# QUASI-UMBILICAL AFFINE HYPERSURFACES CONGRUENT TO THEIR CENTRE MAP

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Abstract. In this paper, we study strictly convex affine hypersurfaces centroaffinely congruent to their centre map, in the case when the shape operator has two distinct eigenvalues: one of multiplicity 1, and one nonzero of multiplicity $n-1$. We show how to construct them from $(n-1)$-dimensional affine hyperspheres.

## 1. Introduction

In [1], the authors introduced the notion of *centre map* for a centroaffine hypersurface and studied affine hypersurfaces centroaffinely congruent to their centre map, completely solving the problem for positive definite surfaces.

The solution to this problem is known in higher dimensions for positive definite improper affine hyperspheres [4] (i.e. for which the shape operator $S$ identically vanishes), and for generic hypersurfaces [5] (i.e. for which $S$ has $n$ different, nonzero eigenvalues). In this paper, we investigate the intermediate case of positive definite quasi-umbilical hypersurfaces, i.e. when $S$ has two distinct eigenvalues: $\lambda_0$, of multiplicity 1, and $\lambda_1$ of multiplicity $n-1$.
More precisely, we prove the

**Theorem 1.1.** *Let $f : M^n \to \mathbf{R}^{n+1}$ be an affine immersion centroaffinely congruent to its centre map $c$. Assume that both $f$ and $c$ are centroaffine, that the Blaschke metric $h$ is positive definite, and that $f$ is quasi-umbilical, with the multiple eigenvalue $\lambda_1 \neq 0$.*
*Then such a hypersurface exists iff $\lambda_0 + \lambda_1 < 0$, and in that case $(M, h)$ is locally isometric to a warped product $\mathbf{R} \times_{e^F} N^{n-1}$. Moreover,*

- *if $(n+2)\lambda_0 + n\lambda_1 \neq 0$, then there exists a proper affine hypersphere $g_2 : N \to \mathbf{R}^n$ such that, up to an affine transformation of $\mathbf{R}^{n+1}$,*

$$f(t, \vec{u}) = \left( t^{-2K_1} g_2(\vec{u}) \ , \ \frac{t^N}{N} \right), \tag{1.1}$$

*where $K_1$ and $N$ are constants related to the $\lambda_i$'s.*

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- *if* $(n+2)\lambda_0 + n\lambda_1 = 0$*, then, up to an affine transformation of* $\mathbf{R}^{n+1}$*,*

$$f(t,\vec{u}) = \left( t^{-2K_1} \,,\; t^{-2K_1}\,\vec{u} \,,\; \varphi_0\, t^{-2K_1}\big(\mathcal{F}(\vec{u}) - \frac{1}{2K_1}\log t\big) \right), \tag{1.2}$$

*where* $\mathcal{F}$ *is a solution of the Monge–Ampère equation, and* $K_1, \varphi_0$ *are constants.*

*The converse also holds.*

The hypersurfaces in Theorem 1.1 are similar to those described in [6], where hypersurfaces with pointwise $SO(n-1)$-symmetry are studied. The shape operator and difference tensor in that paper have indeed the same form as the one we get under the assumptions of Theorem 1.1, the proof of which follows in part that of [6, Theorem 3.1].

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## 2. Preliminaries and notations

Let us now very briefly recall some basic notions of affine geometry (see [3] for details) and introduce the relevant notations.

Let $f : M \to \mathbf{R}^{n+1}$ be a non-degenerate immersion of an $n$-dimensional oriented manifold $M$ into $\mathbf{R}^{n+1}$, with its Blaschke structure. Let us denote by

- $D$ the standard flat affine connection on $\mathbf{R}^{n+1}$,
- $\xi$ the affine normal of $f$,
- $\nabla$ the induced equiaffine connection on $M$,
- $h$ the equiaffine metric on $M$,
- $S$ the shape operator of $f$.

The above quantities are related by the following relations, for all vector fields $X$ and $Y$ on $M$:

$$D_X f_* Y = f_* \nabla_X Y + h(X,Y)\xi,$$
$$D_X \xi = -f_* SX.$$

(We will often drop the symbol $f_*$ in the sequel.)

The standard volume form det on $\mathbf{R}^{n+1}$ induces a volume form $\omega$ on $M$, defined as $\omega(X_1,\dots,X_n) = \det(X_1,\dots,X_n,\xi)$, and, $\xi$ being the affine normal,

$$\omega^2(X_1,\dots,X_n) = \det(h_{ij}), \quad \text{where } h_{ij} = h(X_i,X_j).$$

We will also denote by

- $\widehat{\nabla}$ the Levi-Civita connection of the metric $h$,
- $K$ the *difference tensor*, defined by

$$K(X,Y) = K_X Y = \nabla_X Y - \widehat{\nabla}_X Y.$$

Recall [3, Proposition II.4.1] that

$$h(K_X(Y), Z) = -\frac{1}{2}(\nabla h)(X, Y, Z) \tag{2.1}$$

and also that the apolarity condition $\nabla\omega = 0$ can be expressed as $\operatorname{tr} K_X = 0$ for any vector field $X$ on $M$.

For all $u \in M$, the position vector $f(u)$ can be decomposed as

$$f(u) = f_* Z_u + \rho(u)\xi_u,$$

where $Z$ is a vector field on $M$ and $\rho$ the affine support function of $f$.

We now recall the definition of the *centre map*, which has been introduced in [1].

**Definition 2.1.** The *centre map* of an immersion $f : M \to \mathbf{R}^{n+1}$ is the map $c : M \to \mathbf{R}^{n+1}$ defined for all $u \in M$ by

$$c(u) = f(u) - \rho(u)\xi_u = f_* Z_u.$$

It follows that

$$c_* X = f_*(\operatorname{id} + \rho S)X - (X\rho)\xi,$$

hence the centre map of an immersion $f$ is itself an immersion iff

$$\ker(\operatorname{id} + \rho S) \cap \ker d\rho = \{0\}.$$

From now on, we will assume that the immersion $f$ is *centroaffine*, i.e. that the position vector is everywhere transversal to the tangent space, and that the centre map $c$ of $f$ is centroaffine, too, which amounts to

$$\dim\langle f_* Z^*_u\,,\ f_*(\operatorname{id} + \rho S)X - (X\rho)\xi_u \mid X \in T_u M\rangle = n + 1, \tag{2.2}$$

where we have used the notation $Z^* = \rho^{-1} Z$.

We are interested in immersions $f$ which are centroaffinely congruent to their centre map $c$.

The following result has been established in [1, Propositions 4.1, 4.2]:

**Proposition 2.2** (Furuhata–Vrancken)**.** *Let $f : M \to \mathbf{R}^{n+1}$ be an affine immersion whose centre map $c$ is a centroaffine immersion. Then $f$ is centroaffinely congruent with $c$ iff there exist a nowhere vanishing function $\rho$ and a vector field $Z^*$ on $M$ satisfying the following system of equations for all vector fields $X, Y$ on $M$:*

$$X(\rho) = -\rho\, h(X, Z^*), \tag{2.3}$$

$$(\nabla_X S)Y = h(X, Z^*)SY + h(Y, Z^*)SX - h(X, Y)SZ^*, \tag{2.4}$$

$$(\nabla h)(X, Y, Z^*) = -2\rho^{-1} h(X, Y) - 2h(X, SY) - h(X, Y)h(Z^*, Z^*), \tag{2.5}$$

$$\nabla_X Z^* = h(X, Z^*)Z^* + \rho^{-1}X + SX. \tag{2.6}$$

Using the apolarity condition, (2.1), and (2.5), we get

$$\rho^{-1} = -\frac{1}{n}\operatorname{tr} S - \frac{1}{2}h(Z^*, Z^*), \tag{2.7}$$

hence we can reformulate (2.5) as

$$(\nabla h)(X,Y,Z^*) = \frac{2}{n}\operatorname{tr} S\, h(X,Y) - 2h(X,SY). \tag{2.8}$$

## 3. Preliminary computations

Let $f : M \to \mathbf{R}^{n+1}$ be an immersion whose centre map $c$ is itself a centroaffine immersion, centroaffinely congruent to $f$.

We also assume that the metric $h$ induced by $f$ is positive definite. From the Ricci equation, there exists a local $h$-orthonormal basis $\{X_0, X_1, \ldots, X_{n-1}\}$ of eigenvectors for the shape operator $S$.
If we denote by $\lambda_0, \ldots, \lambda_{n-1}$ the corresponding eigenvalues, then the Codazzi equation for $S$ in this basis reads:

$$X_i(\lambda_j)X_j + \sum_{k=0}^{n-1}(\lambda_j - \lambda_k)\Gamma_{ij}^k X_k = X_j(\lambda_i)X_i + \sum_{k=0}^{n-1}(\lambda_i - \lambda_k)\Gamma_{ji}^k X_k, \tag{3.1}$$

where $\Gamma_{ij}^k$ denote the Christoffel symbols of the equiaffine connection $\nabla$ of $f$.

Writing $Z^* = \sum_{i=0}^{n-1} a_i X_i$, we get from (2.3) that $X_i(\rho) = -\rho a_i$. By [1, Proposition 4.3], there exist constants $\nu_j$ such that $\rho\lambda_j = \nu_j$. Applying $X_i$ to this equality, we obtain

$$X_i(\lambda_j) = a_i \lambda_j. \tag{3.2}$$

We now restrict to the quasi-umbilical case, i.e. when $S$ has two distinct eigenvalues:

- $\lambda_0$, with eigenspace $\langle X_0 \rangle$,
- $\lambda_1$, nonzero, with eigenspace $\langle X_1, \ldots, X_{n-1} \rangle$.

For $i, j = 1, \ldots, n-1$, (3.1) now simplifies to

$$X_i(\lambda_1)X_j + (\lambda_1 - \lambda_0)\Gamma_{ij}^0 X_0 = X_j(\lambda_1)X_i + (\lambda_1 - \lambda_0)\Gamma_{ji}^0 X_0.$$

Therefore $X_i(\lambda_1) = 0$ for $i = 1, \ldots, n-1$, so by (3.2), $a_i = 0$, i.e.

$$Z^* = a_0 X_0.$$

Let us now introduce the two constants

$$K_0 = \frac{\lambda_0}{\lambda_0 - \lambda_1}, \qquad K_1 = \frac{\lambda_1}{\lambda_0 - \lambda_1}.$$

Using (3.1), the Codazzi equation for $h$, and the apolarity condition, we get

**Lemma 3.1.** *For $i, j = 1, \dots, n-1$, one has*

$$\nabla_{X_0} X_0 = -\frac{(n-1)}{2} a_0 (K_0 + K_1) X_0,$$
$$\nabla_{X_0} X_i = \frac{a_0}{2}(K_0 + K_1) X_i + \sum_{k \neq 0, i} \Gamma_{0i}^k X_k,$$
$$\nabla_{X_i} X_0 = a_0 K_1 X_i,$$
$$\nabla_{X_i} X_j = \delta_{ij} a_0 K_0 X_0 + \sum_{k=1}^{n-1} \Gamma_{ij}^k X_k.$$

From (2.5) we have, for $i = 1, \dots, n-1$,

$$-a_0^2 (K_0 + K_1) = \nabla h(X_i, X_i, Z^*) = -2\rho^{-1} - 2\lambda_1 - a_0^2,$$
$$(n-1) a_0^2 (K_0 + K_1) = \nabla h(X_0, X_0, Z^*) = -2\rho^{-1} - 2\lambda_0 - a_0^2,$$

and from (2.6),

$$(\rho^{-1} + \lambda_1) X_i = \nabla_{X_i} a_0 X_0 = X_i(a_0) X_0 + a_0^2 K_1 X_i,$$
$$\big(a_0^2 + (\rho^{-1} + \lambda_0)\big) X_0 = \nabla_{X_0} a_0 X_0 = \left( X_0(a_0) - \frac{n-1}{2} a_0^2 (K_0 + K_1) \right) X_0,$$

so we deduce that

$$X_0(a_0) = \frac{a_0^2}{2}, \tag{3.3}$$
$$\rho^{-1} + \lambda_1 = a_0^2 K_1, \tag{3.4}$$
$$a_0^2(\lambda_0 + \lambda_1) = -\frac{2}{n}(\lambda_0 - \lambda_1)^2. \tag{3.5}$$

**Remark 3.2.** Equation (3.5) shows that we must have $\lambda_0 + \lambda_1 < 0$, as stated in Theorem 1.1.

**Lemma 3.3.** *Under the assumptions of Theorem 1.1, the centre map of $f$ is a centroaffine immersion.*

*Proof.* We know from (2.2) that $c$ is a centroaffine immersion iff

$$\dim\langle f_* Z_u^* \, , \; f_*(\mathrm{id} + \rho S) X - (X\rho)\xi_u \mid X \in T_u M \rangle = n + 1$$

iff the $n+1$ vectors

$$a_0 X_0 \, , \; (1 + \rho\lambda_1) X_1 \, , \; \dots \, , \; (1 + \rho\lambda_1) X_{n-1} \, , \; (1 + \rho\lambda_0) X_0 + \rho a_0 \xi$$

are linearly independent iff $1 + \rho\lambda_1 \neq 0$.

If $\rho^{-1} = -\lambda_1$, then we would get from (2.7) that $a_0^2 = \frac{2}{n}(\lambda_1 - \lambda_0)$. This and (3.5) would imply that $\lambda_1 = 0$, a contradiction. □

A short computation using Lemma 3.1 leads to the following

**Lemma 3.4.**

- *For* $i, j \geq 1$, $\widehat{\nabla}_{X_i} X_j = \frac{1}{2}\delta_{ij} a_0 X_0 + \sum_{k=1}^{n-1} \widehat{\Gamma}_{ij}^k X_k$ *where* $\widehat{\Gamma}_{ij}^k$ *denote the Christoffel symbols of the Levi-Civita connection* $\widehat{\nabla}$.
- $\widehat{\nabla}_{X_0} X_0 = 0$.
- *The difference tensor* $K_{X_0}$ *takes the form*

$$K_{X_0} = \begin{pmatrix} -\frac{n-1}{2}\, a_0(K_0 + K_1) & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \frac{1}{2}\, a_0(K_0 + K_1)\, \mathrm{id}_{n-1} & \\ 0 & & & \end{pmatrix}.$$

**Remark 3.5.** From Lemma 3.4, we see that the form of $K_{X_0}$, as well as that of the shape operator $S$, is the same as in [6].

## 4. Warped products

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds. Using the appropriate projections, any vector $V$ tangent to $M_1 \times M_2$ can be decomposed as $V = V_1 + V_2$, with $V_i$ tangent to $M_i$ $(i = 1, 2)$.

Recall that the *warped metric* $g_1 \times_{e^F} g_2$ on $M_1 \times M_2$ is defined by

$$g(V, W) = g_1(V_1, W_1) + e^{2F} g_2(V_2, W_2),$$

where $F$ is a function on $M_1 \times M_2$ depending only on $M_1$.

The manifold $M_1 \times M_2$, endowed with this metric, is a Riemannian manifold, denoted by $M_1 \times_{e^F} M_2$.

We will now use the following special case of a theorem of Nölker [2]:

**Proposition 4.1.** *Let* $(M, g)$ *be a Riemannian manifold with Levi-Civita connection* $\widehat{\nabla}$*, whose tangent bundle splits into two orthogonal distributions* $\mathcal{N}_1$ *and* $\mathcal{N}_2$*. Assume that there exists* $H \in \mathcal{N}_1$ *such that for all* $X, Y \in \mathcal{N}_1$, $U, V \in \mathcal{N}_2$*, one has*

$$\widehat{\nabla}_X Y \in \mathcal{N}_1,$$

$$g(\widehat{\nabla}_U V, Z) = g(U, V) g(H, Z) \qquad \text{for all } Z \in \mathcal{N}_1.$$

*Assume further that* $U(|H|) = 0$ *for all* $U \in \mathcal{N}_2$*. Then* $(M, g)$ *is locally isometric to a warped product* $M_1 \times_{e^F} M_2$*, with* $M_i$ *integral manifolds of* $\mathcal{N}_i$*.*
*Moreover, one has* $\operatorname{grad} F = -H$.

So from Lemma 3.4, we get that the Riemannian manifold $(M, h)$ is locally isometric to a warped product $\mathbf{R} \times_{e^F} N^{n-1}$, with the induced metric $h_N$ on $N$ given by $h_N(X_i, X_j) = e^{-2F}\delta_{ij}$ $(i, j = 1, \dots, n-1)$, and $H = \frac{1}{2}\, a_0 X_0$.

We now choose coordinates local coordinates $u_1, \dots, u_{n-1}$ on $N$, and a local coordinate $t$ on $\mathbf{R}$ such that $X_0 = \partial_t$.

## 5. Proof of Theorem 1.1: case $(n+2)\lambda_0 + n\lambda_1 \neq 0$

We construct two maps $g_i : M \to \mathbf{R}^{n+1}$ $(i = 1, 2)$ of the form $g_i = \alpha_i \xi + \beta_i X_0$ such that $D_{X_i} g_1 = D_{X_0} g_2 = 0$.

A straightforward computation using Lemma 3.1 and (3.5) leads to

**Lemma 5.1.** *The map $g_1 = a_0 K_1 \xi + \lambda_1 X_0$ satisfies*

$$D_{X_i} g_1 = 0 \qquad (i = 1, \dots, n-1),$$
$$D_{X_0} g_1 = -\frac{a_0}{2}\left((n+1)K_1 + (n-1)K_0\right) g_1.$$

Hence, there exist a function $c(t)$ and a constant vector $C_0$ such that $g_1(t) = c(t)C_0$.

**Lemma 5.2.** *There exists a map $g_2 = \alpha_2 \xi + \beta_2 X_0$ such that $D_{X_0} g_2 = 0$.*

*Proof.* Let us denote by $\nabla^N$ the restriction of $\nabla$ to $\langle X_1, \dots, X_{n-1}\rangle$.

For $i, j = 1, \dots, n-1$, we have from Lemma 3.1

$$D_{X_i} X_j = f_* \nabla^N_{X_i} X_j + \delta_{ij}(a_0 K_0 X_0 + \xi).$$

The map $\phi = a_0 K_0 X_0 + \xi$ satisfies

$$\phi_* X_i = D_{X_i} \phi = (a_0^2 K_0 K_1 - \lambda_1) X_i = -\zeta(t) X_i$$

and from (3.5) we also have

$$\phi_* X_0 = D_{X_0} \phi = a_0 K_0 \phi.$$

Hence we can find a function $\alpha_2(t)$ with $D_{X_0}(\alpha_2 \phi) = 0$. This function has to satisfy

$$X_0(\alpha_2) = -a_0 K_0 \alpha_2, \tag{5.1}$$

so for the map $g_2 = \alpha_2 \phi$, we get $D_{X_0} g_2 = 0$ and $D_{X_i} g_2 = \eta(t) X_i$, with $\eta = \alpha_2(a_0^2 K_0 K_1 - \lambda_1) = -\alpha_2 \zeta$. □

Notice for further use that by (3.5),

$$\zeta = \frac{\lambda_1}{n}\left(\frac{(n+2)\lambda_0 + n\lambda_1}{\lambda_0 + \lambda_1}\right), \tag{5.2}$$

hence the condition in the title of this section reads $\zeta \neq 0$.

**Proposition 5.3.** *When $\zeta \neq 0$, the map $g_2$ is an immersion of $N$ as a proper affine hypersphere in some hyperplane of $\mathbf{R}^{n+1}$.*

*Proof.* We have

$$\begin{aligned} D_{X_j} D_{X_i}\ g_2 &= \eta(t) D_{X_j} X_i \\ &= \eta(t)[f_* \nabla^N_{X_j} X_i + \delta_{ij}\phi] \\ &= g_{2*}(\nabla^N_{X_j} X_i) + \delta_{ij}\eta(t)\phi \\ &= g_{2*}(\nabla^N_{X_j} X_i) - \delta_{ij}\zeta(t) g_2. \end{aligned}$$

When $\zeta \neq 0$, $g_2$ can be viewed as an immersion of $N$ into $\mathbf{R}^{n+1}$. The above computation shows that $g_2$ actually lies in some fixed hyperplane of $\mathbf{R}^{n+1}$, namely

$\mathcal{H} = \langle X_1(p), X_2(p), \ldots, X_{n-1}(p), g_2(p)\rangle$ for some given point $p$. Hence $g_2$ is an immersion of $N$ into $\mathcal{H}$, and the position vector is transversal to $g_{2*}(N)$. From Lemma 3.1, we see that the difference tensor $K^N$ satisfies the apolarity condition, hence $g_2$ is (possibly up to a constant factor) the affine normal of $g_2$, which is therefore a proper affine hypersphere in $\mathcal{H}$. □

**Remark 5.4.** When $\zeta \neq 0$, the vector field $g_1$ is transversal to $\mathcal{H}$.

*Proof.* One has

$$\begin{aligned} & a_0K_1\xi + \lambda_1 X_0 = \frac{\lambda}{\alpha_2} g_2 + \sum_{i=1}^{n-1} a_i X_i \\ \text{iff}\quad & a_0K_1\xi + \lambda_1 X_0 = \lambda(\xi + a_0K_0X_0) + \sum_{i=1}^{n-1} a_i X_i \end{aligned}$$

iff $a_i = 0$ for $i = 1, \ldots, n-1$, $\lambda = a_0K_1$, and $\lambda_1 = a_0^2K_0K_1$, i.e. $\zeta = 0$. □

From $X_0(a_0) = \frac{a_0^2}{2}$, we get $a_0 = -\frac{2}{t}$. Hence, (5.1) gives

$$\begin{aligned} c' &= -\frac{a_0}{2}\big((n+1)K_1 + (n-1)K_0\big)\, c \\ &= \frac{1}{t}\big((n+1)K_1 + (n-1)K_0\big)\, c, \end{aligned}$$

whence

$$c(t) = n_1 t^{(n+1)K_1 + (n-1)K_0} \tag{5.3}$$

for some constant $n_1$.

Solving

$$\begin{cases} g_1 = a_0K_1\xi + \lambda_1 X_0, \\ g_2 = \alpha_2\xi + \alpha_2 a_0 K_0 X_0 \end{cases}$$

for $X_0$, we get

$$X_0 = \frac{a_0K_1}{\eta} g_2 + \frac{c}{\zeta} C_0. \tag{5.4}$$

Hence $\frac{\partial f}{\partial t} = \frac{a_0K_1}{\zeta} g_2 + \frac{c}{\zeta} C_0$, which, after an appropriate affine transformation (putting $C_0$ in the $e_{n+1}$-direction), gives the following expression for $f$:

$$f(t, \vec{u}) = \big(\gamma_1(t) g_2(\vec{u}), \gamma_2(t)\big),$$

where $\vec{u} = (u_1, \ldots, u_{n-1})$ and

$$\gamma_1(t) = \int \frac{a_0K_1}{\eta}(t)\, dt, \qquad \gamma_2(t) = \int \frac{c(t)}{\zeta(t)}\, dt.$$

Let us now explicitly compute $\gamma_1$ and $\gamma_2$.

By (3.2), we know that the eigenvalues $\lambda_i$ only depend on $t$, with $\lambda_i' = -\frac{2}{t}\lambda_i$, hence $\lambda_i = \frac{l_i}{t^2}$ with $l_i$ constant.

Since $\zeta = \lambda_1 - a_0^2 K_0 K_1$, we get $\zeta = \frac{\zeta_0}{t^2}$, where by (5.2), $\zeta_0 = \frac{l_1}{n}\left(\frac{(n+2)l_0+nl_1}{l_0+l_1}\right)$.

Using (5.3), we have

$$\frac{c}{\zeta}(t) = \frac{n_1}{\zeta_0} t^{(n-1)K_1+(n+1)K_0}.$$

Notice that $(n-1)K_1 + (n+1)K_0 \neq -1$. Otherwise, $K_0 + K_1 = -\frac{2}{n}$, hence, by (3.5), $a_0^2 = \lambda_0 - \lambda_1$, i.e. $a_0^2 K_1 = \lambda_1$. But by (3.4), $a_0^2 K_1 = \rho^{-1} + \lambda_1$, a contradiction. So we get

$$\gamma_2(t) = \frac{n_1}{\zeta_0}\,\frac{t^N}{N},$$

where $N = (n-2)K_1 + (n+2)K_0 \neq 0$.

On the other hand, $\eta = -\alpha_2\zeta$, where, from (5.1), $\alpha_2 = n_2 t^{2K_0}$, with $n_2$ constant. Hence $\eta = -n_2\zeta_0 t^{2(K_0-1)} = -n_2\zeta_0 t^{2K_1}$.

It is easy to check that $\eta' = -a_0 K_1 \eta$, hence $\gamma_1 = \frac{1}{\eta} = -\frac{1}{n_2\zeta_0} t^{-2K_1}$. So we have

$$f(t,\vec{u}) = \left(-\frac{1}{n_2\zeta_0} t^{-2K_1} g_2(\vec{u})\ ,\ \frac{n_1}{\zeta_0}\frac{t^N}{N}\right). \tag{5.5}$$

Let us now check that the hypersurfaces described in (5.5) do indeed satisfy the assumptions of Theorem 1.1.

One has

$$\begin{aligned} \partial_t &= (\gamma_1' g_2, \gamma_2'), \\ \partial_{u_i} &= \big(\gamma_1 g_{2*}(\partial_{u_i}), 0\big), \\ \xi &= \frac{1}{\alpha_2} g_2 - a_0 K_0 \partial_t, \end{aligned}$$

so that

$$\begin{aligned} D_{\partial_t}\partial_t &= \big(\frac{\gamma_2''}{\gamma_2'} + a_0 K_0\big)\partial_t + \xi, \\ D_{\partial_t}\partial_{u_i} &= \frac{\gamma_1'}{\gamma_1}\partial_{u_i}, \\ D_{\partial_{u_j}}\partial_{u_i} &= \nabla^N_{\partial_{u_j}}\partial_{u_i} + e^{2F} h_N(\partial_{u_i},\partial_{u_j})(a_0 K_0 \partial_t + \xi). \end{aligned}$$

Hence

$$\begin{aligned} h(\partial_t,\partial_t) &= 1, \\ h(\partial u_i,\partial u_j) &= e^{2F} h_N(\partial u_i,\partial u_j), \\ h(\partial_t,\partial u_i) &= 0, \end{aligned}$$

with $h_N$ the positive definite metric induced on $N$ by $g_2$.

We see that $h$ is positive definite and that $\det h = e^{2(n-1)F} \det h_N$.
On the other hand,

$$\begin{aligned}
\omega(\partial_t\,,\ \partial u_1\,,\ \dots\,,\ \partial_{u_{n-1}}) &= \det\left(\partial_t\,,\ \partial u_1\,,\ \dots\,,\ \partial_{u_{n-1}}\,,\ \frac{1}{\alpha_2}g_2 - a_0K_0\partial_t\right) \\
&= \det\left(\partial_t\,,\ \partial u_1\,,\ \dots\,,\ \partial_{u_{n-1}}\,,\ \frac{1}{\alpha_2}g_2\right) \\
&= \begin{vmatrix} \frac{a_0K_1}{\eta}\, g_2 & \frac{1}{\eta}\, g_{2*}(\partial_{u_1}) & \dots & \frac{1}{\eta}\, g_{2*}(\partial_{u_{n-1}}) & \frac{1}{\alpha_2}\, g_2 \\ \dfrac{c}{\zeta} & 0 & \dots & 0 & 0 \end{vmatrix} \\
&= (-1)^{n+2}\frac{c}{\zeta}\ \det\left(\frac{1}{\eta}\, g_{2*}(\partial_{u_1})\,, \dots\,, \frac{1}{\eta} g_{2*}(\partial_{u_{n-1}})\,, \frac{1}{\alpha_2} g_2\right) \\
&= (-1)^n \frac{c}{\alpha_2\zeta\eta^{n-1}}\ \det\Big(g_{2*}(\partial_{u_1})\,,\ \dots\,,\ g_{2*}(\partial_{u_{n-1}})\,,\ g_2\Big) \\
&= (-1)^{n+1}\frac{c}{\eta^n}\sqrt{\det h_N}.
\end{aligned}$$

For $\xi$ to be the affine normal, we have to check that

$$\omega^2(\partial_t, \partial_{u_1}, \dots, \partial_{u_{n-1}}) = \det h = e^{2(n-1)F} \det h_N. \tag{5.6}$$

Since $\operatorname{grad} f = -\frac{a_0}{2}\, X_0$, $e^F = e_0 t$ for some constant $e_0$, and (5.6) reads:

$$\frac{c^2}{\eta^{2n}} = \frac{n_1^2}{n_2^{2n}\zeta_0^{2n}}\, t^{2(n-1)} = e_0^{2(n-1)} t^{2(n-1)}, \tag{5.7}$$

which does hold after adjusting the integration constants $n_1, n_2, e_0$.

A straightforward computation shows that $D_{X_0}\xi = -\lambda_0 X_0$ and $D_{X_i}\xi = -\lambda_1 X_i$ for $i = 1, \dots, n-1$.

Let us now check that $f$ is indeed congruent to its centre map $c_f$. By definition, $c_f = f_* Z = \rho f_* Z^*$. From (2.3) we deduce $\rho = \rho_0 t^2$, with $\rho_0$ a constant. So

$$\begin{aligned}
c_f &= -2\rho_0 t X_0 \\
&= \left(\frac{4\rho_0 K_1}{\eta} g_2, -2\rho_0 \frac{n_1}{\zeta_0} t^N\right).
\end{aligned}$$

On the other hand, by (5.5)

$$f = \left(\frac{1}{\eta} g_2, \frac{n_1}{\zeta_0}\frac{t^N}{N}\right),$$

hence $c_f = Af$, with

$$A = \begin{pmatrix} & & & 0 \\ & 4\rho_0 K_1\ \mathrm{id}_n & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & -2\rho_0 N \end{pmatrix}.$$

## 6. Proof of Theorem 1.1: case $(n+2)\lambda_0 + n\lambda_1 = 0$

In this case, we have $\zeta = 0$ (cf. (5.2)).

As in the case $\zeta \neq 0$, we have, for $i, j = 1, \dots, n-1$,

$$D_{X_i} X_j = f_* \nabla^N_{X_i} X_j + \delta_{ij}(a_0 K_0 X_0 + \xi). \tag{6.1}$$

The map $\phi = a_0 K_0 X_0 + \xi$ satisfies

$$\phi_* X_i = D_{X_i}\phi = 0$$

and

$$\phi_* X_0 = D_{X_0}\phi = a_0 K_0 \phi = -\frac{2}{t} K_0 \phi,$$

so that $\phi = t^{-2K_0}\phi_0$ with $\phi_0$ a constant vector.

Since $(n+2)K_0 + nK_1 = 0$, one has

$$D_{X_0} X_0 = \frac{a_0}{2}(K_0 + K_1) X_0 + \phi, \tag{6.2}$$

hence $f(t, \vec{u})$ takes the form

$$f(t, \vec{u}) = g_0(\vec{u})\gamma_1(t) + g_1(\vec{u})1 + \alpha(t)\phi_0$$

and

$$X_0 = \gamma_1'(t)\, g_0(\vec{u}) + \alpha'(t)\phi_0.$$

From $D_{\partial u_i} X_0 = a_0 K_1 \partial u_i$, we deduce

- $\gamma_1' = a_0 K_1 \gamma_1$, i.e. $\gamma_1 = \gamma_0 t^{-2K_1}$ ($\gamma_0$ constant),
- $g_1(\vec{u})$ is constant,

and $\alpha(t)\phi_0$ is a solution of (6.2) iff $\alpha(t)$ satisfies

$$\alpha''(t) = \frac{a_0}{2}(K_0 + K_1)\alpha'(t) + t^{-2K_0},$$

i.e.

$$\alpha''(t) + \frac{1}{t}(K_0 + K_1)\alpha'(t) = t^{-2K_0}.$$

The general solution to this equation is

$$\alpha(t) = -\frac{t^{-2K_1}}{2K_1}\log t - B\frac{t^{-2K_1}}{2K_1} + C,$$

where $B$ and $C$ are constants. Hence, up to a translation,

$$f(t, \vec{u}) = \gamma_1 g_0(\vec{u}) - \frac{t^{-2K_1}}{2K_1}(\log t + B)\phi_0 \tag{6.3}$$

and

$$X_0 = a_0 K_1 \gamma_1\, g_0(\vec{u}) + t^{-(K_0+K_1)}(\log t + B)\phi_0.$$

We now show that $g_0$ is an improper affine hypersphere in some hyperplane of $\mathbf{R}^{n+1}$.

We first show that the $n+1$ vectors $g_0(\vec{u}), \partial_{u_i} g_0(\vec{u}), \phi_0$ $(i = 1, \ldots, n-1)$ are linearly independent. Indeed, denoting by $(h_{ij}) = h(\partial_{u_i}, \partial_{u_j})$ $(i, j = 0, \ldots, n-1)$, we know that $\det(\partial_t, \partial_{u_1}, \ldots, \partial_{u_i}, \xi) = \sqrt{\det(h_{ij})} \neq 0$.

Since $\xi = \phi - a_0 K_0 \partial_t$ and $\partial_{u_i} = \gamma_1 \partial_{u_i} g_0$,

$$\begin{aligned}\det(\partial_t, \partial_{u_1}, \ldots, \partial_{u_{n-1}}, \xi) &= \det(\partial_t \,,\, \gamma_1 \partial_{u_1} g_0 \,,\, \ldots \,,\, \gamma_1 \partial_{u_{n-1}} g_0 \,,\, \phi) \\ &= \det(a_0 K_1 \gamma_1 g_0 \,,\, \gamma_1 \partial_{u_1} g_0 \,,\, \ldots \,,\, \gamma_1 \partial_{u_{n-1}} g_0 \,,\, t^{-2K_0} \phi_0),\end{aligned}$$

hence $\det(g_0, \partial_{u_1} g_0, \ldots, \partial_{u_{n-1}} g_0, \phi_0) \neq 0$.

Let us now fix a point $p_0$ in $N$ and choose a frame in $\mathbf{R}^{n+1}$ such that

$$\begin{aligned} g_0(p_0) =& (1, 0, \ldots, 0), \\ \partial_{u_i} g_0(p_0) =& \underset{(1 \text{ in } (i+1)\text{st position})}{(0, \ldots, 1, \ldots, 0)}, \qquad i = 1, \ldots, n-1, \\ \phi_0 =& (0, \ldots, 0, \varphi_0) \qquad (\varphi_0 \text{ a constant}). \end{aligned}$$

From (6.1),

$$D_{\partial_{u_j}} D_{\partial_{u_i}} g_0 = g_{0*} \nabla^N_{\partial_{u_j}} \partial_{u_i} + h^N(\partial_{u_i}, \partial_{u_j}) \phi_0. \tag{6.4}$$

This equation has a unique solution satisfying the initial conditions $g_0(p_0)$ and $\partial_{u_i} g_0(p_0)$. Looking at the first component of (6.4), we see that $g_0$ lies in the hyperplane $\mathcal{H} \equiv x_0 = 1$.

A straightforward computation shows that, since $(n-2)K_0 + nK_1 = 0$,

$$\omega(\partial_{u_1} g_0, \ldots, \partial_{u_{n-1}} g_0, \phi) = \sqrt{\det h_N}.$$

Moreover, $D_{\partial_{u_i}} \phi = 0$, hence $g_0$ is an improper affine hypersphere in $\mathcal{H}$, with affine normal $\phi$. It is well kwown that any such map is locally the graph of a function $\mathcal{F} : N \to \mathbf{R}$ solution of the Monge–Ampère equation $\det\left(\frac{\partial^2 \mathcal{F}}{\partial_{u_i} \partial_{u_j}}\right) = 1$, so that by (6.3),

$$f(t, \vec{u}) = \left( t^{-2K_1} \,,\, t^{-2K_1} \,\vec{u} \,,\, \varphi_0 \, t^{-2K_1} \big( \mathcal{F}(\vec{u}) - \frac{1}{2K_1} \log t \big) \right). \tag{6.5}$$

We also have

$$X_0 = \left( \gamma_1' \,,\, \gamma_1' \vec{u} \,,\, \varphi_0 \left( \gamma_1' \mathcal{F}(\vec{u}) + 2K_1 t^{-2K_1 - 1} \frac{\log t}{2K_1} - \frac{t^{-2K_1 - 1}}{2K_1} \right) \right),$$

with $\gamma_1' = -\frac{2}{t} \, K_1 \, t^{-2K_1}$.

Recall that the centre map is given by $c_f = -2\rho_0 t X_0$, hence $c_f = Af$ with

$$A = \begin{pmatrix} 4\rho_0 K_1 & 0 & \dots & \dots & 0 \\ 0 & 4\rho_0 K_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 4\rho_0 K_1 & 0 \\ \frac{\rho_0\varphi_0}{K_1} & 0 & \dots & 0 & 4\rho_0 K_1 \end{pmatrix}.$$

## References


[1] H. Furuhata, L. Vrancken, The center map of an affine immersion, *Results Math.* **49** (2006), no. 3–4, 201–217.

[2] S. Nölker, Isometric immersions of warped products, *Differential Geom. Appl.* **6** (1996), no. 1, 1–30.

[3] K. Nomizu, T. Sasaki, *Affine differential geometry*, Cambridge University Press, Cambridge, 1994.

[4] H. Trabelsi, Improper affine hyperspheres with self-congruent center map, *Monatsh. Math.* **152** (2007), no. 1, 73–81.

[5] H. Trabelsi, Generic affine hypersurfaces with self congruent center map, *Results Math.* **51** (2007), no. 1–2, 127–140.

[6] K. Schoels, Affine hypersurfaces admitting a pointwise $SO(n-1)$ symmetry, *Results Math.*, to appear.



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