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Quasi-plane waves for spin 1 field in Lobachevsky space and a generalized helicity operator

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Abstract

Spin 1 particle is investigated in 3-dimensional curved space of constant negative curvature. An extended helicity operator is defined and the variables are separated in a tetrad-based 10-dimensional Duffin–Kemmer equation in quasi cartesian coordinates. The problem is solved exactly in hypergeometric functions, the quantum states are determined by three quantum numbers. It is shown that Lobachevsky geometry acts effectively as a medium with simple reflecting properties. Transition to a massless case of electromagnetic field is performed.

This article continues a series of previous papers on constricting exact solutions of the wave equations for fields of different spins on the geometrical background of 3-dimensional spatial models of constant positive or negative curvature [1–4]. Present paper is devoted to the (massive and massless) case of spin 1 field in the Lobachevsky spacial model, the treatment is based on matrix Duffin–Kemmer formalism applied in special quasi-cartesian coordinate of the Lobachevsky space.

In Lobachevsky space-time, let us use quasi-cartesian coordinates and corresponding tetrad

$$x^a = (t, x, y, z) \; , \qquad dS^2 = dt^2 - e^{-2z}(dx^2 + dy^2) - dz^2 \; ,$$

$$e^{\beta}_{(a)} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & e^{z} & 0 & 0 \\ 0 & 0 & e^{z} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| , \qquad e_{(a)\beta} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -e^{-z} & 0 & 0 \\ 0 & 0 & -e^{-z} & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| . \qquad (1)$$

Christoffel symbols are $\Gamma^{0}_{\beta\sigma} = 0$, $\Gamma^{i}_{00} = 0$, $\Gamma^{i}_{0j} = 0$, and

$$x^i = (x, y, z) \; , \qquad \Gamma^{i}{}_{jk} = \frac{1}{2}\, g^{il}\, (-\partial_l \, g_{jk} + \partial_j \, g_{lk} + \partial_k \, g_{lj}) \; ,$$

$$\Gamma^{x}{}_{jk} = \left| \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right| , \; \Gamma^{y}{}_{jk} = \left| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array} \right| , \; \Gamma^{z}{}_{jk} = \left| \begin{array}{ccc} e^{-2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 0 \end{array} \right| .$$

The Ricci rotations coefficients are

$$\gamma_{011} = \gamma_{021} = \gamma_{031} = 0 \, , \; \gamma_{012} = \gamma_{022} = \gamma_{032} = 0 \, , \; \gamma_{013} = \gamma_{023} = \gamma_{033} = 0 \, ,$$

$$\gamma_{231} = 0 \, , \; \gamma_{311} = -1 \, , \; \gamma_{121} = 0 \, , \; \gamma_{232} = 1 \, , \; \gamma_{312} = 0 \, , \; \gamma_{122} = 0 \, ,$$

$$\gamma_{233} = 0 \; , \; \gamma_{313} = 0 \; , \; \gamma_{123} = 0 \; .$$

Duffin–Kemmer equation [5]

$$\left[i\beta^{c} \left(\; e^{\beta}_{(c)} \partial_{\beta} + \frac{1}{2} J^{ab} \gamma_{abc} \right) - M \right] \Psi = 0 \qquad (2)$$

in the above tetrsd takes the form

$$\left[+i\beta^{0} \frac{\partial}{\partial t} + i\beta^{1} e^{z} \frac{\partial}{\partial x} + i\beta^{2} e^{z} \frac{\partial}{\partial y} + i \left(\beta^{3} \frac{\partial}{\partial z} - \beta^{1} J^{31} + \beta^{2} J^{23} \right) - M \right] \Psi = 0 \; . \qquad (3)$$

We will search solutions in the form of quasi-plane waves

$$\Psi = e^{-i\epsilon t} \; e^{iax} \; e^{iby} \left| \begin{array}{c} \Phi_{0}(z) \\ \Phi_{j}(z) \\ E_{j}(z) \\ H_{j}(z) \end{array} \right| , \qquad (4)$$

eq. (3) gives

$$\left[\epsilon\beta^{0} - a\beta^{1} e^{z} - b\beta^{2} e^{z} + i \left(\beta^{3} \frac{\partial}{\partial z} - \beta^{1} J^{31} + \beta^{2} J^{23} \right) - M \right] \Psi = 0 \, . \qquad (5)$$

Below we will use the cyclic basis for Duffin–Kemmer matrices, then a third projection of the spin is diagonal matrix

$$\beta^{0} = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| , \qquad \beta^{i} = \left| \begin{array}{cccc} 0 & 0 & e_{i} & 0 \\ 0 & 0 & 0 & \tau_{i} \\ -e_{i}^{+} & 0 & 0 & 0 \\ 0 & -\tau_{i} & 0 & 0 \end{array} \right| ;$$

$$e_{1} = \frac{1}{\sqrt{2}} (-i, \; 0, \; i) \; , \qquad e_{2} = \frac{1}{\sqrt{2}} (1, \; 0, \; 1) \; , \qquad e_{3} = (0, i, 0) \; ;$$

$$\tau_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}, \; \tau_2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{vmatrix}, \; \tau_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1, \end{vmatrix} = s_3 \; ;$$

$$J^{12} = \beta^1 \beta^2 - \beta^2 \beta^1 = -i \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_3 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_3 \end{vmatrix} = -iS_3 \; ,$$

$$J^{13} = \beta^1 \beta^3 - \beta^3 \beta^1 = i \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_2 & 0 & 0 \\ 0 & 0 & \tau_2 & 0 \\ 0 & 0 & 0 & \tau_2 \end{vmatrix} = iS_2 \; ,$$

$$J^{23} = \beta^2 \beta^3 - \beta^3 \beta^2 = -i \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & 0 \\ 0 & 0 & \tau_1 & 0 \\ 0 & 0 & 0 & \tau_1 \end{vmatrix} = -iS_1 \; ;$$

$$-\beta^1 J^{31} + \beta^2 J^{23} = i \begin{vmatrix} 0 & 0 & (e_1\tau_2 - e_2\tau_1) & 0 \\ 0 & 0 & 0 & (\tau_1\tau_2 - \tau_2\tau_1) \\ 0 & 0 & 0 & 0 \\ 0 & -(\tau_1\tau_2 - \tau_2\tau_1) & 0 & 0 \end{vmatrix} .$$

Eq. (5) in block form reads

$$\Bigg[\epsilon \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} - ae^z \begin{vmatrix} 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & \tau_1 \\ -e_1^+ & 0 & 0 & 0 \\ 0 & -\tau_1 & 0 & 0 \end{vmatrix} -$$

$$-be^z \begin{vmatrix} 0 & 0 & e_2 & 0 \\ 0 & 0 & 0 & \tau_2 \\ -e_2^+ & 0 & 0 & 0 \\ 0 & -\tau_2 & 0 & 0 \end{vmatrix} + i(\begin{vmatrix} 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & \tau_3 \\ -e_3^+ & 0 & 0 & 0 \\ 0 & -\tau_3 & 0 & 0 \end{vmatrix} \frac{\partial}{\partial z} +$$

$$+i \begin{vmatrix} 0 & 0 & (e_1\tau_2 - e_2\tau_1) & 0 \\ 0 & 0 & 0 & (\tau_1\tau_2 - \tau_2\tau_1) \\ 0 & 0 & 0 & 0 \\ 0 & -(\tau_1\tau_2 - \tau_2\tau_1) & 0 & 0 \end{vmatrix} \Bigg) - M \Bigg] \begin{vmatrix} \Phi_0 \\ \Phi_j \\ E_j \\ H_j \end{vmatrix} = 0, \qquad (6a)$$

which is equivalent to

$$-a\, e^z\, e_1\, \vec{E} - b\, e^z\, e_2\, \vec{E} + i\, e_3\, \frac{\partial \vec{E}}{\partial z} - (e_1\tau_2 - e_2\, \tau_1)\, \vec{E} - M\, \Phi_0 = 0\, ,$$

$$\epsilon\, I\, \vec{E} - a\, e^z\, \tau_1\, \vec{H} - b\, e^z\, \tau_2\, \vec{H} + i\, \tau_3\, \frac{\partial \vec{H}}{\partial z} - (\tau_1 \tau_2 - \tau_2\, \tau_1)\, \vec{H} - M\, \vec{\Phi} = 0\,,$$

$$-\epsilon\, I\, \vec{\Phi} + a\, e^z\, e_1^+\, \Phi_0 + b\, e^z\, e_2^+\, \Phi_0 - i\, e_3^+\, \frac{\partial \Phi_0}{\partial z} - M\, \vec{E} = 0\,,$$

$$a\, e^z\, \tau_1\, \vec{\Phi} + b\, e^z\, \tau_2\, \vec{\Phi} - i\, \tau_3\, \frac{\partial \vec{\Phi}}{\partial z} + (\tau_1 \tau_2 - \tau_2\, \tau_1)\, \vec{\Phi} - M\, \vec{H} = 0\,. \qquad (6b)$$

In explicit form, 10 equations are (let $\gamma = 1/\sqrt{2}$)

$$i\gamma\, a\, e^z\, (E_1 - E_3) - \gamma\, b\, e^z\, (E_1 + E_3) - \frac{dE_2}{dz} + 2\, E_2 - M\, \Phi_0 = 0\,, \qquad (7a)$$

$$i\, \epsilon\, E_1 - \gamma\, a\, e^z\, H_2 + i\gamma\, b\, e^z\, H_2 + i\, \frac{dH_1}{dz} - i\, H_1 - M\, \Phi_1 = 0\,,$$

$$i\, \epsilon\, E_2 - \gamma a\, e^z\, (H_1 + H_3) - i\gamma\, b\, e^z\, (H_1 - H_3) - M\, \Phi_2 = 0\,,$$

$$i\, \epsilon\, E_3 - \gamma a\, e^z\, H_2 - i\gamma\, b\, e^z\, H_2 - i\, \frac{dH_3}{dz} + i\, H_3 - M\, \Phi_3 = 0\,, \qquad (7b)$$

$$-i\, \epsilon\, \Phi_1 + i\gamma\, a\, e^z\, \Phi_0 + \gamma b\, e^z\, \Phi_0 - M\, E_1 = 0\,,$$

$$-i\, \epsilon\, \Phi_2 - \frac{d\Phi_0}{dz} - M\, E_2 = 0\,,$$

$$-i\, \epsilon\, \Phi_3 - i\gamma\, a\, e^z\, \Phi_0 + \gamma b\, e^z\, \Phi_0 - M\, E_3 = 0\,, \qquad (7c)$$

$$\gamma a\, e^z\, \Phi_2 - i\gamma\, b\, e^z\, \Phi_2 - i\, \frac{d\Phi_1}{dz} + i\, \Phi_1 - M\, H_1 = 0\,,$$

$$\gamma a\, e^z\, (\Phi_1 + \Phi_3) + i\gamma\, b\, e^z\, (\Phi_1 - \Phi_3) - M\, H_2 = 0\,,$$

$$\gamma a\, e^z\, \Phi_2 + i\gamma\, b\, e^z\, \Phi_2 + i\, \frac{d\Phi_3}{dz} - i\, \Phi_3 - M\, H_3 = 0\,. \qquad (7d)$$

After evident regrouping they look

$$\gamma(ia - b) e^z\, E_1 - \gamma(ia + b) e^z E_3 - (\frac{d}{dz} - 2)\, E_2 - M\, \Phi_0 = 0\,, \qquad (8a)$$

$$i\, \epsilon\, E_1 - \gamma(a - ib)\, e^z\, H_2 + i\, (\frac{d}{dz} - 1)\, H_1 - M\, \Phi_1 = 0\,,$$

$$i\, \epsilon\, E_2 - \gamma(a + ib)\, e^z\, H_1 - \gamma(a - ib)\, e^z\, H_3 - M\, \Phi_2 = 0\,,$$

$$i\, \epsilon\, E_3 - \gamma(a + ib)\, e^z\, H_2 - i(\frac{d}{dz} - 1)\, H_3 - M\, \Phi_3 = 0\,, \qquad (8b)$$

$$-i\,\epsilon\,\Phi_1 + \gamma\,(b+ia)\,e^z\,\Phi_0 - M\,E_1 = 0\,,$$
$$-i\,\epsilon\,\Phi_2 - \frac{d\Phi_0}{dz} - M\,E_2 = 0\,,$$
$$-i\,\epsilon\,\Phi_3 + \gamma(b-ia)\,e^z\,\Phi_0 - M\,E_3 = 0\,, \tag{8c}$$

$$\gamma(a-ib)\,e^z\,\Phi_2 - i(\frac{d}{dz} - 1)\,\Phi_1 - M\,H_1 = 0\,,$$
$$\gamma(a+ib)\,e^z\,\Phi_1 + \gamma(a-ib)\,e^z\,\Phi_3 - M\,H_2 = 0\,,$$
$$\gamma(a+ib)\,e^z\,\Phi_2 + i\,(\frac{d}{dz} - 1)\,\Phi_3 - M\,H_3 = 0\,. \tag{8d}$$

To find explicit solutions of the system, it is convenient to diagonalize additionally a generalized helicity operator (it explicit form can be proposed by analogy reasons with the case of similar operator in Minkowski space – after that it is the matter of simple calculation to verify that it commutes with the wave operator[1]):

$$\Psi_0(z) = (\Phi_0(z), \Phi_j(z), E_j(z), H_j(z))\,,$$
$$\left[ae^z S^1 + be^z S^2 - i(S^3\frac{d}{dz} - S^1 J^{31} + S^2 J^{23})\right]\Psi_z = \sigma\Psi(z)\;,$$
$$-S^1 J^{31} + S^2 J^{23} = -S^3 \tag{9a}$$

in block form it reads

$$\left[ae^z \left|\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & \tau_1 & 0 & 0 \\ 0 & 0 & \tau_1 & 0 \\ 0 & 0 & 0 & \tau_1 \end{array}\right| + be^z \left|\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & \tau_2 & 0 & 0 \\ 0 & 0 & \tau_2 & 0 \\ 0 & 0 & 0 & \tau_2 \end{array}\right| - \right.$$

$$\left. -i \left|\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & \tau_3 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_3 \end{array}\right| (\frac{d}{dz} - 1)\right] \left|\begin{array}{c} \Phi_0(z) \\ \Phi_j(z) \\ E_j(z) \\ H_j(z) \end{array}\right| = \sigma \left|\begin{array}{c} \Phi_0(z) \\ \Phi_j(z) \\ E_j(z) \\ H_j(z) \end{array}\right|\,, \tag{9b}$$

so that

$$0 = \sigma\,\Phi_0\,,$$

[1]To avoid misunderstanding, it should be noted that below we use an helicity operator which differs in the facto i from usual, so its eigenvalues are imaginary or zero.

$$a\,e^{z}\,\tau_1\,\vec{\Phi}+b\,e^{z}\,\tau_2\,\vec{\Phi}-i\,\tau_3\,(\frac{d}{dz}-1)\vec{\Phi}=\sigma\,\vec{\Phi}\,,$$

$$a\,e^{z}\,\tau_1\,\vec{E}+b\,e^{z}\,\tau_2\,\vec{E}-i\,\tau_3\,\frac{d\vec{E}}{dz}=\sigma\,\vec{E}\,,$$

$$a\,e^{z}\,\tau_1\,\vec{H}+b\,e^{z}\,\tau_2\,\vec{H}-i\,\tau_3\,(\frac{d}{dz}-1)\vec{H}=\sigma\,\vec{H}\;. \tag{9c}$$

After simple calculations we get 10 equations

$$0=\sigma\,\Phi_0\,,$$

$$\gamma\,(a-ib)\,e^{z}\,\Phi_2-i\,(\frac{d}{dz}-1)\Phi_1=\sigma\,\Phi_1\,,$$

$$\gamma(a+ib)\,e^{z}\,\Phi_1+i\gamma\,(a-ib)\,e^{z}\,\Phi_3=\sigma\,\Phi_2\,,$$

$$\gamma(a+ib)\,e^{z}\,\Phi_2+i\,(\frac{d}{dz}-1)\Phi_3=\sigma\,\Phi_3\,,$$

$$\gamma\,(a-ib)\,e^{z}\,E_2-i\,\frac{dE_1}{dz}=\sigma\,E_1\,,$$

$$\gamma(a+ib)\,e^{z}\,E_1+\gamma\,(a-ib)\,e^{z}\,E_3=\sigma\,E_2\,,$$

$$\gamma(a+ib)\,e^{z}\,E_2+i\,\frac{dE_3}{dz}=\sigma\,E_3\,,$$

$$\gamma\,(a-ib)\,e^{z}\,H_2-i\,(\frac{d}{dz}-1)H_1=\sigma\,H_1\,,$$

$$\gamma(a+ib)\,e^{z}\,H_1+\gamma\,(a-ib)\,e^{z}\,H_3=\sigma\,H_2\,,$$

$$\gamma(a+ib)\,e^{z}\,H_2+i\,(\frac{d}{dz}-1)H_3=\sigma\,H_3\,.$$

It is more convenient to rewrite them in the form

$$0=\sigma\,\Phi_0\,, \tag{10a}$$

$$-i\,(\frac{d}{dz}-1)\Phi_1=\sigma\,\Phi_1-\gamma\,(a-ib)\,e^{z}\,\Phi_2\,,$$

$$\gamma(a+ib)\,e^{z}\,\Phi_1+\gamma\,(a-ib)\,e^{z}\,\Phi_3=\sigma\,\Phi_2\,,$$

$$+i\,(\frac{d}{dz}-1)\Phi_3=\sigma\,\Phi_3-\gamma(a+ib)\,e^{z}\,\Phi_2\,, \tag{10b}$$

$$-i\,\frac{dE_1}{dz}=\sigma\,E_1-\gamma\,(a-ib)\,e^{z}\,E_2\,,$$

$$\gamma(a+ib)\,e^{z}\,E_1+\gamma\,(a-ib)\,e^{z}\,E_3=\sigma\,E_2\,,$$

$$+i\,\frac{dE_3}{dz} = \sigma\,E_3 - \gamma(a+ib)\,e^z\,E_2\,, \tag{10c}$$

$$-i\,(\frac{d}{dz}-1)H_1 = \sigma\,H_1 - \gamma\,(a-ib)\,e^z\,H_2\,,$$

$$\gamma(a+ib)\,e^z\,H_1 + \gamma\,(a-ib)\,e^z\,H_3 = \sigma\,H_2\,,$$

$$+i\,(\frac{d}{dz}-1)H_3 = \sigma\,H_3 - \gamma(a+ib)\,e^z\,H_2\,. \tag{10d}$$

Note that we see three similar groups of equations for Φ_j, E_j, H_i respectively. Let us examine one of them, foe instance

$$\gamma\,(a-ib)\,e^z\,H_2 - i\,(\frac{d}{dz}-1)H_1 = \sigma\,H_1\,,$$

$$\gamma(a+ib)\,e^z\,H_1 + \gamma\,(a-ib)\,e^z\,H_3 = \sigma\,H_2\,,$$

$$\gamma(a+ib)\,e^z\,H_2 + i\,(\frac{d}{dz}-1)H_3 = \sigma\,H_3\,. \tag{11}$$

The case $\sigma = 0$ reduces to (functions H_1 and H_3 turn to be proportional to each other)

$$(a+ib)H_1 = -(a-ib)H_3\,,$$

$$H_2 = +\frac{ie^{-z}}{\gamma(a-ib)}(\frac{d}{dz}-1)H_1 = -\frac{ie^{-z}}{\gamma(a+ib)}(\frac{d}{dz}-1)H_3\;. \tag{12}$$

When $\sigma \neq 0$, one can exclude H_2, then the first ant the third equations provide us with the linear differential system for H_1 and H_3:

$$(a-ib)\,e^{2z}\;[(a+ib)H_1 + (a-ib)H_3] - i\sigma\,(\frac{d}{dz}-1)H_1 = 2\,\sigma^2\,H_1\,,$$

$$(a+ib)\,e^{2z}\;[(a+ib)H_1 + (a-ib)H_3] + i\sigma\,(\frac{d}{dz}-1)H_3 = 2\,\sigma^2\,H_3\,. \tag{13}$$

One may observe the symmetry in (13):

$$H_1 \Longrightarrow H_3\;, \qquad \sigma \Longrightarrow -\sigma\;, \qquad b \Longrightarrow -b\;.$$

From (13) we get

$$H_3 = \frac{2i\,\sigma}{e^{2z}\,(a-i\,b)^2}\,\left(\frac{d}{dz} - 1 - i\,\sigma\right)\,H_1 - \frac{a^2+b^2}{(a-i\,b)^2}\,H_1\,,$$

$$\frac{d^2H_1}{dz^2} - 4\,\frac{dH_1}{dz} - \left[e^{2z}\,(a^2+b^2) - 4 - (i+\sigma)^2\right]\,H_1 = 0\,, \tag{14a}$$

and the second symmetrical variant

$$H_1 = \frac{-2i\,\sigma}{e^{2z}\,(a+i\,b)^2}\left(\frac{d}{dz} - 1 + i\,\sigma\right) H_3 - \frac{a^2+b^2}{(a+i\,b)^2}\,H_3\,,$$

$$\frac{d^2 H_3}{dz^2} - 4\,\frac{dH_3}{dz} - \left[e^{2z}\,(a^2+b^2) - 4 - (i-\sigma)^2\right]\,H_3 = 0\,. \qquad (14b)$$

Let us translate eqs. (14a) to other variable

$$i\sqrt{a^2+b^2}e^z = Z\;, \qquad \frac{d}{dz} = Z\frac{d}{dZ},$$

then the second order equations reads

$$\left[Z^2\frac{d^2}{dZ^2} - 3Z\frac{d}{dZ} + Z^2 + 4 - (1-i\sigma)^2\right] H_1 = 0 \qquad (15a)$$

or differently $H_1(Z) = Z^2 h_1(Z)$:

$$\left[\frac{d^2}{dZ^2} + \frac{1}{Z}\frac{d}{dZ} + 1 - \frac{(1-i\sigma)^2}{Z^2}\right] h_1 = 0\;, \qquad (15b)$$

which is the Bessel equations with solutions

$$h_1 = J_{\pm\mu}(Z)\;, \qquad \mu = 1 - i\sigma\;. \qquad (15c)$$

Expression h_3 (14a) is defined by

$$Z^2 h_3 = \frac{a+ib}{a-ib}\left[-2i\sigma(Z\frac{d}{dZ} + 1 - i\sigma) - Z^2\right] h_1 \qquad (15d)$$

Similar formulas can be produced for (14b):

$$H_3 = e^{2Z} h_3$$

and

$$\left[\frac{d^2}{dZ^2} + \frac{1}{Z}\frac{d}{dZ} + 1 - \frac{(1+i\sigma)^2}{Z^2}\right] h_3 = 0\;; \qquad (16a)$$

$$h_3 = J_{\pm\nu}(Z)\;, \qquad \nu = 1 + i\sigma\;; \qquad (16b)$$

$$Z^2 h_1 = \frac{a-ib}{a+ib}\left[+2i\sigma(Z\frac{d}{dZ} + 1 + i\sigma) - Z^2\right] h_3\;. \qquad (16c)$$

Now we are to joint together the main equations (8) and equations (11).

First, let us consider the case of non-zero σ. At this, from the very beginning, one must assume $\Phi_0 = 0$. Eqs. (8) giv

$$\gamma(ia-b)e^{z}\, E_1 - \gamma(ia+b)e^{z} E_3 - (\frac{d}{dz} - 2)\, E_2 = 0\,, \qquad (17a)$$

$$i\,\epsilon\, E_1 - \sigma\, H_1 - M\,\Phi_1 = 0\,,$$

$$i\,\epsilon\, E_2 - \sigma\, H_2 - M\,\Phi_2 = 0\,,$$

$$i\,\epsilon\, E_3 - \sigma\, H_3 - M\,\Phi_3 = 0\,, \qquad (17b)$$

$$-i\,\epsilon\,\Phi_1 - M\, E_1 = 0\,, \qquad -i\,\epsilon\,\Phi_2 - M\, E_2 = 0\,, \qquad -i\,\epsilon\,\Phi_3 - M\, E_3 = 0\,, \qquad (17c)$$

$$\sigma\,\Phi_1 - M\, H_1 = 0\,, \qquad \sigma\,\Phi_2 - M\, H_2 = 0\,, \qquad \sigma\,\Phi_3 - M\, H_3 = 0\,. \qquad (17d)$$

On can exclude E_j from eq.

$$\gamma(ia-b)e^{z}\,\Phi_1 - \gamma(ia+b)e^{z}\Phi_3 - (\frac{d}{dz} - 2)\,\Phi_2 = 0\,. \qquad (18)$$

This relation coincides with the Lorentz condition (when $\Phi_0 = 0$). Indeed, let us start with the Lorentz condition in tensor form

$$\nabla_\beta \Phi^{\beta(cart)} = 0 \qquad \Longrightarrow \qquad \nabla_\beta (e^{(b)\beta}\Phi^{cart}_{(b)} = 0 \qquad \Longrightarrow$$

$$\frac{\partial \Phi_{(b)^{cart}}}{\partial x^\beta}\; e^{(b)\beta} + \Phi^{cart}_{(b)}\nabla_\beta e^{(b)\beta} = 0\;, \qquad (19a)$$

or

$$\frac{\partial \Phi^{cart}_{(b)}}{\partial x^\beta}\; e^{(b)\beta} + \Phi^{cart}_{(b)}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\beta}\sqrt{-g}e^{(b)\beta} = 0\;. \qquad (19b)$$

Allowing for relations (1), eq. (19b) reduces to

$$\frac{\partial \Phi^{cart}_{(0)}}{\partial t} - e^{z}\frac{\partial \Phi^{cart}_{(1)}}{\partial x} - e^{z}\frac{\partial \Phi^{cart}_{(2)}}{\partial y} - (\frac{\partial}{\partial z} - 2)\Phi^{cart}_{(3)} = 0\;, \qquad (19c)$$

or with the substitution

$$(\Phi^{cart}_a) = e^{-i\epsilon t}e^{iax}e^{iby}\left|\begin{array}{c} \Phi^{cart}_0(z) \\ \Phi^{cart}_1(z) \\ \Phi^{cart}_2(z) \\ \Phi^{cart}_3(z) \end{array}\right|$$

we arrive at

$$-i\epsilon\Phi^{cart}_{(0)} - iae^z\Phi^{cart}_{(1)} - ibe^z\Phi^{cart}_{(2)} - (\frac{d}{dz} - 2)\Phi^{cart}_{(3)} = 0\ . \qquad (20a)$$

Translating the last relation to the variables of cyclic basis

$$\Phi_2 = \Phi^{cart}_{(3)}\ , \qquad \Phi_3 - \Phi_1 = \sqrt{2}\Phi^{cart}_{(1)}\ , \qquad \Phi_3 + \Phi_1 = \sqrt{2}i\ \Phi^{cart}_{(2)}\ ,$$

we get

$$-i\epsilon\Phi_0 - iae^z\frac{\Phi_3 - \Phi_1}{\sqrt{2}} - ibe^z\frac{\Phi_3 + \Phi_1}{\sqrt{2}i} - (\frac{d}{dz} - 2)\Phi_2 = 0\ ,$$

that is

$$-i\epsilon\Phi_0 + \frac{ia - b}{\sqrt{2}}e^z\Phi_1 - \frac{ia + b}{\sqrt{2}}e^z\Phi_3 - (\frac{d}{dz} - 2)\Phi_2 = 0\ . \qquad (20b)$$

Eq. (20b) when $\Phi_0 = 0$ coincides with (18)

$$\gamma(ia - b)e^z\,\Phi_1 - \gamma(ia + b)e^z\Phi_3 - (\frac{d}{dz} - 2)\,\Phi_2 = 0\,.$$

Therefore, eq. (18) (or($17a$)) is identity that is valid automatically due to the structure of the wave equation for spin 1 field. Remaining 9 equations ($17b, c, d$) lead us to linear constrains between 9 nontrivial constituents:

$$i\,\epsilon\,E_j - \sigma\,H_j - M\,\Phi_j = 0\,,$$

$$-i\,\epsilon\,\Phi_j - M\,E_j = 0\,,$$

$$\sigma\,\Phi_j - M\,H_j = 0\,, \qquad (21a)$$

that is

$$\sigma = \pm\sqrt{\epsilon^2 - M^2}\ , \qquad \Phi_0 = 0\ ,$$

$$E_j = -\frac{i\epsilon}{M}\Phi_j\ , \qquad H_j = \frac{\sigma}{M}\Phi_j\ . \qquad (21b)$$

Now, we are to proceed with the case $\sigma = 0$. Eigenvalue helicity equations here are

$$0 = 0\,, \qquad (22a)$$

$$-i\,(\frac{d}{dz} - 1)\Phi_1 = -\gamma\,(a - ib)\,e^z\,\Phi_2\,,$$

$$\gamma(a + ib)\,e^z\,\Phi_1 + \gamma\,(a - ib)\,e^z\,\Phi_3 = 0\,,$$

$$+i\,(\frac{d}{dz}-1)\Phi_3 = -\gamma(a+ib)\,e^z\,\Phi_2\,, \qquad (22b)$$

$$-i\,\frac{dE_1}{dz} = -\gamma\,(a-ib)\,e^z\,E_2\,,$$

$$\gamma(a+ib)\,e^z\,E_1 + \gamma\,(a-ib)\,e^z\,E_3 = 0\,,$$

$$+i\,\frac{dE_3}{dz} = -\gamma(a+ib)\,e^z\,E_2\,, \qquad (22c)$$

$$-i\,(\frac{d}{dz}-1)H_1 = -\gamma\,(a-ib)\,e^z\,H_2\,,$$

$$\gamma(a+ib)\,e^z\,H_1 + \gamma\,(a-ib)\,e^z\,H_3 = 0\,,$$

$$+i\,(\frac{d}{dz}-1)H_3 = -\gamma(a+ib)\,e^z\,H_2\,. \qquad (22d)$$

Taking into consideration (22) from (8) we get

$$\gamma(ia-b)e^z\,E_1 - \gamma(ia+b)e^z E_3 - (\frac{d}{dz}-2)\,E_2 - M\,\Phi_0 = 0\,, \qquad (23a)$$

$$i\,\epsilon\,E_1 - M\,\Phi_1 = 0\,,$$

$$i\,\epsilon\,E_2 - M\,\Phi_2 = 0\,,$$

$$i\,\epsilon\,E_3 - M\,\Phi_3 = 0\,, \qquad (23b)$$

$$-i\,\epsilon\,\Phi_1 + \gamma\,(b+ia)\,e^z\,\Phi_0 - M\,E_1 = 0\,,$$

$$-i\,\epsilon\,\Phi_2 - \frac{d\Phi_0}{dz} - M\,E_2 = 0\,,$$

$$-i\,\epsilon\,\Phi_3 + \gamma(b-ia)\,e^z\,\Phi_0 - M\,E_3 = 0\,, \qquad (23c)$$

$$-M\,H_1 = 0\,, \qquad -M\,H_2 = 0\,, \qquad -M\,H_3 = 0\,. \qquad (23d)$$

Excluding E_j in (23c), we derive

$$(\epsilon^2 - M^2)\Phi_1 - \gamma\epsilon(a-ib)\,e^z\,\Phi_0 = 0\,,$$

$$(\epsilon^2 - M^2)\Phi_2 - i\epsilon\frac{d}{dz}\Phi_0 = 0\,,$$

$$(\epsilon^2 - M^2)\Phi_3 + \gamma\epsilon(a+ib)\,e^z\,\Phi_0 = 0\,. \qquad (24a)$$

Now let us recall eqs. (12)

$$(a+ib)\Phi_1 + (a-ib)\Phi_3 = 0\,,$$

$$\Phi_2 = +\frac{ie^{-z}}{\gamma(a-ib)}(\frac{d}{dz}-1)\Phi_1 = -\frac{ie^{-z}}{\gamma(a+ib)}(\frac{d}{dz}-1)\Phi_3 \; .$$

Thus, we obtain equation for independent variables

$$\Phi_1 = +\frac{\gamma\epsilon(a-ib)}{(\epsilon^2 - M^2)}\, e^z\, \Phi_0 \, ,$$

$$\Phi_3 = -\frac{\gamma\epsilon(a+ib)}{\epsilon^2 - M^2} e^z \Phi_0 \, ,$$

$$\Phi_2 = \frac{i\epsilon}{(\epsilon^2 - M^2)}\frac{d}{dz}\Phi_0 \, . \qquad (24c)$$

Substituting them into eq. $(23a)$

$$\gamma(ia-b)e^z\,\Phi_1 - \gamma(ia+b)e^z\Phi_3 - (\frac{d}{dz}-2)\,\Phi_2 - i\epsilon\Phi_0 = 0\, ,$$

we arrive at

$$\gamma(ia-b)e^z\,\frac{\gamma\epsilon(a-ib)}{(\epsilon^2 - M^2)}\,e^z\,\Phi_0 + \gamma(ia+b)\frac{\gamma\epsilon(a+ib)}{\epsilon^2 - M^2}e^z\Phi_0 e^z -$$

$$-(\frac{d}{dz}-2)\,\frac{i\epsilon}{(\epsilon^2 - M^2)}\frac{d}{dz}\Phi_0 - i\epsilon\Phi_0 = 0\, ,$$

that is

$$\left[(\frac{d}{dz}-2)\,\frac{d}{dz} + (\epsilon^2 - M^2) - e^{2z}(a^2+b^2)\right]\Phi_0 = 0\, . \qquad (25)$$

It is of the type $(14a)$ and it can be solved in Bessel functions.

In the end, one may mention that in massless case instead of (23) we would have

$$0 = 0\, ,$$

$$i\,\epsilon\,E_1 = 0\, , \qquad i\,\epsilon\,E_2 = 0\, , \qquad i\,\epsilon\,E_3 = 0\, ,$$

$$-i\,\epsilon\,\Phi_1 + \gamma\;(b+ia)\,e^z\,\Phi_0 = 0\, ,$$

$$-i\,\epsilon\,\Phi_2 - \frac{d\Phi_0}{dz} = 0\, , \qquad -i\,\epsilon\,\Phi_3 + \gamma(b-ia)\,e^z\,\Phi_0 = 0\, ,$$

$$H_1 = 0\, , \qquad H_2 = 0\, , \qquad H_3 = 0\, . \qquad (26)$$

These equations describe gauge solution of gradient type, with vanishing electromagnetic tensor.

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References

[1] E.M. Ovsiyuk, V.M. Red'kov On simulating a medium with special reflecting properties by Lobachevsky geometry (One exactly solvable electromagnetic problem). http://arxiv.org/abs/1109.0126.

[2] E.M. Ovsiyuk. Quasi-plane waves for a particle with spin 1/2 on the background of Lobachevsky geometry: simulating of a special medium. http://arxiv.org/abs/1203.4288.

[3] E.M. Ovsiyuk, V.M. Red'kov. On solutions of the Maxwell equations in quasicartesian coordinates in Lobachebsky space. Proceedings of the National Academy of Sciences of Belarus. Series of physicalmathematical sciences. 2009. no 4. P. 99–105.

[4] V.M. Red'kov, E.M. Ovsiyuk, V.V. Kisel Spin 1 particle on 4-dimensional sphere: extended helicity operator, separation of the variables, and exact solutions. http://arxiv.org/abs/1203.5003.

[5] V.M. Red'kov. The fields of the particles in a Riemannian space and the Lorentz group. Minsk, 2009.