Stationary distributions for a class of generalized Fleming-Viot processes [1]

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We identify stationary distributions of generalized Fleming-Viot processes with jump mechanisms specified by certain beta laws together with a parameter measure. Each of these distributions is obtained from normalized stable random measures after a suitable biased transformation followed by mixing by the law of a Dirichlet random measure with the same parameter measure. The calculations are based primarily on the well-known relationship to measure-valued branching processes with immigration.

1 Introduction

In the study of population genetics models, it is of great importance to identify their stationary distributions. Such identifications provide us basic information of possible equilibriums of the models and are needed prior to quantitative discussions on statistical inference. Since [5], [11] and [1], theory of generalized Fleming-Viot processes has served as a new area to be cultivated and has been developed considerably. (See [3] for an exposition.) In view of such progress, it seems that we are in a position to explore the aforementioned problems for some appropriate subclass of those models. In this respect, it would be natural to think of the one-dimensional Wright-Fisher diffusion with mutation as a prototype. This celebrated process is prescribed by its generator

$$A := \frac{1}{2}x(1-x)\frac{d^2}{dx^2} + \frac{1}{2}[c_1(1-x) - c_2 x]\frac{d}{dx}, \quad x \in [0,1], \tag{1.1}$$

where c_1 and c_2 are positive constants interpreted as mutation rates. The stationary distribution is a beta distribution

$$B_{c_1,c_2}(dx) := \frac{\Gamma(c_1+c_2)}{\Gamma(c_1)\Gamma(c_2)} x^{c_1-1}(1-x)^{c_2-1}dx, \tag{1.2}$$

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where $\Gamma(\cdot)$ is the gamma function. In addition, the process associated with (1.1) admits an infinite-dimensional generalization known as the Fleming-Viot process with parent-independent mutation, whose stationary distribution is identified with the law of a Dirichlet random measure. With these situations in mind, we pose problems which are described roughly as follows.

(I) Find a jump process on [0,1] whose generator and stationary distribution extend (1.1) and (1.2), respectively.

(II) Establish an analogous generalization for the Fleming-Viot process case.

Since the problem (I) is rather obscure, it may be worth showing now the generator we will believe to give an 'answer'. For each $\alpha \in (0,1)$, define an operator A_α by

$$\begin{aligned} & A_\alpha G(x) \\ = \; & \int_0^1 \frac{B_{1-\alpha,1+\alpha}(du)}{u^2}\left[xG((1-u)x+u)+(1-x)G((1-u)x)-G(x)\right] \\ & + \int_0^1 \frac{B_{1-\alpha,\alpha}(du)}{(\alpha+1)u}\left[c_1G((1-u)x+u)+c_2G((1-u)x)-(c_1+c_2)G(x)\right], \end{aligned} \tag{1.3}$$

where G are smooth functions on $[0,1]$. Observe that $A_\alpha G(x) \to AG(x)$ as $\alpha \uparrow 1$. It should be noted that A_α is a one-dimensional version of the generator of the process studied in [2] if $c_1 = c_2 = 0$. See also [9] and [10]. The reader, however, is cautioned that our notation α is in conflict with that of these papers, in which α plays the same role as $\alpha + 1$ in our notation. (We adopt such notation in order for the formulae below to be simpler.) As will be discussed later for more general case, (1.3) defines a Markov process on $[0,1]$ and our main concern is its equilibrium state. It will be shown in the forthcoming section that a unique stationary distribution of the process governed by (1.3) is identified with

$$P_{\alpha,(c_1,c_2)}(dx) := \Gamma(\alpha+1)\int_0^1 B_{c_1,c_2}(dy)E_{\alpha,y}\left[(Y_1+Y_2)^{-\alpha}; \frac{Y_1}{Y_1+Y_2} \in dx\right], \tag{1.4}$$

where $E_{\alpha,y}$ denotes the expectation with respect to (Y_1, Y_2) with law determined by $\log E_{\alpha,y}[e^{-\lambda_1 Y_1 - \lambda_2 Y_2}] = -y\lambda_1^\alpha - (1-y)\lambda_2^\alpha$ $(\lambda_1, \lambda_2 \geq 0)$. Again we see that (1.4) with $\alpha = 1$ reduces to (1.2).

In principle, the problems (I) and (II) can be considered in a unified way. Actually our argument will be crucially based on a well-known relationship (see e.g. [2], [10]) between measure-valued branching processes with immigration (henceforth MBI-processes) and generalized Fleming-Viot processes associated with a natural generalization of (1.3). Namely, the generators of the former and the latter are connected with each other through quite a simple identity, which has a one-dimensional version of course. Nevertheless, we shall discuss (I) and (II) separately. This is mainly because the key identity will turn out to yield a correct answer only for certain restricted cases and in one-dimension one can avoid to use it (although the mathematical structure behind is not revealed clearly).

The organization of this paper is as follows. Section 2 is devoted to derivation of (1.4) by purely analytic argument. Exploiting the relationship to MBI-processes, we show in Section 3 that the above mentioned answer to (I) has a natural generalization which settles (II). The irreversibility of the processes we consider is discussed in Section 4.

2 The one-dimensional model

Let $0 < \alpha < 1$, $c_1 > 0$ and $c_2 > 0$ be given. The purpose of this section is to show that (1.4) is a unique stationary distribution of the process with generator (1.3). Analytically we shall prove that a probability measure P on $[0, 1]$ satisfying

$$\int_0^1 A_\alpha G(x)P(dx) = 0 \quad \text{for all } G(x) = \varphi_n(x) := x^n \text{ with } n = 1, 2, \ldots \tag{2.1}$$

is uniquely identified with (1.4). Actual starting point of the calculations below is

$$\int_0^1 A_\alpha G(x)P(dx) = 0 \quad \text{for all } G(x) = G_t(x) := (1 + tx)^{-1} \text{ with } t > 0. \tag{2.2}$$

The equivalence of (2.1) and (2.2) is a consequence of uniform estimates

$$|A_\alpha\varphi_n(x)| \le \left(1 + \frac{c_1 + c_2}{\alpha + 1}\right) 2^n, \qquad n = 1, 2, \ldots,$$

which can be shown by observing that

$$\begin{aligned}
&c_1((1-u)x+u)^n + c_2((1-u)x)^n - (c_1+c_2)x^n \\
&= c_1\left[((1-u)x+u)^n - ((1-u)x+ux)^n\right] + c_2x^n\left[(1-u)^n - ((1-u)+u)^n\right] \\
&= c_1\sum_{k=1}^n \binom{n}{k}(1-u)^{n-k}x^{n-k}u^k(1-x^k) - c_2x^n\sum_{k=1}^n\binom{n}{k}(1-u)^{n-k}u^k \\
&= \sum_{k=1}^n\binom{n}{k}(1-u)^{n-k}u^k\left[c_1x^{n-k} - (c_1+c_2)x^n\right] \qquad (2.3)
\end{aligned}$$

and in particular

$$x((1-u)x+u)^n + (1-x)((1-u)x)^n - x^n = \sum_{k=2}^n\binom{n}{k}(1-u)^{n-k}u^k(x^{n-k+1} - x^n).$$

We prepare a simple lemma in order to calculate $A_\alpha G_t$.

Lemma 2.1 *Assume that $b > 0$ and $a + b > 0$.*
(i) It holds that for any $\theta_1 > 0$ and $\theta_2 > 0$

$$\int_0^1 \frac{B_{\theta_1,\theta_2}(du)}{(au+b)^{\theta_1+\theta_2}} = (a+b)^{-\theta_1}b^{-\theta_2}. \tag{2.4}$$

(ii) In addition, suppose that $a' \ne a$ and $a' + b > 0$. Then

$$\int_0^1 \frac{B_{1-\alpha,1+\alpha}(du)}{(au+b)(a'u+b)} = \frac{1}{\alpha(a-a')b^{1+\alpha}}\left[(a+b)^\alpha - (a'+b)^\alpha\right]. \tag{2.5}$$

(2.4) is a one-dimensional version of the well-known formula due to [4], which is sometimes referred to as the Markov-Krein identity. (See e.g. [18] or (3.5) below.) We will give a self-contained proof showing essence of a proof for general case.

Proof of Lemma 2.1. The proof of (2.4) is simply done by noting that

$$(a+b)^{-\theta_1}b^{-\theta_2} = \int_0^\infty \frac{dz_1}{\Gamma(\theta_1)} z_1^{\theta_1-1}e^{-(a+b)z_1} \int_0^\infty \frac{dz_2}{\Gamma(\theta_2)} z_2^{\theta_2-1}e^{-bz_2}$$

and then by change of variables to $u := z_1/(z_1+z_2), v := z_1+z_2$. (2.5) can be deduced from (2.4) with $\theta_1 = 1-\alpha$ and $\theta_2 = \alpha$ since $B_{1-\alpha,1+\alpha}(du) = B_{1-\alpha,\alpha}(du)(1-u)/\alpha$ and

$$\frac{1-u}{(au+b)(a'u+b)} = \frac{1}{(a-a')b}\left(\frac{a+b}{au+b} - \frac{a'+b}{a'u+b}\right).$$

■

We proceed to calculate $A_\alpha G_t$.

Lemma 2.2 *For any $t > 0$ and $x \in [0,1]$,*

$$A_\alpha G_t(x) = t \cdot \frac{(1+t)^\alpha - 1}{\alpha} \cdot \frac{x(1-x)}{(1+tx)^{2+\alpha}} - \frac{t}{\alpha+1} \cdot \frac{c_1(1-x)(1+t)^{\alpha-1} - c_2 x}{(1+tx)^{1+\alpha}}. \quad (2.6)$$

Proof. By straightforward calculations

$$\begin{aligned} & c_1 G_t((1-u)x+u) + c_2 G_t((1-u)x) - (c_1+c_2)G_t(x) \\ = & -\frac{tu}{1+tx}\left[\frac{c_1(1-x)}{1+t(1-u)x+tu} - \frac{c_2 x}{1+t(1-u)x}\right]. \end{aligned}$$

Replacing c_1 and c_2 by x and $1-x$, respectively, we get

$$\begin{aligned} & xG_t((1-u)x+u) + (1-x)G_t((1-u)x) - G_t(x) \\ = & \frac{t^2u^2x(1-x)}{1+tx} \cdot \frac{1}{(1+t(1-u)x+tu)(1+t(1-u)x)}. \end{aligned}$$

Plugging these equalities into (1.3) with $G = G_t$, we deduce (2.6) with the help of Lemma 2.1. The details are left to the reader. ■

Next, we are going to characterize stationary distributions P in terms of

$$S_\alpha(t) := \int_0^1 \frac{P(dx)}{(1+tx)^\alpha}, \qquad t \geq 0, \quad (2.7)$$

which is a variant of the generalized Stieltjes transform of order α.

Proposition 2.3 *Let P be a stationary distribution of the process associated with (1.3). Then S_α defined by (2.7) satisfies for all $t > 0$*

$$\begin{aligned} & \frac{(1+t)^\alpha - 1}{\alpha}(1+t)S_\alpha''(t) \quad (2.8) \\ & + \left[\left(c_1 + 1 + \frac{1}{\alpha}\right)(1+t)^\alpha + c_2\left(1+\frac{1}{\alpha}\right)\right] S_\alpha'(t) + \alpha c_1 (1+t)^{\alpha-1} S_\alpha(t) = 0. \end{aligned}$$

Proof. By virtue of Lemma 2.2, (2.2) is equivalent to that for all $t > 0$

$$-\frac{(1+t)^\alpha - 1}{\alpha}\int_0^1 \frac{x(1-x)}{(1+tx)^{2+\alpha}}P(dx)$$
$$+\frac{c_1}{\alpha+1}(1+t)^{\alpha-1}\int_0^1 \frac{1-x}{(1+tx)^{1+\alpha}}P(dx) - \frac{c_2}{\alpha+1}\int_0^1 \frac{x}{(1+tx)^{1+\alpha}}P(dx) = 0.$$

(2.8) follows by substituting the equalities

$$-\int_0^1 \frac{x(1-x)}{(1+tx)^{2+\alpha}}P(dx) = \frac{1+t}{\alpha(\alpha+1)}S''_\alpha(t) + \frac{1}{\alpha}S'_\alpha(t),$$

$$\int_0^1 \frac{1-x}{(1+tx)^{1+\alpha}}P(dx) = \frac{1+t}{\alpha}S'_\alpha(t) + S_\alpha(t)$$

and

$$\int_0^1 \frac{x}{(1+tx)^{1+\alpha}}P(dx) = -\frac{1}{\alpha}S'_\alpha(t).$$

■

We now derive (1.4) as a unique stationary distribution we are looking for. Recall that for each $y \in (0,1)$ we denote by $E_{\alpha,y}$ the expectation with respect to the two-dimensional random variable (Y_1, Y_2) with joint law determined by

$$E_{\alpha,y}[e^{-\lambda_1 Y_1 - \lambda_2 Y_2}] = e^{-y\lambda_1^\alpha - (1-y)\lambda_2^\alpha}, \qquad \lambda_1, \lambda_2 \geq 0.$$

By using $t^{-\alpha} = \Gamma(\alpha)^{-1}\int_0^\infty dv v^{\alpha-1}e^{-vt}$ $(t > 0)$, observe that

$$E_{\alpha,y}\left[(tY_1 + Y_2)^{-\alpha}\right] = \frac{1}{\Gamma(\alpha+1)} \cdot \frac{1}{1 + (t^\alpha - 1)y}, \qquad t \geq 0. \tag{2.9}$$

In particular, $E_{\alpha,y}\left[(Y_1+Y_2)^{-\alpha}\right] = 1/\Gamma(\alpha+1)$ and hence

$$P_{\alpha,(c_1,c_2)}(dx) = \Gamma(\alpha+1)\int_0^1 B_{c_1,c_2}(dy)E_{\alpha,y}\left[(Y_1+Y_2)^{-\alpha}; \frac{Y_1}{Y_1+Y_2} \in dx\right] \tag{2.10}$$

defines a probability measure on $[0,1]$. Although for each $y \in (0,1)$ an expression of the distribution function

$$[0,1] \ni x \mapsto \Gamma(\alpha+1)E_{\alpha,y}\left[(Y_1+Y_2)^{-\alpha}; \frac{Y_1}{Y_1+Y_2} \leq x\right]$$

is given as the formula (3.2) in [19], we do not have any explicit form concerning $P_{\alpha,(c_1,c_2)}$ except the case $c_1 + c_2 = 1$. (See Remark (ii) at the end of this section.)

The main result of this section is the following.

Theorem 2.4 *The process associated with (1.3) has a unique stationary distribution, which coincides with* $P_{\alpha,(c_1,c_2)}$.

Proof. Let P and S_α be as in Proposition 2.3. Put

$$T_\alpha(u) = S_\alpha\left((u+1)^{1/\alpha} - 1\right)$$

for $u \geq 0$. It is direct to see that the equation (2.8) is transformed into a hypergeometric equation of the form

$$u(u+1)T_\alpha''(u) + [(c_1+c_2) + (c_1+2)\,u]\,T_\alpha'(u) + c_1 T_\alpha(u) = 0, \quad u > 0. \tag{2.11}$$

Clearly $T_\alpha(0) = S_\alpha(0) = 1$. In addition,

$$T_\alpha'(0) = S_\alpha'(0)/\alpha = -\int_0^1 P(dx)x = -c_1/(c_1+c_2),$$

where the last equality follows from (2.1) with $n = 1$. These facts together imply that

$$T_\alpha(u) = \int_0^1 \frac{B_{c_1,c_2}(dy)}{1+uy}, \qquad u \geq 0$$

or

$$S_\alpha(t) = \int_0^1 \frac{B_{c_1,c_2}(dy)}{1+\{(1+t)^\alpha - 1\}\,y}, \qquad t \geq 0.$$

(See e.g. Sections 7.2 and 9.1 in [13].) Combining this with

$$\frac{1}{1+\{(1+t)^\alpha - 1\}\,y} = \Gamma(\alpha+1)\int_0^1 \frac{1}{(1+tx)^\alpha} E_{\alpha,y}\left[(Y_1+Y_2)^{-\alpha}; \frac{Y_1}{Y_1+Y_2} \in dx\right],$$

which is immediate from (2.9), we arrive at

$$S_\alpha(t) = \int_0^1 \frac{P_{\alpha,(c_1,c_2)}(dx)}{(1+tx)^\alpha}, \qquad t \geq 0 \tag{2.12}$$

in view of (2.10). Therefore, we conclude that $P = P_{\alpha,(c_1,c_2)}$ and the proof of Theorem 2.4 is complete. ■

Remarks. (i) In the case where $c_1 + c_2 > 1$, an alternative expression of $P_{\alpha,(c_1,c_2)}$ exists:

$$P_{\alpha,(c_1,c_2)}(dx) = \Gamma(\alpha+1)(c_1+c_2-1)E\left[(Z_1+Z_2)^{-\alpha}; \frac{Z_1}{Z_1+Z_2} \in dx\right] =: \widetilde{P}_{\alpha,(c_1,c_2)}(dx), \tag{2.13}$$

where Z_1 and Z_2 are independent random variables with Laplace transforms

$$E[e^{-\lambda Z_i}] = \exp\left[-c_i \log(1+\lambda^\alpha)\right], \qquad \lambda \geq 0. \tag{2.14}$$

This reflects the fact that the solution to (2.11) with the same initial conditions $T_\alpha(0) = 1$ and $T_\alpha'(0) = -c_1/(c_1+c_2)$ admits another integral expression of the form

$$T_\alpha(u) = \int_0^1 \frac{B_{1,c_1+c_2-1}(dy)}{(1+uy)^{c_1}}, \qquad u \geq 0$$

and accordingly by (2.12)

$$\int_0^1 \frac{P_{\alpha,(c_1,c_2)}(dx)}{(1+tx)^\alpha} = \int_0^1 \frac{B_{1,c_1+c_2-1}(dy)}{\left[1+\{(1+t)^\alpha-1\}\,y\right]^{c_1}}, \qquad t \geq 0. \tag{2.15}$$

On the other hand, it is not difficult to show that (2.15) with $\widetilde{P}_{\alpha,(c_1,c_2)}$ in place of $P_{\alpha,(c_1,c_2)}$ holds, too. In fact, we prove in Lemma 3.4 below a generalization of the coincidence (2.13) in the setting of random measures. Also, the role of Z_1 and Z_2 will be made clear in connection with branching processes with immigration related closely to the process generated by (1.3). (Compare (2.14) with (3.8) below.)
(ii) It will be shown in Remark after Lemma 3.4 below that $P_{\alpha,(c_1,c_2)} = B_{\alpha c_1,\alpha c_2}$ holds whenever $c_1+c_2=1$. At least at formal level, this would be seen by letting $c_1+c_2 \downarrow 1$ in (2.15) and then by making use of (2.4).
(iii) In contrast with the case of the Wright-Fisher diffusion mentioned in Introduction, $P_{\alpha,(c_1,c_2)}$ with $0<\alpha<1$ is not a reversible distribution for the generator (1.3) at least in case $c_1 \neq c_2$. This will be seen in Section 4.

3 The measure-valued process case

The main subject of this section is an extension of Theorem 2.4 to a class of generalized Fleming-Viot processes. But the strategy will be different from that in the previous section, and so an alternative proof of Theorem 2.4 will be given as a by-product. To discuss in the setting of measure-valued process, we need new notation. Let E be a compact metric space having at least two distinct points and $C(E)$ (resp. $B_+(E)$) the set of continuous (resp. non-negative, bounded Borel) functions on E. Define $\mathcal{M}(E)$ to be the totality of finite Borel measures on E, and we equip $\mathcal{M}(E)$ with the weak topology. Denote by $\mathcal{M}(E)^\circ$ the set of non-null elements of $\mathcal{M}(E)$. The set $\mathcal{M}_1(E)$ of Borel probability measures on E is regarded as a subspace of $\mathcal{M}(E)$. We also use notation $\langle \eta, f\rangle := \int_E f(r)\eta(dr)$. For each $r \in E$, let δ_r denote the delta distribution at r. Given a probability measure Q, we write also $E^Q[\cdot]$ for the expectation with respect to Q.

Let $0<\alpha<1$ and $m \in \mathcal{M}(E)$ be given. As a natural generalization of the process generated by (1.3), we shall discuss in this section an $\mathcal{M}_1(E)$-valued Markov process associated with

$$\begin{aligned} &\mathcal{A}_{\alpha,m}\Phi(\mu) \qquad\qquad (3.1)\\ &:= \int_0^1 \frac{B_{1-\alpha,1+\alpha}(du)}{u^2}\int_E \mu(dr)\left[\Phi((1-u)\mu+u\delta_r)-\Phi(\mu)\right] \\ &\quad + \int_0^1 \frac{B_{1-\alpha,\alpha}(du)}{(\alpha+1)u}\int_E m(dr)\left[\Phi((1-u)\mu+u\delta_r)-\Phi(\mu)\right], \qquad \mu \in \mathcal{M}_1(E), \end{aligned}$$

where Φ is in the linear span $\mathcal{F}_0$ of functions of the form $\mu \mapsto \langle \mu, f_1\rangle \cdots \langle \mu, f_n\rangle$ with $f \in C(E)$ and n being a positive integer. We note that discussed in [10] is the case where $E=[0,1]$ and $m = c\delta_0$ for some $c>0$. As far as the martingale problem for $\mathcal{A}_{\alpha,m}$ in (3.1) is concerned, we can establish the well-posedness by modifying some existing arguments. Indeed, the existence can be shown through a limit theorem for suitably generalized Moran particle systems by modifying those considered in the proof of Theorem 2.1 (especially (2.2)) of [11], which took account of the jump mechanism describing simultaneous reproduction (sampling) only, so that simultaneous movement (mutation) of particles to a random location (type) distributed according to $\nu(dr) := m(dr)/m(E)$ is allowed. Also, the uniqueness follows by the duality argument employing a function-valued process which generalizes the one in the proof of Theorem 2.1 of [11] so that the transitions from functions $\prod_{i=1}^n f_i(r_i)$ to $\prod_{i\in I}\langle \nu, f_i\rangle \prod_{j\notin I} f_j(r_j)$ are allowed for any nonempty subset I of $\{1,\ldots,n\}$. (The precise transition rates are implicit in (3.22) below.) For simplicity, we call the $\mathcal{A}_{\alpha,m}$-process the Markov process governed by the generator $\mathcal{A}_{\alpha,m}$.

The following argument is based primarily on the relationship between the $\mathcal{A}_{\alpha,m}$-process and a suitable MBI-process, which takes values in $\mathcal{M}(E)$. More precisely, the generator, say $\mathcal{L}_{\alpha,m}$, of the latter will be chosen so that for some constant $C > 0$

$$\mathcal{L}_{\alpha,m}\Psi(\eta) = C\eta(E)^{-\alpha}\mathcal{A}_{\alpha,m}\Phi\left(\eta(E)^{-1}\eta\right), \qquad \eta \in \mathcal{M}(E)^\circ, \tag{3.2}$$

where $\Phi \in \mathcal{F}_0$ is arbitrary and $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$. In the case of Fleming-Viot process (which corresponds to $\alpha = 1$ formally), such a relation is well-known. For instance, it played a key role in [16]. As for the generalized Fleming-Viot process, factorizations of the form (3.2) have been shown in [2] for $m = 0$ (the null measure) and in [10] for degenerate measures m. From now on, suppose that $m \in \mathcal{M}(E)^\circ$. To exploit (3.2) in the study of stationary distributions, we further require the MBI-process associated with $\mathcal{L}_{\alpha,m}$ to be ergodic, i.e., to have a unique stationary distribution, say $\widetilde{Q}_{\alpha,m}$, supported on $\mathcal{M}(E)^\circ$. Once these requirements are fulfilled, (3.2) suggests that

$$\widetilde{P}_{\alpha,m}(\cdot) := E^{\widetilde{Q}_{\alpha,m}}\left[\eta(E)^{-\alpha}; \eta(E)^{-1}\eta \in \cdot\right] / E^{\widetilde{Q}_{\alpha,m}}\left[\eta(E)^{-\alpha}\right] \tag{3.3}$$

would give a stationary distribution of the $\mathcal{A}_{\alpha,m}$-process provided that $\eta(E)^{-\alpha}$ is integrable with respect to $\widetilde{Q}_{\alpha,m}$. This conditional answer may be modified to be a general one, which must be consistent with the one-dimensional result (1.4).

To describe the answer, we need both the α-stable random measure with parameter measure m and the Dirichlet random measure with parameter measure m, whose laws on $\mathcal{M}(E)^\circ$ and $\mathcal{M}_1(E)$ are denoted by $Q_{\alpha,m}$ and $\mathcal{D}_m$, respectively. These infinite dimensional laws are determined uniquely by the identities

$$\int_{\mathcal{M}(E)^\circ} Q_{\alpha,m}(d\eta)e^{-\langle \eta, f\rangle} = e^{-\langle m, f^\alpha\rangle} \tag{3.4}$$

and

$$\int_{\mathcal{M}_1(E)} \mathcal{D}_m(d\mu)\langle \mu, 1+f\rangle^{-m(E)} = e^{-\langle m, \log(1+f)\rangle}, \tag{3.5}$$

where $f \in B_+(E)$ is arbitrary. A random measure with law $Q_{\alpha,m}$ is constructed from a Poisson random measure on $(0,\infty)\times E$. (See also Definition 6 in [18].) Observe from (3.4) that $E^{Q_{\alpha,m}}[\eta(E)^{-\alpha}] = 1/(m(E)\Gamma(\alpha+1))$. As in [8], $\mathcal{D}_m$ is defined originally to be the law of a random measure whose arbitrary finite-dimensional distributions are Dirichlet distributions with parameters specified by m. The useful identity (3.5) is due to [4] and reduces to (2.4) in one-dimension. We now state the main result of this paper.

Theorem 3.1 *For any $m \in \mathcal{M}(E)^\circ$, the $\mathcal{A}_{\alpha,m}$-process has a unique stationary distribution, which is identified with*

$$P_{\alpha,m}(\cdot) := \Gamma(\alpha+1)\int_{\mathcal{M}_1(E)} \mathcal{D}_m(d\mu) E^{Q_{\alpha,\mu}}\left[\eta(E)^{-\alpha}; \eta(E)^{-1}\eta \in \cdot\right]. \tag{3.6}$$

The proof will be divided into three steps. As mentioned earlier, we first find an ergodic MBI-process whose generator satisfies (3.2) and show, under necessary integrability condition, that $\widetilde{P}_{\alpha,m}$ in (3.3) gives a stationary distribution of the $\mathcal{A}_{\alpha,m}$-process. (In fact, the condition will turn out to be that $m(E) > 1$. This motivates us to make reparametrization $m =: \theta\nu$ with $\theta > 0$ and $\nu \in \mathcal{M}_1(E)$.) Second, for each $\nu \in \mathcal{M}_1(E)$, we prove that $\widetilde{P}_{\alpha,\theta\nu} = P_{\alpha,\theta\nu}$ for any $\theta > 1$. As the last step, we extends stationarity of $P_{\alpha,\theta\nu}$ with respect to $\mathcal{A}_{\alpha,\theta\nu}$ to all $\theta > 0$ by interpreting the condition of stationarity as certain recursion equations among moment measures which are seen to be real analytic in $\theta > 0$. Also, the recursion equations will be shown to yield uniqueness of the stationary distribution.

For the first step, we prove in the next proposition that the MBI-process with the following generator is a desired one:

$$\begin{aligned}
&\mathcal{L}_{\alpha,m}\Psi(\eta) \\
&:= \frac{\alpha+1}{\Gamma(1-\alpha)}\int_0^\infty \frac{dz}{z^{2+\alpha}}\int_E \eta(dr)\left[\Psi(\eta+z\delta_r) - \Psi(\eta) - z\frac{\delta\Psi}{\delta\eta}(r)\right] - \frac{1}{\alpha}\langle \eta, \frac{\delta\Psi}{\delta\eta}\rangle \\
&\quad + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{dz}{z^{1+\alpha}}\int_E m(dr)\left[\Psi(\eta+z\delta_r) - \Psi(\eta)\right],
\end{aligned} \tag{3.7}$$

where Ψ is in the class $\mathcal{F}$ of functions of the form $\eta \mapsto F(\langle\eta, f_1\rangle, \ldots, \langle\eta, f_n\rangle)$ for some $F \in C_b^2(\mathbf{R}^n)$, $f_i \in C(E)$ and a positive integer n, and $\frac{\delta\Psi}{\delta\eta}(r) = \frac{d}{d\epsilon}\Psi(\eta+\epsilon\delta_r)\Big|_{\epsilon=0}$. Up to this first order differential term, the operator (3.7) for $E = [0,1]$ and $m = c\delta_0$ with $c > 0$ is the same as the one discussed in Lemma 12 of [10], in which the factorization (3.2) has been proved. Thus, our main observation in the next proposition is that, keeping the validity of (3.2), such an extra term yields the ergodicity. Note that the generator (3.7) is a special case of the one discussed in Chapter 9 of [14]. (See (9.25) there for an expression of the generator.) In particular, a unique solution to the martingale problem for $\mathcal{L}_{\alpha,m}$ defines an $\mathcal{M}(E)$-valued Markov process, which henceforth we call the $\mathcal{L}_{\alpha,m}$-process. Intuitively, because of absence of 'motion process', the law of this process is considered as continuum convolution of the continuous-state branching process with immigration (CBI-process) studied in [12]. (See (3.10) below.)

Proposition 3.2 *Let $m \in \mathcal{M}(E)^\circ$. Then $\mathcal{L}_{\alpha,m}$ in (3.7) and $\mathcal{A}_{\alpha,m}$ in (3.1) together satisfy (3.2) with $C = \Gamma(\alpha+2)$ and $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$ for any $\Phi \in \mathcal{F}_0$. Moreover,* *the $\mathcal{L}_{\alpha,m}$-process has a unique stationary distribution $\widetilde{Q}_{\alpha,m}$ with Laplace functional*

$$\int_{\mathcal{M}(E)^\circ} \widetilde{Q}_{\alpha,m}(d\eta)e^{-\langle \eta,f\rangle} = e^{-\langle m,\log(1+f^\alpha)\rangle}, \qquad f \in B_+(E). \tag{3.8}$$

A random measure with law $\widetilde{Q}_{\alpha,m}$ may be called a Linnik random measure since it is an infinite-dimensional analogue of the random variable with law sometimes referred to as a (non-symmetric) Linnik distribution, whose Laplace transform appeared already in (2.14). It is obtained by subordinating to an α-stable subordinator by a gamma process. (See e.g. Example 30.8 in [15].) (3.8) clearly shows an analogous structure underlying, i.e.,

$$\widetilde{Q}_{\alpha,m}(\cdot) = \int_{\mathcal{M}(E)^\circ} \mathcal{G}_m(d\eta)Q_{\alpha,\eta}(\cdot),$$

where $\mathcal{G}_m$ is the law of the standard gamma process on (E,m). (See Definition 5 in [18]). It is also obvious from (3.8) that, as $\alpha \uparrow 1$, $\widetilde{Q}_{\alpha,m}$ converges to $\mathcal{G}_m$. In addition, one can see that

$$\lim_{\alpha\uparrow 1} \mathcal{L}_{\alpha,m}\Psi(\eta) = \langle \eta, \frac{\delta^2\Psi}{\delta\eta^2}\rangle - \langle \eta, \frac{\delta\Psi}{\delta\eta}\rangle + \langle m, \frac{\delta\Psi}{\delta\eta}\rangle =: \mathcal{L}_m\Psi(\eta)$$

for 'nice' functions Ψ, where $\frac{\delta^2\Psi}{\delta\eta^2}(r) = \frac{d^2}{d\epsilon^2}\Psi(\eta+\epsilon\delta_r)\Big|_{\epsilon=0}$. This is a special case of generator of MBI-processes discussed in Section 3 of [17]. It has been proved there that $\mathcal{G}_m$ is a reversible stationary distribution of the process associated with $\mathcal{L}_m$.

Proof of Proposition 3.2. As already remarked, if the term $-\alpha^{-1}\langle \eta, \frac{\delta\Psi}{\delta\eta}\rangle$ in (3.7) would vanish, (3.2) can be shown by essentially the same calculations as in the proof of Lemma 12 in [10]. (In fact, the change of variable $z =: \eta(E)u/(1-u)$ in the integrals with respect to dz in (3.7) almost suffices for our purpose.) So, for the proof of (3.2), we only need to observe that $\langle \eta, \frac{\delta\Psi}{\delta\eta}\rangle = 0$ for Ψ of the form $\Psi(\eta) = \Phi(\eta(E)^{-1}\eta)$ with $\Phi \in \mathcal{F}_0$. But this is readily done by giving a specific form of Φ.

The argument regarding ergodicity is based on a well-known formula for Laplace functional of transition functions. (See (9.18) in [14] for much more general case than ours.) To write it down, we need only auxiliary functions called Ψ-semigroup [12] because there is no 'motion process'. These functions form a one-parameter family $\{\psi(t,\cdot)\}_{t\geq 0}$ of non-negative functions on $[0,\infty)$ and are determined by the equation

$$\frac{\partial\psi}{\partial t}(t,\lambda) = -\frac{1}{\alpha}\psi(t,\lambda)^{1+\alpha} - \frac{1}{\alpha}\psi(t,\lambda), \qquad \psi(0,\lambda) = \lambda \tag{3.9}$$

with $\lambda \geq 0$ being arbitrary. An explicit expression is found in Example 3.1 of [14]:

$$\psi(t,\lambda) = \frac{e^{-t/\alpha}\lambda}{[1+(1-e^{-t})\lambda^\alpha]^{1/\alpha}}.$$

Let $\{\eta_t : t \geq 0\}$ be an $\mathcal{L}_{\alpha,m}$-process, and for each $\eta \in \mathcal{M}(E)$ denote by E_η the expectation with respect to $\{\eta_t : t \geq 0\}$ starting at η. Then for any $f \in B_+(E)$ and $t \geq 0$

$$E_\eta\left[e^{-\langle \eta_t,f\rangle}\right] = \exp\left[-\langle \eta, V_t f\rangle - \int_0^t \langle m, (V_s f)^\alpha\rangle ds\right], \tag{3.10}$$

where $V_t f(r) = \psi(t, f(r))$. As $t \to \infty$ the right side converges to

$$\exp\left[-\int_0^\infty \langle m, (V_t f)^\alpha \rangle dt\right] = \exp\left[-\langle m, \log(1+f^\alpha)\rangle\right]$$

since by (3.9)

$$\frac{d}{dt}\log\left(1+(V_t f(r))^\alpha\right) = -(V_t f(r))^\alpha.$$

This shows the ergodicity required and completes the proof. ■

Proposition 3.3 *Suppose that $m(E) > 1$ and let $\widetilde{Q}_{\alpha,m}$ be as in Proposition 3.2. Then*

$$E^{\widetilde{Q}_{\alpha,m}}\left[\eta(E)^{-\alpha}\right] = (\Gamma(\alpha+1)(m(E)-1))^{-1}.$$

Moreover,

$$\widetilde{P}_{\alpha,m}(\cdot) = \Gamma(\alpha+1)(m(E)-1)E^{\widetilde{Q}_{\alpha,m}}\left[\eta(E)^{-\alpha}; \eta(E)^{-1}\eta \in \cdot\right] \tag{3.11}$$

is a stationary distribution of the $\mathcal{A}_{\alpha,m}$-process.

Proof. The first assertion is shown by using $t^{-\alpha} = \Gamma(\alpha)^{-1}\int_0^\infty dv v^{\alpha-1}e^{-vt}$ $(t > 0)$ and (3.8) with $f \equiv v$. Indeed, these equalities together with Fubini's theorem yield

$$\begin{aligned} E^{\widetilde{Q}_{\alpha,m}}\left[\eta(E)^{-\alpha}\right] &= \Gamma(\alpha)^{-1}\int_0^\infty dv v^{\alpha-1}\exp\left[-m(E)\log(1+v^\alpha)\right] \\ &= \Gamma(\alpha+1)^{-1}\int_0^\infty dz \exp\left[-m(E)\log(1+z)\right] \\ &= \Gamma(\alpha+1)^{-1}(m(E)-1)^{-1}. \end{aligned}$$

For the proof of stationarity of (3.11) with respect to $\mathcal{A}_{\alpha,m}$, it suffices to show that

$$\int_{\mathcal{M}_1(E)} \widetilde{P}_{\alpha,m}(d\mu)\mathcal{A}_{\alpha,m}\Phi(\mu) = 0 \tag{3.12}$$

for any Φ of the form $\Phi(\mu) = \langle \mu, f_1\rangle \cdots \langle \mu, f_n\rangle$ with $f_i \in C(E)$ and n being a positive integer. Without any loss of generality we can assume that $f_i(E) \subset [0,1]$ for each $i = 1, \ldots, n$. Furthermore, we only have to consider the case where $f_1 = \cdots = f_n =: f$ because the coefficients of monomial $t_1 \cdots t_n$ in $\langle \mu, t_1 f_1 + \cdots + t_n f_n\rangle^n$ equals $n!\langle \mu, f_1\rangle \cdots \langle \mu, f_n\rangle$. Thus, we let $\Phi(\mu) = \langle \mu, f\rangle^n$ with $f(E) \subset [0,1]$. Because of the basic relation (3.2) and (3.11) together, (3.12) is rewritten into

$$\int_{\mathcal{M}(E)^\circ} \widetilde{Q}_{\alpha,m}(d\eta)\mathcal{L}_{\alpha,m}\Psi(\eta) = 0, \tag{3.13}$$

where $\Psi(\eta) = \langle \eta, f\rangle^n \langle \eta, 1\rangle^{-n}$. The main difficulty comes from the fact that Ψ does not belong to $\mathcal{F}$. For each $\epsilon > 0$, introduce $\Psi_\epsilon(\eta) := \langle \eta, f\rangle^n(\langle \eta, 1\rangle + \epsilon)^{-n}$ and observe that $\Psi_\epsilon \in \mathcal{F}$. Thanking to Proposition 3.2, we then have (3.13) with Ψ_ϵ in place of Ψ provided that $\mathcal{L}_{\alpha,m}\Psi_\epsilon$ is bounded. Thus, the proof of (3.13) reduces to showing the following two assertions:

(i) For every $\epsilon > 0$, $\mathcal{L}^{(1)}_{\alpha,m}\Psi_\epsilon$, $\mathcal{L}^{(2)}_{\alpha,m}\Psi_\epsilon$ and $\mathcal{L}^{(3)}_{\alpha,m}\Psi_\epsilon$ are bounded functions on $\mathcal{M}(E)$.
(ii) It holds that for each $k \in \{1, 2, 3\}$

$$\lim_{\epsilon\downarrow 0}\int_{\mathcal{M}(E)^\circ}\widetilde{Q}_{\alpha,m}(d\eta)\mathcal{L}^{(k)}_{\alpha,m}\Psi_\epsilon(\eta) = \int_{\mathcal{M}(E)^\circ}\widetilde{Q}_{\alpha,m}(d\eta)\mathcal{L}^{(k)}_{\alpha,m}\Psi(\eta). \tag{3.14}$$

Here, $\mathcal{L}_{\alpha,m} = \mathcal{L}^{(1)}_{\alpha,m} + \mathcal{L}^{(2)}_{\alpha,m} + \mathcal{L}^{(3)}_{\alpha,m}$, and the operators $\mathcal{L}^{(1)}_{\alpha,m}$, $\mathcal{L}^{(2)}_{\alpha,m}$ and $\mathcal{L}^{(3)}_{\alpha,m}$ correspond respectively to the first, second and last term on the right side of (3.7).

First, we consider $\mathcal{L}^{(2)}_{\alpha,m}$. Observe that

$$\begin{aligned}\frac{\delta\Psi_\epsilon}{\delta\eta}(r) &= \frac{nf(r)\langle\eta, f\rangle^{n-1}}{(\langle\eta,1\rangle+\epsilon)^n} - \frac{n\langle\eta, f\rangle^n}{(\langle\eta,1\rangle+\epsilon)^{n+1}} \\ &= \frac{n\left(f(r)\langle\eta,1\rangle - \langle\eta, f\rangle + \epsilon f(r)\right)\langle\eta, f\rangle^{n-1}}{(\langle\eta,1\rangle+\epsilon)^{n+1}},\end{aligned} \tag{3.15}$$

from which it follows that $\alpha\mathcal{L}^{(2)}_{\alpha,m}\Psi_\epsilon(\eta) = -n\epsilon\Psi_\epsilon(\eta)/(\langle\eta,1\rangle+\epsilon)$. Hence $\mathcal{L}^{(2)}_{\alpha,m}\Psi_\epsilon$ is a bounded function on $\mathcal{M}(E)$ and $\mathcal{L}^{(2)}_{\alpha,m}\Psi_\epsilon(\eta) \to 0 = \mathcal{L}^{(2)}_{\alpha,m}\Psi(\eta)$ boundedly as $\epsilon \downarrow 0$. This proves that (i) and (ii) hold true for $\mathcal{L}^{(2)}_{\alpha,m}$.

In calculating $\mathcal{L}^{(3)}_{\alpha,m}\Psi_\epsilon$, (3.15) is useful since $\frac{d}{dz}\Psi_\epsilon(\eta + z\delta_r) = \frac{\delta\Psi_\epsilon}{\delta(\eta+z\delta_r)}(r)$. Indeed, by Fubini's theorem

$$\begin{aligned}\int_0^\infty \frac{dz}{z^{1+\alpha}}\left[\Psi_\epsilon(\eta+z\delta_r) - \Psi_\epsilon(\eta)\right] &= \int_0^\infty \frac{dz}{z^{1+\alpha}}\int_0^z dw\frac{\delta\Psi_\epsilon}{\delta(\eta+w\delta_r)}(r) \\ &= \frac{1}{\alpha}\int_0^\infty w^{-\alpha}dw\frac{\delta\Psi_\epsilon}{\delta(\eta+w\delta_r)}(r),\end{aligned} \tag{3.16}$$

and combining with (3.15) yields

$$\begin{aligned}&\left|\int_0^\infty \frac{dz}{z^{1+\alpha}}\left[\Psi_\epsilon(\eta+z\delta_r) - \Psi_\epsilon(\eta)\right]\right| \\ &\le \frac{1}{\alpha}\int_0^\infty w^{-\alpha}dw\frac{n\left|f(r)\langle\eta+w\delta_r,1\rangle - \langle\eta+w\delta_r, f\rangle + \epsilon f(r)\right|\langle\eta+w\delta_r, f\rangle^{n-1}}{(\langle\eta+w\delta_r,1\rangle+\epsilon)^{n+1}} \\ &\le \frac{n}{\alpha}\int_0^\infty w^{-\alpha}dw\frac{1}{\langle\eta,1\rangle+w+\epsilon} \\ &= \frac{n}{\alpha}\int_0^\infty w^{-\alpha}dw\int_0^\infty dv e^{-v(\langle\eta,1\rangle+w+\epsilon)} \\ &= n\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\alpha}(\langle\eta,1\rangle+\epsilon)^{-\alpha}.\end{aligned} \tag{3.17}$$

This shows not only that $\mathcal{L}^{(3)}_{\alpha,m}\Psi_\epsilon$ is bounded but also

$$|\mathcal{L}^{(3)}_{\alpha,m}\Psi_\epsilon(\eta)| \le n\Gamma(\alpha)\cdot\frac{\langle m,1\rangle}{\langle\eta,1\rangle^\alpha},$$

which is integrable with respect to $\widetilde{Q}_{\alpha,m}$ as proved already. It can be seen also from (3.15) and (3.16) that $\mathcal{L}^{(3)}_{\alpha,m}\Psi_\epsilon$ converges pointwise to $\mathcal{L}^{(3)}_{\alpha,m}\Psi$ as $\epsilon \downarrow 0$. By Lebesgue's dominated convergence theorem we have proved (3.14) for $\mathcal{L}^{(3)}_{\alpha,m}$.

The final task is to deal with $\mathcal{L}^{(1)}_{\alpha,m}\Psi_\epsilon$. Similarly to (3.16)

$$\begin{aligned} I_\epsilon(\eta,r) &:= \int_0^\infty \frac{dz}{z^{2+\alpha}}\left[\Psi_\epsilon(\eta+z\delta_r)-\Psi_\epsilon(\eta)-z\frac{\delta\Psi_\epsilon}{\delta\eta}(r)\right] \\ &= \int_0^\infty \frac{dz}{z^{2+\alpha}}\int_0^z dw\left[\frac{\delta\Psi_\epsilon}{\delta(\eta+w\delta_r)}(r)-\frac{\delta\Psi_\epsilon}{\delta\eta}(r)\right] \\ &= \frac{1}{1+\alpha}\int_0^\infty \frac{dw}{w^{1+\alpha}}\left[\frac{\delta\Psi_\epsilon}{\delta(\eta+w\delta_r)}(r)-\frac{\delta\Psi_\epsilon}{\delta\eta}(r)\right]. \end{aligned}$$

By (3.15) $\frac{\delta\Psi_\epsilon}{\delta(\eta+w\delta_r)}(r)-\frac{\delta\Psi_\epsilon}{\delta\eta}(r)$ equals

$$\begin{aligned} &\frac{(\langle\eta,1\rangle+\epsilon)^{n+1}n\left(f(r)\langle\eta,1\rangle-\langle\eta,f\rangle+\epsilon f(r)\right)\left[\langle\eta+w\delta_r,f\rangle^{n-1}-\langle\eta,f\rangle^{n-1}\right]}{(\langle\eta,1\rangle+w+\epsilon)^{n+1}(\langle\eta,1\rangle+\epsilon)^{n+1}} \\ &+\frac{\left[(\langle\eta,1\rangle+\epsilon)^{n+1}-(\langle\eta,1\rangle+w+\epsilon)^{n+1}\right]n\left(f(r)\langle\eta,1\rangle-\langle\eta,f\rangle+\epsilon f(r)\right)\langle\eta,f\rangle^{n-1}}{(\langle\eta,1\rangle+w+\epsilon)^{n+1}(\langle\eta,1\rangle+\epsilon)^{n+1}}. \end{aligned}$$

Moreover, we have bounds

$$\begin{aligned} \left|\langle\eta+w\delta_r,f\rangle^{n-1}-\langle\eta,f\rangle^{n-1}\right| &= \left|\int_0^w dv(n-1)f(r)\langle\eta+v\delta_r,f\rangle^{n-2}\right| \\ &\le w(n-1)(\langle\eta,1\rangle+w)^{n-2} \end{aligned}$$

and

$$\begin{aligned} \left|(\langle\eta,1\rangle+\epsilon)^{n+1}-(\langle\eta,1\rangle+w+\epsilon)^{n+1}\right| &= (n+1)\int_0^w dv(\langle\eta,1\rangle+v+\epsilon)^n \\ &\le w(n+1)(\langle\eta,1\rangle+w+\epsilon)^n. \end{aligned}$$

Consequently

$$\begin{aligned} \left|\frac{\delta\Psi_\epsilon}{\delta(\eta+w\delta_r)}(r)-\frac{\delta\Psi_\epsilon}{\delta\eta}(r)\right| &\le w\frac{n(\langle\eta,1\rangle+\epsilon)^{n+2}(n-1)(\langle\eta,1\rangle+w)^{n-2}}{(\langle\eta,1\rangle+w+\epsilon)^{n+1}(\langle\eta,1\rangle+\epsilon)^{n+1}} \\ &\quad+w\frac{(n+1)(\langle\eta,1\rangle+w+\epsilon)^n n(\langle\eta,1\rangle+\epsilon)\langle\eta,1\rangle^{n-1}}{(\langle\eta,1\rangle+w+\epsilon)^{n+1}(\langle\eta,1\rangle+\epsilon)^{n+1}} \\ &\le w\frac{2n^2}{(\langle\eta,1\rangle+w+\epsilon)(\langle\eta,1\rangle+\epsilon)}. \end{aligned}$$

Therefore, analogous calculations to those in (3.17) lead to

$$\begin{aligned} \left|\mathcal{L}^{(1)}_{\alpha,m}\Psi_\epsilon(\eta)\right| &= \left|\frac{\alpha+1}{\Gamma(1-\alpha)}\int_E I_\epsilon(\eta,r)\eta(dr)\right| \\ &\le 2n^2\Gamma(\alpha)(\langle\eta,1\rangle+\epsilon)^{-\alpha}\cdot\frac{\langle\eta,1\rangle}{\langle\eta,1\rangle+\epsilon}. \end{aligned}$$

This makes it possible to argue as in the case of $\mathcal{L}^{(3)}_{\alpha,m}\Psi_\epsilon$ to verify (i) and (ii) for $\mathcal{L}^{(1)}_{\alpha,m}$. We complete the proof of Proposition 3.3. ■

Next, we show the coincidence of two distributions (3.3) (or (3.11)) and (3.6).

Lemma 3.4 *If $m(E) > 1$, then $\widetilde{P}_{\alpha,m}$ in (3.3) coincides with $P_{\alpha,m}$ in (3.6).*

Proof. It suffices to show that for any $f \in B_+(E)$

$$\widetilde{I}(f) := \int_{\mathcal{M}_1(E)} \widetilde{P}_{\alpha,m}(d\mu)\langle \mu, 1+f\rangle^{-\alpha} = \int_{\mathcal{M}_1(E)} P_{\alpha,m}(d\mu)\langle \mu, 1+f\rangle^{-\alpha} =: I(f).$$

In view of (3.11), $\widetilde{I}(f)/(\Gamma(\alpha+1)(m(E)-1))$ equals

$$\begin{aligned}
& E^{\widetilde{P}_{\alpha,m}}\left[\langle \eta, 1\rangle^{-\alpha}\left(1+\langle \eta, 1\rangle^{-1}\langle \eta, f\rangle\right)^{-\alpha}\right] \;=\; E^{\widetilde{P}_{\alpha,m}}\left[\langle \eta, 1+f\rangle^{-\alpha}\right] && (3.18)\\
&= \Gamma(\alpha)^{-1}\int_0^\infty dv v^{\alpha-1}\exp\left[-\langle m, \log(1+v^\alpha(1+f)^\alpha)\rangle\right]\\
&= \Gamma(\alpha+1)^{-1}\int_0^\infty dz \exp\left[-\langle m, \log(1+z(1+f)^\alpha)\rangle\right]\\
&= \frac{1}{\Gamma(\alpha+1)}\int_0^1 du(1-u)^{-2}\exp\left[-\langle m, \log(1+\frac{u}{1-u}(1+f)^\alpha)\rangle\right]\\
&= \frac{1}{\Gamma(\alpha+1)}\int_0^1 du(1-u)^{m(E)-2}\exp\left[-\langle m, \log(1+u((1+f)^\alpha-1))\rangle\right]\\
&= \frac{1}{\Gamma(\alpha+1)}\int_0^1 du(1-u)^{m(E)-2}\int_{\mathcal{M}_1(E)}\mathcal{D}_m(d\mu)\langle \mu, 1+u((1+f)^\alpha-1)\rangle^{-m(E)},
\end{aligned}$$

where the last equality follows from (3.5). Hence, by applying Fubini's theorem and (2.4)

$$\begin{aligned}
\widetilde{I}(f) &= \int_{\mathcal{M}_1(E)}\mathcal{D}_m(d\mu)\int_0^1 \frac{B_{1,m(E)-1}(du)}{\langle \mu, 1+u((1+f)^\alpha-1)\rangle^{m(E)}}\\
&= \int_{\mathcal{M}_1(E)}\mathcal{D}_m(d\mu)\langle \mu, (1+f)^\alpha\rangle^{-1}.
\end{aligned}$$

On the other hand, it is obvious from (3.6) that

$$I(f) = \int_{\mathcal{M}_1(E)}\mathcal{D}_m(d\mu)\int_{\mathcal{M}_1(E)}\mathcal{D}_\mu^{(\alpha,\alpha)}(d\mu')\langle \mu', 1+f\rangle^{-\alpha}, \tag{3.19}$$

where in general, for $\theta > -\alpha$, $\mathcal{D}_\mu^{(\alpha,\theta)}$ is the law of the two-parameter generalization of Dirichlet random measure with parameter (α, θ) and parameter measure μ, i.e.,

$$\mathcal{D}_\mu^{(\alpha,\theta)}(\cdot) = \frac{\Gamma(\theta+1)}{\Gamma(\frac{\theta}{\alpha}+1)}\int_{\mathcal{M}_1(E)} E^{Q_{\alpha,\mu}}\left[\eta(E)^{-\theta}; \eta(E)^{-1}\eta \in \cdot\right].$$

(See e.g. Section 5 of [18].) By Theorem 4 in [18] (or equivalently by similar calculations to (3.18))

$$\int_{\mathcal{M}_1(E)}\mathcal{D}_\mu^{(\alpha,\alpha)}(d\mu')\langle \mu', 1+f\rangle^{-\alpha} = \langle \mu, (1+f)^\alpha\rangle^{-1} \tag{3.20}$$

and therefore $I(f) = \widetilde{I}(f)$ as desired. ∎

Remark. For any $\nu \in \mathcal{M}_1(E)$, we have $P_{\alpha,\nu} = \mathcal{D}_{\alpha\nu}$. Indeed, noting that (3.19) and (3.20) require no assumption on the value of $m(E)$, observe that by (3.19) and (3.20) with $m = \nu$

$$\begin{aligned}
\int_{\mathcal{M}_1(E)} P_{\alpha,\nu}(d\mu)\langle \mu, 1+f\rangle^{-\alpha} &= \int_{\mathcal{M}_1(E)} \mathcal{D}_\nu(d\mu)\langle \mu, (1+f)^\alpha\rangle^{-1} \\
&= \exp[-\langle \nu, \log\{(1+f)^\alpha\}\rangle] \\
&= \exp[-\langle \alpha\nu, \log(1+f)\rangle] \\
&= \int_{\mathcal{M}_1(E)} \mathcal{D}_{\alpha\nu}(d\mu)\langle \mu, 1+f\rangle^{-\alpha},
\end{aligned}$$

where (3.5) has been applied twice. (A one-dimensional version of the above identity is mentioned in Remark (ii) at the end of Section 2.) What we have just seen is rewritten as

$$\int_{\mathcal{M}_1(E)} \mathcal{D}_\nu(d\mu)\mathcal{D}_\mu^{(\alpha,\alpha)}(\cdot) = \mathcal{D}_{\alpha\nu}(\cdot),$$

which is a special case of

$$\int_{\mathcal{M}_1(E)} \mathcal{D}_\nu^{(\beta,\theta/\alpha)}(d\mu)\mathcal{D}_\mu^{(\alpha,\theta)}(\cdot) = \mathcal{D}_\nu^{(\alpha\beta,\theta)}(\cdot), \qquad \beta \in [0,1),\ \theta > -\alpha\beta.$$

Here notice that, in case $\beta = 0$, $\mathcal{D}_\nu^{(0,\theta)} = \mathcal{D}_{\theta\nu}$ by definition. This generalization can be proved analogously by virtue of the two-parameter generalization of (3.5). (See e.g. Theorem 4 in [18].)

We close this section with the proof of Theorem 3.1, in which we write $\theta\nu$ ($\theta > 0$, $\nu \in \mathcal{M}_1(E)$) for the parameter measure m.

Proof of Theorem 3.1. Let $\nu \in \mathcal{M}_1(E)$ be given. We first show that, for arbitrary $\theta > 0$, $P_{\alpha,\theta\nu}$ is a stationary distribution of the $\mathcal{A}_{\alpha,\theta\nu}$-process. For the same reason as in the proof of Proposition 3.3 (cf. (3.12)), it is sufficient to prove that

$$\int_{\mathcal{M}_1(E)} P_{\alpha,\theta\nu}(d\mu)\mathcal{A}_{\alpha,\theta\nu}\Phi(\mu) = 0 \tag{3.21}$$

for Φ of the form $\Phi(\mu) = \langle \mu, f\rangle^n$ with $f \in C(E)$ and n being a positive integer. Since Proposition 3.3 and Lemma 3.4 together imply that (3.21) holds true for any $\theta > 1$, it is enough to show that the left side of (3.21) defines a real analytic function of $\theta > 0$. We claim that

$$\begin{aligned}
\mathcal{A}_{\alpha,\theta\nu}\Phi(\mu) &= \frac{1}{\Gamma(n)}\sum_{k=2}^n \binom{n}{k}(1-\alpha)_{k-2}(\alpha+1)_{n-k}(\langle \mu, f^k\rangle\langle \mu, f\rangle^{n-k} - \langle \mu, f\rangle^n) \\
&\quad + \frac{\theta}{(\alpha+1)\Gamma(n)}\sum_{k=1}^n \binom{n}{k}(1-\alpha)_{k-1}(\alpha)_{n-k}(\langle \nu, f^k\rangle\langle \mu, f\rangle^{n-k} - \langle \mu, f\rangle^n) \\
&= \frac{1}{\Gamma(n)}\sum_{k=2}^n \binom{n}{k}(1-\alpha)_{k-2}(\alpha+1)_{n-k}\langle \mu, f^k\rangle\langle \mu, f\rangle^{n-k} \\
&\quad + \frac{\theta}{(\alpha+1)\Gamma(n)}\sum_{k=1}^n \binom{n}{k}(1-\alpha)_{k-1}(\alpha)_{n-k}\langle \nu, f^k\rangle\langle \mu, f\rangle^{n-k} \qquad (3.22) \\
&\quad - \frac{(\alpha+1)_{n-1}}{(\alpha+1)\Gamma(n)}(\theta+n-1)\langle \mu, f\rangle^n,
\end{aligned}$$

where $(a)_b = \Gamma(a+b)/\Gamma(a)$. The first equality can be observed by similar calculations to those in (2.3), i.e., by noting that

$$((1-u)\langle\mu, f\rangle + uf(r))^n - \langle\mu, f\rangle^n = \sum_{k=1}^{n} \binom{n}{k} u^k(1-u)^{n-k}\langle\mu, f\rangle^{n-k}(f(r)^k - \langle\mu, f\rangle^k),$$

and the second one can be shown with the help of Leibniz's formula $(\phi_1\phi_2)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}\phi_1^{(n-k)}\phi_2^{(k)}$ for $\phi_1(t) = (1-t)^{-a}$ and $\phi_2(t) = (1-t)^{-b}$ with $(a,b) = (\alpha+1, -\alpha-1)$ or $(a,b) = (\alpha, -\alpha)$. In view of (3.22), it is clear that the proof reduces to verifying real analyticity of $\int P_{\alpha,\theta\nu}(d\mu)\langle\mu, f_1\rangle \cdots \langle\mu, f_n\rangle$ in θ for arbitrary $f_1, \ldots, f_n \in C(E)$.

To this end, we shall exploit the following identity which is deduced from (3.19) and (3.20):

$$\int_{\mathcal{M}_1(E)} P_{\alpha,\theta\nu}(d\mu)\langle\mu, 1+f\rangle^{-\alpha} = \int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu)\langle\mu, (1+f)^\alpha\rangle^{-1}, \tag{3.23}$$

where $f \in B_+(E)$ is arbitrary. Clearly this remains true for all bounded Borel functions f on E such that $\inf_{r\in E} f(r) > -1$. Therefore, for any $t_1, \ldots, t_n \in \mathbf{R}$ with $|t_1| + \cdots + |t_n|$ being sufficiently small, (3.23) for $f = -\sum_{i=1}^{n} t_i f_i$ is valid, that is, $I(t_1, \cdots, t_n) = J(t_1, \cdots, t_n)$, where

$$I(t_1, \ldots, t_n) = \int_{\mathcal{M}_1(E)} P_{\alpha,\theta\nu}(d\mu)\left(1 - \langle\mu, \sum_{i=1}^{n} t_i f_i\rangle\right)^{-\alpha} \tag{3.24}$$

and

$$J(t_1, \ldots, t_n) = \int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu)\langle\mu, \left(1 - \sum_{i=1}^{n} t_i f_i\right)^{\alpha}\rangle^{-1}. \tag{3.25}$$

Noting that $(1-t)^{-\alpha} = 1 + \sum_{k=1}^{\infty}(\alpha)_k t^k/k!$ as long as $|t|$ is small enough, we see from (3.24) that the coefficient of monomial $t_1 \cdots t_n$ in the expansion of $I(t_1, \ldots, t_n)$ is given by

$$(\alpha)_n \int_{\mathcal{M}_1(E)} P_{\alpha,\theta\nu}(d\mu)\langle\mu, f_1\rangle \cdots \langle\mu, f_n\rangle. \tag{3.26}$$

To find the corresponding coefficient for $J(t_1, \ldots, t_n)$, define

$$h_\alpha(t) = 1 - (1-t)^\alpha = \alpha\sum_{l=1}^{\infty}(1-\alpha)_{l-1}t^l/l!$$

and observe from (3.25) that $J(t_1, \ldots, t_n)$ equals

$$\begin{aligned}
&\int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu)\langle\mu, 1 - h_\alpha\left(\sum_{i=1}^{n} t_i f_i\right)\rangle^{-1} \\
&\quad = 1 + \sum_{k=1}^{\infty}\int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu)\langle\mu, h_\alpha\left(\sum_{i=1}^{n} t_i f_i\right)\rangle^k \\
&\quad = 1 + \sum_{k=1}^{\infty}\alpha^k\int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu)\sum_{l_1,\ldots,l_k=1}^{\infty}\prod_{j=1}^{k}\left\{\frac{(1-\alpha)_{l_j-1}}{l_j!}\langle\mu, \left(\sum_{i=1}^{n} t_i f_i\right)^{l_j}\rangle\right\}.
\end{aligned}$$

One can see that the coefficient of monomial $t_1 \cdots t_n$ in the expansion of $J(t_1, \ldots, t_n)$ can be expressed as

$$\sum_{k=1}^{n} \alpha^k k! \sum_{\gamma \in \pi(n,k)} \int_{\mathcal{M}_1(E)} \mathcal{D}_{\theta\nu}(d\mu) \prod_{j=1}^{k} \left\{ \frac{(1-\alpha)_{|\gamma_j|-1}}{\gamma_j!} \langle \mu, \prod_{i \in \gamma_j} f_i \rangle \right\}, \tag{3.27}$$

where $\pi(n,k)$ is the set of partitions γ of $\{1, \ldots, n\}$ into k unordered nonempty subsets $\gamma_1, \ldots, \gamma_k$ and $|\cdot|$ stands for the cardinality. By Lemma 2.2 of [6] (or equivalently by Lemma 2.4 of [7]), each integral in the above sum is a real analytic function of $\theta > 0$. Hence, so is the integral in (3.26) and the stationarity of $P_{\alpha,\theta\nu}$ with respect to $\mathcal{A}_{\alpha,\theta\nu}$ follows.

It remains to prove the uniqueness of stationary distribution P of the $\mathcal{A}_{\alpha,\theta\nu}$-process for each $\theta > 0$. But this is an immediate consequence of (3.21) with P in place of $P_{\alpha,\theta\nu}$ and (3.22), which together determine uniquely $\int P(d\mu)\langle \mu, f\rangle^n$ and hence the nth moment measure

$$M_n(dr_1 \cdots dr_n) := \int_{\mathcal{M}_1(E)} P(d\mu)\mu(dr_1) \cdots \mu(dr_n)$$

for any $n = 1, 2, \ldots$. This completes the proof of Theorem 3.1. ■

It is not clear whether we can derive from (3.27) an extension of the Ewens sampling formula in some explicit and informative form. (See Remarks after the proof of Lemma 2.2 in [6].) In view of $P_{\alpha,m}(\cdot) = \int \mathcal{D}_m(d\mu)\mathcal{D}_\mu^{(\alpha,\alpha)}(\cdot)$, one might think that Pitman's sampling formula would be applicable. But it is not the case since $\mathcal{D}_m(\mu$ is discrete.$) = 1$. The expression (3.11) might be rather useful for such a purpose.

4 Irreversibility

In this section we discuss reversibility of our processes. In contrast with the Fleming-Viot diffusion case, we guess that for any $0 < \alpha < 1$ and non-degenerate m the $\mathcal{A}_{\alpha,m}$-process would be irreversible. Unfortunately, the following result does not give an affirmative answer in all cases. However, this does not suggest any possibility of the reversibility in the exceptional case, which is believed to be dealt with a different choice of test functions.

Theorem 4.1 *Let $m \in \mathcal{M}(E)^\circ$ be given. Assume that either of the following two conditions holds.*
(i) The support of m has at least three distinct points.
(ii) The support of m has exactly two points, say r_1 and r_2, and $m(\{r_1\}) \neq m(\{r_2\})$.
Then the stationary distribution $P_{\alpha,m}$ of the $\mathcal{A}_{\alpha,m}$-process is not a reversible distribution of it.

Proof. As in the proof of Theorem 3.1, we write $\theta\nu$ instead of m. Thus, $\theta > 0$ and $\nu \in \mathcal{M}_1(E)$. Recall that an equivalent condition to the reversibility of $P_{\alpha,\theta\nu}$ with respect to $\mathcal{A}_{\alpha,\theta\nu}$ is the symmetry

$$E\left[\Phi \mathcal{A}_{\alpha,\theta\nu}\Phi'\right] = E\left[\Phi' \mathcal{A}_{\alpha,\theta\nu}\Phi\right], \qquad \Phi, \Phi' \in \mathcal{F}_0,$$

in which $E[\cdot]$ stands for the expectation with respect to $P_{\alpha,\theta\nu}$. (See the proof of Theorem 2.3 in [6].) In the rest of the proof we suppress the suffix '$\alpha, \theta\nu$' for simplicity. Let $f \in C(E)$ be given and define $\Phi_n(\mu) = \langle \mu, f\rangle^n$ for each positive integer n. We are going to calculate

$$\Delta := E\left[\Phi_2 \mathcal{A}\Phi_1\right] - E\left[\Phi_1 \mathcal{A}\Phi_2\right]. \tag{4.1}$$

For this purpose, observe from (3.22) that

$$\mathcal{A}\Phi_1(\mu) = \frac{\theta}{\alpha+1}(\langle \nu, f\rangle - \langle \mu, f\rangle), \tag{4.2}$$

$$\mathcal{A}\Phi_2(\mu) = \langle \mu, f^2\rangle + \frac{2\alpha\theta}{\alpha+1}\langle \nu, f\rangle\langle \mu, f\rangle + \frac{(1-\alpha)\theta}{\alpha+1}\langle \nu, f^2\rangle - (\theta+1)\langle \mu, f\rangle^2 \tag{4.3}$$

and

$$\begin{aligned}\Gamma(3)\mathcal{A}\Phi_3(\mu) &= 3(\alpha+1)\langle \mu, f^2\rangle\langle \mu, f\rangle + (1-\alpha)\langle \mu, f^3\rangle \\ &\quad + \frac{\theta}{\alpha+1}\cdot 3\alpha(\alpha+1)\langle \nu, f\rangle\langle \mu, f\rangle^2 + \frac{\theta}{\alpha+1}\cdot 3(1-\alpha)\alpha\langle \nu, f^2\rangle\langle \mu, f\rangle \\ &\quad + \frac{\theta}{\alpha+1}\cdot (1-\alpha)(2-\alpha)\langle \nu, f^3\rangle - (\alpha+2)(\theta+2)\langle \mu, f\rangle^3. \end{aligned} \tag{4.4}$$

Combining (4.2) with the stationarity $E[\mathcal{A}\Phi_1] = 0$, we get $E[\langle \mu, f\rangle] = \langle \nu, f\rangle$. Therefore, it is possible to deduce from (4.3) and $E[\mathcal{A}\Phi_2] = 0$

$$(\theta+1)E[\langle \mu, f\rangle^2] = \frac{2\alpha\theta}{\alpha+1}\langle \nu, f\rangle^2 + \left(1 + \frac{(1-\alpha)}{\alpha+1}\theta\right)\langle \nu, f^2\rangle.$$

More generally

$$(\theta+1)E[\langle \mu, f\rangle\langle \mu, g\rangle] = \frac{2\alpha\theta}{\alpha+1}\langle \nu, f\rangle\langle \nu, g\rangle + \left(1 + \frac{(1-\alpha)}{\alpha+1}\theta\right)\langle \nu, fg\rangle, \tag{4.5}$$

where $g \in C(E)$ is also arbitrary. In the rest of the proof we assume that $\langle \nu, f\rangle = 0$. This makes the calculations below considerably simple. By (4.5)

$$M_{1,2} := E[\langle \mu, f\rangle\langle \mu, f^2\rangle] = \frac{(\alpha+1) + (1-\alpha)\theta}{(\alpha+1)(\theta+1)}\langle \nu, f^3\rangle. \tag{4.6}$$

The equality $E[\mathcal{A}\Phi_3] = 0$ together with (4.4) implies that

$$(\alpha+2)(\theta+2)E[\langle \mu, f\rangle^3] = 3(\alpha+1)M_{1,2} + (1-\alpha)\left(1 + \frac{2-\alpha}{\alpha+1}\theta\right)\langle \nu, f^3\rangle. \tag{4.7}$$

These preliminaries help us calculate Δ in (4.1) as follows. By (4.3) and (4.4)

$$\begin{aligned}\Delta &= E\left[\langle\mu,f\rangle^2\left(-\frac{\theta}{\alpha+1}\langle\mu,f\rangle\right)\right]-E\left[\langle\mu,f\rangle\left(\langle\mu,f^2\rangle-(\theta+1)\langle\mu,f\rangle^2\rangle\right)\right]\\ &= \frac{(\alpha+1)+\alpha\theta}{\alpha+1}E[\langle\mu,f\rangle^3]-M_{1,2}\end{aligned}$$

and hence (4.7) yields

$$\begin{aligned}&(\alpha+1)(\alpha+2)(\theta+2)\Delta\\ &= [(\alpha+1)+\alpha\theta]\left[3(\alpha+1)M_{1,2}+(1-\alpha)\left(1+\frac{2-\alpha}{\alpha+1}\theta\right)\langle\nu,f^3\rangle\right]\\ &\quad -(\alpha+1)(\alpha+2)(\theta+2)M_{1,2}\\ &= (\alpha+1)(\alpha-1)(2\theta+1)M_{1,2}+[(\alpha+1)+\alpha\theta](1-\alpha)\left(1+\frac{2-\alpha}{\alpha+1}\theta\right)\langle\nu,f^3\rangle.\end{aligned}$$

Plugging (4.6) into this expression, we obtain

$$(\alpha+1)(\alpha+2)(\theta+2)\Delta=\frac{1-\alpha}{(\alpha+1)(\theta+1)}U(\alpha,\theta)\langle\nu,f^3\rangle,$$

where

$$\begin{aligned}U(\alpha,\theta) &= -(\alpha+1)(2\theta+1)[(\alpha+1)+(1-\alpha)\theta]\\ &\quad +[(\alpha+1)+\alpha\theta](\theta+1)[(\alpha+1)+(2-\alpha)\theta]\\ &= \alpha\theta^2[(\alpha+4)+(2-\alpha)\theta] \;=:\; V(\alpha,\theta).\end{aligned}$$

(The second equality between quadratic functions of α is verified by checking that $U(-1,\theta)=-3\theta^2(\theta+1)=V(-1,\theta)$, $U(0,\theta)=0=V(0,\theta)$ and $U(1,\theta)=\theta^2(\theta+5)=V(1,\theta)$.) Consequently, whenever $\langle\nu,f\rangle=0$, we have

$$\Delta=\frac{\alpha(1-\alpha)\theta^2[(\alpha+4)+(2-\alpha)\theta]}{(\alpha+1)^2(\alpha+2)(\theta+1)(\theta+2)}\langle\nu,f^3\rangle.$$

Thus, all that remains is to construct an $f\in C(E)$ such that $\langle\nu,f\rangle=0$ and $\langle\nu,f^3\rangle>0$. Because of the assumption, we can choose a closed subset E_0 of E such that $0<\nu(E_0)<1/2$. Indeed, in the case (ii) this is trivial while in the case (i) there exist disjoint closed subsets E_1,E_2 and E_3 of E such that $\nu(E_1)\nu(E_2)\nu(E_3)>0$ and so $0<\nu(E_i)<1/2$ for some $i\in\{1,2,3\}$. Letting g denote the indicator function of E_0, we observe that

$$\begin{aligned}\langle\nu,(g-\langle\nu,g\rangle)^3\rangle &= \langle\nu,g^3\rangle-3\langle\nu,g^2\rangle\langle\nu,g\rangle+3\langle\nu,g\rangle\langle\nu,g\rangle^2-\langle\nu,g\rangle^3\\ &= \nu(E_0)-3\nu(E_0)^2+2\nu(E_0)^3\\ &= \nu(E_0)(1-\nu(E_0))(1-2\nu(E_0))>0.\end{aligned}$$

Finally, the required f exists since g can be approximated boundedly and pointwise by a sequence of functions in $C(E)$. The proof of the theorem is complete. ∎

It is worth noting that the exceptional case of Theorem 4.1 corresponds to a subclass of the one-dimensional case discussed in Section 1, more specifically, the process generated by (1.3) with $c_1 = c_2$. There is no reason why this class is so special in respect of the reversibility, and it seems that such a 'spatial symmetry' makes it more subtle to see the asymmetry in time. The actual difficulty in showing the irreversibility for these processes along similar lines to the above proof is that expressions of $E[\Phi_{n_1}\mathcal{A}\Phi_{n_2}]$ with $n_1 + n_2 \geq 4$ as functions of α and θ are too complicated to handle.

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