RATIONAL CUBIC FOURFOLDS CONTAINING A PLANE WITH NONTRIVIAL CLIFFORD INVARIANT

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Abstract. We isolate a general class of smooth rational cubic fourfolds X containing a plane whose associated quadric surface bundle does not have a rational section. Equivalently, the Brauer class β of the even Clifford algebra over the discriminant cover—a K3 surface S of degree 2—associated to the quadric bundle, is nontrivial. These fourfolds provide nontrivial examples verifying Kuznetsov's conjecture on the rationality of cubic fourfolds containing a plane. Indeed, using homological projective duality for grassmannians, one obtains another K3 surface S' of degree 14 and a nontrivial twisted derived equivalence $\mathbf{A}_X \cong \mathrm{D}^{\mathrm{b}}(S, \beta) \cong \mathrm{D}^{\mathrm{b}}(S')$, where $\mathbf{A}_X$ is Kuznetsov's residual category associated to the cubic hypersurface X.

Introduction

Let k be a field and X a *cubic fourfold*, i.e., a smooth cubic hypersurface $X \subset \mathbb{P}^5_k$. Determining the (geometric) k-rationality of X is a classical question in algebraic geometry. Some classes of geometrically rational cubic fourfolds have been described by Fano [6], Tregub [22], and Beauville–Donagi [2]. In particular, *pfaffian cubic fourfolds*, defined by pfaffians of skew-symmetric 6×6 matrices of linear forms, are rational. The closure of their locus forms a divisor in the moduli space $\mathcal{C}$ of cubic fourfolds. When $k = \mathbb{C}$, Hassett [8] describes, via lattice theory on the Hodge structure of X, a countable set of divisors $\mathcal{C}_d$ in $\mathcal{C}$. In particular, $\mathcal{C}_{14}$ is the closure of the locus of pfaffian cubics and $\mathcal{C}_8$ is the locus of cubic fourfolds containing a plane. Hassett [9] proves that there are countably many divisors of $\mathcal{C}_8$ consisting of rational cubic fourfolds. Nevertheless, it is expected that the general cubic fourfold (as well as the general cubic fourfold containing a plane) is nonrational. At present, however, not a single cubic fourfold is provably nonrational.

Recently, derived categories and their semiorthogonal decompositions have been brought to bear on the rationality problem for cubic fourfolds. In particular, for a smooth cubic fourfold X, Kuznetsov [17] has established a semiorthogonal decomposition of the bounded derived category

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathbf{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2) \rangle.$$

The category $\mathbf{A}_X$ has the remarkable property of being a 2-Calabi–Yau category, which roughly means that it is a noncommutative deformation of the derived category of a K3 surface. Based on evidence from known cases as well as general categorical considerations, Kuznetsov has conjectured that the category $\mathbf{A}_X$ contains all the information about the rationality of X.

Conjecture (Kuznetsov)**.** *A smooth complex cubic fourfold X is rational if and only if there exists a K3 surface S and an equivalence $\mathbf{A}_X \cong \mathrm{D}^{\mathrm{b}}(S)$.*

Supporting evidence for the conjecture comes from three classes of examples: singular cubic fourfolds [17, §5], pfaffian cubics fourfolds [15, Thm. 2], and smooth cubic fourfolds containing a plane whose associated even Clifford algebra has trivial Brauer class [17, §4]. In each of these classes, the cubic fourfolds considered are known to be rational and the associated K3 surfaces can be explicitly described. For example, using homological projective duality, Kuznetsov [15] shows that associated to a pfaffian cubic fourfold X, there is a smooth K3 surface S of degree 14 and an equivalence $\mathbf{A}_X \cong \mathrm{D}^{\mathrm{b}}(S)$ (see also Theorem 10).

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If X contains a plane P, a further geometric description of $\mathbf{A}_X$ is available. Indeed, X is birational to the total space of a quadric surface bundle $\pi\colon \widetilde{X} \to \mathbb{P}^2$ by blowing up (and projecting from) the plane. The degeneration divisor of π is a sextic curve $D \subset \mathbb{P}^2$ with *discriminant cover* $f\colon S \to \mathbb{P}^2$ branched along D. Let $\mathscr{C}_0$ be the isomorphism class of the even Clifford algebra associated to π, see [16] or [1, §2]. Since the base is regular, the center of $\mathscr{C}_0$ is isomorphic to $f_*\mathscr{O}_S$, by [13, IV Prop. 4.8.3]. Denote by $\mathscr{B}_0$ the associated $\mathscr{O}_S$-algebra. Now assume that $\pi\colon \widetilde{X} \to \mathbb{P}^2$ has simple degeneration, i.e., the degeneration divisor $D \subset \mathbb{P}^2$ is a smooth closed subscheme of codimension ≤ 1 above which the fibers are (singular) quadrics with isolated singularities. This is equivalent to X not containing another plane meeting P. In this case, S is a smooth K3 surface of degree 2 and the $\mathscr{O}_S$-algebra $\mathscr{B}_0$ is Azumaya by [16, Prop. 3.13]. We call the Brauer class $\beta \in \mathrm{Br}(S)$ of $\mathscr{B}_0$ the *Clifford invariant* of X. By [1, Thm. B.3] (also see [17, Lemma 4.2]), β is also the Brauer class associated to the $\mathbb{P}^1$-bundle that is the connected part of the Stein factorization of the relative Hilbert scheme of lines in the fibers of $\pi\colon \widetilde{X} \to \mathbb{P}^2$. In this context, the Clifford invariant is also studied in [24], [18], and [11]. Via mutations, Kuznetsov [17, Thm. 4.3] shows that there is an equivalence $\mathbf{A}_X \cong \mathrm{D}^{\mathrm{b}}(S, \beta)$ to the bounded derived category of S twisted by β. Kuznetsov's conjecture can then be restated for cubic fourfolds containing a plane.

Conjecture (Kuznetsov "containing a plane")**.** *Let X be a smooth complex cubic fourfold containing a plane whose associated quadric surface bundle has simple degeneration. Let S be the associated K3 surface of degree 2. Then X is rational if and only if there exists a K3 surface S' and an equivalence $\mathrm{D}^{\mathrm{b}}(S, \beta) \cong \mathrm{D}^{\mathrm{b}}(S')$.*

By classical results in the algebraic theory of quadratic forms (see [1, Thm. 2.24]), β is trivial if and only if $\pi\colon \widetilde{X} \to \mathbb{P}^2$ has a rational section, equivalently $\widetilde{X}$ (hence X) is $k(\mathbb{P}^2)$-rational (a stronger condition than k-rationality). By Springer's theorem on zeros of quadratic forms in odd degree extensions, this is equivalent to $\widetilde{X} \to S$ having an odd degree rational multisection, a condition investigated in [9, Thm. 3.1] and [17, Prop. 4.7]. In particular, if $\beta \in \mathrm{Br}(S)$ is trivial then X is k-rational and Kuznetsov's conjecture is verified. This should be understood as the *trivial case* of Kuznetsov's conjecture on cubic fourfolds containing a plane. Some of the geometric insights of this conjecture have been highlighted by Macrì–Stellari [18], who show how the Fano variety of lines in X can be described in terms of a moduli space of stable objects in $\mathrm{D}^{\mathrm{b}}(S, \beta)$.

The purpose of this note is to showcase a class of smooth geometrically rational cubic fourfolds containing a plane whose associated Clifford invariant β is nontrivial and yet Kuznetsov's conjecture is verified nonetheless. While the generic smooth cubic fourfold containing a plane (generic means off a union of countably many codimension one subvarieties of $\mathcal{C}_8$) has nontrivial Clifford invariant, it is not *a priori* clear if any are rational, though this is indicated in [9, Rem. 4.3].

Theorem 1. *Let X be a smooth complex cubic fourfold containing a plane whose associated quadric surface bundle has simple degeneration. Let S be the associated K3 surface of degree 2. If S has Picard rank 2 and even Néron–Severi discriminant then the Clifford invariant $\beta \in \mathrm{Br}(S)[2]$ of X is nontrivial.*

Furthermore, there exist smooth pfaffian cubic fourfolds satisfying these hypotheses. In particular, such cubic fourfolds give rise to K3 surfaces S and S' of degrees 2 and 14, respectively, and a nontrivial twisted derived equivalence $\mathrm{D}^{\mathrm{b}}(S, \beta) \cong \mathrm{D}^{\mathrm{b}}(S')$.

The nontriviality of the Brauer class in the above result is mostly proved via Hodge theory (see Proposition 3) while the existence statement is proved by providing an explicit numerical example (see Theorem 6), which we found with the aid of the `Magma` [4] adapting some of the computational techniques developed in [11]. We also prove the existence of smooth cubic fourfolds in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ whose associated K3 surface of degree 2 has Picard rank 2 and odd Néron–Severi discriminant, and for which the Clifford invariant is trivial.

We are guided by Hassett [9, Rem. 4.3], who suggests that examples of rational cubic fourfolds containing a plane with associated nontrivial Clifford invariant ought to lie in $\mathcal{C}_{14} \cap \mathcal{C}_8$. We note that the locus of smooth pfaffian (hence rational) cubic fourfolds containing a plane is a dense subset of $\mathcal{C}_{14} \cap \mathcal{C}_8$. Our example with nontrivial Clifford invariant affirm Hassett's suggestion and also provides a *nontrivial* verification of Kuznetsov's conjecture on cubic fourfolds containing a plane (i.e., where the derived equivalence is twisted by a nontrivial Brauer class). In particular, such examples provide occurrences of nontrivial twisted derived equivalences between K3 surfaces of degree 2 and 14.

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1. Hodge theoretic preliminaries

In this section, by means of straightforward lattice-theoretic calculations, we describe a class of cubic fourfolds containing a plane with nontrivial Clifford invariant.

If (H, q) is a $\mathbb{Z}$-lattice and $A \subset H$, then the orthogonal complement $A^\perp = \{v \in H : q(v, A) = 0\}$ is a *saturated* sublattice (i.e., $A^\perp = A^\perp \otimes_\mathbb{Z} \mathbb{Q} \cap H$) and is thus a *primitive* sublattice (i.e., $H/A^\perp$ is torsion free). Denote by $d(H, q) \in \mathbb{Z}$ the lattice *discriminant* or determinant of the Gram matrix.

Let X be a smooth cubic fourfold over $\mathbb{C}$. The integral Hodge conjecture holds for X (by [20], [27], cf. [26, Thm. 18]) and we denote by $A(X) = H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$ the lattice of integral middle Hodge classes, which are all algebraic.

Now suppose that X contains a plane P and let $\pi\colon \widetilde{X} \to \mathbb{P}^2$ be the quadric surface bundle defined by blowing up and projecting away from P. Let h be the hyperplane class associated to the embedding $X \subset \mathbb{P}^5$. The *transcendental* lattice $T(X)$, the *nonspecial cohomology* lattice K, and the *primitive cohomology* lattice $H^4(X, \mathbb{Z})_0$ are the orthogonal complements (with respect to the cup product polarization q_X) of $A(X)$, $\langle h^2, P\rangle$, and $\langle h^2\rangle$ inside $H^4(X, \mathbb{Z})$, respectively. Thus $T(X) \subset K \subset H^4(X, \mathbb{Z})_0$. We have that $T(X) = K$ for a "very generic" cubic fourfold, cf. proof of [24, Prop. 2]. There are natural polarized Hodge structures on $T(X)$, K, and $H^4(X, \mathbb{Z})_0$ given by restriction from $H^4(X, \mathbb{Z})$.

Similarly, let S be a smooth integral projective surface over $\mathbb{C}$ and $\mathrm{NS}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$ its Néron–Severi lattice. Let $h_1 \in \mathrm{NS}(S)$ be a fixed class with $h_1.h_1 \neq 0$. The *transcendental* lattice $T(S)$ and the *primitive cohomology* $H^2(S, \mathbb{Z})_0$ are the orthogonal complements (with respect to the cup product polarization q_S) of $\mathrm{NS}(S)$ and $\langle h_1\rangle$ inside $H^2(S, \mathbb{Z})$, respectively. If $f\colon S \to \mathbb{P}^2$ is a double cover, then we take $h_1 = f^*h$, where h is a hyperplane class.

Let $F(X)$ be the Fano variety of lines in X and $W \subset F(X)$ the divisor consisting of lines meeting P. Then W is identified with the relative Hilbert scheme of lines in the fibers of π. Its Stein factorization $W \xrightarrow{p} S \to \mathbb{P}^2$ displays W as a smooth conic bundle over a K3 surface S of degree 2, which is a double cover of $\mathbb{P}^2$ branched along a sextic curve. Then the Abel–Jacobi map

$$\Phi\colon H^4(X, \mathbb{Z}) \to H^2(W, \mathbb{Z})$$

becomes an isomorphism of $\mathbb{Q}$-Hodge structures $\Phi\colon H^4(X, \mathbb{Q}) \to H^2(W, \mathbb{Q})(-1)$, see [24, Prop. 1]. Finally, $p\colon W \to S$ is a smooth conic bundle and there is an injective (see [25, Lemma 7.28]) morphism of Hodge structures $p^*\colon H^2(S, \mathbb{Z}) \to H^2(W, \mathbb{Z})$. Let $h_1 \in \mathrm{NS}(S)$ be the pullback of the hyperplane class from $\mathbb{P}^2$. First, we recall a result of Voisin [24, Prop. 2].

Proposition 1. *Let X be a smooth cubic fourfold containing a plane whose associated quadric surface bundle has simple degeneration. Then $\Phi(K) \subset p^*H^2(S, \mathbb{Z})_0(-1)$ is a polarized Hodge substructure of index 2.*

Proof. That $\Phi(K) \subset p^*H^2(S, \mathbb{Z})_0$ is an inclusion of index 2 is proved in [24, Prop. 2]. We now verify that the inclusion respects the Hodge filtrations. The Hodge filtration of $\Phi(K) \otimes_\mathbb{Z} \mathbb{C}$ is that induced from $H^2(W,\mathbb{C})(-1)$ since Φ is an isomorphism of $\mathbb{Q}$-Hodge structures. On the other hand, $p\colon W \to S$ is a smooth conic bundle so $R^1 p_* \mathbb{C} = 0$, hence $p^*\colon H^2(S,\mathbb{C}) \to H^2(W,\mathbb{C})$ is injective by the Leray spectral sequence, and thus $p^* H^{p,2-p}(S) = p^* H^2(S,\mathbb{C}) \cap H^{p,2-p}(W)$. Thus the Hodge filtration of $p^*H^2(S,\mathbb{C})(-1)$ is also that induced from $H^2(W,\mathbb{C})(-1)$, and similarly for primitive cohomology. Thus the inclusion is a morphism of Hodge structures. Finally, by [24, Prop. 2], we have that $b_X(x,y) = -b_S(\Phi(x),\Phi(y))$ for $x,y \in K$, and thus the inclusion $\Phi(K) \subset p^*H^2(S,\mathbb{Z})_0(-1)$ preserves the polarization. □

By abuse of notation (of which we are already guilty), for $x \in K$, we will consider $\Phi(x)$ as an element of $p^*H^2(S,\mathbb{Z})_0(-1)$ without explicitly mentioning so.

Corollary 2. *Let X be a smooth cubic fourfold containing a plane whose associated quadric surface bundle has simple degeneration. Then $\Phi(T(X)) \subset p^*T(S)$ is a sublattice of index ϵ dividing 2. In particular,* $\operatorname{rk} A(X) = \operatorname{rk} \operatorname{NS}(S) + 1$ *and* $d(A(X)) = 2^{2(\epsilon-1)} d(\operatorname{NS}(S))$.

Proof. By the saturation property, $T(X)$ and $T(S)$ coincide with the orthogonal complement of $A(X) \cap K$ in K and $\operatorname{NS}(S) \cap H^2(S,\mathbb{Z})_0$ in $H^2(S,\mathbb{Z})_0$, respectively. Now, for $x \in T(X)$ and $b \in \operatorname{NS}(S)_0$, we have

$$q_S(\Phi(x), b) = -\frac{1}{2}\Phi(x).g.p^*b = -\frac{1}{2} q_X(x, {}^t\Phi(g.p^*b)) = 0$$

by [24, Lemme 3] and the fact that ${}^t\Phi(g.p^*b) \in A(X)$ (here $g \in H^2(W,\mathbb{Z})$ is the pullback of the hyperplane class from the canonical grassmannian embedding), which follows since ${}^t\Phi\colon H^4(W,\mathbb{Z}) \cong H_2(W,\mathbb{Z}) \to H_4(X,\mathbb{Z}) \cong H^4(X,\mathbb{Z})$ preserves the Hodge structure by the same argument as in the proof of Proposition 1. Therefore $\Phi(T(X)) \subset p^*T(S)$.

Since $T(X) \subset K$ and $T(S) \subset H^2(S,\mathbb{Z})_0$ are saturated (hence primitive) sublattices, an application of the snake lemma shows that $p^*T(S)/\Phi(T(X)) \subset p^*H^2(S,\mathbb{Z})_0/\Phi(K) \cong \mathbb{Z}/2\mathbb{Z}$, hence the index ϵ of $\Phi(T(X))$ in $p^*T(S)$ divides 2.

We now verify the final claims. We have $\operatorname{rk} K = \operatorname{rk} H^2(X,\mathbb{Z}) - 2 = \operatorname{rk} T(X) + \operatorname{rk} A(X) - 2$ and $\operatorname{rk} H^2(S,\mathbb{Z})_0 = \operatorname{rk} H^2(S,\mathbb{Z}) - 1 = \operatorname{rk} T(S) + \operatorname{rk} \operatorname{NS}(S) - 1$ (since P, h^2, and h_1 are anisotropic vectors, respectively), while $\operatorname{rk} K = \operatorname{rk} H^2(S,\mathbb{Z})_0$ and $\operatorname{rk} T(X) = \operatorname{rk} T(S)$ by Proposition 1 and the above, respectively. The claim concerning the discriminant then follows by standard lattice theory. □

Finally, we provide a general class of smooth cubic fourfolds containing a plane with nontrivial Clifford invariant.

Proposition 3. *Let X be a smooth cubic fourfold containing a plane P whose associated quadric surface bundle has simple degeneration. If $A(X)$ has rank 3 and even discriminant then the Clifford invariant $\beta \in \operatorname{Br}(S)[2]$ of X is nontrivial. In particular, this is the case if the associated K3 surface S of degree 2 has Picard rank 2 and even Néron–Severi discriminant.*

Proof. The Clifford invariant $\beta \in \operatorname{Br}(S)[2]$ associated to the quadric surface bundle $\pi\colon \widetilde{X} \to \mathbb{P}^2$ is trivial if and only if π has a rational section, see [14, Thm. 6.3] or [21, 2 Thm. 14.1, Lemma 14.2]. Such a section exists if and only if there exists an algebraic cycle $R \in A(X)$ such that $R.(h^2-P) = 1$, see [9, Thm. 3.1] or [17, Prop. 4.7].

Suppose that such a R exists and consider the sublattice $\langle h^2, h^2-P, R\rangle \subset A(X)$. Its intersection form has Gram matrix

	h^2	h^2-P	R
h^2	3	2	x
h^2-P	2	4	1
R	x	1	y

for some $x, y \in \mathbb{Z}$. The determinant of this matrix is always odd, so this lattice cannot be a finite index sublattice of $A(X)$, which has even discriminant by hypothesis. Hence no such 2-cycle R exists and thus β is nontrivial. The final claim follows directly from Corollary 2. □

This proves the first part of Theorem 1. The second part, the existence of smooth pfaffian cubic fourfolds satisfying these hypotheses, is guaranteed by the example developed in §2. In Remark 9, we show that there also exist smooth cubic fourfolds in $\mathcal{C}_8 \cap \mathcal{C}_{14}$ whose associated K3 surface of degree 2 has odd Néron–Severi discriminant.

We end this section with an example providing explicit geometric conditions for the hypotheses of Proposition 3 to hold, and a remark showing that, assuming Kuznetsov conjecture, if a cubic fourfold containing a plane is rational, then the Picard number of the associated degree 2 K3 surface is at least 2.

Example 4. Let $C \subset \mathbb{P}^2$ be a conic tangent to the discriminant curve $D \subset \mathbb{P}^2$ along its intersection. Consider the pull back of the conic C to S via the discriminant double cover $f\colon S \to \mathbb{P}^2$. Then f^*C has two components C_1 and C_2. The sublattice of the Néron–Severi lattice of S generated by $h_1 = (C_1 + C_2)/2$ and C_1 has intersection form with Gram matrix

$$\begin{array}{ccc} & h_1 & C_1 \\ h_1 & 2 & 2 \\ C_1 & 2 & -2 \end{array}$$

having determinant -8. If S has Picard rank 2, then the entire Néron–Severi lattice is in fact generated by h_1 and C_1 (see [5, §2] for further details). We can then apply Proposition 3 to conclude the nontriviality of the Clifford invariant.

Remark 5. If X is a smooth cubic fourfold containing a plane such that the associated $K3$ surface S of degree 2 is smooth with Picard rank 1, then the Clifford invariant is nontrivial. Indeed, the lattice $A(X)$ has Gram matrix

$$\begin{array}{ccc} & h^2 & h^2 - P \\ h^2 & 3 & 2 \\ h^2 - P & 2 & 4 \end{array}$$

showing that $R.(h^2 - P)$ is even for every $R \in A(X)$, hence the Clifford invariant is nontrivial by an application of the criteria in [9, Thm. 3.1] or [17, Prop. 4.7] (cf. the proof of Proposition 3). A stronger statement, more in the spirit of Kuznetsov's conjecture, is that there exists no $K3$ surface S' with $\mathbf{A}_X \cong \mathrm{D}^{\mathrm{b}}(S')$ (see [17, Prop. 4.8]). In particular the "very generic" cubic fourfold containing a plane should be nonrational, according to Kuznetsov's conjecture. Our examples (see Theorem 6 and Remark 9) show that there exist rational cubic fourfolds containing a plane with associated S of Picard rank 2 and with both nontrivial and trivial Clifford invariant.

2. The example

In this section, we give an example of a smooth pfaffian cubic fourfold containing a plane and satisfying the hypotheses of Proposition 3, hence having nontrivial Clifford algebra.

Theorem 6. *Let A be the 6×6 antisymmetric matrix*

$$\begin{pmatrix} 0 & y+u & x+y+u & u & z & y+u+v \\ & 0 & x+y+z & x+z+u+w & y+z+u+v+w & x+y+z+u+v+w \\ & & 0 & x+y+u+w & x+y+u+v+w & x+y+z+v+w \\ & & & 0 & x+u+v+w & x+u+w \\ & & & & 0 & z+u+w \\ & & & & & 0 \end{pmatrix}$$

of linear forms in $\mathbb{Q}[x,y,z,u,v,w]$ and let $X \subset \mathbb{P}^5$ be the cubic fourfold defined by the vanishing of the pfaffian of A:

$$\begin{aligned} &(x-4y-z)u^2 + (-x-3y)uv + (x-3y)uw + (x-2y-z)vw - 2yv^2 + xw^2 \\ &+ (2x^2 + xz - 4y^2 + 2z^2)u + (x^2 - xy - 3y^2 + yz - z^2)v + (2x^2 + xy + 3xz - 3y^2 + yz)w \\ &+ x^3 + x^2y + 2x^2z - xy^2 + xz^2 - y^3 + yz^2 - z^3. \end{aligned}$$

Then:

a) X *is smooth, rational, and contains the plane* $P = \{x = y = z = 0\}$.
b) *The degeneration divisor* $D \subset \mathbb{P}^2$ *of the associated quadric surface bundle* $\pi\colon \widetilde{X} \to \mathbb{P}^2$ *is the sextic curve*

$$\begin{aligned} &x^6 + 6x^5y + 12x^5z + x^4y^2 + 22x^4yz + 28x^3y^3 - 38x^3y^2z + 46x^3yz^2 + 4x^3z^3 \\ &\quad + 24x^2y^4 - 4x^2y^3z - 37x^2y^2z^2 - 36x^2yz^3 - 4x^2z^4 + 48xy^4z - 24xy^3z^2 \\ &\quad + 34xy^2z^3 + 4xyz^4 + 20y^5z + 20y^4z^2 - 8y^3z^3 - 11y^2z^4 - 4yz^5. \end{aligned}$$

This curve is smooth; in particular, π *has simple degeneration and the discriminant cover is a smooth K3 surface* S *of degree 2.*

c) *The Clifford invariant of* X *is represented by the unramified quaternion algebra* (b, ac) *over the function field of* S*, where*

$$a = x - 4y - z, \quad b = x^2 + 14xy - 23y^2 - 8yz,$$

and

$$c = 3x^3 + 2x^2y - 4x^2z + 8xyz + 3xz^2 - 16y^3 - 11y^2z - 8yz^2 - z^3,$$

d) *The conic* $C \subset \mathbb{P}^2$ *defined by the vanishing of* $x^2 + yz$ *is tangent to the degeneration divisor* D *at six points (five of which are distinct).*
e) *The Clifford invariant of* X *is geometrically nontrivial.*

Proof. Verifying smoothness of X and D is a straightforward application of the jacobian criterion, while the inclusion $P \subset X$ is checked by inspecting the expression for $\mathrm{pf}(A)$: every monomial is divisible by x, y or z. Rationality comes from being a pfaffian cubic fourfold; see [22]. The smoothness of D implies that π has simple degeneration; [11, Rem. 7.1], or [1, Rem. 2.6]. This establishes parts *a)* and *b)*. To prove *c)*, we first establish a handy general formula.

Lemma 7. *Let* K *be a field of characteristic* $\neq 2$ *and* q *a nondegenerate quadratic form of rank 4 over* K *with discriminant extension* L/K*. For* $1 \leq r \leq 4$ *denote by* m_r *the determinant of the leading principal* $r \times r$ *minor of the symmetric Gram matrix of* q*. Then the class* $\beta \in \mathrm{Br}(L)$ *of the even Clifford algebra of* q *is the quaternion algebra* $(-m_2, -m_1m_3)$.

Proof. On $n \times n$ matrices M over K, symmetric gaussian elimination is the following operation:

$$M = \begin{pmatrix} a & v^t \\ v & A \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & A - a^{-1}vv^t \end{pmatrix}$$

where $a \in K^\times$, $v \in K^{n-1}$ is a column vector, and A is an $(n-1) \times (n-1)$ matrix over K. Then $m_1 = a$ and the element in the first row and column of $A - a^{-1}vv^t$ is precisely m_2/m_1. By induction, M can be diagonalization, using symmetric gaussian elimination, to the matrix

$$\mathrm{diag}(m_1, m_2/m_1, \ldots, m_n/m_{n-1}).$$

For q of rank 4 with symmetric Gram matrix M, we have

$$q = \langle m_1 \rangle \otimes \langle 1, m_2, m_1m_2m_3, m_1m_2m_4 \rangle$$

so that over $L = K(\sqrt{m_4})$, we have that $q|_L = \langle m_1 \rangle \otimes \langle 1, m_2, m_1m_3, m_1m_2m_3 \rangle$, which is similar to the norm form of the quaternion L-algebra $(-m_2, -m_1m_3)$. Thus $C_0(q)$ is Brauer equivalent to $(-m_2, -m_1m_3)$ over L. □

Since S is regular, $\mathrm{Br}(S) \to \mathrm{Br}(k(S))$ is injective (see [7, Cor. 1.10]). Since $\beta|_{k(S)}$ is represented by $\mathscr{B}_0|_{k(S)}$, which is Brauer equivalent to the even Clifford algebra $C_0(q|_{k(S)})$ of $q|_{k(\mathbb{P}^2)}$, we can perform our calculations in the function field $k(S)$. The symmetric Gram matrix in our example is

$$\begin{pmatrix} 2(x-4y-z) & -x-3y & x-3y & 2x^2+xz-4y^2+2z^2 \\ & 2(-2y) & x-2y-z & x^2-xy-3y^2+yz-z^2 \\ & & 2x & 2x^2+xy+3xz-3y^2+yz \\ & & & 2(x^3+x^2y+2x^2z-xy^2+xz^2-y^3+yz^2-z^3) \end{pmatrix}$$

which has $m_1 = 2a$, $m_2 = -b$, and $m_3 = -2c$ as claimed, so that part *c)* follows from the above lemma.

Part *d)* follows since we can write the equation for the degeneration divisor as $(x^2 + yz)f + g^2$ where

$$\begin{aligned} f &= x^4 + 6x^3y + 12x^3z + x^2y^2 + 21x^2yz - 25x^2z^2 + 28xy^3 \\ &\quad - 24xy^2z + 34xyz^2 + 4xz^3 + 20y^4 - 5y^3z - 8y^2z^2 - 11yz^3 - 4z^4 \\ g &= 2xy^2 + 5y^2z - 5x^2z \end{aligned}$$

hence the conic $C \subset \mathbb{P}^2$ defined by $x^2 + yz$ is tangent to D along a zero-dimensional scheme of length 6 that is everywhere non-reduced.

Finally, to prove *e)*, that the Clifford invariant of X is geometrically nontrivial, it suffices to verify that $\rho(S) = 2$, by Proposition 3 and Example 4.

The surface S is the smooth sextic in $\mathbb{P}(1,1,1,3) = \operatorname{Proj} \mathbb{Q}[x,y,z,w]$ given by

$$w^2 = p(x,y,z),$$

where $p(x,y,z)$ is the homogeneous polynomial of degree six in Theorem 6*b)*. In these coordinates, the discriminant double cover $f\colon S \to \mathbb{P}^2$ is simply the restriction to S of the projection $P(1,1,1,3) \dashrightarrow \mathbb{P}^2$. Let $C \subset \mathbb{P}^2$ be the conic from Theorem 6*d)*. As discussed in the proof of Theorem 1, the curve f^*C consists of two (-2)-curves C_1 and C_2. Explicitly, we have

$$\begin{aligned} C_1\colon x^2 + yz = w - (2xy^2 + 5y^2z - 5x^2z) = 0, \\ C_2\colon x^2 + yz = w + (2xy^2 + 5y^2z - 5x^2z) = 0. \end{aligned}$$

These curves generate a non-saturated sublattice of $\mathrm{NS}(\overline{S})$ of rank 2. Hence $\rho(\overline{S}) \geq 2$.

We show next that $\rho(\overline{S}) \leq 2$. Write $S_3/\mathbb{F}_3$ for the reduction mod 3 of S. Let $\ell \neq 3$ be a prime and write $\phi(t)$ for the characteristic polynomial of the action of absolute Frobenius on $H^2_{\text{ét}}(\overline{S}_3, \mathbb{Q}_\ell)$. Then $\rho(\overline{S}_3)$ is bounded above by the number of roots of $\phi(t)$ that are of the form 3ζ, where ζ is a root of unity [23, Prop. 2.3]. Combining the Lefschetz trace formula with Newton's identities and the functional equation that $\phi(t)$ satisfies, it is possible calculate $\phi(t)$ from knowledge of $\#S(\mathbb{F}_{3^n})$ for $1 \leq n \leq 11$; see [23] for details.

Let $\widetilde{\phi}(t) = 3^{-22}\phi(3t)$, so that the number of roots of $\widetilde{\phi}(t)$ that are roots of unity gives an upper bound for $\rho(\overline{S}_3)$. Using `Magma`, we compute

$$\begin{aligned} \widetilde{\phi}(t) = \frac{1}{3}(t-1)^2(3t^{20} + t^{19} + t^{17} + t^{16} + 2t^{15} + 3t^{14} + t^{12} + 3t^{11} \\ + 2t^{10} + 3t^9 + t^8 + 3t^6 + 2t^5 + t^4 + t^3 + t + 3) \end{aligned}$$

The roots of the degree 20 factor are not integral, and hence they are not roots of unity. We conclude that $\rho(\overline{S}_3) \leq 2$. By [23], we have $\rho(\overline{S}) \leq \rho(\overline{S}_3)$, so $\rho(\overline{S}) \leq 2$. It follows that that S has Picard number 2. This concludes the proof of the theorem. □

Remark 8. We remark that, contrary to the situation in [11], the transcendental class $\beta \in \mathrm{Br}(S)[2]$ is actually constant when evaluated on $S(\mathbb{Q})$. Indeed, using elimination theory, we find that the odd primes p of bad reduction of S are 5, 23, 263, 509, 1117, 6691, 3342589, 197362715625311, and 4027093318108984867401313726363. For each odd prime of bad reduction, we compute that the singular locus of $S_{\overline{\mathbb{F}}_p}$ consists of a single ordinary double point. By [10, Prop. 4.1, Lemma 4.2], the local invariant map associated to β is constant on $S(\mathbb{Q}_p)$ for odd primes p of bad reduction. By an adaptation of [10, Lemma 4.4], the local invariant map is also constant at odd primes of good reduction.

At the real place, we prove that $S(\mathbb{R})$ is connected, hence the local invariant map is constant. To this end, recall that the set of real points of a smooth hypersurface of even degree in $\mathbb{P}^2(\mathbb{R})$ consists of a disjoint union of ovals (i.e., topological circles, in the language of real algebraic geometry, whose complement is homeomorphic to a union of a disk and a Möbius band). In particular, $\mathbb{P}^2(\mathbb{R}) \smallsetminus D(\mathbb{R})$

has a unique nonorientable connected component R. By graphing an affine chart of $D(\mathbb{R})$, we find that the point $(1:0:0)$ is contained in R. We compute that the map projecting from $(1:0:0)$ has four real critical values, hence $D(\mathbb{R})$ consists of two ovals. These ovals are not nested, as can be seen by inspecting the graph of $D(\mathbb{R})$ in an affine chart. The Gram matrix of the quadratic form, specialized at $(1:0:0)$, has positive determinant, hence by local constancy, the equation for D is positive over the entire component R and negative over the interiors of the two ovals (since D is smooth). In particular, the map $f\colon S(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$ has empty fibers over the interiors of the two ovals and nonempty fibers over $R \subset \mathbb{P}^2(\mathbb{R})$ where it restricts to a nonsplit unramified cover of degree 2, which must be the orientation double cover of R since $S(\mathbb{R})$ is orientable (the Kähler form on S defines an orientation). In particular, $S(\mathbb{R})$ is connected.

This shows that β is constant on $S(\mathbb{Q})$. We believe that the local invariant map is also constant at the prime 2, though this must be checked with a brute force computation.

Remark 9. One may wonder if the Clifford invariant $\beta \in \mathrm{Br}(S)[2]$ of X is nontrivial *for every* cubic fourfold $X \in \mathcal{C}_8 \cap \mathcal{C}_{14}$ such that $A(X)$ has rank 3. However, we find and example where this is not the case. On the other hand, we still do not know if any pfaffian cubic fourfold containing a plane has trivial Clifford invariant.

Every cubic fourfold X in $\mathcal{C}_{14}$ contains a rational quartic scroll. For $X \in \mathcal{C}_8 \cap \mathcal{C}_{14}$ write P and T for the class of the plane and a quartic scroll, respectively. The intersection form of the sublattice $\langle h^2, P, T\rangle \subset A(X)$ has Gram matrix

$$\begin{array}{cccc} & h^2 & P & T \\ h^2 & 3 & 1 & 4 \\ P & 1 & 3 & x \\ T & 4 & x & 10 \end{array} \tag{1}$$

for some $x \in \mathbb{Z}$. There may, a priori, be restrictions on the value of x.

We will use recent work of Mayanskiy [19, Thm. 6.1, Rem. 6.3] to show that there exists a cubic fourfold $X \in \mathcal{C}_8 \cap \mathcal{C}_{14}$ such that $A(X) = \langle h^2, P, T\rangle$ has intersection form with Gram matrix (1) and $x = 1$. In this case, the Clifford invariant $\beta \in \mathrm{Br}(S)[2]$ is trivial by an application of the criteria in [9, Thm. 3.1] or [17, Prop. 4.7] (cf. the proof of Proposition 3), as $(T+P).(h^2-P) = 1$.

Let $A = h^2\mathbb{Z} \oplus P\mathbb{Z} \oplus T\mathbb{Z}$ and (A, q) be the lattice whose bilinear form has Gram matrix (1) with $x = 1$. The lattice (A, q) is saturated since $d(A, q) = 37$ is squarefree. We now proceed to verify the conditions of [19, Thm. 6.1]:

a) $q(h^2, h^2) = 3$;

b) The vectors $(1, -3, 0)$ and $(0, -4, 1)$ form a basis for $A_0 = \langle h^2\rangle^\perp$ and $q|_{A_0}$ is even.

c) For $\delta = (a, -3a-4b, b) \in A_0$, we have
$$q(\delta, \delta) = 2(12a^2 + 33ab + 25b^2),$$
which does not take the values 2 or 6.

d) For $\delta(x, y, z) \in A$, we have
$$q(h^2, \delta)^2 = 9x^2 + y^2 + 16z^2 + \text{terms with even coefficients},$$
$$q(\delta, \delta) = 3x^2 + 3y^2 + 10z^2 + \text{terms with even coefficients},$$
so that $q(h^2, \delta)^2 - q(\delta, \delta)$ takes even values for all $\delta \in A$.

e) The discriminant group A^*/A is $\mathbb{Z}/37\mathbb{Z}$, thus $l(A^*/A) = 1$ in the terminology of [19]. We then appeal to [19, Rem. 6.3].

We conclude that the lattice (A, q) arises as $(A(X), q_X)$ for some smooth cubic fourfold X. As (A, q) contains saturated rank 2 sublattices containing h^2 with determinants 8 (the upper left 2×2 minor) and 14 (the outer 2×2 minor), we have that $X \in \mathcal{C}_8$ and $X \in \mathcal{C}_{14}$, respectively. However, while $X \in \mathcal{C}_8 \cap \mathcal{C}_{14}$, we do not know whether X is a pfaffian cubic fourfold, since the locus of pfaffian cubic fourfolds is only a dense subset of $\mathcal{C}_{14}$. In any case, since β is trivial, it is known that X is rational and verifies Kuznetsov's conjecture.

3. The twisted derived equivalence

On the categorical side, homological projective duality (HPD) for grassmannians can be used to obtain a significant semiorthogonal decomposition for pfaffian cubic fourfolds.

Theorem 10 ([15]). *Let W be a $\mathbb{C}$-vector space of dimension 6. Then the universal cubic pfaffian variety in $\mathbb{P}(\wedge^2 W^\vee) = \mathbb{P}^{14}$ is homologically projectively dual to the grassmannian $\mathrm{Gr}(2, W)$. In particular, a smooth pfaffian cubic fourfold X admits a semiorthogonal decomposition*

$$\mathrm{D^b}(X) = \langle \mathrm{D^b}(S'), \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2)\rangle,$$

where S' is a smooth K3 surface of degree 14.

Proof. The statement is the sum of [15, Thm. 1] and [15, Thm. 2]. The decomposition is due to the fact that any pfaffian cubic fourfold is a linear intersection of the universal pfaffian in $\mathbb{P}^{14}$ with a 6-dimensional linear subspace. On the other hand, the K3 surface S' is the intersection of the grassmannian $\mathrm{Gr}(2, 6)$ with a $\mathbb{P}^8$ in the dual space $\check{\mathbb{P}}^{14}$. To be precise, the semiorthogonal decomposition described by Kuznetsov is

$$\mathrm{D^b}(X) = \langle \mathscr{O}_X(-3), \mathscr{O}_X(-2), \mathscr{O}_X(-1), \mathrm{D^b}(S')\rangle.$$

To obtain the equivalence $\mathbf{A}_X \cong \mathrm{D^b}(S')$, it is enough to act on $\mathrm{D^b}(X)$ by the autoequivalence $-\otimes \mathscr{O}_X(3)$ and then mutate the image of $\mathrm{D^b}(S')$ to the left with respect to its left orthogonal: this gives a category equivalent to $\mathrm{D^b}(S')$ (see [3]) which is the left orthogonal complement of $\langle \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2)\rangle$, that is $\mathbf{A}_X$. □

Now consider the smooth pfaffian cubic fourfold X containing a plane from Theorem 6. By combining results of [17] and [15], we obtain the following corollary of Theorem 6.

Corollary 11. *Let $X \subset \mathbb{P}^5$ be the smooth cubic fourfold of Theorem 6. Then the remarkable category $\mathbf{A}_X$ is equivalent both to $\mathrm{D^b}(S, \beta)$ and to $\mathrm{D^b}(S')$. In particular, there is a nontrivial twisted derived equivalence $\mathrm{D^b}(S, \beta) \cong \mathrm{D^b}(S')$*

Remark 12. By [12, Rem. 7.10] there is no equivalence $\mathrm{D^b}(S, \beta) \cong \mathrm{D^b}(S)$, so X provides an example of a smooth rational cubic fourfold containing a plane satisfying Kuznetsov's conjecture, but not via the derived category $\mathrm{D^b}(S)$ of the K3 surface of degree 2 associated to the quadric fibration.

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