ON LUSTERNIK-SCHNIRELMANN CATEGORY OF CONNECTED SUMS

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Abstract. In this paper we estimate the Lusternik-Schnirelmann category of the connected sum of two manifolds through their categories. We achieve a more general result regarding the category of a quotient space X/A where A is a suitable subspace of X.

1. Introduction

1.1. **Definition.** The Lusternik-Schnirelmann category (LS category) of a space X is the smallest nonnegative integer n such that there exists $\{A_0, A_1, ..., A_n\}$, an open cover of X with each A_i contractible in X. This is denoted by $n = \operatorname{cat} X$.

Following this definition, spaces with LS category 0 are contractible.

The goal of the paper is to prove the inequality

$$\max\{\operatorname{cat} M, \operatorname{cat} N\} - 1 \leq \operatorname{cat}(M\#N) \leq \max\{\operatorname{cat} M, \operatorname{cat} N\}, \tag{1.1}$$

where M and N are closed manifolds.

To prove the inequality, we consider a more general problem about the relation of $\operatorname{cat} X$ and $\operatorname{cat}(X/A)$. This problem is indeed more general: in fact, put $X = M\#N$ and A be an $(n-1)$ sphere that separates M and N (with removed discs). Then $X/A = M \vee N$ and $\operatorname{cat}(M \vee N) = \max\{\operatorname{cat} M, \operatorname{cat} N\}$.

All spaces are assumed to be CW spaces.

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2. Preliminaries

2.1. **Definition.** For a path-connected space X with basepoint x_0, we define PX to be the set of all continuous functions $\gamma : I \to X$ satisfying $\gamma(0) = x_0$ topologized by the compact-open topology.

We then define $p : PX \to X$ given by $p(\gamma) = \gamma(1)$, a fibration with base space X and fiber $\Omega(X)$, the loop space of X.

Given $f : Y \to X$ and $g : Z \to X$ we can define $Y *_X Z = \{(y, z, t) \in Y * Z | f(y) = g(z)\}$ and $(f *_X g) : Y *_X Z \to X$ by $(f *_X g)(y, z, t) = f(y)$.

From this, we define P_nX to be the fiberwise join of n copies of PX over $p : PX \to X$ defined above and denote the fiberwise map as

$$p_n^X : P_nX \to X. \tag{2.1}$$

Note that the (homotopy) fiber of p_n^X is $\Omega(X)^{*n}$, the n-fold join of $\Omega(X)$.

We need the following theorem of Schwarz, see [2, 16].

2.2. **Theorem.** *The inequality* $\operatorname{cat}(X) \le n$ *holds iff there exists a section* $s : X \to P_{n+1}X$ *to* $p : P_{n+1}X \to X$.

2.3. **Remark.** These claims are well-known, we list them here for reference.
(1) $\operatorname{cat}(X \vee Y) = \max\{\operatorname{cat} X, \operatorname{cat} Y\}$;
(2) $\operatorname{cat}(X \cup Y) \le \operatorname{cat}(X) + \operatorname{cat}(Y) + 1$;
(3) $\operatorname{cat}(X/A) - 1 \le \operatorname{cat} X$. This follows from the fact that X/A has homotopy type of $X \cup CA$, the union of X with the cone over A, and item (2).

It should be noted that Berstein and Hilton explored the changes in category of a space after attaching a cone in [1] following Hilton's exploration of what's now known as the Hilton-Hopf invariant in [8].

3. Main results

3.1. **Proposition.** *The inequality* (1.1) *holds whenever* $\max\{\operatorname{cat} M, \operatorname{cat} N\} \le 2$

Proof. If $\operatorname{cat} M = 1 = \operatorname{cat} N$ then M and N are homotopy spheres, and so $M \# N$ is. Conversely, if $\operatorname{cat} M \# N = 1$ then M and N must be homotopy spheres. Thus, we proved that $\operatorname{cat}(M \# N) = \max\{\operatorname{cat} M, \operatorname{cat} N\}$ if $\max\{\operatorname{cat} M, \operatorname{cat} N\} \le 2$. □

3.2. **Theorem.** *Suppose* X *is an* n*-dimensional space with* m*-connected subspace* A*, with* $3 \le \operatorname{cat}(X/A) \le k$*, and* $k + m - 1 \ge n$*. Then* $\operatorname{cat} X \le k$*.*

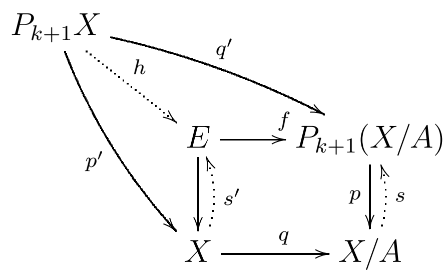
Proof. For sake of simplicity, put $p = p_{k+1}^{X/A}$ and $p' = p_{k+1}^X$, cf. (2.1). As $\operatorname{cat}(X/A) \le k$, and by Theorem 2.2, there exists the following section s with $ps = 1_{X/A}$.

$$\begin{array}{c} P_{k+1}(X/A) \\ p \big\downarrow \big\uparrow s \\ X/A \end{array}$$

Now we consider the collapsing map $q : X \to X/A$, and get the fiber-pullback diagram.

$$\begin{array}{ccccc} E & \xrightarrow{f} & P_{k+1}(X/A) & = & P_{k+1}(X/A) \\ \big\downarrow & & p \big\downarrow & & \big\uparrow s \\ X & \longrightarrow & X/A & = & X/A \end{array} \tag{3.1}$$

Now consider $P_{k+1}X$. We already have $p' : P_{k+1}X \to X$, and the collapsing map $q : X \to X/A$ induces a map $q' : P_{k+1}X \to P_{k+1}(X/A)$. Since $pq' = gp'$ and the square is the pull-back diagram, we get a map $h : P_{k+1} \to X$ such that the following diagram commutes.



Recall that our goal is to prove $\operatorname{cat} X \le k$. Because of Schwarz's Theorem 2.2, it suffices to construct a section of p'. To do this, it suffices in turn to construct a section of the map $h : P_{k+1}(X) \to E$. Moreover, since $\dim X = n$, it suffices to construct a section of h over the n-skeleton $E^{(n)}$ of E, i.e., to construct a map $\phi : E^{(n)} \to P_{k+1}(X)$ with $h\phi = 1_E$.

By homotopy excision [7, Prop. 4.28], and because A is m-connected, the quotient map $q : X \to X/A$ induces isomorphisms $q_* : \pi_n(X) \to \pi_n(X/A)$ for $n \le m$ and epimorphism for $n = m+1$. So, $\pi_n(\Omega X) \to \pi_n(\Omega(X/A))$ is an isomorphism for $n \le (m-1)$ and epimorphism for $n = m$. Therefore $(\Omega X)^{*(k+1)} \to (\Omega(X/A))^{*(k+1)}$ is an isomorphism for $n \le m+k$ because of [3, Prop. 5.7].

The long exact sequence of homotopy groups for a fibration yields the following commutative diagram

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \pi_i((\Omega X)^{*(k+1)}) & \longrightarrow & \pi_i(P_{k+1}X) & \longrightarrow & \pi_i(X) & \longrightarrow & \dots \\ & & \downarrow \cong & & \vdots\, h_* & & \downarrow \cong & & \\ \dots & \longrightarrow & \pi_i((\Omega X)^{*(k+1)}) & \longrightarrow & \pi_i(E) & \longrightarrow & \pi_i(X) & \longrightarrow & \dots \end{array}$$

By the 5-lemma, the map h_* is an isomorphism for $i \le (m+k-1)$ and epimorphism for $n = m+k$. So by Whitehead's theorem, there exists a map $\phi : E^{(n)} \to P_{k+1}X$. Now, the composition $(\phi \circ s')$ is a section to $p' : P_{k+1} \to X$. Thus $\operatorname{cat} X \le k$. □

Combining this with the previous Remark 2.3 gives the following inequality:

$$\operatorname{cat}(X/A) - 1 \le \operatorname{cat}(X) \le \operatorname{cat}(X/A)$$

under the dimension-connectivity conditions from Theorem 3.2.

Consider the case where $X = M\#N$, the connected sum of n-dimensional manifolds M and N, and $A = S^{n-1}$ is the separating sphere between M and N. Then $X/A = M \vee N$, and $\operatorname{cat}(X/A) = \max\{\operatorname{cat} M, \operatorname{cat} N\}$. We have A is $(n-2)$-connected, and we can assume $\operatorname{cat} M, \operatorname{cat} N \ge 3$ because of Proposition 3.1. Then $\operatorname{cat}(X/A) \ge 3$, and so as $(n-2)+3-1 \ge n$, we are in the case of Theorem 3.2 and get the following corollary.

3.3. **Corollary.** *There is a double inequality*

$$\max\{\operatorname{cat} M, \operatorname{cat} N\} - 1 \le \operatorname{cat}(M\#N) \le \max\{\operatorname{cat} M, \operatorname{cat} N\}.$$

Proof. Consider the case where $X = M\#N$, the connected sum of n-dimensional manifolds M and N, and $A = S^{n-1}$ is the separating sphere between M and N. Then $X/A = M \vee N$, and $\operatorname{cat}(X/A) = \max\{\operatorname{cat} M, \operatorname{cat} N\}$. We have A is $(n-2)$-connected, and we can assume

$\operatorname{cat} M, \operatorname{cat} N \geq 2$ because of Proposition 3.1. Then $\operatorname{cat}(X/A) \geq 3$, and so as $(n-2)+3-1 \geq n$, we are in the case of Theorem 3.2 and get the corollary. □

3.4. **Remark.** In [6], an upper bound is given for the LS category of a double mapping cylinder. If we consider the connected sum of n-manifolds M and N as such a double mapping cylinder, then the following inequality is obtained:

$$\operatorname{cat} M\#N \leq \min\{1 + \operatorname{cat} M' + \operatorname{cat} N', 1 + \max\{\operatorname{cat} M', \operatorname{cat} N'\}\}. \tag{3.2}$$

Here M' and N' are $M \setminus \mathrm{pt}$ and $N \setminus \mathrm{pt}$, respectively.
Rivadeneyra proved in [12] that category of a manifold without bondary does not increase when a point is removed. If the categories of M and N do decrease by one when a point is removed, then (3.2) has already established the main result here. However, in [11] a closed manifold is constructed so that the category remains unchanged after the deletion of a point, and Theorem 3.2 gives an improvement of the category estimate for such a case.

3.5. **Remark.** It is unknown if there is an example of two manifolds M and N such that $\operatorname{cat} M\#N = \max\{\operatorname{cat} M, \operatorname{cat} N\} - 1$.

4. Connected sum and Toomer invariant

4.1. **Definition.** The Toomer invariant of X $e(X)$ is the least integer k for which the map $p_n^* : H^*(X) \to H^*(P_n(X))$ is injective, see [2]. It follows that $e(X) \leq \operatorname{cat} X$.

4.2. **Proposition.** *For closed and oriented manifolds M and N, $e(M\#N) \geq \max\{e(M), e(N)\}$.*

Proof. Consider $f : M\#N \to M$ the collapsing map onto M. Then we have the following diagram.

$$\begin{array}{ccc} H^*(P_nM) & \longrightarrow & H^*(P_n(M\#N)) \\ \uparrow & & \uparrow \\ H^*(M) & \overset{f^*}{\rightarrowtail} & H^*(M\#N) \end{array}$$

This map has degree 1, and so $f^* : H^*(M) \to H^*(M\#N)$ is injective [13, Theorem V, 2.13]. Also suppose $p_n^* : H^*(M\#N) \to H^*(P_n(M\#N)$ is injective. Consider $u \in H^*(M)$. As f^* and p_n^* are injective, $p_n^*(u) \in H^*(P_nM)$ is nonzero, and so $p_n^* : H^*(M) \to H^*(P_nM)$ is injective, and similarly for N. And so $e(M\#N) \geq \max\{e(M), e(N)\}$. □

4.3. **Proposition.** *For closed, oriented manifolds M and N, if $\operatorname{cat} M = e(M)$ and $\operatorname{cat} N = e(N)$, then $\operatorname{cat}(M\#N) = \max\{\operatorname{cat} M, \operatorname{cat} N\}$.*

Proof. Combining the assumptions $e(M) = \operatorname{cat} M$ and $e(N) = \operatorname{cat} N$ with the inequality $\max\{e(M), e(N)\} \leq e(M\#N) \leq \operatorname{cat}(M\#N) \leq \max\{\operatorname{cat} M, \operatorname{cat} N\}$, we have the claim. □

Rudyak asked if the existence of a map $f : M \to N$, of degree 1, implies the inequality $\operatorname{cat} M \geq \operatorname{cat} N$ [14], [2, Open problem 2.48]. While not achieving the full result, he was able to prove some partial results. In particular it follows from the same injective property of f^* (4.2) that $e(M) \geq e(N)$, when such a map exists [14].

4.4. **Remark.** We know $e(M \times S^n) \geq e(M)+1$, and there exist examples where $\operatorname{cat}(M \times S^n) = \operatorname{cat} M$ for suitable M and n, [9], [10].

5. Rationalizations

Here we assume X to be simply connected and denote by $X_{\mathbb{Q}}$ the rationalization of X, see [5],[15]. We define $e_{\mathbb{Q}}(X)$ to be the least integer n such that the nth fibration $P_nX \to X$ induces an injection in cohomology with coefficients in $\mathbb{Q}$. For X simply connected and of finite type, we have that $e_{\mathbb{Q}}(X) = e(X_{\mathbb{Q}})$, [2].

5.1. **Proposition.** *For simply connected, CW spaces X and Y, $(X \vee Y)_{\mathbb{Q}} \cong X_{\mathbb{Q}} \vee Y_{\mathbb{Q}}$.*

Proof. In the following diagram, the map l is the localization map of $X \vee Y$, and k is given by the wedge of localization maps on X and Y. The map j exists by the universal property of $(X \vee Y)_{\mathbb{Q}}$, and induces isomorphisms in homology. Hence $X_{\mathbb{Q}} \vee Y_{\mathbb{Q}} \cong (X \vee Y)_{\mathbb{Q}}$

$$\begin{array}{ccc} X \vee Y & \xrightarrow{\;l\;} & (X \vee Y)_{\mathbb{Q}} \\ & {\scriptstyle k}\searrow & \downarrow{\scriptstyle j} \\ & & X_{\mathbb{Q}} \vee Y_{\mathbb{Q}} \end{array}$$

□

In [4] it is shown that for a closed, simply connected manifold M, $e(M) = e_{\mathbb{Q}}(M) = \operatorname{cat}(M_{\mathbb{Q}})$, and hence $\operatorname{cat} M_{\mathbb{Q}} \leq \operatorname{cat} M$.

5.2. **Proposition.** For M and N, closed and simply connected manifolds, $\operatorname{cat}(M\#N)_{\mathbb{Q}} = \max\{\operatorname{cat} M_{\mathbb{Q}}, \operatorname{cat} N_{\mathbb{Q}}\}$.

Proof. As M and N are closed and simply connected, $M\#N$ is closed and simply connected, and $e_{\mathbb{Q}}(M\#N) = \operatorname{cat}(M\#N)_{\mathbb{Q}}$. Combining (3.3) and (4.2) establishes on the left hand side,

$$\max\{\operatorname{cat} M_{\mathbb{Q}}, \operatorname{cat} N_{\mathbb{Q}}\} = \max\{e_{\mathbb{Q}}(M), e_{\mathbb{Q}}(N)\} \leq e_{\mathbb{Q}}(M\#N) = \operatorname{cat}(M\#N)_{\mathbb{Q}}.$$

While on the right hand side we have,

$$\operatorname{cat}(M\#N)_{\mathbb{Q}} = \operatorname{cat}_{\mathbb{Q}}(M\#N) \leq \max(\operatorname{cat}_{\mathbb{Q}} M, \operatorname{cat}_{\mathbb{Q}} N) = \max\{\operatorname{cat} M_{\mathbb{Q}}, \operatorname{cat} N_{\mathbb{Q}}\},$$

where the middle inequality comes from (3.3). □

Returning to Rudyak's question on a possible relation between degree and category, we can settle it in the rational context.

5.3. **Proposition.** For closed and simply connected m-manifolds M and N with $f : M \to N$ of nonzero degree, we have $\operatorname{cat} M_{\mathbb{Q}} \geq \operatorname{cat} N_{\mathbb{Q}}$.

Proof. It suffices to show $e_{\mathbb{Q}}(M) \geq e_{\mathbb{Q}}(N)$. That is, suppose $p^* : H^*(M;\mathbb{Q}) \to H^*(P_n(M)\mathbb{Q})$ in the following diagram is injective.

$$\begin{array}{ccc} H^*(P_n(M);\mathbb{Q}) & \longleftarrow & H^*(P_n(N);\mathbb{Q}) \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ H^*(M;\mathbb{Q}) & \xleftarrow{\;f^*\;} & H^*(N;\mathbb{Q}) \end{array}$$

By [13, V.2.13], the map f^* is injective. Since p^* and f^* are injective, the composition $p^* \circ f^*$ is injective, and $p^* : H^*(N;\mathbb{Q}) \to H^*(P_n(N);\mathbb{Q})$ is injective. Thus $e_{\mathbb{Q}}(N) \leq n$.

□

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