-space of the bispectrum $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A, -)$. We raise a question that there is a category of commutative $\mathbb{C}$-algebras $\Re$ such that the fibrant simplicial set $\mathcal{K}^\sigma(\mathbb{C}, \mathbb{C})$ has homotopy type of $\Omega^\infty\Sigma^\infty S^0$. The question is justified by a recent result of Levine [21] saying that over an algebraically closed field F of characteristic zero the homotopy groups of weight zero of the motivic sphere spectrum evaluated at F are isomorphic to the stable homotopy groups of the classical sphere spectrum. The role of the motivic sphere spectrum in our context plays the bispectrum $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty r\mathbb{C}$.

We finish the paper by proving that the triangulated category of K-motives $DK(\Re)$ is fully faithfully embedded into the homotopy category of $(S^1, \mathbb{G})$-bispectra. We also construct an explicit fibrant $(S^1, \mathbb{G})$-bispectrum $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}^{st}(A, -)$, obtained from fibrant S^1-spectra $\mathbb{K}^{st}(\sigma^n A, -)$, $n \geqslant 0$, by stabilization in σ-direction. The following result says that the bispectrum $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}^{st}(A, -)$ is $(2, 1)$-periodic and represents stable algebraic Kasparov K-theory (cf. Voevodsky [26, 6.8-6.9]).

Theorem. *For any algebras $A, B \in \Re$ and any integers p, q there is an isomorphism*

$$\pi_{p,q}(\Theta^\infty_{\mathbb{G}} \mathbb{KG}^{st}(A, B)) \cong \mathbb{K}^{st}_{p-2q}(A, B).$$

As a consequence of the theorem one has that for any algebra $B \in \Re$ and any integers p, q there is an isomorphism

$$\pi_{p,q}(\Theta^\infty_{\mathbb{G}} \mathbb{KG}^{st}(k, B)) \cong KH_{p-2q}(B).$$

Thus the bispectrum $\mathbb{KG}^{st}(k, B)$ yields a model for homotopy K-theory.

Throughout the paper k is a fixed commutative ring with unit and Alg_k is the category of non-unital k-algebras and non-unital k-homomorphisms. If there is no likelihood of confusion, we replace $\otimes_k$ by $\otimes$. If $\mathcal{C}$ is a category and A, B are objects of $\mathcal{C}$, we shall often write $\mathcal{C}(A, B)$ to denote the Hom-set $\mathrm{Hom}_{\mathcal{C}}(A, B)$.

In general, we shall not be very explicit about set-theoretical foundations, and we shall tacitly assume we are working in some fixed universe $\mathbb{U}$ of sets. Members of $\mathbb{U}$ are then called *small sets*, whereas a collection of members of $\mathbb{U}$ which does not itself belong to $\mathbb{U}$ will be referred to as a *large set* or a *proper class*.

2. Preliminaries

In this section we collect basic facts about admissible categories of algebras and triangulated categories associated with them. We mostly follow [9, 10].

2.1. *Algebraic homotopy*

Following Gersten [13] a category of non-unital k-algebras $\Re$ is *admissible* if it is a full subcategory of Alg_k and

1. R in $\Re$, I a (two-sided) ideal of R then I and R/I are in $\Re$;
2. if R is in $\Re$, then so is $R[x]$, the polynomial algebra in one variable;
3. given a cartesian square

$$\begin{array}{ccc} D & \xrightarrow{\rho} & A \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

   in Alg_k with A, B, C in $\Re$, then D is in $\Re$.

$\Re$ is said to be *tensor closed* if $k \in \Re$ and $A \otimes B \in \Re$ for all $A, B \in \Re$. Observe that $\Re$ is a symmetric monoidal category in this case with k a monoidal unit.

Observe that every algebra which is isomorphic to an algebra from $\Re$ belongs to $\Re$. One may abbreviate 1, 2, and 3 by saying that $\Re$ is closed under operations of taking ideals, homomorphic images, polynomial extensions in a finite number of variables, and pullbacks. For instance, the category of commutative k-algebras CAlg_k is admissible.

Recall that an algebra A is *square zero* if $A^2 = 0$. If we regard every k-module M as a non-unital k-algebra with trivial multiplication $m_1 \cdot m_2 = 0$ for all $m_1, m_2 \in M$, then $\mathrm{Mod}\, k$ is an admissible category of k-algebras coinciding with the category of square zero algebras.

If R is an algebra then the polynomial algebra $R[x]$ admits two homomorphisms onto R

$$R[x] \underset{\partial_x^1}{\overset{\partial_x^0}{\rightrightarrows}} R$$

where

$$\partial_x^i|_R = 1_R, \quad \partial_x^i(x) = i, \quad i = 0, 1.$$

Of course, $\partial_x^1(x) = 1$ has to be understood in the sense that $\Sigma r_n x^n \mapsto \Sigma r_n$.

Definition. Two homomorphisms $f_0, f_1 : S \to R$ are *elementary homotopic*, written $f_0 \sim f_1$, if there exists a homomorphism

$$f : S \to R[x]$$

such that $\partial_x^0 f = f_0$ and $\partial_x^1 f = f_1$. A map $f : S \to R$ is called an *elementary homotopy equivalence* if there is a map $g : R \to S$ such that fg and gf are elementary homotopic to id_R and id_S respectively.

For example, let A be a $\mathbb{Z}_{n\geqslant 0}$-graded algebra, then the inclusion $A_0 \to A$ is an elementary homotopy equivalence. The homotopy inverse is given by the projection $A \to A_0$. Indeed, the map $A \to A[x]$ sending a homogeneous element $a_n \in A_n$ to $a_n t^n$ is a homotopy between the composite $A \to A_0 \to A$ and the identity id_A.

The relation "elementary homotopic" is reflexive and symmetric [13, p. 62]. One may take the transitive closure of this relation to get an equivalence relation (denoted by the symbol "$\simeq$"). Following notation of Gersten [14], the set of equivalence classes of morphisms $R \to S$ is written $[R, S]$.

Lemma 2.1 (Gersten [14]). *Given morphisms in* Alg_k

$$R \xrightarrow{f} S \underset{g'}{\overset{g}{\rightrightarrows}} T \xrightarrow{h} U$$

such that $g \simeq g'$*, then* $gf \simeq g'f$ *and* $hg \simeq hg'$*.*

Thus homotopy behaves well with respect to composition and we have category $\mathcal{H}(\mathrm{Alg}_k)$, the *homotopy category of k-algebras*, whose objects are k-algebras and $\mathrm{Hom}_{\mathcal{H}(\mathrm{Alg}_k)}(R, S) = [R, S]$. The homotopy category of an admissible category of algebras $\Re$ will be denoted by $\mathcal{H}(\Re)$. Call a homomorphism $s : A \to B$ an *I-weak equivalence* if its image in $\mathcal{H}(\Re)$ is an isomorphism.

The diagram in Alg_k

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a short exact sequence if f is injective ($\equiv \mathrm{Ker}\, f = 0$), g is surjective, and the image of f is equal to the kernel of g.

Definition. An algebra R is *contractible* if $0 \sim 1$; that is, if there is a homomorphism $f : R \to R[x]$ such that $\partial_x^0 f = 0$ and $\partial_x^1 f = 1_R$.

For example, every square zero algebra $A \in \mathrm{Alg}_k$ is contractible by means of the homotopy $A \to A[x]$, $a \in A \mapsto ax \in A[x]$. Therefore every k-module, regarded as a k-algebra with trivial multiplication, is contractible.

Following Karoubi and Villamayor [20] we define ER, the *path algebra* on R, as the kernel of $\partial_x^0 : R[x] \to R$, so $ER \to R[x] \overset{\partial_x^0}{\to} R$ is a short exact sequence in Alg_k. Also $\partial_x^1 : R[x] \to R$ induces a surjection

$$\partial_x^1 : ER \to R$$

and we define the *loop algebra* ΩR of R to be its kernel, so we have a short exact sequence in Alg_k

$$\Omega R \to ER \overset{\partial_x^1}{\to} R.$$

We call it the *loop extension* of R. Clearly, ΩR is the intersection of the kernels of ∂_x^0 and ∂_x^1. By [13, 3.3] ER is contractible for any algebra R.

2.2. *Categories of fibrant objects*

Definition. Let $\mathcal{A}$ be a category with finite products and a final object e. Assume that $\mathcal{A}$ has two distinguished classes of maps, called *weak equivalences* and *fibrations*. A map is called a *trivial fibration* if it is both a weak equivalence and a fibration. We define a *path space* for an object B to be an object B^I together with maps

$$B \xrightarrow{s} B^I \xrightarrow{(d_0,d_1)} B \times B,$$

where s is a weak equivalence, (d_0, d_1) is a fibration, and the composite is the diagonal map.

Following Brown [2], we call $\mathcal{A}$ a *category of fibrant objects* or a *Brown category* if the following axioms are satisfied.

(A) Let f and g be maps such that gf is defined. If two of f, g, gf are weak equivalences then so is the third. Any isomorphism is a weak equivalence.

(B) The composite of two fibrations is a fibration. Any isomorphism is a fibration.

(C) Given a diagram

$$A \xrightarrow{u} C \xleftarrow{v} B,$$

with v a fibration (respectively a trivial fibration), the pullback $A \times_C B$ exists and the map $A \times_C B \to A$ is a fibration (respectively a trivial fibration).

(D) For any object B in $\mathcal{A}$ there exists at least one path space B^I (not necessarily functorial in B).

(E) For any object B the map $B \to e$ is a fibration.

2.3. *The triangulated category* $D(\Re, \mathfrak{F})$

In what follows we denote by $\mathfrak{F}$ the class of k-split surjective algebra homomorphisms. We shall also refer to $\mathfrak{F}$ as *fibrations*.

Let $\mathfrak{W}$ be a class of weak equivalences in an admissible category of algebras $\Re$ containing homomorphisms $A \to A[t]$, $A \in \Re$, such that the triple $(\Re, \mathfrak{F}, \mathfrak{W})$ is a Brown category.

Definition. The *left derived category* $D^-(\Re, \mathfrak{F}, \mathfrak{W})$ of $\Re$ with respect to $(\mathfrak{F}, \mathfrak{W})$ is the category obtained from $\Re$ by inverting the weak equivalences.

By [10] the family of weak equivalences in the category $\mathcal{H}\mathfrak{R}$ admits a calculus of right fractions. The left derived category $D^-(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ (possibly "large") is obtained from $\mathcal{H}\mathfrak{R}$ by inverting the weak equivalences. The left derived category $D^-(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ is left triangulated (see [9, 10] for details) with Ω a loop functor on it.

There is a general method of stabilizing Ω (see Heller [15]) and producing a triangulated (possibly "large") category $D(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ from the left triangulated structure on $D^-(\mathfrak{R},\mathfrak{F},\mathfrak{W})$.

An object of $D(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ is a pair (A,m) with $A\in D^-(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ and $m\in\mathbb{Z}$. If $m,n\in\mathbb{Z}$ then we consider the directed set $I_{m,n}=\{k\in\mathbb{Z}\mid m,n\leqslant k\}$. The morphisms between (A,m) and $(B,n)\in D(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ are defined by

$$D(\mathfrak{R},\mathfrak{F},\mathfrak{W})[(A,m),(B,n)]:=\operatorname{colim}_{k\in I_{m,n}}D^-(\mathfrak{R},\mathfrak{F},\mathfrak{W})(\Omega^{k-m}(A),\Omega^{k-n}(B)).$$

Morphisms of $D(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ are composed in the obvious fashion. We define the *loop automorphism* on $D(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ by $\Omega(A,m):=(A,m-1)$. There is a natural functor $S:D^-(\mathfrak{R},\mathfrak{F},\mathfrak{W})\to D(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ defined by $A\longmapsto(A,0)$.

$D(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ is an additive category [9, 10]. We define a triangulation $\mathcal{T}r(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ of the pair $(D(\mathfrak{R},\mathfrak{F},\mathfrak{W}),\Omega)$ as follows. A sequence

$$\Omega(A,l)\to(C,n)\to(B,m)\to(A,l)$$

belongs to $\mathcal{T}r(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ if there is an even integer k and a left triangle of representatives $\Omega(\Omega^{k-l}(A))\to\Omega^{k-n}(C)\to\Omega^{k-m}(B)\to\Omega^{k-l}(A)$ in $D^-(\mathfrak{R},\mathfrak{F},\mathfrak{W})$. Then the functor S takes left triangles in $D^-(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ to triangles in $D(\mathfrak{R},\mathfrak{F},\mathfrak{W})$. By [9, 10] $\mathcal{T}r(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ is a triangulation of $D(\mathfrak{R},\mathfrak{F},\mathfrak{W})$ in the classical sense of Verdier [25].

By an $\mathfrak{F}$-*extension* or just *extension* in $\mathfrak{R}$ we mean a short exact sequence of algebras

$$(E):A\to B\overset{\alpha}{\to}C$$

such that $\alpha\in\mathfrak{F}$. Let $\mathcal{E}$ be the class of all $\mathfrak{F}$-extensions in $\mathfrak{R}$.

Definition. Following Cortiñas–Thom [3] a *($\mathfrak{F}$-)excisive homology theory* on $\mathfrak{R}$ with values in a triangulated category $(\mathcal{T},\Omega)$ consists of a functor $X:\mathfrak{R}\to\mathcal{T}$, together with a collection $\{\partial_E:E\in\mathcal{E}\}$ of maps $\partial_E^X=\partial_E\in\mathcal{T}(\Omega X(C),X(A))$. The maps ∂_E are to satisfy the following requirements.

(1) For all $E\in\mathcal{E}$ as above,

$$\Omega X(C)\xrightarrow{\partial_E}X(A)\xrightarrow{X(f)}X(B)\xrightarrow{X(g)}X(C)$$

is a distinguished triangle in $\mathcal{T}$.

(2) If

$$\begin{array}{ccccc}(E): & A & \xrightarrow{f} & B & \xrightarrow{g} & C\\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma}\\ (E'): & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'\end{array}$$

is a map of extensions, then the following diagram commutes

$$\begin{array}{ccc} \Omega X(C) & \xrightarrow{\partial_E} & X(A) \\ {\scriptstyle \Omega X(\gamma)}\downarrow & & \downarrow{\scriptstyle X(\alpha)} \\ \Omega X(C') & \xrightarrow[\partial_{E'}]{} & X(A). \end{array}$$

We say that the functor $X : \mathfrak{R} \to \mathcal{T}$ is *homotopy invariant* if it maps homotopic homomomorphisms to equal maps, or equivalently, if for every $A \in \mathrm{Alg}_k$, X maps the inclusion $A \subset A[t]$ to an isomorphism.

Denote by $\mathfrak{W}_\triangle$ the class of homomorphisms f such that $X(f)$ is an isomorphism for any excisive, homotopy invariant homology theory $X : \mathfrak{R} \to \mathcal{T}$. We shall refer to the maps from $\mathfrak{W}_\triangle$ as *stable weak equivalences.* The triple $(\mathfrak{R}, \mathfrak{F}, \mathfrak{W}_\triangle)$ is a Brown category. In what follows we shall write $D^-(\mathfrak{R}, \mathfrak{F})$ and $D(\mathfrak{R}, \mathfrak{F})$ to denote $D^-(\mathfrak{R}, \mathfrak{F}, \mathfrak{W}_\triangle)$ and $D(\mathfrak{R}, \mathfrak{F}, \mathfrak{W}_\triangle)$ respectively, dropping $\mathfrak{W}_\triangle$ from notation.

By [10] the canonical functor

$$\mathfrak{R} \to D(\mathfrak{R}, \mathfrak{F})$$

is the universal excisive, homotopy invariant homology theory on $\mathfrak{R}$.

3. Homotopy theory of algebras

Let $\mathfrak{R}$ be a *small* admissible category of algebras. We shall work with various model category structures for the category of simplicial functors on $\mathfrak{R}$. We mostly adhere to [9, 11].

3.1. *Bousfield localization*

Recall from [16] that if $\mathcal{M}$ is a model category and S a set of maps between cofibrant objects, we shall produce a new model structure on $\mathcal{M}$ in which the maps S are weak equivalences. The new model structure is called the *Bousfield localization* or just localization of the old one. Since all model categories we shall consider are simplicial one can use the simplicial mapping object instead of the homotopy function complex for the localization theory of $\mathcal{M}$.

Definition. Let $\mathcal{M}$ be a simplicial model category and let S be a set of maps between cofibrant objects.

(1) An *S-local object* of $\mathcal{M}$ is a fibrant object X such that for every map $A \to B$ in S, the induced map of $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence of simplicial sets.
(2) An *S-local equivalence* is a map $A \to B$ such that $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence for every S-local object X.

In words, the S-local objects are the ones which see every map in S as if it were a weak equivalence. The S-local equivalences are those maps which are seen as weak equivalences by every S-local object.

Theorem 3.1 (Hirschhorn [16]). *Let $\mathcal{M}$ be a cellular, simplicial model category and let S be a set of maps between cofibrant objects. Then there exists a new model structure on $\mathcal{M}$ in which*

1. *the weak equivalences are the S-local equivalences;*
2. *the cofibrations in $\mathcal{M}/S$ are the same as those in $\mathcal{M}$;*
3. *the fibrations are the maps having the right-lifting-property with respect to cofibrations which are also S-local equivalences.*

Left Quillen functors from $\mathcal{M}/S$ to $\mathcal{D}$ are in one to one correspondence with left Quillen functors $\Phi : \mathcal{M} \to \mathcal{D}$ such that $\Phi(f)$ is a weak equivalence for all $f \in S$. In addition, the fibrant objects of $\mathcal{M}$ are precisely the S-local objects, and this new model structure is again cellular and simplicial.

The model category whose existence is guaranteed by the above theorem is called *S-localization* of $\mathcal{M}$. The underlying category is the same as that of $\mathcal{M}$, but there are more trivial cofibrations (and hence fewer fibrations). We sometimes use $\mathcal{M}/S$ to denote the S-localization.

Note that the identity maps yield a Quillen pair $\mathcal{M} \rightleftarrows \mathcal{M}/S$, where the left Quillen functor is the map $\mathrm{id} : \mathcal{M} \to \mathcal{M}/S$.

3.2. *The categories of pointed simplicial functors $U_\bullet\Re$*

Throughout this paper we work with a model category $U_\bullet\Re$. To define it, we first enrich $\Re$ over pointed simplicial sets $\mathbb{S}_\bullet$. Given an algebra $A \in \Re$, denote by rA the representable functor $\mathrm{Hom}_\Re(A, -)$. Let $\Re_\bullet$ have the same objects as $\Re$ and whose pointed simplicial sets of morphisms are $rA(B) = \mathrm{Hom}_\Re(A, B)$ pointed at zero. Denote by $U_\bullet\Re$ the category of $\mathbb{S}_\bullet$-enriched functors from $\Re_\bullet$ to $\mathbb{S}_\bullet$. One easily checks that $U_\bullet\Re$ can be regarded as the category of covariant pointed simplicial functors $X : \Re \to \mathbb{S}_\bullet$ such that $X(0) = *$.

By [8, 4.2] we define the projective model structure on $U_\bullet\Re$. This is a proper, simplicial, cellular model category with weak equivalences and fibrations being defined objectwise, and cofibrations being those maps having the left lifting property with respect to trivial fibrations.

The class of projective cofibrations for $U_\bullet\Re$ is generated by the set

$$I_{U_\bullet\Re} \equiv \{rA \wedge (\partial\Delta^n \subset \Delta^n)_+\}^{n\geqslant 0}$$

indexed by $A \in \Re$. Likewise, the class of acyclic projective cofibrations is generated by

$$J_{U_\bullet\Re} \equiv \{rA \wedge (\Lambda^k_n \subset \Delta^n)_+\}^{n>0}_{0\leqslant k\leqslant n}.$$

Given $\mathcal{X}, \mathcal{Y} \in U_\bullet\Re$ the pointed function complex $\mathrm{Map}_\bullet(\mathcal{X}, \mathcal{Y})$ is defined as

$$\mathrm{Map}_\bullet(\mathcal{X}, \mathcal{Y})_n = \mathrm{Hom}_{U_\bullet\Re}(\mathcal{X} \wedge \Delta^n_+, \mathcal{Y}), \quad n \geqslant 0.$$

By [8, 2.1] there is a natural isomorphism of pointed simplicial sets

$$\mathrm{Map}_\bullet(rA, \mathcal{X}) \cong \mathcal{X}(A)$$

for all $A \in \Re$ and $\mathcal{X} \in U_\bullet\Re$.

Recall that the model category $U\Re$ of functors from $\Re$ to unpointed simplicial sets $\mathbb{S}$ is defined in a similar fashion (see [9]). Since we mostly work with spectra in this paper, the category of spectra associated with $U_\bullet \Re$ is technically more convenient the category of spectra associated with $U\Re$.

3.3. *The model categories* $U_\bullet \Re_I, U_\bullet \Re_J, U_\bullet \Re_{I,J}$

Let $I = \{i = i_A : r(A[t]) \to r(A) \mid A \in \Re\}$, where each i_A is induced by the natural homomorphism $i : A \to A[t]$. Recall that a functor $F : \Re \to \mathbb{S}_\bullet/Spectra$ is *homotopy invariant* if $F(A) \to F(A[t])$ is a weak equivalence for all $A \in \Re$. Consider the projective model structure on $U_\bullet \Re$. We shall refer to the I-local equivalences as (projective) I-weak equivalences. The resulting model category $U_\bullet \Re/I$ will be denoted by $U_\bullet \Re_I$. Notice that any objectwise fibrant homotopy invariant functor $F \in U_\bullet \Re$ is an I-local object, hence fibrant in $U_\bullet \Re_I$.

Let us introduce the class of excisive functors on $\Re$. They look like flasque presheaves on a site defined by a cd-structure in the sense of Voevodsky [27, section 3].

Definition. A simplicial functor $\mathcal{X} \in U\Re$ is called *excisive* with respect to $\mathfrak{F}$ if for any cartesian square in $\Re$

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & C \end{array}$$

with f a fibration (call such squares *distinguished*) the square of simplicial sets

$$\begin{array}{ccc} \mathcal{X}(D) & \longrightarrow & \mathcal{X}(A) \\ \downarrow & & \downarrow \\ \mathcal{X}(B) & \longrightarrow & \mathcal{X}(C) \end{array}$$

is a homotopy pullback square. It immediately follows from the definition that every excisive object takes $\mathfrak{F}$-extensions in $\Re$ to homotopy fibre sequences of simplicial sets.

Let α denote a distinguished square in $\Re$

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

Let us apply the simplicial mapping cylinder construction cyl to α and form the pushouts:

$$\begin{array}{ccccc} rC & \longrightarrow & \mathrm{cyl}(rC \to rA) & \longrightarrow & rA \\ \downarrow & & \downarrow & & \downarrow \\ rB & \longrightarrow & \mathrm{cyl}(rC \to rA) \coprod_{rC} rB & \longrightarrow & rD \end{array}$$

Note that $rC \to \mathrm{cyl}(rC \to rA)$ is a projective cofibration between (projective) cofibrant objects of $U_\bullet \Re$. Thus $s(\alpha) \equiv \mathrm{cyl}(rC \to rA) \coprod_{rC} rB$ is (projective) cofibrant [17, 1.11.1]. For the same reasons, applying the simplicial mapping cylinder to $s(\alpha) \to rD$ and setting $t(\alpha) \equiv \mathrm{cyl}\big(s(\alpha) \to rD\big)$ we get a projective cofibration

$$\mathrm{cyl}(\alpha)\colon s(\alpha) \longrightarrow t(\alpha).$$

Let $J^{\mathrm{cyl}(\alpha)}_{U_\bullet\Re}$ consists of all pushout product maps

$$s(\alpha)\wedge\Delta^n_+\coprod\nolimits_{s(\alpha)\wedge\partial\Delta^n_+}t(\alpha)\wedge\partial\Delta^n_+\longrightarrow t(\alpha)\wedge\Delta^n_+$$

and let $J\equiv J_{U_\bullet\Re}\cup J^{\mathrm{cyl}(\alpha)}_{U_\bullet\Re}$. It is directly verified that $\mathcal{X}\in U_\bullet\Re$ is J-local if and only if it has the right lifting property with respect to J. Also, $\mathcal{X}$ is J-local if and only if it is objectwise fibrant and excisive [9, 4.3].

Finally, let us introduce the model category $U_\bullet\Re_{I,J}$. It is, by definition, the Bousfield localization of $U_\bullet\Re$ with respect to $I\cup J$. The weak equivalences (trivial cofibrations) of $U_\bullet\Re_{I,J}$ will be referred to as (projective) (I,J)-weak equivalences ((projective) (I,J)-trivial cofibrations). By [9, 4.5] a functor $\mathcal{X}\in U_\bullet\Re$ is (I,J)-local if and only if it is objectwise fibrant, homotopy invariant and excisive.

Remark. The model category $U_\bullet\Re_{I,J}$ can also be regarded as a kind of unstable motivic model category associated with $\Re$. Indeed, the construction of $U_\bullet\Re_{I,J}$ is similar to Morel–Voevodsky's unstable motivic theory for smooth schemes Sm/F over a field F [22]. If we replace the family I by

$$I'=\{X\times\mathbb{A}^1\xrightarrow{pr}X\mid X\in Sm/F\}$$

and the family of distinguished squares by the family of elementary Nisnevich squares and get the corresponding family J' associated to it, then one of the equivalent models for Morel–Voevodsky's unstable motivic theory is obtained by Bousfield localization of simplicial presheaves with respect to $I'\cup J'$.

For this reason, $U_\bullet\Re_{I,J}$ can also be called the category of (pointed) motivic spaces, where each algebra A is identified with the pointed motivic space rA. One can also refer to (I,J)-weak equivalences as motivic weak equivalences.

3.4. *Monoidal structure on $U_\bullet\Re$*

In this section we mostly follow [24, section 2.1] Suppose $\Re$ is tensor closed, that is $k, A\otimes B\in\Re$ for all $A,B\in\Re$. We introduce the monoidal product $\mathcal{X}\otimes\mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$ in $U_\bullet\Re$ by the formulas

$$\mathcal{X}\otimes\mathcal{Y}(A)\equiv\operatorname*{colim}_{A_1\otimes A_2\to A}\mathcal{X}(A_1)\wedge\mathcal{Y}(A_2).$$

The colimit is indexed on the category with objects $\alpha\colon A_1\otimes A_2\to A$ and maps pairs of maps $(\varphi,\psi)\colon(A_1,A_2)\to(A_1',A_2')$ such that $\alpha'(\psi\otimes\varphi)=\alpha$. By functoriality of colimits it follows that $\mathcal{X}\otimes\mathcal{Y}$ is in $U_\bullet\Re$.

The tensor product can also be defined by the formula

$$\mathcal{X}\otimes\mathcal{Y}(A)=\int^{A_1,A_2\in\Re}\big(\mathcal{X}(A_1)\wedge\mathcal{Y}(A_2)\big)\wedge\mathrm{Hom}_\Re(A_1\otimes A_2,A).$$

This formula is obtained from a theorem of Day [7], which also asserts that the triple $(U_\bullet\Re,\otimes,r(k))$ forms a closed symmetric monoidal category.

The internal Hom functor, right adjoint to $\mathcal{X}\otimes-$, is given by

$$\underline{\mathrm{Hom}}(\mathcal{X},\mathcal{Y})(A)=\int_{B\in\Re}\mathrm{Map}_\bullet(\mathcal{X}(B),\mathcal{Y}(A\otimes B)),$$

where $\mathrm{Map}_\bullet$ stands for the function complex in $\mathbb{S}_\bullet$.

So there exist natural isomorphisms

$$\underline{\mathrm{Hom}}(\mathcal{X}\otimes\mathcal{Y},\mathcal{Z})\cong\underline{\mathrm{Hom}}\big(\mathcal{X},\underline{\mathrm{Hom}}(\mathcal{Y},\mathcal{Z})\big)$$

and

$$\underline{\mathrm{Hom}}(r(k),\mathcal{Z})\cong\mathcal{Z}.$$

Concerning smash products of representable functors, one has a natural isomorphism

$$rA\otimes rB\cong r(A\otimes B),\quad A,B\in\Re.$$

Note as well that for pointed simplicial sets K and L, one has $K\otimes L=K\wedge L$.

We recall a pointed simplicial set tensor and cotensor structure on $U_\bullet\Re$. If $\mathcal{X}$ and $\mathcal{Y}$ are in $U_\bullet\Re$ and K is a pointed simplicial set, the tensor $\mathcal{X}\otimes K$ is given by

$$\mathcal{X}\otimes K(A)\equiv\mathcal{X}(A)\wedge K$$

and the cotensor $\mathcal{Y}^K$ in terms of the ordinary function complex

$$\mathcal{Y}^K(A)\equiv\mathrm{Map}_\bullet\big(K,\mathcal{Y}(A)\big).$$

The function complex $\mathrm{Map}_\bullet(\mathcal{X},\mathcal{Y})$ of $\mathcal{X}$ and $\mathcal{Y}$ is defined by setting

$$\mathrm{Map}_\bullet(\mathcal{X},\mathcal{Y})_n\equiv\mathrm{Hom}_{U_\bullet\Re}(\mathcal{X}\otimes\Delta^n_+,\mathcal{Y}).$$

By the Yoneda lemma there exists a natural isomorphism of pointed simplicial sets

$$\mathrm{Map}_\bullet(rA,\mathcal{Y})\cong\mathcal{Y}(A).$$

Using these definitions $U_\bullet\Re$ is enriched in pointed simplicial sets $\mathbb{S}_\bullet$. Moreover, there are natural isomorphisms of pointed simplicial sets

$$\mathrm{Map}_\bullet(\mathcal{X}\otimes K,\mathcal{Y})\cong\mathrm{Map}_\bullet\big(K,\mathrm{Map}_\bullet(\mathcal{X},\mathcal{Y})\big)\cong\mathrm{Map}_\bullet(\mathcal{X},\mathcal{Y}^K).$$

It is also useful to note that

$$\underline{\mathrm{Hom}}(\mathcal{X},\mathcal{Y})(A)=\mathrm{Map}_\bullet\big(\mathcal{X},\mathcal{Y}(A\otimes -)\big).$$

and

$$\underline{\mathrm{Hom}}(rB,\mathcal{Y})=\mathcal{Y}(-\otimes B).$$

It can be shown similar to [24, 3.10; 3.43; 3.89] that the model categories $U_\bullet\Re$, $U_\bullet\Re_I$, $U_\bullet\Re_J$, $U_\bullet\Re_{I,J}$ are monoidal.

4. Unstable algebraic Kasparov K-theory

Let $\mathcal{U}$ be an arbitrary category and let $\Re$ be an admissible category of k-algebras. Suppose that there are functors $F:\Re\to\mathcal{U}$ and $\widetilde{T}:\mathcal{U}\to\Re$ such that $\widetilde{T}$ is left adjoint to F. We denote $\widetilde{T}FA, A\in\Re$, and the counit map $\widetilde{T}FA\to A$ by TA and η_A respectively. If $X\in\mathrm{Ob}\mathcal{U}$ then the unit map $X\to F\widetilde{T}X$ is denoted by i_X. We note that the composition

$$FA\xrightarrow{i_{FA}}F\widetilde{T}FA\xrightarrow{F\eta_A}FA$$

equals 1_{FA} for every $A\in\Re$, and hence $F\eta_A$ splits in $\mathcal{U}$. We call an admissible category of k-algebras *T-closed* if $TA\in\Re$ for all $A\in\Re$.

Lemma 4.1. *Suppose $\mathcal{U}$ is either a full subcategory of sets or a full subcategory of k-modules. Suppose as well that $F:\Re\to\mathcal{U}$ is the forgetful functor. Then for every $A\in\Re$ the algebra TA is contractible, i.e. there is a contraction $\tau:TA\to TA[x]$ such that $\partial^0_x\tau=0,\partial^1_x\tau=1$. Moreover, the contraction is functorial in A.*

Proof. Consider a map $u : FTA \to FTA[x]$ sending an element $b \in FTA$ to $bx \in FTA[x]$. By assumption, u is a morphisms of $\mathcal{U}$. The desired contraction τ is uniquely determined by the map $u \circ i_{FA} : FA \to FTA[x]$. By using elementary properties of adjoint functors, one can show that $\partial_x^0 \tau = 0, \partial_x^1 \tau = 1$. □

Throughout this paper, whenever we deal with a T-closed admissible category of k-algebras $\Re$ we assume fixed an underlying category $\mathcal{U}$, which is a full subcategory of $\operatorname{Mod} k$.

Examples. (1) Let $\Re = \mathrm{Alg}_k$. Given an algebra A, consider the algebraic tensor algebra

$$TA = A \oplus A \otimes A \oplus A^{\otimes^3} \oplus \cdots$$

with the usual product given by concatenation of tensors. In Cuntz's treatment of bivariant K-theory [4, 5, 6], tensor algebras play a prominent role.

There is a canonical k-linear map $A \to TA$ mapping A into the first direct summand. Every k-linear map $s : A \to B$ into an algebra B induces a homomorphism $\gamma_s : TA \to B$ defined by

$$\gamma_s(x_1 \otimes \cdots \otimes x_n) = s(x_1)s(x_2)\cdots s(x_n).$$

$\Re$ is plainly T-closed.

(2) If $\Re = \mathrm{CAlg}_k$ then

$$T(A) = Sym(A) = \oplus_{n \geqslant 1} S^n A, \quad S^n A = A^{\otimes n}/\langle a_1 \otimes \cdots \otimes a_n - a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \rangle,$$

the symmetric algebra of A, and $\Re$ is T-closed.

Given a small T-closed admissible category of k-algebras $\Re$, we denote by $Sp(\Re)$ the category of S^1-spectra in the sense of Hovey [18] associated with the model category $U_\bullet \Re_{I,J}$. Recall that a spectrum consists of sequences $\mathcal{E} \equiv (\mathcal{E}_n)_{n \geqslant 0}$ of pointed simplicial functors in $U_\bullet \Re$ equipped with structure maps $\sigma_n^{\mathcal{E}} : \Sigma \mathcal{E}_n \to \mathcal{E}_{n+1}$ where $\Sigma = - \wedge S^1$ is the suspension functor. A map $f : \mathcal{E} \to \mathcal{F}$ of spectra consists of compatible maps $f_n : \mathcal{E}_n \to \mathcal{F}_n$ in the sense that the diagrams

$$\begin{array}{ccc} \Sigma\mathcal{E}_n & \xrightarrow{\sigma_n^{\mathcal{E}}} & \mathcal{E}_{n+1} \\ {\scriptstyle \Sigma f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma\mathcal{F}_n & \xrightarrow{\sigma_n^{\mathcal{F}}} & \mathcal{F}_{n+1} \end{array}$$

commute for all $n \geqslant 0$. The category $Sp(\Re)$ is endowed with the stable model structure (see [18] for details).

Given an algebra $A \in \Re$, we denote by $\Sigma^\infty rA$ the suspension spectrum associated with the functor rA pointed at zero. By definition, $(\Sigma^\infty rA)_n = rA \wedge S^n$ with obvious structure maps.

In order to define one of the main spectra of the paper $\mathcal{R}(A)$ associated to an algebra $A \in \Re$, we have to recall some definitions from [11].

For any $B \in \Re$ we define a simplicial algebra

$$B^\Delta : [n] \mapsto B^{\Delta^n} := B[t_0, \ldots, t_n]/\langle 1 - \sum_i t_i \rangle \quad (\cong B[t_1, \ldots, t_n]).$$

The face and degeneracy operators $\partial_i : B[\Delta^n] \to B[\Delta^{n-1}]$ and $s_i : B[\Delta^n] \to B[\Delta^{n+1}]$ are given by

$$\partial_i(t_j) \text{ (resp. } s_i(t_j)) = \begin{cases} t_j \text{ (resp. } t_j), & j < i \\ 0 \text{ (resp. } t_j + t_{j+1}), & j = i \\ t_{j-1} \text{ (resp. } t_{j+1}), & i < j \end{cases}$$

We have that $B^\Delta \cong B \otimes k^\Delta$ and B^Δ is pointed at zero.

For any pointed simplicial set $X \in \mathbb{S}_\bullet$, we denote by $B^\Delta(X)$ the simplicial algebra $\mathrm{Map}_\bullet(X, B^\Delta)$. The simplicial algebra associated to any unpointed simplicial set and any simplicial algebra is defined in a similar way. By $\mathbb{B}^\Delta(X)$ we shall mean the pointed simplicial ind-algebra

$$B^\Delta(X) \to B^\Delta(\mathrm{sd}^1 X) \to B^\Delta(\mathrm{sd}^2 X) \to \cdots$$

In particular, one defines the "path space" simplicial ind-algebra $P\mathbb{B}^\Delta$. We shall also write $\mathbb{B}^\Delta(\Omega^n)$ to denote $\mathbb{B}^\Delta(S^n)$, where $S^n = S^1 \wedge \cdots \wedge S^1$ is the simplicial n-sphere. For any $A \in \Re$ we denote by $\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}^\Delta(\Omega^n))$ the colimit of the sequence in $\mathbb{S}_\bullet$

$$\mathrm{Hom}_{\mathrm{Alg}_k}(A, B^\Delta(S^n)) \to \mathrm{Hom}_{\mathrm{Alg}_k}(A, B^\Delta(\mathrm{sd}^1 S^n)) \to \mathrm{Hom}_{\mathrm{Alg}_k}(A, B^\Delta(\mathrm{sd}^2 S^n)) \to \cdots$$

The natural simplicial map $d_1 : P\mathbb{B}^\Delta(\Omega^n) \to \mathbb{B}^\Delta(\Omega^n)$ has a natural k-linear splitting described below. Let $\mathbf{t} \in P\Bbbk^\Delta(\Delta^1 \times \overset{n}{\cdots} \times \Delta^1)_0$ stand for the composite map

$$\mathrm{sd}^m(\Delta^1 \times \overset{n+1}{\cdots} \times \Delta^1) \xrightarrow{pr} \mathrm{sd}^m \Delta^1 \to \Delta^1 \xrightarrow{t} k^\Delta,$$

where pr is the projection onto the $(n+1)$th direct factor Δ^1 and $t = t_0 \in k^{\Delta^1}$. The element $\mathbf{t}$ can be regarded as a 1-simplex of the unital ind-algebra $\Bbbk^\Delta(\Delta^1 \times \overset{n}{\cdots} \times \Delta^1)$ such that $\partial_0(\mathbf{t}) = 0$ and $\partial_1(\mathbf{t}) = 1$. Let $\imath : \mathbb{B}^\Delta(\Omega^n) \to (\mathbb{B}^\Delta(\Omega^n))^{\Delta^1}$ be the natural inclusion. Multiplication with $\mathbf{t}$ determines a k-linear map $(\mathbb{B}^\Delta(\Omega^n))^{\Delta^1} \xrightarrow{\mathbf{t}\cdot} P\mathbb{B}^\Delta(\Omega^n)$. Now the desired k-linear splitting $\mathbb{B}^\Delta(\Omega^n) \xrightarrow{\upsilon} P\mathbb{B}^\Delta(\Omega^n)$ of simplicial ind-modules is defined as

$$\upsilon := \mathbf{t} \cdot \imath.$$

If we consider $\mathbb{B}^\Delta(\Omega^n)$ as a $(\mathbb{Z}_{\geqslant 0} \times \Delta)$-diagram in $\Re$, then there is a commutative diagram of extensions for $(\mathbb{Z}_{\geqslant 0} \times \Delta)$-diagrams

$$\begin{array}{ccccc} J\mathbb{B}^\Delta(\Omega^n) & \longrightarrow & T\mathbb{B}^\Delta(\Omega^n) & \longrightarrow & \mathbb{B}^\Delta(\Omega^n) \\ \downarrow{\scriptstyle \xi_\upsilon} & & \downarrow & & \| \\ \mathbb{B}^\Delta(\Omega^{n+1}) & \longrightarrow & P\mathbb{B}^\Delta(\Omega^n) & \xrightarrow{d_1} & \mathbb{B}^\Delta(\Omega^n), \end{array}$$

where the map ξ_υ is uniquely determined by the k-linear splitting υ. For every element $f \in \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}^\Delta(\Omega^n))$ one sets:

$$\varsigma(f) := \xi_\upsilon \circ J(f) \in \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, \mathbb{B}^\Delta(\Omega^{n+1})).$$

The spectrum $\mathcal{R}(A)$ is defined at every $B \in \Re$ as the sequence of spaces pointed at zero

$$\mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}^\Delta), \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(JA, \mathbb{B}^\Delta), \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^2A, \mathbb{B}^\Delta), \ldots$$

By [11, section 2] each $\mathcal{R}(A)_n(B)$ is a fibrant simplicial set and

$$\Omega^k \mathcal{R}(A)(B) = \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(A, \mathbb{B}^\Delta(\Omega^k)).$$

Each structure map $\sigma_n : \mathcal{R}(A)_n \wedge S^1 \to \mathcal{R}(A)_{n+1}$ is defined at B as adjoint to the map $\varsigma : \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}^\Delta) \to \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^{n+1}A, \mathbb{B}^\Delta(\Omega))$.

For every $A \in \Re$ there is a natural map in $Sp(\Re)$

$$i : \Sigma^\infty rA \to \mathcal{R}(A)$$

functorial in A.

Definition. (see [11]) (1) Given two k-algebras $A, B \in \Re$, the *unstable algebraic Kasparov K-theory* of (A, B) is the space $\mathcal{K}(A, B)$ defined as the fibrant space

$$\mathrm{colim}_n \, \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(J^n A, \mathbb{B}^\Delta(\Omega^n)),$$

where the colimit maps are given by ξ_v-s. Its homotopy groups will be denoted by $\mathcal{K}_n(A, B)$, $n \geqslant 0$. The simplicial functor $\mathcal{K}(A, -)$ is fibrant in $U_\bullet(\Re)_{I,J}$ by [11, section 4]. Also, there is a natural isomorphism of simplicial sets

$$\mathcal{K}(A, B) \cong \Omega\mathcal{K}(JA, B).$$

In particular, $\mathcal{K}(A, B)$ is an infinite loop space with $\mathcal{K}(A, B)$ simplicially isomorphic to $\Omega^n\mathcal{K}(J^n A, B)$ (see [11, 5.1]).

(2) The *unstable algebraic Kasparov KK-theory spectrum* of (A, B) consists of the sequence of spaces

$$\mathcal{K}(A, B), \mathcal{K}(JA, B), \mathcal{K}(J^2A, B), \ldots$$

together with the natural isomorphisms $\mathcal{K}(J^n A, B) \cong \Omega\mathcal{K}(J^{n+1}A, B)$. It forms an Ω-spectrum which we also denote by $\mathbb{K}^{unst}(A, B)$. Its homotopy groups will be denoted by $\mathbb{K}_n^{unst}(A, B)$, $n \in \mathbb{Z}$. We sometimes write $\mathbb{K}(A, B)$ instead of $\mathbb{K}^{unst}(A, B)$, dropping "unst" from notation. Observe that $\mathbb{K}_n(A, B) \cong \mathcal{K}_n(A, B)$ for any $n \geqslant 0$ and $\mathbb{K}_n(A, B) \cong \mathcal{K}_0(J^{-n}A, B)$ for any $n < 0$.

There is a natural map of spectra

$$j : \mathcal{R}(A) \to \mathbb{K}(A, -).$$

By [11, section 6] it is a stable equivalence and $\mathbb{K}(A, -)$ is a fibrant object of $Sp(\Re)$. In fact for any algebra $B \in \Re$ the map

$$j : \mathcal{R}(A)(B) \to \mathbb{K}(A, B)$$

is a stable equivalence of ordinary spectra.

The following theorem is crucial in our analysis. It states that $\mathbb{K}(A, -)$ is a fibrant replacement of $\Sigma^\infty rA$ in $Sp(\Re)$.

Theorem 4.2. *Given $A \in \Re$ the map $i : \Sigma^\infty rA \to \mathcal{R}(A)$ is a level (I, J)-weak equivalence, and therefore the composite map*

$$\Sigma^\infty rA \xrightarrow{i} \mathcal{R}(A) \xrightarrow{j} \mathbb{K}(A, -)$$

is a stable equivalence in $Sp(\Re)$, functorial in A.

Proof. Recall that for any functor F from rings to simplicial sets, $Sing(F)$ is defined at each ring R as the diagonal of the bisimplicial set $F(R[\Delta])$. The map $i_0 : (\Sigma^\infty rA)_0 \to \mathcal{R}(A)_0$ equals $rA \to Ex^\infty \circ Sing(rA)$, which is an I-weak equivalence by [9, 3.8]. Let us show that $i_1 : rA \wedge S^1 \to \mathcal{R}(A)_1 = Ex^\infty \circ Sing(r(JA))$ is an (I, J)-weak equivalence. It is fully determined by the element $\rho_A : JA \to \Omega A$, which is a zero simplex of $\Omega(Ex^\infty \circ Sing(r(JA))(A))$, coming from the adjunction isomorphism

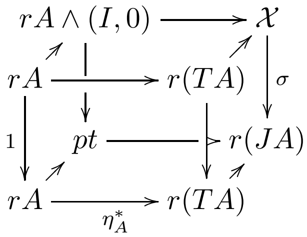
$$\mathrm{Map}_\bullet(rA \wedge S^1, Ex^\infty \circ Sing(r(JA))) \cong \Omega(Ex^\infty \circ Sing(r(JA))(A)).$$

Let $(I,0)$ denote $\Delta[1]$ pointed at 0. Consider a commutative diagram of cofibrant objects in $U_\bullet\Re$

$$\begin{array}{ccccc} rA & \overset{\nu}{\rightarrowtail} & rA \wedge (I,0) & \twoheadrightarrow & rA \wedge S^1 \\ {\scriptstyle \eta_A^*}\downarrow & & \downarrow & & \| \\ r(TA) & \rightarrowtail & \mathcal{X} & \overset{\alpha}{\twoheadrightarrow} & rA \wedge S^1 \end{array}$$

where the left square is pushout, the left map is induced by the canonical homomorphism $\eta_A : TA \to A$, ν is induced by the natural inclusion $d^0 : \Delta[0] \to \Delta[1]$. Lemma 4.1 implies $r(TA)$ is weakly equivalent to zero in $U_\bullet\Re_I$. It follows that α is an I-weak equivalence.

By the universal property of pullback diagrams there is a unique morphism $\sigma : \mathcal{X} \to r(JA)$ whose restriction to $r(TA)$ equals ι_A^*, where $\iota_A = \operatorname{Ker}\eta_A$, which makes the diagram



commutative. Since the upper and the lower squares are homotopy pushouts in $U_\bullet\Re_{I,J}$ and $rA \wedge (I,0)$ is weakly equivalent to zero, it follows from [16, 13.5.10] that σ is an (I,J)-weak equivalence. Therefore the composite map, we shall denote it by ρ,

$$\mathcal{X} \overset{\sigma}{\to} r(JA) \to \mathcal{R}(A)_1$$

is an (I,J)-weak equivalence, where the right map is the natural I-weak equivalence.

Let $\mathcal{R}(A)_1[x] \in U_\bullet\Re$ be the simplicial functor defined as

$$\mathcal{R}(A)_1[x](B) = \mathrm{Hom}_{\mathrm{Alg}_k^{\mathrm{ind}}}(JA, \mathbb{B}^\Delta[x]) = Ex^\infty \circ \mathrm{Hom}_{\mathrm{Alg}_k}(JA, B[x]^\Delta), \quad B \in \Re.$$

There is a natural map $s : \mathcal{R}(A)_1 \to \mathcal{R}(A)_1[x]$, induced by the monomorphism $B \to B[x]$ at each B. It follows from [9, 3.2] that this map is a weak equivalence in $U_\bullet\Re$. The evaluation homomorphisms $\partial_x^0, \partial_x^1 : B[x] \to B$ induce a map $(\partial_x^0, \partial_x^1) : \mathcal{R}(A)_1[x] \to \mathcal{R}(A)_1 \times \mathcal{R}(A)_1$, whose composition with s is the diagonal map $\mathcal{R}(A)_1 \to \mathcal{R}(A)_1 \times \mathcal{R}(A)_1$. We see that $\mathcal{R}(A)_1[x]$ is a path object for the projectively fibrant object $\mathcal{R}(A)_1$.

If we constructed a homotopy $H : \mathcal{X} \to \mathcal{R}(A)_1[x]$ such that $\partial_x^0 H = i_1\alpha$ and $\partial_x^1 H = \rho$ it would follow that $i_1\alpha$, being homotopic to the (I,J)-weak equivalence ρ, is an (I,J)-weak equivalence. Since as well α is an (I,J)-weak equivalence, then so would be i_1.

The desired map H is uniquely determined by maps $h_1 : r(TA) \to \mathcal{R}(A)_1[x]$ and $h_2 : rA \wedge (I,0) \to \mathcal{R}(A)_1[x]$ such that $h_1\eta_A^* = h_2\nu$ defined as follows. The map h_1 is uniquely determined by the homomorphism $JA \to TA[x]$ which is the composition of ι_A and the contraction homomorphism $\tau : TA \to TA[x]$, functorial in A, that exists by Lemma 4.1. The map h_2 is uniquely determined by the one-simplex $JA \to A[\Delta^1][x]$ of $Ex^\infty \circ \mathrm{Hom}_{\mathrm{Alg}_k}(JA, A[x]^\Delta)$ which is the composition of $\rho_A : JA \to \Omega A = (t^2 -$

$t)A[t] \subset A[\Delta^1]$ and the homomorphism $\omega : A[\Delta^1] \to A[\Delta^1][x]$ sending the variable t to $1-(1-t)(1-x)$.

Thus we have shown that

$$i_1 : rA \wedge S^1 \to \mathcal{R}(A)_1$$

is an (I,J)-weak equivalence. It follows that the composite map

$$rA \wedge S^1 \xrightarrow{i_0 \wedge S^1} \mathcal{R}(A)_0 \wedge S^1 \xrightarrow{\sigma_0} \mathcal{R}(A)_1,$$

which is equal to i_1, is an (I,J)-weak equivalence. Hence σ_0 is an (I,J)-weak equivalence, because $i_0 \wedge S^1$ is an I-weak equivalence. More generally, one gets that every structure map

$$\mathcal{R}(A)_n \wedge S^1 \xrightarrow{\sigma_n} \mathcal{R}(A)_{n+1}$$

is an (I,J)-weak equivalence.

By induction, assume that $i_n : rA \wedge S^n \to \mathcal{R}(A)_n$ is an (I,J)-weak equivalence. Then $i_n \wedge S^1$ is an (I,J)-weak equivalence, and hence so is $i_{n+1} = \sigma_n \circ (i_n \wedge S^1)$. ☐

Denote by $SH_{S^1}(\Re)$ the stable homotopy category of $Sp(\Re)$. Since the endofunctor $-\wedge S^1$ is an equivalence on $SH_{S^1}(\Re)$ by [18], it follows from [17, Ch. 7] that $SH_{S^1}(\Re)$ is a triangulated category. Moreover, it is compactly generated with compact generators $\{(\Sigma^\infty rA)[n]\}_{A\in\Re, n\in\mathbb{Z}}$.

Corollary 4.3. *$\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}$ forms a family of compact generators for $SH_{S^1}(\Re)$. Moreover, there is a natural isomorphism*

$$SH_{S^1}(\Re)(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}_n(A,B)$$

for all $A, B \in \Re$ and $n \in \mathbb{Z}$.

Denote by $\mathcal{S}$ the full subcategory of $SH_{S^1}(\Re)$ whose objects are $\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}$. The next statement gives another description of the triangulated category $D(\Re,\mathfrak{F})$.

Theorem 4.4. *The category $\mathcal{S}$ is triangulated. Moreover, there is a contravariant equivalence of triangulated categories*

$$T : D(\Re,\mathfrak{F}) \to \mathcal{S}.$$

Proof. By [10] the natural functor

$$j : \Re \to D(\Re,\mathfrak{F})$$

is a universal excisive homotopy invariant homology theory. Consider the homology theory

$$t : \Re \to SH_{S^1}(\Re)^{\mathrm{op}}$$

that takes an algebra $A \in \Re$ to $\Sigma^\infty rA$. It is homotopy invariant and excisive, hence there is a unique triangulated functor

$$T : D(\Re,\mathfrak{F}) \to SH_{S^1}(\Re)^{\mathrm{op}}$$

such that $t = T \circ j$. If we apply T to the loop extension

$$\Omega A \to EA \to A,$$

we get an isomorpism

$$T(\Omega A) \cong \Sigma^\infty rA[1],$$

which is functorial in A.

It follows from Comparison Theorem B of [11] and Corollary 4.3 that T is full and faithful. Every object of $\mathcal{S}$ is plainly equivalent to the image of an object in $D(\Re,\mathfrak{F})$. $\square$

Recall from [11] that we can vary $\Re$ in the following sense. If $\Re'$ is another T-closed admissible category of algebras containing $\Re$, then $D(\Re,\mathfrak{F})$ is a full subcategory of $D(\Re',\mathfrak{F})$.

5. Morita stable algebraic Kasparov K-theory

If A is an algebra and $n>0$ is a positive integer, then there is a natural inclusion $\iota : A \to M_nA$ of algebras, sending A to the upper left corner of M_nA. Throughout this section we assume that $\Re$ is a small T-closed admissible category of k-algebras with $M_nk\in\Re$, $n\geqslant 1$. Then $M_nA\in\Re$ for any $A\in\Re$ and $M_n(f)\in\mathfrak{F}$ for any $f\in\mathfrak{F}$.

Denote by $U_\bullet\Re^{mor}_{I,J}$ the model category obtained from $U_\bullet\Re_{I,J}$ by Bousfield localization with respect to the family of maps of cofibrant objects

$$\{r(M_nA)\to rA \mid A\in\Re, n>0\}.$$

Let $Sp_{mor}(\Re)$ be the stable model category of S^1-spectra associated with $U_\bullet\Re^{mor}_{I,J}$. Observe that it is also obtained from $Sp(\Re)$ by Bousfield localization with respect to the family of maps of cofibrant objects in $Sp(\Re)$

$$\{F_s(r(M_nA))\to F_s(rA) \mid A\in\Re, n>0, s\geqslant 0\}.$$

Here $F_s : U_\bullet\Re^{mor}_{I,J}\to Sp_{mor}(\Re)$ is the canonical functor adjoint to the evaluation functor $Ev_s : Sp_{mor}(\Re)\to U_\bullet\Re^{mor}_{I,J}$.

Definition. (see [9]) (1) The *Morita stable algebraic Kasparov K-theory* of two algebras $A,B\in\Re$ is the space

$$\mathcal{K}^{mor}(A,B)=\operatorname{colim}(\mathcal{K}(A,B)\to\mathcal{K}(A,M_2k\otimes B)\to\mathcal{K}(A,M_3k\otimes B)\to\cdots).$$

Its homotopy groups will be denoted by $\mathcal{K}^{mor}_n(A,B)$, $n\geqslant 0$.

(2) A functor $X:\Re\to\mathbb{S}/(Spectra)$ is *Morita invariant* if each morphism $X(A)\to X(M_nA)$, $A\in\Re, n>0$, is a weak equivalence.

(3) An excisive, homotopy invariant homology theory $X:\Re\to\mathcal{T}$ is *Morita invariant* if each morphism $X(A)\to X(M_nA)$, $A\in\Re, n>0$, is an isomorphism.

(4) The *Morita stable algebraic Kasparov K-theory spectrum* of $A,B\in\Re$ is the Ω-spectrum

$$\mathbb{K}^{mor}(A,B)=(\mathcal{K}^{mor}(A,B),\mathcal{K}^{mor}(JA,B),\mathcal{K}^{mor}(J^2A,B),\ldots).$$

Denote by $SH^{mor}_{S^1}(\Re)$ the (stable) homotopy category of $Sp_{mor}(\Re)$. It is a compactly generated triangulated category with compact generators $\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}$. Let $\mathcal{S}_{mor}$ be the full subcategory of $SH^{mor}_{S^1}(\Re)$ whose objects are $\{\Sigma^\infty rA[n]\}_{A\in\Re,n\in\mathbb{Z}}$.

Recall from [10] the definition of the triangulated category $D_{mor}(\Re,\mathfrak{F})$. Its objects are those of $\Re$ and the set of morphisms between two algebras $A,B\in\Re$ is defined as the colimit of the sequence of abelian groups

$$D(\Re,\mathfrak{F})(A,B)\to D(\Re,\mathfrak{F})(A,M_2B)\to D(\Re,\mathfrak{F})(A,M_3B)\to\cdots$$

There is a canonical functor $\Re\to D_{mor}(\Re,\mathfrak{F})$. It is the universal excisive, homotopy invariant and Morita invariant homology theory on $\Re$.

Theorem 5.1. *Given $A \in \Re$ the composite map*

$$\Sigma^\infty rA \xrightarrow{i} \mathcal{R}(A) \xrightarrow{j} \mathbb{K}(A,-) \to \mathbb{K}^{mor}(A,-) \quad (1)$$

is a stable equivalence in $Sp_{mor}(\Re)$, functorial in A. In particular, there is a natural isomorphism

$$SH^{mor}_{S^1}(\Re)(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}^{mor}_n(A,B)$$

for all $A, B \in \Re$ and $n \in \mathbb{Z}$. Furthermore, the category $\mathcal{S}_{mor}$ is triangulated and there is a contravariant equivalence of triangulated categories

$$T : D_{mor}(\Re, \mathfrak{F}) \to \mathcal{S}_{mor}.$$

Proof. Let $\mathcal{S}^c$ and $\mathcal{S}^c_{mor}$ be the categories of compact objects in $SH_{S^1}(\Re)$ and $SH^{mor}_{S^1}(\Re)$ respectively. Denote by $\mathcal{R}$ the full triangulated subcategory of $\mathcal{S}$ generated by objects

$$\{\operatorname{cone}(\Sigma^\infty r(M_nA) \to \Sigma^\infty rA)[k] \mid A \in \Re, n > 0, k \in \mathbb{Z}\}.$$

Let $\mathcal{R}^c$ be the thick closure of $\mathcal{R}$ in $SH_{S^1}(\Re)$. It follows from [23, 2.1] that the natural functor

$$\mathcal{S}^c/\mathcal{R}^c \to \mathcal{S}^c_{mor}$$

is full and faithful and $\mathcal{S}^c_{mor}$ is the thick closure of $\mathcal{S}^c/\mathcal{R}^c$.

We claim that the natural functor

$$\mathcal{S}/\mathcal{R} \to \mathcal{S}^c/\mathcal{R}^c \quad (2)$$

is full and faithful. For this consider a map $\alpha : X \to Y$ in $\mathcal{S}^c$ such that its cone Z is in $\mathcal{R}^c$ and $Y \in \mathcal{S}$. We can find $Z' \in \mathcal{R}^c$ such that $Z \oplus Z'$ is isomorphic to an object $W \in \mathcal{R}$. Construct a commutative diagram in $\mathcal{S}^c$

$$\begin{array}{ccccccc} U & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & \Sigma U \\ \downarrow{\scriptstyle s} & & \| & & \downarrow{\scriptstyle p} & & \downarrow \\ X & \xrightarrow{\alpha} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X, \end{array}$$

where p is the natural projection. We see that αs is such that its cone W belongs to $\mathcal{R}$. Standard facts for Gabriel–Zisman localization theory imply (2) is a fully faithful embedding. It also follows that

$$\mathcal{S}_{mor} = \mathcal{S}/\mathcal{R}.$$

We want to compute Hom sets in $\mathcal{S}/\mathcal{R}$. For this observe first that there is a contravariant equivalence of triangulated categories

$$\tau : D(\Re, \mathfrak{F})/\mathfrak{U} \to \mathcal{S}_{mor},$$

where $\mathfrak{U}$ is the smallest full triangulated subcategory of $D(\Re, \mathfrak{F})$ containing

$$\{\operatorname{cone}(A \xrightarrow{\iota} M_nA) \mid A \in \Re, n > 0\}.$$

This follows from Theorem 4.4.

By construction, every excisive homotopy invariant Morita invariant homology theory $\Re \to \mathcal{T}$ factors through $D(\Re, \mathfrak{F})/\mathfrak{U}$. Since $\Re \to D_{mor}(\Re, \mathfrak{F})$ is a universal excisive homotopy invariant Morita invariant homology theory [10], we see that there exists a triangle equivalence of triangulated categories

$$D_{mor}(\Re, \mathfrak{F}) \simeq D(\Re, \mathfrak{F})/\mathfrak{U}.$$

So there is a natural contravariant triangle equivalence of triangulated categories

$$T : D_{mor}(\Re, \mathfrak{F}) \to \mathcal{S}_{mor}.$$

Using this and [11, 9.9], there is a natural isomorphism

$$\mathcal{S}_{mor}(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}_n^{mor}(A, B)$$

for all $A, B \in \Re$ and $n \in \mathbb{Z}$. The fact that (1) is a stable equivalence in $Sp_{mor}(\Re)$ is now obvious. □

6. Stable algebraic Kasparov K-theory

If A is an algebra one sets $M_\infty A = \cup_n M_n A$. There is a natural inclusion $\iota : A \to M_\infty A$ of algebras, sending A to the upper left corner of $M_\infty A$. Throughout the section we assume that $\Re$ is a small tensor closed and T-closed admissible category of k-algebras with $M_\infty(k) \in \Re$. It follows that $M_\infty A \cong A \otimes M_\infty(k) \in \Re$ for all $A \in \Re$.

Denote by $U_\bullet \Re_{I,J}^\infty$ the model category obtained from $U_\bullet \Re_{I,J}$ by Bousfield localization with respect to the family of maps of cofibrant objects

$$\{r(M_\infty A) \to rA \mid A \in \Re\}.$$

Observe that $U_\bullet \Re_{I,J}^\infty$ is a monoidal model category whenever $\Re$ is T-closed.

Let $Sp_\infty(\Re)$ be the stable model category of S^1-spectra associated with $U_\bullet \Re_{I,J}^\infty$. Observe that it is also obtained from $Sp(\Re)$ by Bousfield localization with respect to the family of maps of cofibrant objects in $Sp(\Re)$

$$\{F_s(r(M_\infty A)) \to F_s(rA) \mid A \in \Re, s \geqslant 0\}.$$

Definition. (see [11]) (1) The *stable algebraic Kasparov K-theory* of two algebras $A, B \in \Re$ is the space

$$\mathcal{K}^{st}(A, B) = \operatorname{colim}(\mathcal{K}(A, B) \to \mathcal{K}(A, M_\infty k \otimes B) \to \mathcal{K}(A, M_\infty k \otimes M_\infty k \otimes B) \to \cdots).$$

Its homotopy groups will be denoted by $\mathcal{K}_n^{st}(A, B)$, $n \geqslant 0$.

(2) A functor $X : \Re \to \mathbb{S}/(Spectra)$ is *stable* or *M_∞-invariant* if $X(A) \to X(M_\infty A)$ is a weak equivalence for all $A \in \Re$.

(3) An excisive, homotopy invariant homology theory $X : \Re \to \mathcal{T}$ is *stable* or *M_∞-invariant* if $X(A) \to X(M_\infty A)$ is an isomorphism for all $A \in \Re$.

(4) The *stable algebraic Kasparov K-theory spectrum* for $A, B \in \Re$ is the Ω-spectrum

$$\mathbb{K}^{st}(A, B) = (\mathcal{K}^{st}(A, B), \mathcal{K}^{st}(JA, B), \mathcal{K}^{st}(J^2A, B), \ldots).$$

Denote by $SH_{S^1}^\infty(\Re)$ the (stable) homotopy category of $Sp_\infty(\Re)$. It is a compactly generated triangulated category with compact generators $\{\Sigma^\infty rA[n]\}_{A \in \Re, n \in \mathbb{Z}}$. Let $\mathcal{S}_\infty$ be the full subcategory of $SH_{S^1}^\infty(\Re)$ whose objects are $\{\Sigma^\infty rA[n]\}_{A \in \Re, n \in \mathbb{Z}}$.

Recall from [10] the definition of the triangulated category $D_{st}(\Re, \mathfrak{F})$. Its objects are those of $\Re$ and the set of morphisms between two algebras $A, B \in \Re$ is defined as the colimit of the sequence of abelian groups

$$D(\Re, \mathfrak{F})(A, B) \to D(\Re, \mathfrak{F})(A, M_\infty k \otimes_k B) \to D(\Re, \mathfrak{F})(A, M_\infty k \otimes_k M_\infty k \otimes_k B) \to \cdots$$

There is a canonical functor $\Re \to D_{st}(\Re, \mathfrak{F})$. It is the universal excisive, homotopy invariant and stable homology theory on $\Re$.

The proof of the next result is like that of Theorem 5.1.

Theorem 6.1. *Given $A \in \Re$ the composite map*

$$\Sigma^\infty rA \xrightarrow{i} \mathcal{R}(A) \xrightarrow{j} \mathbb{K}(A,-) \to \mathbb{K}^{st}(A,-) \tag{3}$$

is a stable equivalence in $Sp_\infty(\Re)$, functorial in A. In particular, there is a natural isomorphism

$$SH^\infty_{S^1}(\Re)(\Sigma^\infty rB[n], \Sigma^\infty rA) \cong \mathbb{K}^{st}_n(A,B)$$

for all $A, B \in \Re$ and $n \in \mathbb{Z}$. Furthermore, the category $\mathcal{S}_\infty$ is triangulated and there is a contravariant equivalence of triangulated categories

$$T : D_{st}(\Re, \mathfrak{F}) \to \mathcal{S}_\infty.$$

Let ΓA, $A \in \mathrm{Alg}_k$, be the algebra of $\mathbb{N} \times \mathbb{N}$-matrices which satisfies the following two properties.

(i) The set $\{a_{ij} \mid i,j \in \mathbb{N}\}$ is finite.
(ii) There exists a natural number $N \in \mathbb{N}$ such that each row and each column has at most N nonzero entries.

$M_\infty A \subset \Gamma A$ is an ideal. We put

$$\Sigma A = \Gamma A / M_\infty A.$$

We note that ΓA, ΣA are the cone and suspension rings of A considered by Karoubi and Villamayor in [20, p. 269], where a different but equivalent definition is given. By [3] there are natural ring isomorphisms

$$\Gamma A \cong \Gamma k \otimes A, \quad \Sigma A \cong \Sigma k \otimes A.$$

We call the short exact sequence

$$M_\infty A \rightarrowtail \Gamma A \twoheadrightarrow \Sigma A$$

the *cone extension*. By [3] $\Gamma A \twoheadrightarrow \Sigma A$ is a split surjection of k-modules.

Let τ be the k-algebra which is unital and free on two generators α and β satisfying the relation $\alpha\beta = 1$. By [3, 4.10.1] the kernel of the natural map

$$\tau \to k[t^{\pm 1}]$$

is isomorphic to $M_\infty k$. We set $\tau_0 = \tau \oplus_{k[t^{\pm 1}]} \sigma$.

Let A be a k-algebra. We get an extension

$$M_\infty A \longrightarrow \tau A \longrightarrow A[t^{\pm 1}],$$

and an analogous extension

$$M_\infty A \longrightarrow \tau_0 A \longrightarrow \sigma A. \tag{4}$$

Definition. We say that an admissible category of k-algebras $\Re$ is τ_0-*closed* (respectively Γ-*closed*) if $\tau_0 A \in \Re$ (respectively $\Gamma A \in \Re$) for all $A \in \Re$.

Cuntz [4, 5, 6] constructed a triangulated category kk^{lca} whose objects are the locally convex algebras. Later Cortiñas–Thom [3] construct in a similar fashion a triangulated category kk whose objects are all k-algebras Alg_k. If we suppose that $\Re$ is also Γ-closed, then one can define a full triangulated subcategory $kk(\Re)$ of kk whose objects are those of $\Re$.

It can be shown similar to [9, 7.4] or [10, 9.4] that there is an equivalence of triangulated categories

$$D_{st}(\Re, \mathfrak{F}) \xrightarrow{\sim} kk(\Re).$$

An important computational result of Cortiñas–Thom [3] states that there is an isomorphism of graded abelian groups

$$\bigoplus_{n\in\mathbb{Z}} kk(\Re)(k, \Omega^n A) \cong \bigoplus_{n\in\mathbb{Z}} KH_n(A),$$

where the right hand side is the homotopy K-theory of $A \in \Re$ in the sense of Weibel [28].

Summarizing the above arguments together with Theorem 6.1 we obtain the following

Theorem 6.2. *Suppose $\Re$ is Γ-closed. Then there is a contravariant equivalence of triangulated categories*

$$kk(\Re) \to \mathcal{S}_\infty.$$

Moreover, there is a natural isomorphism

$$SH^{\infty}_{S^1}(\Re)(\Sigma^\infty rA[n], \Sigma^\infty r(k)) \cong KH_n(A).$$

for any $A \in \Re$ and any integer n.

7. K-motives of algebras

Throughout the section we assume that $\Re$ is a small tensor closed and T-closed admissible category of k-algebras with $M_\infty(k) \in \Re$. It follows that $M_\infty A :\cong A \otimes M_\infty(k) \in \Re$ for all $A \in \Re$.

In this section we define and study the triangulated category of K-motives. It shares lots of common properties with the category of K-motives for algebraic varieties constructed in [12].

Since $\Re$ is tensor closed, it follows that $U_\bullet\Re^{\infty}_{I,J}$ is a monoidal model category. Let $Sp^{\Sigma}_{\infty}(\Re)$ be the monoidal category of symmetric spectra in the sense of Hovey [18] associated to $U_\bullet\Re^{\infty}_{I,J}$.

Definition. The *category of K-motives* $DK(\Re)$ is the stable homotopy category of $Sp^{\Sigma}_{\infty}(\Re)$. The *$K$-motive* $M_K(A)$ of an algebra $A \in \Re$ is the image of A in $DK(\Re)$, that is $M_K(A) = \Sigma^\infty rA$. Thus one has a canonical contravariant functor

$$M_K : \Re \to DK(\Re)$$

sending algebras to their K-motives.

The following proposition follows from standard facts for monoidal model categories.

Proposition 7.1. *$DK(\Re)$ is a symmetric monoidal compactly generated triangulated category with compact generators $\{M_K(A)\}_{A\in\Re}$. For any two algebras $A, B \in \Re$ one has a natural isomorphism*

$$M_K(A) \otimes M_K(B) \cong M_K(A \otimes B).$$

Furthermore, any extension of algebras in $\Re$

$$(E): \quad A \to B \to C$$

induces a triangle in $DK(\Re)$

$$M_K(E): \quad M_K(C) \to M_K(B) \to M_K(A) \xrightarrow{+} .$$

There is a pair of adjoint functors

$$V : Sp_\infty(\Re) \leftrightarrows Sp_\infty^\Sigma(\Re) : U,$$

where U is the right Quillen forgetful functor. These form a Quillen equivalence. In particular, the induced functors

$$V : SH_{S^1}^\infty(\Re) \leftrightarrows DK(\Re) : U$$

are equivalences of triangulated categories. It follows from Proposition 7.1 that $SH_{S^1}^\infty(\Re)$ is a symmetric monoidal category and

$$\Sigma^\infty rA \otimes \Sigma^\infty rB \cong \Sigma^\infty r(A \otimes B)$$

for all $A, B \in \Re$. Moreover,

$$V(\Sigma^\infty rA) \cong M_K(A)$$

for all $A \in \Re$.

Summarizing the above arguments together with Theorem 6.1 we get the following

Theorem 7.2. *For any two algebras* $A, B \in \Re$ *and any integer* n *one has a natural isomorphism of abelian groups*

$$DK(\Re)(M_K(B)[n], M_K(A)) \cong \mathbb{K}_n^{st}(A, B).$$

If $\mathcal{T}$ *is the full subcategory of* $DK(\Re)$ *spanned by* K*-motives of algebras* $\{M_K(A)\}_{A \in \Re}$ *then* $\mathcal{T}$ *is triangulated and there is an equivalence of triangulated categories*

$$D_{st}(\Re, \mathfrak{F}) \to \mathcal{T}^{\mathrm{op}}$$

sending an algebra $A \in \Re$ *to its* K*-motive* $M_K(A)$*.*

The next result is a reminiscence of a similar result for K-motives of algebraic varieties in the sense of [12] identifying the K-motive of the point with algebraic K-theory.

Corollary 7.3. *Suppose* $\Re$ *is* Γ*-closed. Then for any algebra* A *and any integer* n *one has a natural isomorphism of abelian groups*

$$DK(\Re)(M_K(A)[n], M_K(k)) \cong KH_n(A),$$

where the right hand side is the n*-th homotopy* K*-theory group in the sense of Weibel* [28]*.*

Proof. This follows from Theorem [10, 10.6] and the preceding theorem. □

We finish the section by showing that the category $kk(\Re)$ of Cortiñas–Thom [3] can be identified with the K-motives of algebras.

Theorem 7.4. *Suppose* $\Re$ *is* Γ*-closed. Then there is a natural equivalence of triangulated categories*

$$kk(\Re) \xrightarrow{\sim} \mathcal{T}^{\mathrm{op}}$$

sending an algebra $A \in \Re$ *to its* K*-motive* $M_K(A)$*.*

Proof. This follows from Theorem 7.2 and the fact that $D_{st}(\Re, \mathfrak{F})$ and $kk(\Re)$ are triangle equivalent (see [9, 7.4] or [10, 9.4]). □

The latter theorem shows in particular that $kk(\Re)$ is embedded into the compactly generated triangulated category of K-motives $DK(\Re)$ and generates it.

8. The $\mathbb{G}$-stable theory

Motivic stable homotopy theory over a field is homotopy theory of T-spectra, where $T = S^1 \wedge \mathbb{G}_m$ (see [26, 19]). There are various equivalent definitions of the theory, one of which is given by $(S^1, \mathbb{G}_m)$-bispectra. In our context the role of the motivic space $\mathbb{G}_m$ plays $\sigma = (t-1)k[t^{\pm 1}]$. Its simplicial functor $r(\sigma)$ is denoted by $\mathbb{G}$. In this section we define the stable category of $(S^1, \mathbb{G})$-bispectra and construct an explicit fibrant replacement of the $(S^1, \mathbb{G})$-bispectrum $\Sigma^\infty_{\mathbb{G}} \Sigma^\infty rA$ of an algebra A. One can as well define a Quillen equivalent category of T-spectra, where $T = S^1 \wedge \mathbb{G}$, and compute an explicit fibrant replacement for the T-spectrum of an algebra. However we prefer to work with $(S^1, \mathbb{G})$-bispectra rather than T-spectra in order to study K-motives of algebras in terms of associated $(S^1, \mathbb{G})$-bispectra (see the next section).

Throughout the section we assume that $\Re$ is a small tensor closed and T-closed admissible category of k-algebras. We have that $\sigma A := A \otimes \sigma \in \Re$ for all $A \in \Re$.

Recall that $U_\bullet \Re_{I,J}$ is a monoidal model category. It follows from [18, 6.3] that $Sp(\Re)$ is a $U_\bullet \Re_{I,J}$-model category. In particular

$$- \otimes \mathbb{G} : Sp(\Re) \to Sp(\Re)$$

is a left Quillen endofunctor.

By definition, a $(S^1, \mathbb{G})$-*bispectrum* or *bispectrum* $\mathcal{E}$ is given by a sequence $(E_0, E_1, \ldots)$, where each E_j is a S^1-spectrum of $Sp(\Re)$, together with bonding morphisms $\varepsilon_n : E_n \wedge \mathbb{G} \to E_{n+1}$. Maps are sequences of maps in $Sp(\Re)$ respecting the bonding morphisms. We denote the category of bispectra by $Sp_{\mathbb{G}}(\Re)$. It can be regarded as the category of $\mathbb{G}$-spectra on $Sp(\Re)$ in the sense of Hovey [18].

$Sp_{\mathbb{G}}(\Re)$ is equipped with the stable $U_\bullet \Re_{I,J}$-model structure in which weak equivalences are defined by means of bigraded homotopy groups. The bispectrum object $\mathcal{E}$ determines a sequence of maps of S^1-spectra

$$E_0 \xrightarrow{\tilde{\varepsilon}_0} \Omega_{\mathbb{G}} E_1 \xrightarrow{\Omega_{\mathbb{G}}(\tilde{\varepsilon}_1)} \Omega^2_{\mathbb{G}} E_2 \to \cdots$$

where $\Omega_{\mathbb{G}}$ is the functor $\underline{\mathrm{Hom}}(\mathbb{G}, -)$ and $\tilde{\varepsilon}_n$-s are adjoint to the structure maps of $\mathcal{E}$. We define $\pi_{p,q}\mathcal{E}$ in A-sections as the colimit

$$\mathrm{colim}_l \big(\mathrm{Hom}_{SH_{S^1}(\Re)}(S^{p-q}, \Omega^{q+l}_{\mathbb{G}} JE_l(A)) \to \mathrm{Hom}_{SH_{S^1}(\Re)}(S^{p-q}, \Omega^{q+l+1}_{\mathbb{G}} JE_{l+1}(A)) \to \cdots\big)$$

once $\mathcal{E}$ has been replaced up to levelwise equivalence by a levelwise fibrant object $J\mathcal{E}$ so that the "loop" constructions make sense. We shall also refer to $\pi_{*,q}\mathcal{E}$ as homotopy groups of weight q.

By definition, a map of bispectra is a weak equivalence in $Sp_{\mathbb{G}}(\Re)$ if it induces an isomorphism on bigraded homotopy groups. We denote the homotopy category of $Sp_{\mathbb{G}}(\Re)$ by $SH_{S^1,\mathbb{G}}(\Re)$. It is a compactly generated triangulated category.

In order to define the main $(S^1, \mathbb{G})$-bispectrum of this section, denoted by $\mathbb{KG}(A, -)$, we should first establish some facts for algebra homomorphisms.

Suppose $A, C \in \Re$, then one has a commutative diagram

$$\begin{array}{ccccc} J(A \otimes C) & \rightarrowtail & T(A \otimes C) & \xrightarrow{\eta_{A\otimes C}} & A \otimes C \\ \downarrow{\scriptstyle \gamma_{A,C}} & & \downarrow & & \| \\ JA \otimes C & \rightarrowtail & T(A) \otimes C & \xrightarrow{\eta_A \otimes C} & A \otimes C \end{array}$$

in which $\gamma_{A,C}$ is uniquely determined by the split monomorphism $i_A \otimes C : A \otimes C \to T(A) \otimes C$.

One sets $\gamma^0_{A,C} := 1_{A\otimes C}$. We construct inductively

$$\gamma^n_{A,C} : J^n(A \otimes C) \to J^n(A) \otimes C, \quad n \geqslant 1.$$

Namely, $\gamma^{n+1}_{A,C}$ is the composite

$$J^{n+1}(A \otimes C) \xrightarrow{J(\gamma^n_{A,C})} J(J^n(A) \otimes C) \xrightarrow{\gamma_{J^n A,C}} J^{n+1}(A) \otimes C.$$

Given $n \geqslant 0$ we define a map

$$t_n = t^{A,C}_n : \mathcal{K}(J^n A, -) \to \mathcal{K}(J^n(A \otimes C), - \otimes C) = \underline{\mathrm{Hom}}(rC, \mathcal{K}(J^n(A \otimes C), -))$$

as follows. Let $B \in \Re$ and $(\alpha : J^{n+m}A \to \mathbb{B}(\Omega^m)) \in \mathcal{K}(J^n A, B)$. We set $t_n(\alpha) \in \mathcal{K}(J^n(A \otimes C), B \otimes C)$ to be the composite

$$J^{n+m}(A \otimes C) \xrightarrow{\gamma^{n+m}_{A,C}} J^{n+m}(A) \otimes C \xrightarrow{\alpha\otimes C} \mathbb{B}^{\Delta}(\Omega^m) \otimes C \overset{\tau}{\cong} (\mathbb{B} \otimes \mathbb{C})^{\Delta}(\Omega^m).$$

Here τ is a canonical isomorphism (see [3, 3.1.3]) and $(\mathbb{B} \otimes \mathbb{C})^{\Delta}$ stands for the simplicial ind-algebra

$$[m, \ell] \mapsto \mathrm{Hom}_{\mathbb{S}}(\mathrm{sd}^m \Delta^{\ell}, (B \otimes C)^{\Delta}) = (B \otimes C)^{\mathrm{sd}^m \Delta^{\ell}} \cong k^{\mathrm{sd}^m \Delta^{\ell}} \otimes (B \otimes C).$$

One has to verify that t_n is consistent with maps

$$\mathrm{Hom}_{\mathrm{Alg}^{\mathrm{ind}}_k}(J^{n+m}A, \mathbb{B}^{\Delta}(\Omega^m)) \xrightarrow{\varsigma} \mathrm{Hom}_{\mathrm{Alg}^{\mathrm{ind}}_k}(J^{n+m+1}A, \mathbb{B}^{\Delta}(\Omega^{m+1})).$$

More precisely, we must show that the map

$$J^{n+m+1}(A\otimes C) \xrightarrow{J(\gamma^{n+m}_{A,C})} J(J^{n+m}A\otimes C) \xrightarrow{J(\alpha\otimes 1)} J(\mathbb{B}^{\Delta}(\Omega^m)\otimes C) \overset{J\tau}{\cong} J((\mathbb{B}\otimes\mathbb{C})^{\Delta}(\Omega^m)) \xrightarrow{\xi_v} (\mathbb{B}\otimes\mathbb{C})^{\Delta}(\Omega^{m+1})$$

is equal to the map

$$J^{n+m+1}(A\otimes C) \xrightarrow{\gamma^{n+m+1}_{A,C}} J^{n+m+1}A\otimes C \xrightarrow{J\alpha\otimes 1} J(\mathbb{B}^{\Delta}(\Omega^m))\otimes C \xrightarrow{\xi_v\otimes 1} \mathbb{B}^{\Delta}(\Omega^{m+1})\otimes C \overset{\tau}{\cong} (\mathbb{B}\otimes\mathbb{C})^{\Delta}(\Omega^{m+1}).$$

The desired property follows from commutativity of the diagram

$$\begin{array}{ccccccc}
J^{n+m+1}(A \otimes C) & & & & & & \\
\Big\downarrow {\scriptstyle J(\gamma^{n+m}_{A,C})} & & & & & & \\
J(J^{n+m}A \otimes C) & \xrightarrow{\gamma_{J^{n+m}A,C}} & J^{n+m+1}A \otimes C & \rightarrowtail & TJ^{n+m}A \otimes C & \twoheadrightarrow & J^{n+m}A \otimes C \\
\Big\downarrow {\scriptstyle J(\alpha\otimes 1)} & & \Big\downarrow {\scriptstyle J(\alpha)\otimes 1} & & \Big\downarrow & & \Big\downarrow {\scriptstyle \alpha\otimes 1} \\
J(\mathbb{B}^{\Delta}(\Omega^m) \otimes C) & \longrightarrow & J(\mathbb{B}^{\Delta}(\Omega^m)) \otimes C & \rightarrowtail & T(\mathbb{B}^{\Delta}(\Omega^m)) \otimes C & \twoheadrightarrow & \mathbb{B}^{\Delta}(\Omega^m) \otimes C \\
\Big\| & & \Big\downarrow {\scriptstyle \xi_v\otimes 1} & & \Big\downarrow & & \Big\| \\
J(\mathbb{B}^{\Delta}(\Omega^m) \otimes C) & \longrightarrow & \mathbb{B}^{\Delta}(\Omega^{m+1}) \otimes C & \rightarrowtail & P(\mathbb{B}^{\Delta}(\Omega^m)) \otimes C & \twoheadrightarrow & \mathbb{B}^{\Delta}(\Omega^m) \otimes C \\
\Big\downarrow {\scriptstyle J\tau} & & \Big\downarrow {\scriptstyle \tau} & & \Big\downarrow {\scriptstyle \tau} & & \Big\downarrow {\scriptstyle \tau} \\
J((\mathbb{B} \otimes \mathbb{C})^{\Delta}(\Omega^m)) & \xrightarrow[\xi_v]{} & (\mathbb{B} \otimes \mathbb{C})^{\Delta}(\Omega^{m+1}) & \rightarrowtail & P(\mathbb{B} \otimes \mathbb{C})^{\Delta}(\Omega^m) & \twoheadrightarrow & (\mathbb{B} \otimes \mathbb{C})^{\Delta}(\Omega^m)
\end{array}$$

We see that t_n is well defined. We claim that the collection of maps $(t_n)_n$ defines a map of S^1-spectra

$$t : \mathbb{K}(A, B) \to \mathbb{K}(A \otimes C, B \otimes C).$$

We have to check that for each $n \geqslant 0$ the diagram

$$\begin{array}{ccc} \mathcal{K}(J^nA,B) & \xrightarrow{\cong} & \Omega\mathcal{K}(J^{n+1}A,B) \\ \downarrow{t_n} & & \downarrow{\Omega t_{n+1}} \\ \mathcal{K}(J^n(A\otimes C),B\otimes C) & \xrightarrow{\cong} & \Omega\mathcal{K}(J^{n+1}(A\otimes C),B\otimes C) \end{array}$$

is commutative. But this directly follows from the definition of the horizontal maps and arguments above made for t_n-s.

If we replace C by σ we get that the array

$$\begin{array}{cccccc} & \vdots & \vdots & \vdots & \\ \mathbb{K}(\sigma^2A,B): & \mathcal{K}(\sigma^2A,B) & \mathcal{K}(J\sigma^2A,B) & \mathcal{K}(J^2\sigma^2A,B) & \cdots \\ \mathbb{K}(\sigma A,B): & \mathcal{K}(\sigma A,B) & \mathcal{K}(J\sigma A,B) & \mathcal{K}(J^2\sigma A,B) & \cdots \\ \mathbb{K}(A,B): & \mathcal{K}(A,B) & \mathcal{K}(JA,B) & \mathcal{K}(J^2A,B) & \cdots \end{array}$$

together with structure maps

$$\mathbb{K}(\sigma^nA,-)\otimes\mathbb{G}\to\mathbb{K}(\sigma^{n+1}A,-)$$

defined as adjoint maps to

$$t:\mathbb{K}(\sigma^nA,-)\to\underline{\mathrm{Hom}}(\mathbb{G},\mathbb{K}(\sigma^{n+1}A,-))$$

form a $(S^1,\mathbb{G})$-bispectrum, which we denote by $\mathbb{KG}(A,-)$.

There is a natural map of $(S^1,\mathbb{G})$-bispectra

$$\Gamma:\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA\to\mathbb{KG}(A,-),$$

where $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA$ is the $(S^1,\mathbb{G})$-bispectrum represented by the array

$$\begin{array}{cccc} & \vdots & \vdots & \\ \Sigma^\infty rA\otimes\mathbb{G}^2: & rA\otimes\mathbb{G}^2\ (\cong r(\sigma^2A)) & (rA\wedge S^1)\otimes\mathbb{G}^2\ (\cong r(\sigma^2A)\wedge S^1) & \cdots \\ \Sigma^\infty rA\otimes\mathbb{G}: & rA\otimes\mathbb{G}\ (\cong r(\sigma A)) & (rA\wedge S^1)\otimes\mathbb{G}\ (\cong r(\sigma A)\wedge S^1) & \cdots \\ \Sigma^\infty rA: & rA & rA\wedge S^1 & \cdots \end{array}$$

with obvious structure maps.

By Theorem 4.2 each map

$$\Gamma_n:\Sigma^\infty rA\otimes\mathbb{G}^n\to\mathbb{KG}(A,-)_n=\mathbb{K}(\sigma^nA,-)$$

is a stable weak equivalence in $Sp(\Re)$. By [11] each $\mathbb{K}(\sigma^n A,-)$ is a fibrant object in $Sp(\Re)$. For each $n \geqslant 0$ we set,

$$\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A,-)_n = \operatorname{colim}(\mathbb{K}(\sigma^n A,-) \xrightarrow{t_0} \mathbb{K}(\sigma^{n+1}A,-\otimes\sigma) \xrightarrow{\Omega_{\mathbb{G}}(t_1)} \mathbb{K}(\sigma^{n+2}A,-\otimes\sigma^2) \to \cdots)$$

By specializing a collection of results in [18, section 4] to our setting we have that $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A,-)$ is a fibrant bispectrum and the natural map

$$j : \mathbb{K}\mathbb{G}(A,-) \to \Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A,-)$$

is a weak equivalence in $Sp_{\mathbb{G}}(\Re)$.

We have thus shown that $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A,-)$ is an explicit fibrant replacement for the bispectrum $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA$ of the algebra A. Denote by $\mathcal{K}^\sigma(A,B)$ the $(0,0)$-space of the bispectrum $\Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A,B)$. It is, by construction, the colimit

$$\operatorname{colim}_n \mathcal{K}(\sigma^n A, \sigma^n B).$$

Its homotopy groups will be denoted by $\mathcal{K}^\sigma_n(A,B)$, $n \geqslant 0$.

Theorem 8.1. *Let A be an algebra in $\Re$; then the composite map*

$$j \circ \Gamma : \Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA \to \Theta^\infty_{\mathbb{G}}\mathbb{K}\mathbb{G}(A,-)$$

is a fibrant replacement of $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA$. In particular,

$$SH_{S^1,\mathbb{G}}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rB, \Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA) = \mathcal{K}^\sigma_0(A,B)$$

for all $B \in \Re$.

Remark. Let $SH(F)$ be the motivic stable homotopy theory over a field F. The category $SH_{S^1,\mathbb{G}}(\Re)$ shares a lot of common properties with $SH(F)$. To the author knowledge computation of $SH(F)(\Sigma^\infty_{s,t}X_+, \Sigma^\infty_{s,t}Y_+)$, where $X, Y \in Sm/F$, as well as an explicit construction of a fibrant replacement for $\Sigma^\infty_{s,t}X_+$ is a very hard problem in $\mathbb{A}^1$-topology.

Let F be an algebraically closed field of characteristic zero with an embedding $F \hookrightarrow \mathbb{C}$ and SH be the stable homotopy category of ordinary spectra. Let $c : SH \to SH(F)$ be the functor induced by sending a space to the constant presheaf of spaces on Sm/F. Levine [21] has recently shown that c is fully faithful. This is in fact implied by a result of Levine [21] saying that the Betti realization functor in the sense of Ayoub [1]

$$Re_B : SH(F) \to SH$$

gives an isomorphism

$$Re_{B*} : \pi_{n,0}\mathcal{S}_F(F) \to \pi_n(\mathcal{S})$$

for all $n \in \mathbb{Z}$. Here $\mathcal{S}_F$ is the motivic sphere spectrum in $SH(F)$ and $\mathcal{S}$ is the classical sphere spectrum in SH. These results use recent developments for the spectral sequence associated with the slice filtration of the motivic sphere $\mathcal{S}_F$.

All this justifies to raise the following

Questions. (1) *Is there an admissible category of commutative algebras $\Re$ over the field of complex numbers $\mathbb{C}$ such that the natural functor*

$$c : SH \to SH_{S^1,\mathbb{G}}(\Re),$$

induced by the functor $\mathbb{S} \to U\Re$ sending a simplicial set to the constant simplicial functor on $\Re$, is fully faithful?

(2) *Let* $\Re$ *be an admissible category of commutative* $\mathbb{C}$*-algebras and let* $\mathcal{S}_{\mathbb{C}}$ *be the bispectrum* $\Sigma_{\mathbb{G}}^{\infty}\Sigma^{\infty}r\mathbb{C}$*. Is it true that the homotopy groups of weight zero* $\pi_{n,0}\mathcal{S}_{\mathbb{C}}(\mathbb{C}) = \mathcal{K}_n^{\sigma}(\mathbb{C},\mathbb{C})$*,* $n \geqslant 0$*, are isomorphic to the stable homotopy groups* $\pi_n(\mathcal{S})$ *of the classical sphere spectrum?*

We should also mention that one can define $(S^1,\mathbb{G})$-bispectra by starting at the monoidal category of symmetric spectra $Sp^{\Sigma}(\Re)$ associated with monoidal category $U_{\bullet}(\Re)_{I,J}$ and then stabilize the left Quillen functor $-\otimes\mathbb{G} : Sp^{\Sigma}(\Re) \to Sp^{\Sigma}(\Re)$. One produces the model category $Sp_{\mathbb{G}}^{\Sigma}(\Re)$ of (usual, non-symmetric) $\mathbb{G}$-spectra in $Sp^{\Sigma}(\Re)$. Using Hovey's [18] notation, one has, by definition, $Sp_{\mathbb{G}}^{\Sigma}(\Re) = Sp^{\mathbb{N}}(Sp^{\Sigma}(\Re), -\otimes\mathbb{G})$.

There is a Quillen equivalence

$$V : Sp(\Re) \leftrightarrows Sp^{\Sigma}(\Re) : U$$

as well as a Quillen equivalence

$$V : Sp_{\mathbb{G}}(\Re) \leftrightarrows Sp_{\mathbb{G}}^{\Sigma}(\Re) : U,$$

where U is the forgetful functor (see [18, 5.7]).

If we denote by $SH_{S^1}^{\Sigma}(\Re)$ and $SH_{S^1,\mathbb{G}}^{\Sigma}(\Re)$ the homotopy categories of $Sp^{\Sigma}(\Re)$ and $Sp_{\mathbb{G}}^{\Sigma}(\Re)$ respectively, then one has equivalences of categories

$$V : SH_{S^1}(\Re) \leftrightarrows SH_{S^1}^{\Sigma}(\Re) : U$$

and

$$V : SH_{S^1,\mathbb{G}}(\Re) \leftrightarrows SH_{S^1,\mathbb{G}}^{\Sigma}(\Re) : U.$$

We refer the interested reader to [18, 19] for further details.

9. K-motives and $(S^1,\mathbb{G})$-bispectra

We prove in this section that the triangulated category of K-motives is fully faithfully embedded into the stable homotopy category of $(S^1,\mathbb{G})$-bispectra $SH_{S^1,\mathbb{G}}(\Re)$. In particular, the triangulated category $kk(\Re)$ of Cortiñas–Thom [3] is fully faithfully embedded into $SH_{S^1,\mathbb{G}}(\Re)$ by means of a contravariant functor. As an application we construct an explicit fibrant $(S^1,\mathbb{G})$-bispectrum representing homotopy K-theory in the sense of Weibel [28].

Throughout this section we assume that $\Re$ is a small tensor closed, T-, Γ- and τ_0-closed admissible category of k-algebras. It follows that $\sigma A, \Sigma A, M_{\infty}A \in \Re$ for all $A \in \Re$.

Let $Sp_{\infty,\mathbb{G}}^{\Sigma}(\Re)$ denote the model category of (usual, non-symmetric) $\mathbb{G}$-spectra in $Sp_{\infty}^{\Sigma}(\Re)$. Using Hovey's notation [18] $Sp_{\infty,\mathbb{G}}^{\Sigma}(\Re) = Sp^{\mathbb{N}}(Sp_{\infty}^{\Sigma}(\Re), -\otimes\mathbb{G})$.

Proposition 9.1. *The functor*

$$-\otimes\mathbb{G} : Sp_{\infty}^{\Sigma}(\Re) \to Sp_{\infty}^{\Sigma}(\Re)$$

and the canonical functor

$$F_{0,\mathbb{G}} = \Sigma_{\mathbb{G}}^{\infty} : Sp_{\infty}^{\Sigma}(\Re) \to Sp_{\infty,\mathbb{G}}^{\Sigma}(\Re)$$

are left Quillen equivalences.

Proof. We first observe that $-\otimes\mathbb{G}$ is a left Quillen equivalence on $Sp^{\Sigma}_{\infty}(\Re)$ if and only if so is $-\otimes\Sigma^{\infty}\mathbb{G}$. By [3, section 4] there is an extension

$$M_{\infty}k \rightarrowtail \tau_0 \twoheadrightarrow \sigma.$$

It follows from [3, 7.3.2] that $\Sigma^{\infty}(r(\tau_0)) = 0$ in $DK(\Re)$, and hence $\Sigma^{\infty}(r(\tau_0))$ is weakly equivalent to zero in $Sp^{\Sigma}_{\infty}(\Re)$.

The extension above yields therefore a zig-zag of weak equivalences between cofibrant objects in $Sp^{\Sigma}_{\infty}(\Re)$ from $\Sigma^{\infty}(r(M_{\infty}k))$ to $\Sigma^{\infty}\mathbb{G}\wedge S^1$. Since $\Sigma^{\infty}(r(M_{\infty}k))$ is weakly equivalent to the monoidal unit $\Sigma^{\infty}(r(k))$, we see that $\Sigma^{\infty}(r(k))$ is zig-zag weakly equivalent to $(\Sigma^{\infty}\mathbb{G})\wedge S^1$ in the category of cofibrant objects in $Sp^{\Sigma}_{\infty}(\Re)$.

Since $\Sigma^{\infty}(r(k))$ is a monoidal unit in $Sp^{\Sigma}_{\infty}(\Re)$, then $-\otimes\Sigma^{\infty}(r(k))$ is a left Quillen equivalence on $Sp^{\Sigma}_{\infty}(\Re)$, and hence so is $-\otimes((\Sigma^{\infty}\mathbb{G})\wedge S^1))$. But $-\wedge S^1$ is a left Quillen equivalence on $Sp^{\Sigma}_{\infty}(\Re)$. Therefore $-\otimes\Sigma^{\infty}\mathbb{G}$ is a left Quillen equivalence by [17, 1.3.15].

The fact that the canonical functor

$$F_{0,\mathbb{G}} : Sp^{\Sigma}_{\infty}(\Re) \to Sp^{\Sigma}_{\infty,\mathbb{G}}(\Re)$$

is a left Quillen equivalence now follows from [18, 5.1]. □

Denote the homotopy category of $Sp^{\Sigma}_{\infty,\mathbb{G}}(\Re)$ by $SH^{\Sigma,\infty}_{S^1,\mathbb{G}}(\Re)$.

Corollary 9.2. *The canonical functor*

$$F_{0,\mathbb{G}} = \Sigma^{\infty}_{\mathbb{G}} : DK(\Re) \to SH^{\Sigma,\infty}_{S^1,\mathbb{G}}(\Re)$$

is an equivalence of triangulated categories.

Recall that $Sp^{\Sigma}_{\infty}(\Re)$ is Bousfield localization of $Sp^{\Sigma}(\Re)$ with respect to

$$\{F_s(r(M_{\infty}A)) \to F_s(rA) \mid A \in \Re, s \geqslant 0\}.$$

It follows that the induced triangulated functor

$$DK(\Re) \to SH^{\Sigma}_{S^1}(\Re)$$

is fully faithful.

In a similar fashion, $Sp^{\Sigma}_{\infty,\mathbb{G}}(\Re)$ can be obtained from $Sp^{\Sigma}_{\mathbb{G}}(\Re)$ by Bousfield localization with respect to

$$\{F_{k,\mathbb{G}}(F_s(r(M_{\infty}A))) \to F_{k,\mathbb{G}}(F_s(rA)) \mid A \in \Re, k, s \geqslant 0\}.$$

We summarize all of this together with Proposition 9.1 as follows.

Theorem 9.3. *There is an adjoint pair of triangulated functors*

$$\Phi : SH^{\Sigma}_{S^1,\mathbb{G}}(\Re) \leftrightarrows DK(\Re) : \Psi$$

such that Ψ is fully faithful. Moreover, $\mathcal{T} = \operatorname{Ker}\Phi$ is the localizing subcategory of $SH^{\Sigma}_{S^1,\mathbb{G}}(\Re)$ generated by compact objects

$$\{\operatorname{cone}(F_{k,\mathbb{G}}(F_s(r(M_{\infty}A))) \to F_{k,\mathbb{G}}(F_s(rA))) \mid A \in \Re\}$$

and $DK(\Re)$ is triangle equivalent to $SH^{\Sigma}_{S^1,\mathbb{G}}(\Re)/\mathcal{T}$.

Corollary 9.4. *There is a contravariant fully faithful triangulated functor*

$$kk(\Re) \to SH_{S^1,\mathbb{G}}(\Re).$$

Proof. This follows from Theorems 7.4 and 9.3. □

Let $Sp_{\infty,\mathbb{G}}(\Re)$ denote the model category of $\mathbb{G}$-spectra in $Sp_\infty(\Re)$. Using Hovey's notation [18] $Sp_{\infty,\mathbb{G}}(\Re) = Sp^{\mathbb{N}}(Sp_\infty(\Re), -\otimes\mathbb{G})$. As above, there is a Quillen equivalence

$$V : Sp_{\infty,\mathbb{G}}(\Re) \leftrightarrows Sp^{\Sigma}_{\infty,\mathbb{G}}(\Re) : U,$$

where U is the forgetful functor. It induces an equivalence of triangulated categories

$$V : SH^{\infty}_{S^1,\mathbb{G}}(\Re) \leftrightarrows SH^{\Sigma,\infty}_{S^1,\mathbb{G}}(\Re) : U,$$

where the left hand side is the homotopy category of $Sp_{\infty,\mathbb{G}}(\Re)$.

Given an algebra $A \in \Re$, consider a $(S^1,\mathbb{G})$-bispectrum $\mathbb{KG}^{st}(A,-)$ which we define at each $B \in \Re$ as

$$\operatorname{colim}_n(\mathbb{KG}(A,B) \to \mathbb{KG}(A, M_\infty k \otimes B)) \to \mathbb{KG}(A, M^2_\infty k \otimes B) \to \cdots)$$

It can also be presented as the array

$$\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
\mathbb{K}^{st}(\sigma^2 A, B): & \mathcal{K}^{st}(\sigma^2 A, B) & \mathcal{K}^{st}(J\sigma^2 A, B) & \mathcal{K}^{st}(J^2\sigma^2 A, B) & \cdots \\
\mathbb{K}^{st}(\sigma A, B): & \mathcal{K}^{st}(\sigma A, B) & \mathcal{K}^{st}(J\sigma A, B) & \mathcal{K}^{st}(J^2\sigma A, B) & \cdots \\
\mathbb{K}^{st}(A, B): & \mathcal{K}^{st}(A, B) & \mathcal{K}^{st}(JA, B) & \mathcal{K}^{st}(J^2 A, B) & \cdots
\end{array}$$

It follows from Theorem 6.1 that the canonical map of bispectra

$$\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA \to \mathbb{KG}^{st}(A,-)$$

is a level weak equivalence in $Sp_{\infty,\mathbb{G}}(\Re)$. If we define $\Theta^\infty_{\mathbb{G}}\mathbb{KG}^{st}(A,-)$ similar to the bispectrum $\Theta^\infty_{\mathbb{G}}\mathbb{KG}(A,-)$, then the canonical map

$$j : \mathbb{KG}^{st}(A,-) \to \Theta^\infty_{\mathbb{G}}\mathbb{KG}^{st}(A,-)$$

is a stable equivalence of bispectra.

The following result says that the bispectrum $\mathbb{KG}^{st}(A,-)$ is $(2,1)$-periodic and represents stable algebraic Kasparov K-theory (cf. [26, 6.8-6.9]).

Theorem 9.5. *For any algebras $A, B \in \Re$ and any integers p, q there is an isomorphism of abelian groups*

$$\pi_{p,q}(\mathbb{KG}^{st}(A,B)) \cong \operatorname{Hom}_{SH_{S^1,\mathbb{G}}(\Re)}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rB \otimes S^{p-q} \otimes \mathbb{G}^q, \mathbb{KG}^{st}(A,-)) \cong \mathbb{K}^{st}_{p-2q}(A,B).$$

In particular,

$$\pi_{p,q}(\mathbb{KG}^{st}(A,B)) \cong \pi_{p+2,q+1}(\mathbb{KG}^{st}(A,B)).$$

Proof. As we have shown above, the bispectrum $\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA$ is level weak equivalent to $\mathbb{KG}^{st}(A,-)$ in $Sp_{\infty,\mathbb{G}}(\Re)$. Therefore,

$$\pi_{p,q}(\mathbb{KG}^{st}(A,B)) \cong \operatorname{Hom}_{SH^\infty_{S^1,\mathbb{G}}(\Re)}(\Sigma^\infty_{\mathbb{G}}\Sigma^\infty rB \otimes S^{p-q} \otimes \mathbb{G}^q, \Sigma^\infty_{\mathbb{G}}\Sigma^\infty rA).$$

Corollary 9.2 implies that the right hand side is isomorphic to $DK(\Re)(M_K(B)\otimes S^{p-q}\otimes \mathbb{G}^q, M_K(A))$. On the other hand,

$$DK(\Re)(M_K(B)\otimes S^{p-q}\otimes \mathbb{G}^q, M_K(A)) \cong DK(\Re)(M_K(B)\otimes S^{p-2q}\otimes S^q\otimes \mathbb{G}^q, M_K(A)).$$

The proof of Proposition 9.1 shows that $\Sigma^\infty(S^1\otimes\mathbb{G})$ is isomorphic to the monoidal unit. Therefore,

$$DK(\Re)(M_K(B)\otimes S^{p-2q}\otimes S^q\otimes \mathbb{G}^q, M_K(A)) \cong DK(\Re)(M_K(B)[p-2q], M_K(A)).$$

Our statement now follows from Theorem 7.2. □

The next statement says that the bispectrum $\mathbb{KG}^{st}(k,B)$ gives a model for homotopy K-theory in the sense of Weibel [28] (cf. [26, 6.9]).

Corollary 9.6. *For any algebra $B\in\Re$ and any integers p,q there is an isomorphism of abelian groups*

$$\pi_{p,q}(\mathbb{KG}^{st}(k,B)) \cong KH_{p-2q}(B).$$

Proof. This follows from the preceding theorem and [11, 9.11]. □

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