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Tensors, monads and actions

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Abstract

We exhibit sufficient conditions for a monoidal monad $\mathbb{T}$ on a monoidal category C to induce a monoidal structure on the Eilenberg–Moore category $\mathsf{C}^{\mathbb{T}}$ that classifies bimorphisms. The category of actions in $\mathsf{C}^{\mathbb{T}}$ is then shown to be monadic over the base category C.

0 Introduction

The original motivation for the current work stemmed from the observation that the category C^M of M-actions in a monoidal category $(\mathsf{C}, \otimes, E)$ is *monadic* over its base (see Proposition 3.2.1 below). Monadicity also appears to be preserved when the base itself is an Eilenberg–Moore category: the category of R-modules, seen as a category of R-actions in the category $\mathsf{Set}^{\mathbb{A}} \cong \mathsf{AbGrp}$ of abelian groups, is monadic over Set; the category of actions of an integral quantale in the category $\mathsf{Set}^{\mathbb{P}} \cong \mathsf{Sup}$ of sup-semilattices is monadic over Set [15]; more recently, actions of the unit interval in the category $\mathsf{Set}^{\mathbb{F}} \cong \mathsf{Cnt}$ of continuous lattices have been shown to be monadic over Set [4]. Such a display suggest the following underlying principle:

> The category $(\mathsf{C}^{\mathbb{T}})^M$ of actions of a monoid M in a monoidal Eilenberg–Moore category $\mathsf{C}^{\mathbb{T}}$ is monadic over C.

In order to *define* actions in $\mathsf{C}^{\mathbb{T}}$, we first need a tensor $\boxtimes$ on $\mathsf{C}^{\mathbb{T}}$ that encodes the "bilinear" nature of the action morphism $M \boxtimes X \longrightarrow X$. The providential structure is provided by a *monoidal monad* on $(\mathsf{C}, \otimes, E)$ that facilitates the introduction of C-morphisms that are "$\mathbb{T}$-algebra homomorphisms in each variable", as originally suggested in [10].

Let us say a word on the technical setting we adopted for this work. In [6, 7, 8, 9], Kock presents the fundamentals of symmetric monoidal monads in a context of closed categories. However, closedness does not appear to play an explicit role in the classical construction of the tensor product on AbGrp or Sup. It also seemed reasonable to aim for an action morphism that occurs as an algebraic structure $M \otimes A \longrightarrow A$ on A, rather than as a morphism of monoids $M \longrightarrow [A, A]$ (where $[-, -]$ would designate an internal hom). Hence, we chose to follow [3] and consider a base category whose monoidal structure is neither assumed to be symmetric, nor closed. Alas, the result we needed in *op.cit.* to construct a monoidal structure on $\mathsf{C}^{\mathbb{T}}$ is presented with hypotheses that we found difficult to verify (moreover, the proposed construction of the unit isomorphism in $\mathsf{C}^{\mathbb{T}}$ seems a bit brusque). This lead us to our current version of Theorem 2.6.1 that, in turn, provided the necessary ingredients to prove the sought result in Theorem 3.3.3. Our work is thus structured as follows.

In Section 1, we establish the relevant definitions and notations pertaining to monoidal monads. We also recall that these are fundamentally linked to monoidal structures of Kleisli categories. In Section 2, we recall how bimorphisms and the tensor on $\mathsf{C}^{\mathbb{T}}$ induced by a monoidal monad $\mathbb{T}$ are related. Proposition 2.2.2 then explicits the link between the tensor proposed in [10] with the one studied in [3]. We also go over some useful facts about reflexive coequalizers, and show how these can be tensored in Proposition 2.5.2. This result is crucial to establish associativity of the tensor on $\mathsf{C}^{\mathbb{T}}$, a fact that is proved in Theorem 2.6.1. It then follows that the algebraic functors $\mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{C}^{\mathbb{S}}$ induced by monoidal monad morphisms $\mathbb{S} \longrightarrow \mathbb{T}$ are themselves monoidal. Once $\mathsf{C}^{\mathbb{T}}$ is equipped with the adequate monoidal structure, we turn our attention to actions in Section 3. Our main result is, as mentioned above, that the monadic functors $(\mathsf{C}^{\mathbb{T}})^M \longrightarrow \mathsf{C}^{\mathbb{T}}$ and $\mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{C}$ compose to form yet another monadic functor $(\mathsf{C}^{\mathbb{T}})^M \longrightarrow \mathsf{C}$. We conclude by showing that the classical restriction-of-scalars functor between categories of modules is algebraic.

Throughout the text, we illustrate the various notions introduced with the three examples mentioned above, that is, with structures related to the free abelian group, the powerset, and the filter monads. Example 2.6.3(5) also demonstrates that binary coproducts in $\mathsf{C}^{\mathbb{T}}$ can be interpreted as the tensor induced by binary coproducts in C. Of course, these examples are far from being exhaustive, but we feel that they adequately represent the concepts developed, while hinting at further applications. Other examples can be found in the references, or obtained by expanding upon the given ones.

1 Basic structures

1.1 Monoidal categories. Let C be a monoidal category, with its tensor denoted by $(-)\otimes(-) : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$, its unit by E, and its structure natural isomorphisms by

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)\ , \quad \lambda_X : E \otimes X \longrightarrow X\ , \quad \rho_X : X \otimes E \longrightarrow X\ .$$

When the monoidal category C is symmetric, we denote by $\sigma_{X,Y} : X \otimes Y \longrightarrow X \otimes Y$ the components of its braiding natural isomorphisms (see [13] or [2]). We will denote a monoidal category $(\mathsf{C}, \otimes, \alpha, E, \lambda, \rho)$ or $(\mathsf{C}, \otimes, \alpha, E, \lambda, \rho, \sigma)$ more briefly by $(\mathsf{C}, \otimes, E)$.

1.1.1 Example. The monoidal category recurrent in most of our examples is the category $(\mathsf{Set}, \times, \{\star\})$ of sets and maps with its cartesian structure.

1.2 Monoidal monads. Let $(\mathsf{C}, \otimes, E)$ be a monoidal category. A functor $T : \mathsf{C} \longrightarrow \mathsf{C}$, with a map $\eta_E : E \longrightarrow TE$, and a family of maps $\kappa = (\kappa_{X,Y} : TX \otimes TY \longrightarrow T(X \otimes Y))_{X,Y \in \mathrm{ob}\,\mathsf{C}}$ natural in X and Y, is *monoidal* if (T, η_E, κ) is compatible with the associativity and unitary natural transformations of $(\mathsf{C}, \otimes, E)$, so that one has for all $X, Y, Z \in \mathrm{ob}\,\mathsf{C}$:

(1) $\kappa_{X,Y\otimes Z}\cdot(1_{TX}\otimes\kappa_{Y,Z})\cdot\alpha_{TX,TY,TZ}=T\alpha_{X,Y,Z}\cdot\kappa_{X\otimes Y,Z}\cdot(\kappa_{X,Y}\otimes 1_{TZ})$, that is, the diagram

$$\begin{array}{ccccc}
(TX\otimes TY)\otimes TZ & \xrightarrow{\kappa\otimes 1} & T(X\otimes Y)\otimes TZ & \xrightarrow{\kappa} & T((X\otimes Y)\otimes Z)\\
{\scriptstyle\alpha}\downarrow & & & & \downarrow{\scriptstyle T\alpha}\\
TX\otimes(TY\otimes TZ) & \xrightarrow{1\otimes\kappa} & TX\otimes T(Y\otimes Z) & \xrightarrow{\kappa} & T(X\otimes(Y\otimes Z))
\end{array}$$

commutes;

(2) $T\lambda_X\cdot\kappa_{E,X}\cdot(\eta_E\otimes 1_{TX})=\lambda_{TX}$ and $T\rho_X\cdot\kappa_{X,E}\cdot(1_{TX}\otimes\eta_E)=\rho_{TX}$, that is, the diagrams

$$\begin{array}{ccccc}
E\otimes TX & \xrightarrow{\eta_E\otimes 1} & TE\otimes TX & \xrightarrow{\kappa} & T(E\otimes X)\\
 & {\scriptstyle\lambda}\searrow & & \swarrow{\scriptstyle T\lambda} & \\
 & & TX & &
\end{array}
\quad\text{and}\quad
\begin{array}{ccccc}
TX\otimes E & \xrightarrow{1\otimes\eta_E} & TX\otimes TE & \xrightarrow{\kappa} & T(X\otimes E)\\
 & {\scriptstyle\rho}\searrow & & \swarrow{\scriptstyle T\rho} & \\
 & & TX & &
\end{array}$$

commute.

A monad $\mathbb{T}=(T,\eta,\mu)$ is *monoidal* if the functor $T:\mathsf{C}\longrightarrow\mathsf{C}$ comes with a family of maps $\kappa=(\kappa_{X,Y}:TX\otimes TY\longrightarrow T(X\otimes Y))_{X,Y\in\mathrm{ob}\,\mathsf{C}}$, natural in X and Y, that make (T,η_E,κ) a monoidal functor, and such that the following conditions are satisfied for all $X,Y\in\mathrm{ob}\,\mathsf{C}$:

(3) $\kappa_{X,Y}\cdot(\eta_X\otimes\eta_Y)=\eta_{X\otimes Y}$, that is, the diagram

$$\begin{array}{ccc}
 & X\otimes Y & \\
{\scriptstyle\eta\otimes\eta}\swarrow & & \searrow{\scriptstyle\eta}\\
TX\otimes TY & \xrightarrow{\quad\kappa\quad} & T(X\otimes Y)
\end{array}$$

commutes;

(4) $\mu_{X\otimes Y}\cdot T\kappa_{X,Y}\cdot\kappa_{TX,TY}=\kappa_{X,Y}\cdot(\mu_X\otimes\mu_Y)$, that is, the diagram

$$\begin{array}{ccc}
TTX\otimes TTY & \xrightarrow{T\kappa\cdot\kappa} & TT(X\otimes Y)\\
{\scriptstyle\mu\otimes\mu}\downarrow & & \downarrow{\scriptstyle\mu}\\
TX\otimes TY & \xrightarrow{\quad\kappa\quad} & T(X\otimes Y)
\end{array}$$

commutes.

In the case where $(\mathsf{C},\otimes,E)$ is symmetric, the monoidal monad $(\mathbb{T},\kappa)$ is itself *symmetric* if the following condition is verified for all $X,Y\in\mathrm{ob}\,\mathsf{C}$:

(5) $\kappa_{Y,X}\cdot\sigma_{TX,TY}=T\sigma_{X,Y}\cdot\kappa_{X,Y}$, that is, the diagram

$$\begin{array}{ccc}
TX\otimes TY & \xrightarrow{\kappa} & T(X\otimes Y)\\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle T\sigma}\\
TY\otimes TX & \xrightarrow{\kappa} & T(Y\otimes X)
\end{array}$$

commutes.

1.2.1 Examples.

(1) The identity monad $\mathbb{I} = (1_{\mathsf{C}}, 1, 1)$ on any monoidal category C is a monoidal monad via the identity natural transformation $(1_{X\otimes Y} : X \otimes Y \longrightarrow X \otimes Y)_{X,Y\in \mathrm{ob}\,\mathsf{C}}$. The monoidal monad is symmetric whenever C is.

(2) The free abelian group monad $\mathbb{A} = (A, \sum, (-))$ on $(\mathsf{Set}, \times, \{\star\})$ is a symmetric monoidal monad via the natural transformation κ whose component $\kappa_{X,Y} : AX \times AY \longrightarrow A(X \times Y)$ sends a pair $\big(\sum_{x\in X} n_x \cdot x, \sum_{y\in Y} n_y \cdot y\big)$ (with all coefficients n_i integers) to the element $\sum_{x\in X, y\in Y}(n_x + n_y) \cdot (x, y)$.

(3) The powerset monad $\mathbb{P} = (P, \bigcup, \{-\})$ on Set (with its cartesian structure) is a symmetric monoidal monad via the natural transformation ι whose component $\iota_{X,Y} : PX \times PY \longrightarrow P(X \times Y)$ sends a pair of subsets (A, B) to their product $A \times B \subseteq X \times Y$.

(4) The filter monad $\mathbb{F} = (F, \bigcup\bigcap, \dot{(-)})$ on Set (see for example [12, Section 4.5]) is a symmetric monoidal monad with the natural transformation κ whose component $\kappa_{X,Y} : FX \times FY \longrightarrow F(X \times Y)$ sends a pair of filters $(\mathfrak{f}, \mathfrak{g})$ to the product filter $\mathfrak{f} \times \mathfrak{g}$, spanned by the set $\{A \times B \mid A \in \mathfrak{f}, B \in \mathfrak{g}\}$.

(5) Any monad $\mathbb{T}$ on a category C whose monoidal structure is given by finite coproducts (so $\otimes = +$ and $E = \varnothing$ is the initial object in C) is monoidal with respect to the connecting C-morphisms $\kappa_{A,B} : TA + TB \longrightarrow T(A + B)$.

Monoidal monads correspond to monoidal structures of the Kleisli category, as follows.

1.2.2 Proposition. *Given a monad $\mathbb{T}$ on a monoidal category C, there is a bijective correspondence between the following data:*

(1) *families $\kappa = (\kappa_{X,Y} : TX \otimes TY \longrightarrow T(X \otimes Y))_{X,Y\in \mathrm{ob}\,\mathsf{C}}$ of C-morphisms natural in X and Y making $\mathbb{T}$ a monoidal monad;*

(2) *monoidal structures on the Kleisli category $\mathsf{C}_{\mathbb{T}}$ such that the left adjoint functor $F_{\mathbb{T}} : \mathsf{C} \longrightarrow \mathsf{C}_{\mathbb{T}}$ is strict monoidal.*

Moreover, $(\mathbb{T}, \kappa)$ is symmetric precisely when the corresponding monoidal structure on $\mathsf{C}_{\mathbb{T}}$ is symmetric.

Proof. Given a C-morphism $\kappa_{Z,W}$, and C-morphisms $f : X \longrightarrow TZ$, $g : Y \longrightarrow TW$, one can define the C-morphism $f \otimes_{\mathbb{T}} g := \kappa_{Z,W} \cdot (f \otimes g) : X \otimes Y \longrightarrow T(Z \otimes W)$, thus equipping $\mathsf{C}_{\mathbb{T}}$ with a tensor $\otimes_{\mathbb{T}}$ for which $F_{\mathbb{T}}$ is strict monoidal. Conversely, if $\mathsf{C}_{\mathbb{T}}$ is monoidal with a tensor $\otimes_{\mathbb{T}}$, then strict monoidality of $F_{\mathbb{T}}$ forces equalities $X \otimes_{\mathbb{T}} Y = X \otimes Y$ for all C-objects X and Y, and one can define $\kappa_{X,Y} := 1_{TX} \otimes_{\mathbb{T}} 1_{TY}$. (See for example [3, Proposition 8].) $\square$

1.3 Monoidal monad morphisms. Let $(\mathbb{S}, \iota)$ and $(\mathbb{T}, \kappa)$ be monoidal monads on a monoidal category $(\mathsf{C}, \otimes, E)$. A monad morphism $\phi : \mathbb{S} \longrightarrow \mathbb{T}$ is *monoidal* if, for all $X, Y \in \mathrm{ob}\,\mathsf{C}$, the equality

$\phi_{X\otimes Y} \cdot \iota_{X,Y} = \kappa_{X,Y} \cdot (\phi_X \otimes \phi_Y)$ holds, or equivalently, the diagram

$$\begin{array}{ccc} SX \otimes SY & \xrightarrow{\iota} & S(X \otimes Y) \\ {\scriptstyle \phi\otimes\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ TX \otimes TY & \xrightarrow{\kappa} & T(X \otimes Y) \end{array}$$

commutes.

1.3.1 Examples.

(1) For any monoidal monad $(\mathbb{T}, \kappa)$ on $(\mathsf{C}, \otimes, E)$, the unit $\eta : (\mathbb{I}, 1) \longrightarrow (\mathbb{T}, \kappa)$ is a monoidal monad morphism.

(2) The principal filter monad morphism $\phi : (\mathbb{P}, \iota) \longrightarrow (\mathbb{F}, \kappa)$ from the powerset to the filter monad is monoidal (see Examples 1.2.1(3) and (4)).

The correspondence of Proposition 1.2.2 extends to morphisms.

1.3.2 Proposition. *Given a monad $\mathbb{T}$ on a monoidal category C, there is a bijective correspondence between the following data:*

(1) *monoidal monad morphisms $\phi : \mathbb{S} \longrightarrow \mathbb{T}$;*

(2) *strict monoidal functors $L : \mathsf{C}_{\mathbb{S}} \longrightarrow \mathsf{C}_{\mathbb{T}}$ that commute with the left adjoint functors from C:*

$$\begin{array}{ccc} \mathsf{C}_{\mathbb{S}} & \xrightarrow{\quad L \quad} & \mathsf{C}_{\mathbb{T}} \\ & {\scriptstyle F_{\mathbb{S}}}\nwarrow \quad \nearrow{\scriptstyle F_{\mathbb{T}}} & \\ & \mathsf{C} & . \end{array}$$

Proof. The one-to-one correspondence between monad morphisms and functors between the Kleisli categories is standard (see for example [14, Theorem 2.2.2]): a monad morphism $\phi : \mathbb{S} \longrightarrow \mathbb{T}$ defines a functor $L : \mathsf{C}_{\mathbb{S}} \longrightarrow \mathsf{C}_{\mathbb{T}}$ that is identical on objects and sends a $\mathsf{C}_{\mathbb{S}}$-morphism $f : X \longrightarrow SY$ to the $\mathsf{C}_{\mathbb{T}}$-morphism $\phi_Y \cdot f : X \longrightarrow TY$; conversely, a functor L as in (2) defines a monad morphism $\phi : \mathbb{S} \longrightarrow \mathbb{T}$ via its components $\phi_X := L(1_{SX}) : SX \longrightarrow TY$. One easily verifies that if ϕ is monoidal, then L is strict monoidal, and that the converse holds, too. □

2 The monoidal structure of $\mathsf{C}^{\mathbb{T}}$

The prototypical tensor product that we wish to study is provided by the tensor product of R-modules. The role of this tensor is to classify bilinear maps. In our setting, the monoidal structure of the monad facilitates the introduction of the notion of such a "morphism in each variable" (as suggested in [10] and [7]).

2.1 Bimorphisms. Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C}, \otimes, E)$, and denote by $\mathsf{C}^{\mathbb{T}}$ its category of Eilneberg-Moore algebras. For $\mathbb{T}$-algebras (A, a), (B, b), and (C, c), we say that a C-morphism $f : A \otimes B \longrightarrow C$ is a $\mathsf{C}^{\mathbb{T}}$-*bimorphism*[1] (or simply a *bimorphism*) if the diagram

$$\begin{array}{ccccc} TA \otimes TB & \xrightarrow{\kappa} & T(A \otimes B) & \xrightarrow{Tf} & TC \\ {\scriptstyle a\otimes b}\downarrow & & & & \downarrow{\scriptstyle c} \\ A \otimes B & & \xrightarrow{\quad f \quad} & & C \end{array} \tag{2.1.i}$$

commutes. The set of all bimorphisms $f : (A, a) \otimes (B, b) \longrightarrow (C, c)$ is denoted by $\mathsf{C}^{\mathbb{T}}(A, B; C)$.

2.1.1 Examples.

(1) For the identity monad $\mathbb{I} = (1_{\mathsf{C}}, 1, 1)$ on a monoidal category C, with the identity natural transformation, $\mathsf{C}^{\mathbb{I}}$-bimorphisms are just C-morphisms since $\mathsf{C}^{\mathbb{I}} \cong \mathsf{C}$.

(2) For the free abelian group monad $\mathbb{A}$, the category $\mathsf{Set}^{\mathbb{A}}$ is isomorphic to AbGrp, the category of abelian groups. Via the natural transformation κ of Example 1.2.1(2), a $\mathsf{Set}^{\mathbb{A}}$-bimorphism $f : A \times B \longrightarrow C$ is a map that is additive in each variable.

(3) For the powerset monad $\mathbb{P}$ on Set with the natural transformation ι of Example 1.2.1(3), the Eilneberg-Moore category $\mathsf{Set}^{\mathbb{P}}$ is isomorphic with Sup, the category of complete sup-semilattices. With this interpretation, a $\mathsf{Set}^{\mathbb{P}}$-bimorphism $f : A \times B \longrightarrow C$ is a map that preserves suprema in each variable.

(4) For the filter monad $\mathbb{F}$ on Set with the natural transformation κ of Example 1.2.1(4), $\mathsf{Set}^{\mathbb{F}}$ is isomorphic to the category Cnt of continuous lattices (here, we consider that a sup-semilattice X is continuous if the infima map $\bigwedge : \mathrm{Fil}\, X \longrightarrow X$ from the set $\mathrm{Fil}\, X$ of down-directed sets in X has a right adjoint). In this case, a $\mathsf{Set}^{\mathbb{F}}$-bimorphism $f : A \times B \longrightarrow C$ is a map that preserves suprema and filtered infima in each variable.

The next result shows in what sense a bimorphism captures the idea of a "morphism in each variable".

2.1.2 Proposition. *Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C}, \otimes, E)$. For $\mathbb{T}$-algebras (A, a), (B, b), and (C, c), a C-morphism $f : A \otimes B \longrightarrow C$ is a $\mathsf{C}^{\mathbb{T}}$-bimorphism if and only if both diagrams*

$$\begin{array}{ccccc} A \otimes TB & \xrightarrow{\kappa\cdot(\eta\otimes 1)} & T(A \otimes B) & \xrightarrow{Tf} & TC \\ {\scriptstyle 1\otimes b}\downarrow & & & & \downarrow{\scriptstyle c} \\ A \otimes B & & \xrightarrow{\quad f \quad} & & C \end{array} \quad \textit{and} \quad \begin{array}{ccccc} TA \otimes B & \xrightarrow{\kappa\cdot(1\otimes\eta)} & T(A \otimes B) & \xrightarrow{Tf} & TC \\ {\scriptstyle a\otimes 1}\downarrow & & & & \downarrow{\scriptstyle c} \\ A \otimes B & & \xrightarrow{\quad f \quad} & & C \end{array}$$

commute.

[1] These bimorphisms should not be confused with the C-morphisms that are at the same time monic and epic. Since we do not consider the latter, there is little reason to avoid the bimorphism terminology.

Proof. If f is a $\mathsf{C}^{\mathbb{T}}$-bimorphism, then one can append each of the following commutative diagrams

$$\begin{array}{ccc} A\otimes TB & \xrightarrow{\eta\otimes 1} & TA\otimes TB \\ & {\scriptstyle 1\otimes b}\searrow & \downarrow{\scriptstyle a\otimes b} \\ & & A\otimes B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} TA\otimes B & \xrightarrow{1\otimes \eta} & TA\otimes TB \\ & {\scriptstyle a\otimes 1}\searrow & \downarrow{\scriptstyle a\otimes b} \\ & & A\otimes B \end{array}$$

to the left of (2.1.i) to obtain the respective diagrams of the statement. Conversely, if the two diagrams of the statement commute, then

$$\begin{aligned} f\cdot(a\otimes b) &= f\cdot(1_A\otimes b)\cdot(a\otimes 1_{TB}) \\ &= c\cdot Tf\cdot\kappa_{A,B}\cdot(\eta_A\otimes 1_{TB})\cdot(a\otimes 1_{TB}) \\ &= c\cdot Tf\cdot\kappa_{A,B}\cdot(Ta\otimes T1_B)\cdot(\eta_{TA}\otimes 1_{TB}) \\ &= c\cdot T(f\cdot(a\otimes 1_B))\cdot\kappa_{A,B}\cdot(\eta_{TA}\otimes 1_{TB}) \\ &= c\cdot T(c\cdot Tf\cdot\kappa_{A,B}\cdot(1_{TA}\otimes\eta_B))\cdot\kappa_{A,B}\cdot(\eta_{TA}\otimes 1_{TB}) \\ &= c\cdot Tf\cdot\mu_{A\otimes B}\cdot T\kappa_{A,B}\cdot T(1_{TA}\otimes\eta_B)\cdot\kappa_{A,B}\cdot(\eta_{TA}\otimes 1_{TB}) \\ &= c\cdot Tf\cdot\mu_{A\otimes B}\cdot T\kappa_{A,B}\cdot\kappa_{TA,TB}\cdot(T1_{TA}\otimes T\eta_B)\cdot(\eta_{TA}\otimes 1_{TB}) \\ &= c\cdot Tf\cdot\kappa_{A,B}\cdot(\mu_A\otimes\mu_B)\cdot(\eta_{TA}\otimes T\eta_B) = c\cdot Tf\cdot\kappa_{A,B}\ . \end{aligned}$$

This shows commutativity of (2.1.i). □

2.1.3 Proposition. *Let* $(\mathbb{T},\kappa)$ *be a monoidal monad on a monoidal category* $(\mathsf{C},\otimes,E)$*. Then*

$$\mathsf{C}^{\mathbb{T}}(-,-;-):(\mathsf{C}^{\mathbb{T}}\times\mathsf{C}^{\mathbb{T}})^{\mathrm{op}}\times\mathsf{C}^{\mathbb{T}}\longrightarrow\mathsf{Set}$$

is a functor defined by

$$\mathsf{C}^{\mathbb{T}}(g,h;k)(f):=k\cdot f\cdot(g\otimes h)$$

for all $\mathsf{C}^{\mathbb{T}}$*-morphisms* $g:A'\longrightarrow A$*,* $h:B'\longrightarrow B$*,* $k:C\longrightarrow C'$*, and bimorphisms* $f:A\otimes B\longrightarrow C$*.*

Proof. The given functor is well-defined. Indeed, for C-morphisms f, g, h, k as in the claim (and $\mathbb{T}$-algebras (A,a), (B,b), (C,c), (A',a'), (B',b'), (C',c')), one has

$$\begin{aligned} c'\cdot T(k\cdot f\cdot(g\otimes h))\cdot\kappa_{A',B'} &= k\cdot c\cdot Tf\cdot\kappa_{A,B}\cdot(Tg\otimes Th) \\ &= k\cdot f\cdot(a\otimes b)\cdot(Tg\otimes Th) = k\cdot f\cdot(g\otimes h)\cdot(a'\otimes b')\ , \end{aligned}$$

so that $\mathsf{C}^{\mathbb{T}}(g,h;k)f$ is a bimorphism in $\mathsf{C}^{\mathbb{T}}(A',B';C')$. Functoriality is immediate by monoidality of C. □

2.2 Tensor products. Let $(\mathbb{T},\kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C},\otimes,E)$ such that $\mathsf{C}^{\mathbb{T}}$ has reflexive coequalizers. The *tensor product* $(A\boxtimes B, a\bowtie b)$ of $\mathbb{T}$-algebras (A,a) and (B,b) is given by the following coequalizer $q_{A,B}$

$$T(TA\otimes TB)\underset{T(a\otimes b)}{\overset{\mu\cdot T\kappa}{\rightrightarrows}}T(A\otimes B)\xrightarrow{\quad q\quad}A\boxtimes B$$

in $\mathsf{C}^{\mathbb{T}}$. For $\mathsf{C}^{\mathbb{T}}$-morphisms $g : (A, a) \longrightarrow (A', a')$, $h : (B, b) \longrightarrow (B', b')$, one has

$$\begin{aligned} q_{A',B'} \cdot T(g \otimes h) \cdot \mu_{A\otimes B} \cdot T\kappa_{A,B} &= q_{A',B'} \cdot \mu_{A',B'} \cdot TT(g \otimes h) \cdot T\kappa_{A,B} \\ &= q_{A',B'} \cdot \mu_{A',B'} \cdot T\kappa_{A',B'} \cdot T(Tg \otimes Th) \\ &= q_{A',B'} \cdot T(a' \otimes b') \cdot T(Tg \otimes Th) = q_{A',B'} \cdot T(g \otimes h) \cdot T(a \otimes b) \ , \end{aligned}$$

so there is a unique $\mathsf{C}^{\mathbb{T}}$-morphism $g \boxtimes h : A \boxtimes B \longrightarrow A' \boxtimes B'$ with $(g \boxtimes h) \cdot q_{A,B} = q_{A',B'} \cdot T(g \otimes h)$, and the diagram

$$\begin{array}{ccc} T(A\otimes B) & \xrightarrow{\ q\ } & A \boxtimes B \\ {\scriptstyle T(g\otimes h)}\downarrow & & \downarrow{\scriptstyle g\boxtimes h} \\ T(A'\otimes B') & \xrightarrow{\ q\ } & A' \boxtimes B' \end{array}$$

commutes.

2.2.1 Convention. For the sake of convenience, from now on we say that a diagram of the form

$$X \underset{h}{\overset{g}{\rightrightarrows}} Y \xrightarrow{\ f\ } Z$$

commutes if $f \cdot g = f \cdot h$.

2.2.2 Proposition. *Given a monoidal monad* $(\mathbb{T}, \kappa)$ *on a monoidal category* $(\mathsf{C}, \otimes, E)$ *and* $\mathbb{T}$-*algebras* (A, a), (B, b), *the following statements are equivalent for all* C-*morphisms* $f : T(A \otimes B) \longrightarrow C$:

(i) *the diagram* $T(TA \otimes TB) \underset{T(a\otimes b)}{\overset{\mu\cdot T\kappa}{\rightrightarrows}} T(A \otimes B) \xrightarrow{\ f\ } C$ *commutes;*

(ii) *the diagram* $T(TA \otimes TB) \underset{\mu\cdot T(\kappa\cdot(1\otimes\eta\cdot b))}{\overset{\mu\cdot T(\kappa\cdot(\eta\cdot a\otimes 1))}{\rightrightarrows}} T(A \otimes B) \xrightarrow{\ f\ } C$ *commutes.*

Proof. Suppose that $f \cdot \mu_{A\otimes B} \cdot T\kappa_{A,B} = f \cdot T(a \otimes b)$. Then one immediately obtains

$$f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (\eta_A \cdot a \otimes 1_{TB})) = f \cdot T(a \otimes b) = f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (1_{TA} \otimes \eta_B \cdot b)) \ .$$

Conversely, suppose that $f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (\eta_A \cdot a \otimes 1_{TB})) = f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (1_{TA} \otimes \eta_B \cdot b))$ holds. One first notes

$$\begin{aligned} f \cdot T(a \otimes b) &= f \cdot \mu_{A\otimes B} \cdot T\eta_{A\otimes B} \cdot T(a \otimes b) \\ &= f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (\eta_A \otimes \eta_B) \cdot (a \otimes b)) \\ &= f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (\eta_A \cdot a \otimes 1_{TB}) \cdot (1_{TA} \otimes \eta_B \cdot b)) \\ &= f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (1_{TA} \otimes \eta_B \cdot b) \cdot (1_{TA} \otimes \eta_B \cdot b)) \\ &= f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (1_{TA} \otimes \eta_B \cdot b)) \ , \end{aligned}$$

so

$$f \cdot T(a \otimes 1_B) = f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (1_{TA} \otimes \eta_B)) \qquad \text{and}$$
$$f \cdot T(a \otimes b) = f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (\eta_A \cdot a \otimes 1_{TB}))\ .$$

These equalities, together with the fact that

$$\begin{aligned}\kappa_{A,B} &= \kappa_{A,B} \cdot (\mu_A \otimes \mu_B) \cdot (\eta_{TA} \otimes T\eta_B)\\ &= \mu_{A\otimes B} \cdot T\kappa_{A,B} \cdot \kappa_{TA,TB} \cdot (1_{TTA} \otimes T\eta_B) \cdot (\eta_{TA} \otimes 1_{TB})\\ &= \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (1_{TA} \otimes \eta_B)) \cdot \kappa_{TA,B} \cdot (\eta_{TA} \otimes 1_{TB})\end{aligned}$$

yield

$$\begin{aligned}f \cdot \mu_{A\otimes B} \cdot T\kappa_{A,B} &= f \cdot \mu_{A\otimes B} \cdot T(\mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (1_{TA} \otimes \eta_B)) \cdot \kappa_{TA,B} \cdot (\eta_{TA} \otimes 1_{TB}))\\ &= f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (1_{TA} \otimes \eta_B)) \cdot \mu_{TA\otimes TB} \cdot T(\kappa_{TA,B} \cdot (\eta_{TA} \otimes 1_{TB}))\\ &= f \cdot T(a \otimes 1_B) \cdot \mu_{TA\otimes TB} \cdot T(\kappa_{TA,B} \cdot (\eta_{TA} \otimes 1_{TB}))\\ &= f \cdot \mu_{A\otimes B} \cdot T(\kappa_{A,B} \cdot (\eta_A \cdot a \otimes 1_{TB})) = f \cdot T(a \otimes b)\ ,\end{aligned}$$

as required. □

2.2.3 Corollary. *Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C}, \otimes, E)$ such that $\mathsf{C}^{\mathbb{T}}$ has reflexive coequalizers. Then for $\mathbb{T}$-algebras (A, a) and (B, b), the diagram*

$$T(TA \otimes TB) \underset{\mu\cdot T(\kappa\cdot(1\otimes\eta\cdot b))}{\overset{\mu\cdot T(\kappa\cdot(\eta\cdot a\otimes 1))}{\rightrightarrows}} T(A \otimes B) \xrightarrow{\quad q \quad} A \boxtimes B$$

is a coequalizer in $\mathsf{C}^{\mathbb{T}}$ (with $q_{A,B}$ as defined in 2.2).

Proof. This is an immediate consequence of the previous Proposition. □

2.2.4 Remarks.

(1) Corollary 2.2.3 confirms that the coequalizer suggested to define a tensor in [10, Remark, Section 1] is the same as the one appearing in [3, Proposition 16].

(2) As remarked by Linton [11], the study of bimorphisms trivializes for certain monads. For example, suppose that $\mathbb{T}$ is a monoidal monad on $(\mathsf{Set}, \times, \{\star\})$ whose $\mathbb{T}$-algebras X have two nullary operations $0, 1 : \{\star\} \longrightarrow X$ and a binary one $(-) * (-) : X \times X \longrightarrow X$ such that

$$0 * x = 0 \qquad \text{and} \qquad 1 * x = x$$

for all $x \in X$. If $f : A \times B \longrightarrow C$ is a bimorphism, then

$$f(a, 1) = 1 \qquad \text{and} \qquad f(0, b) = 0$$

for all $a \in A$, $b \in B$, so that $1 = f(0, 1) = 0$. Hence, if $c \in C$, then $c = 1 * c = 0 * c = 0$. That is, the only bimorphisms $f : A \times B \longrightarrow C$ are those for which C is a singleton. (See also Remark 2.3.5.)

2.3 Classifying bimorphisms. A major motivation to the introduction of tensor products in cateogories of R-modules is the classification of bimorphisms. Here, we show that the tensor product of 2.2 plays that role with respect to the bimorphisms of 2.1.

2.3.1 Proposition. *Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C}, \otimes, E)$ such that $\mathsf{C}^{\mathbb{T}}$ has reflexive coequalizers. Then*

$$\mathsf{C}^{\mathbb{T}}(- \boxtimes -, -) : (\mathsf{C}^{\mathbb{T}} \times \mathsf{C}^{\mathbb{T}})^{\mathrm{op}} \times \mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{Set}$$

is a functor.

Proof. Immediate by functoriality of $(- \otimes -)$ and T (see 2.2 and Proposition 2.1.3). □

The relation between bimorphisms and the tensor product is given by the following result.

2.3.2 Lemma. *Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C}, \otimes, E)$. For $\mathbb{T}$-algebras (A, a), (B, b), (C, c), the following statements are equivalent for any C-morphism $f : A \otimes B \longrightarrow C$:*

(i) *f is a bimorphism;*

(ii) *The diagram* $T(TA \otimes TB) \underset{T(a \otimes b)}{\overset{\mu \cdot T\kappa}{\rightrightarrows}} T(A \otimes B) \xrightarrow{c \cdot Tf} C$ *commutes.*

Proof. If f is a bimorphism, then

$$c \cdot Tf \cdot \mu_{A \otimes B} \cdot T\kappa_{A,B} = c \cdot Tc \cdot TTf \cdot T\kappa_{A,B} = c \cdot Tf \cdot T(a \otimes b)\ .$$

Conversely, if the diagram in (ii) commutes, then

$$c \cdot Tf \cdot \kappa_{A,B} = c \cdot Tf \cdot \mu_{A \otimes B} \cdot \eta_{T(A \otimes B)} \cdot \kappa_{A,B} = c \cdot Tf \cdot T(a \otimes b) \cdot \eta_{TA \otimes TB} = f \cdot (a \otimes b)\ ,$$

so f is a bimorphism. □

2.3.3 Lemma. *Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C}, \otimes, E)$ such that $\mathsf{C}^{\mathbb{T}}$ has reflexive coequalizers. For $\mathbb{T}$-algebras (A, a), (B, b), and (C, c), the following statements hold.*

(1) *If $f : A \otimes B \longrightarrow C$ is a bimorphism, then there is a unique $\mathsf{C}^{\mathbb{T}}$-morphism $\overline{f} : A \boxtimes B \longrightarrow C$ such that*

$$\begin{array}{ccc} T(A \otimes B) & \xrightarrow{Tf} & TC \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle c} \\ A \boxtimes B & \xrightarrow{\overline{f}} & C \end{array}$$

commutes.

(2) *If $g : A \boxtimes B \longrightarrow C$ is a $\mathsf{C}^{\mathbb{T}}$-morphism, then $g \cdot q_{A,B} \cdot \eta_{A \otimes B} : A \otimes B \longrightarrow C$ is a bimorphism that induces g, that is, $\overline{g \cdot q_{A,B} \cdot \eta_{A \otimes B}} = g$.*

Proof. Since $c \cdot Tf$ is a $\mathsf{C}^{\mathbb{T}}$-morphism, the first claim follows directly from Lemma 2.3.2 and the universal property of the coequalizer $q_{A,B}$.

Given a $\mathsf{C}^{\mathbb{T}}$-morphism $g : A \boxtimes B \longrightarrow C$, one uses that $q_{A,B}$ is a coequalizer and that $g \cdot q_{A,B} : (T(A \otimes B), \mu_{A\otimes B}) \longrightarrow (C, c)$ is a $\mathsf{C}^{\mathbb{T}}$-morphism to write

$$\begin{aligned} g \cdot q_{A,B} \cdot \eta_{A\otimes B} \cdot (a \otimes b) &= g \cdot q_{A,B} \cdot T(a \otimes b) \cdot \eta_{TA\otimes TB} \\ &= g \cdot q_{A,B} \cdot \mu_{A\otimes B} \cdot T\kappa_{A,B} \cdot \eta_{TA\otimes TB} \\ &= g \cdot q_{A,B} \cdot \mu_{A\otimes B} \cdot \eta_{T(A\otimes B)} \cdot \kappa_{A,B} \\ &= g \cdot q_{A,B} \cdot \mu_{A\otimes B} \cdot T\eta_{A\otimes B} \cdot \kappa_{A,B} \\ &= c \cdot T(g \cdot q_{A,B}) \cdot T\eta_{A\otimes B} \cdot \kappa_{A,B} = c \cdot T(g \cdot q_{A,B} \cdot \eta_{A\otimes B}) \cdot \kappa_{A,B} \ , \end{aligned}$$

showing that $g \cdot q_{A,B} \cdot \eta_{A\otimes B}$ is a bimorphism. Since

$$c \cdot T(g \cdot q_{A,B} \cdot \eta_{A\otimes B}) = g \cdot q_{A,B} \cdot \mu_{A\otimes B} \cdot T\eta_{A\otimes B} = g \cdot q_{A,B} \ ,$$

the $\mathsf{C}^{\mathbb{T}}$-morphism induced by $g \cdot q_{A,B} \cdot \eta_{A\otimes B}$ is indeed g. □

2.3.4 Proposition. *Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C}, \otimes, E)$ such that $\mathsf{C}^{\mathbb{T}}$ has reflexive coequalizers. For $\mathbb{T}$-algebras (A, a), (B, b), and (C, c), there is a bijection*

$$\mathsf{C}^{\mathbb{T}}(A, B; C) \cong \mathsf{C}^{\mathbb{T}}(A \boxtimes B, C)$$

natural in each variable.

Proof. The required bijection is described in Lemma 2.3.3. Indeed, if $g : A \boxtimes B \longrightarrow C$ is a $\mathsf{C}^{\mathbb{T}}$-morphism, then $f = g \cdot q_{A,B} \cdot \eta_{A\otimes B}$ is a bimorphism, to which corresponds the unique $\mathsf{C}^{\mathbb{T}}$-morphism $g : A \boxtimes B \longrightarrow C$. Conversely, if $f : A \otimes B \longrightarrow C$ is a bimorphism, then there is a $\mathsf{C}^{\mathbb{T}}$-morphism $\overline{f} : A \boxtimes B \longrightarrow C$ such that $c \cdot Tf = \overline{f} \cdot q_{A,B}$; according to Lemma 2.3.3, one obtains in return a bimorphism

$$g = \overline{f} \cdot q_{A,B} \cdot \eta_{A\otimes B} = c \cdot Tf \cdot \eta_{A\otimes B} = c \cdot \eta_C \cdot f = f \ .$$

For a $\mathsf{C}^{\mathbb{T}}$-bimorphism $f : A \otimes B \longrightarrow C$, and $\mathsf{C}^{\mathbb{T}}$-morphisms $g : A' \longrightarrow A$, $h : B' \longrightarrow B$, $k : C \longrightarrow C'$, one has

$$k \cdot \overline{f} \cdot (g \boxtimes h) \cdot q_{A',B'} = k \cdot \overline{f} \cdot q_{A,B} \cdot T(g \otimes h) = k \cdot c \cdot Tf \cdot T(g \otimes h) = c' \cdot Tk \cdot Tf \cdot T(g \otimes h) \ ,$$

that is,

$$\overline{k \cdot f \cdot (g \otimes h)} = k \cdot \overline{f} \cdot (g \boxtimes h)$$

(by unicity of the induced $\mathsf{C}^{\mathbb{T}}$-morphism). Hence, the bijection is natural. □

2.3.5 Remark. In the trivial cases where the only bimorphisms are C-morphisms $f : A \otimes B \longrightarrow T$ into the terminal object of C (as in Remark 2.2.4(2)), Proposition 2.3.4 shows that $A \boxtimes B \cong T$: the identity $\mathsf{C}^{\mathbb{T}}$-morphism $1_{A\boxtimes B}$ corresponds to a bimorphism $f : A \otimes B \longrightarrow A \boxtimes B$, so that $A \boxtimes B \cong T$.

2.4 Reflexive coequalizers. We recall here some basic results pertaining to reflexive coequalizers, and thus applying to the coequalizer q defined in 2.2.

2.4.1 Proposition. *A functor $F : \mathsf{X} \times \mathsf{Y} \longrightarrow \mathsf{Z}$ preserves reflexive coequalizers if and only if $F(X, -) : \mathsf{Y} \longrightarrow \mathsf{Z}$ and $F(-, Y) : \mathsf{X} \longrightarrow \mathsf{Z}$ preserves reflexive coequalizers for all $X \in \operatorname{ob}\mathsf{X}$, $Y \in \operatorname{ob}\mathsf{Y}$.*

Proof. The necessity of the statement is immediate since $F(X, g) = F(1_X, g)$ and $F(f, Y) = F(f, 1_Y)$ for all $X \in \operatorname{ob}\mathsf{X}$, $Y \in \operatorname{ob}\mathsf{Y}$. For the sufficiency, suppose that

$$A \underset{f_2}{\overset{f_1}{\rightrightarrows}} X \xrightarrow{\ f\ } P \qquad \text{and} \qquad B \underset{g_2}{\overset{g_1}{\rightrightarrows}} Y \xrightarrow{\ g\ } Q$$

are reflexive coequalizer diagrams in X and Y respectively, and consider a Z-morphism $h : F(X, Y) \longrightarrow Z$ such that $h \cdot F(f_1, g_1) = h \cdot F(f_2, g_2)$. Since g_1 and g_2 are split epimorphisms, one has $h \cdot F(f_1, Y) = h \cdot F(f_2, Y)$, so there exists a unique Z-morphism $\overline{h} : F(P, Y) \longrightarrow Z$ with $\overline{h} \cdot F(f, Y) = h$. Similarly, f_1 and f_2 are split epimorphisms, so $h \cdot F(X, g_1) = h \cdot F(X, g_2)$. Since the inner and outer squares in the diagram

$$\begin{array}{ccccc}
F(X,B) & \underset{F(X,g_2)}{\overset{F(X,g_1)}{\rightrightarrows}} & F(X,Y) & \overset{h}{\searrow} & \\
{\scriptstyle F(f,B)}\downarrow & & \downarrow{\scriptstyle F(f,Y)} & & \\
F(P,B) & \underset{F(P,g_1)}{\overset{F(P,g_2)}{\rightrightarrows}} & F(P,Y) & \xrightarrow{\ \overline{h}\ } & Z
\end{array}$$

commute, one has $\overline{h} \cdot F(P, g_1) = \overline{h} \cdot F(P, g_2)$ because $F(f, B)$ is epic. The universal property of $F(P, g)$ then yields a unique Z-morphism $\overline{\overline{h}} : F(P, Q) \longrightarrow Z$ with $\overline{\overline{h}} \cdot F(P, g) = \overline{h}$. Hence, $\overline{\overline{h}}$ satisfies

$$\overline{\overline{h}} \cdot F(f, g) = \overline{\overline{h}} \cdot F(P, g) \cdot F(f, Y) = \overline{h} \cdot F(f, Y) = h\ .$$

Unicity of the connecting Z-morphism $\overline{\overline{h}}$ follows from the hypothesis that $F(f, Y)$ and $F(P, g)$ are both coequalizers, and therefore epimorphisms. □

We recall that the forgetful functor $\mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{C}$ creates all coequalizers that are preserved by T.

2.4.2 Proposition. *Let $\mathbb{T}$ be a monad on C, and consider $\mathsf{C}^{\mathbb{T}}$-morphisms $f, g : (Z, z) \longrightarrow (Y, y)$. Suppose that the coequalizer $p : Y \longrightarrow X$ of (f, g) exists in C, that Tp is a C-coequalizer of (Tf, Tg), and that TTp is an epimorphism in C. Then there exists a unique C-morphism $x : TX \longrightarrow X$ such that*

$$x \cdot Tp = p \cdot y\ .$$

In fact, x defines a $\mathbb{T}$-algebra structure on X, and p becomes a coequalizer of (f, g) in $\mathsf{C}^{\mathbb{T}}$.

Proof. See [10, Proposition 3]. □

2.4.3 Corollary. *Let $\mathbb{T}$ be a monad on C. If C has reflexive coequalizers and T preserves them, then the forgetful functor $\mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{C}$ creates them.*

In particular, under the stated hypotheses the coequalizer $q_{A,B} : T(A \otimes B) \longrightarrow A \boxtimes B$ of $\mu_{A\otimes B} \cdot T\kappa_{A,B}$ and $T(a \otimes b)$ exists in $\mathsf{C}^{\mathbb{T}}$.

Proof. The result is an immediate application of Proposition 2.4.2. □

2.5 Tensoring coequalizers. The central argument in the proof of the associativity of the tensor $(-) \boxtimes (-)$ is that the coequalizer q facilitates the tensoring of coequalizers in $\mathsf{C}^{\mathbb{T}}$.

2.5.1 Lemma. *Let $\mathbb{T}$ be a monad on a category C. For C-morphisms $r : Z \longrightarrow TY$, $s : Z \longrightarrow Y$, and $p : TY \longrightarrow X$,*

$$(p \cdot \mu_Y \cdot Tr = p \cdot Ts) \implies (p \cdot r = p \cdot \eta_Y \cdot s) \ .$$

Proof. One simply has

$$p \cdot r = p \cdot \mu_Y \cdot \eta_{TY} \cdot r = p \cdot \mu_Y \cdot Tr \cdot \eta_Z = p \cdot Ts \cdot \eta_Z = p \cdot \eta_Y \cdot s \ .$$

□

2.5.2 Proposition. *Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C}, \otimes, E)$, and consider coequalizer diagrams*

$$TZ \underset{Ts}{\overset{\mu \cdot Tr}{\rightrightarrows}} TY \xrightarrow{\ p\ } X \qquad \text{and} \qquad TZ' \underset{Ts'}{\overset{\mu \cdot Tr'}{\rightrightarrows}} TY' \xrightarrow{\ p'\ } X'$$

in $\mathsf{C}^{\mathbb{T}}$ (here, (X, x), (X', x') are $\mathbb{T}$-algebras, and the objects of the form TA are equipped with their free structure μ_A). Suppose that $T(- \otimes -) : \mathsf{C}^{\mathbb{T}} \times \mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{C}^{\mathbb{T}}$ preserves the coequalizer diagram of (p, p') and that $T(Tp \otimes Tp')$ is an epimorphism.

If the coequalizer of $\mu_{Y \otimes Y'} \cdot T(\kappa_{Y,Y'} \cdot (r \otimes r'))$ and $T(s \otimes s')$ exists $\mathsf{C}^{\mathbb{T}}$, then it can be given by $q_{X,X'} \cdot T(p \cdot \eta_Y \otimes p' \cdot \eta_{Y'})$:

$$T(Z \otimes Z') \underset{T(s \otimes s')}{\overset{\mu \cdot T(\kappa \cdot (r \otimes r'))}{\rightrightarrows}} T(Y \otimes Y') \xrightarrow{q \cdot T(p \cdot \eta \otimes p' \cdot \eta)} X \boxtimes X' \ .$$

Proof. By hypothesis, there are coequalizer diagrams

$$T(TZ \otimes TZ') \underset{T(Ts \otimes Ts')}{\overset{T((\mu \cdot Tr) \otimes (\mu \cdot Tr'))}{\rightrightarrows}} T(TY \otimes TY') \xrightarrow{\quad T(p \otimes p') \quad} T(X \otimes X') \ .$$

and

$$T(Z \otimes Z') \underset{T(s \otimes s')}{\overset{\mu \cdot T(\kappa \cdot (r \otimes r'))}{\rightrightarrows}} T(Y \otimes Y') \xrightarrow{\quad t \quad} X \circledast X'$$

in $\mathsf{C}^{\mathbb{T}}$. We proceed to show that $X \circledast X'$ and $X \boxtimes X'$ are isomorphic. Since the diagram

$$T(TZ\otimes TZ') \underset{T(Ts\otimes Ts')}{\overset{T((\mu\cdot Tr)\otimes(\mu\cdot Tr'))}{\rightrightarrows}} T(TY\otimes TY') \xrightarrow{\quad t\cdot\mu\cdot T\kappa \quad} X\circledast X'$$

commutes, there exists a unique $\mathsf{C}^{\mathbb{T}}$-morphism $\overline{t} : T(X \otimes X') \longrightarrow X \circledast X'$ such that

$$\overline{t}\cdot T(p\otimes p') = t\cdot \mu_{Y\otimes Y'}\cdot T\kappa_{Y,Y'}\ . \tag{2.5.ii}$$

We can therefore consider the diagram

$$\begin{array}{ccc} T(TTY\otimes TTY') & \xrightarrow{T(Tp\otimes Tp')} & T(TX\otimes TX') \\ {\scriptstyle T(\mu\otimes\mu)}\downarrow\downarrow{\scriptstyle \mu\cdot T\kappa} & & {\scriptstyle \mu\cdot T\kappa}\downarrow\downarrow{\scriptstyle T(x\otimes x')} \\ T(TY\otimes TY') & \xrightarrow{\quad T(p\otimes p') \quad} & T(X\otimes X') \\ {\scriptstyle \mu\cdot T\kappa}\downarrow & & \downarrow{\scriptstyle \overline{t}} \\ T(Y\otimes Y') & \xrightarrow{\qquad t \qquad} & X\circledast X' \end{array}$$

in which the inner- and outer-upper squares commute, as does the lower one. Moreover, the two left vertical arrows of the large square can be identified:

$$\begin{aligned} \mu_{Y\otimes Y'}\cdot T(\kappa_{Y,Y'}\cdot(\mu_Y\otimes\mu_{Y'})) &= \mu_{Y\otimes Y'}\cdot T(\mu_{Y\otimes Y'}\cdot T\kappa_{Y,Y'}\cdot\kappa_{TY,TY'}) \\ &= \mu_{Y\otimes Y'}\cdot T\kappa_{Y,Y'}\cdot\mu_{TY\otimes TY'}\cdot T\kappa_{TY,TY'}\ . \end{aligned}$$

By hypothesis, $T(Tp \otimes Tp')$ is an epimorphism, so the vertical $\mathsf{C}^{\mathbb{T}}$-morphism $T(TX \otimes TX') \longrightarrow X \circledast X'$ that makes the large square commute is unique, that is,

$$T(TX\otimes TX') \underset{T(x\otimes x')}{\overset{\mu\cdot T\kappa}{\rightrightarrows}} T(X\otimes X') \xrightarrow{\quad \overline{t} \quad} X\circledast X'$$

commutes. Thus, the universal property of $q_{X,X'}$ yields a unique $\mathsf{C}^{\mathbb{T}}$-morphism $\overline{\overline{t}} : X \boxtimes X' \longrightarrow X \circledast X'$ such that

$$\overline{\overline{t}}\cdot q_{X,X'} = \overline{t}\ .$$

Let us verify now that $\overline{\overline{t}}$ is an isomorphism. With (2.5.ii), one obtains

$$\overline{\overline{t}}\cdot q_{X,X'}\cdot T(p\otimes p') = t\cdot\mu_{Y\otimes Y'}\cdot T\kappa_{Y,Y'}\ , \tag{2.5.iii}$$

and therefore

$$t = \overline{\overline{t}}\cdot q_{X,X'}\cdot T(p\otimes p')\cdot T(\eta_Y\otimes\eta_{Y'}) = \overline{\overline{t}}_{X,X'}\cdot q_{X,X'}\cdot T(p\cdot\eta_Y\otimes p'\cdot\eta_{Y'})\ .$$

The equality

$$q_{X,X'}\cdot T(p\cdot\eta_Y\otimes p'\cdot\eta_{Y'})\cdot\mu_{Y\otimes Y'}\cdot T\kappa_{Y,Y'} = q_{X,X'}\cdot T(p\otimes p')\ , \tag{2.5.iv}$$

with Lemma 2.5.1 shows that

$$T(Z\otimes Z') \underset{T(s\otimes s')}{\overset{\mu\cdot T(\kappa\cdot(r\otimes r'))}{\rightrightarrows}} T(Y\otimes Y') \xrightarrow{q\cdot T(p\cdot\eta\otimes p'\cdot\eta)} X\boxtimes X'$$

commutes. The universal property of t then yields a unique $\mathsf{C}^{\mathbb{T}}$-morphism $u : X\circledast X' \longrightarrow X\boxtimes X'$ such that

$$q_{X,X'}\cdot T(p\cdot\eta_Y\otimes p'\cdot\eta_{Y'}) = u\cdot t\ ,$$

and we can consider the following commutative diagram:

$$\begin{array}{ccccc}
T(TY\otimes TY') & \xrightarrow{T(p\otimes p')} & T(X\otimes X') & \xrightarrow{\quad q\quad} & X\boxtimes X' \\
{\scriptstyle \mu\cdot T\kappa}\downarrow & & & \searrow^{\bar{t}} & \downarrow{\scriptstyle \bar{\bar{t}}} \\
T(Y\otimes Y') & & \xrightarrow{\qquad\qquad t\qquad\qquad} & & X\circledast X' \\
{\scriptstyle T(\eta\otimes\eta)}\downarrow & & & & \downarrow{\scriptstyle u} \\
T(TY\otimes TY') & \xrightarrow{T(p\otimes p')} & T(X\otimes X') & \xrightarrow{\quad q\quad} & X\boxtimes X'\ .
\end{array}$$

By using (2.5.iv), one then observes

$$q_{X,X'}\cdot T(p\otimes p') = u\cdot\bar{\bar{t}}\cdot q_{X,X'}\cdot T(p\otimes p')\ ,$$

so $u\cdot\bar{\bar{t}} = 1_{X\boxtimes X'}$ because both $q_{X,X'}$ and $T(p\otimes p')$ are epimorphisms. By exchanging the displayed upper and lower diagrams, one obtains similarly that $\bar{\bar{t}}\cdot u = 1_{X\circledast X'}$, and can conclude that $\bar{\bar{t}}$ is an isomorphism. □

2.5.3 Corollary. *Let* $(\mathbb{T},\kappa)$ *be a monoidal monad on a monoidal category* $(\mathsf{C},\otimes,E)$ *such that* C *has reflexive coequalizers. If* $T(X\otimes -)$ *and* $T(-\otimes Y)$ *preserve reflexive coequalizers in* C *(for all* $X,Y\in \mathrm{ob}\,\mathsf{C}$*), then*

$$T((TA\otimes TB)\otimes TC) \underset{T((a\otimes b)\otimes c)}{\overset{\mu\cdot T(\kappa\cdot(\kappa\otimes 1))}{\rightrightarrows}} T((A\otimes B)\otimes C) \xrightarrow{q\cdot T(q\cdot\eta\otimes 1)} (A\boxtimes B)\boxtimes C$$

and

$$T(TA\otimes(TB\otimes TC)) \underset{T(a\otimes(b\otimes c))}{\overset{\mu\cdot T(\kappa\cdot(1\otimes\kappa))}{\rightrightarrows}} T(A\otimes(B\otimes C)) \xrightarrow{q\cdot T(1\otimes q\cdot\eta)} A\boxtimes(B\boxtimes C)$$

are coequalizer diagrams in $\mathsf{C}^{\mathbb{T}}$ *(for all* $\mathbb{T}$*-algebras* (A,a)*,* (B,b) *and* (C,c)*).*

Proof. We prove the statement for the first diagram, the proof for the second following similarly. For this, we only need to verify that the hypotheses of Proposition 2.5.2 are verified for the reflexive coequalizer diagrams

$$T(TA\otimes TB) \underset{T(a\otimes b)}{\overset{\mu\cdot T\kappa}{\rightrightarrows}} T(A\otimes B) \xrightarrow{\quad q\quad} A\boxtimes B \qquad\text{and}\qquad TTC \underset{Tc}{\overset{\mu}{\rightrightarrows}} TC \xrightarrow{\quad c\quad} C\ .$$

Since $T \cong T(E\otimes -)$ preserves reflexive coequalizers, the hypotheses combined with Corollary 2.4.3 imply that $T(q \otimes 1_C)$ is a coequalizer in $\mathsf{C}^{\mathbb{T}}$; hence, $T(q \otimes c)$ is one too by Proposition 2.4.1. Similarly, $T(Tq \otimes Tc)$ is a reflexive coequalizer in $\mathsf{C}^{\mathbb{T}}$ and consequently an epimorphism. Finally, the coequalizer of

$$\mu_{(A\otimes B)\otimes C} \cdot T(\kappa_{A\otimes B,C} \cdot (\kappa_{A\otimes B} \otimes 1_C)) \qquad \text{and} \qquad T(a \otimes b) \otimes Tc$$

exists in $\mathsf{C}^{\mathbb{T}}$ because it is a reflexive pair (split by $T((\eta_A \otimes \eta_B) \otimes \eta_C)$). $\square$

2.5.4 Corollary. *Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C}, \otimes, E)$ such that C has reflexive coequalizers. If $T(X \otimes -)$ and $T(- \otimes Y)$ preserve reflexive coequalizers in C (for all $X, Y \in \operatorname{ob}\mathsf{C}$), then*

$$T(((A \otimes B) \otimes C) \otimes D) \xrightarrow{q\cdot T(q\cdot\eta\otimes 1)\cdot T((q\cdot\eta\otimes 1)\otimes 1)} ((A \boxtimes B) \boxtimes C) \boxtimes D$$

is an epimorphism in $\mathsf{C}^{\mathbb{T}}$ (for all $\mathbb{T}$-algebras (A, a), (B, b), (C, c), and (D, d)).

Proof. The given morphism is in fact a reflexive coequalizer in $\mathsf{C}^{\mathbb{T}}$, obtained by applying Proposition 2.4.1 to the coequalizer diagram of Corollary 2.5.3 and that of $d : TD \longrightarrow D$. $\square$

2.6 The monoidal Eilenberg–Moore category. Consider a monoidal monad $(\mathbb{T}, \kappa)$ on a monoidal category $(\mathsf{C}, \otimes, E)$. Thanks to Corollaries 2.5.3 and 2.5.4 we can verify that the $\mathsf{C}^{\mathbb{T}}$-morphisms

$$q_{A,B} : T(A \otimes B) \longrightarrow A \boxtimes B$$

induce a monoidal structure on $\mathsf{C}^{\mathbb{T}}$. The hypotheses of the Theorem are chosen to refer to the constituent data T and $(\mathsf{C}, \otimes, E)$, rather than to the constructed category $(\mathsf{C}^{\mathbb{T}}, \boxtimes, TE)$ as in [3, Proposition 22].

2.6.1 Theorem. *Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C}, \otimes, E)$ with reflexive coequalizers. If $T(X \otimes -)$ and $T(- \otimes Y)$ preserve reflexive coequalizers in C (for all $X, Y \in \operatorname{ob}\mathsf{C}$), then $(\mathsf{C}^{\mathbb{T}}, \boxtimes, TE)$ is a monoidal category whose structure morphisms are induced by those of $(\mathsf{C}, \otimes, E)$.*

Moreover, if $(\mathsf{C}, \otimes, E)$ and $(\mathbb{T}, \kappa)$ are symmetric monoidal, then so is $(\mathsf{C}^{\mathbb{T}}, \boxtimes, TE)$.

Proof.

Associativity. By definition of α and κ, both the inner and outer squares in

$$\begin{array}{ccc} T((TA \otimes TB) \otimes TC) & \underset{T((a\otimes b)\otimes c)}{\overset{\mu\cdot T(\kappa\cdot(\kappa\otimes 1))}{\rightrightarrows}} & T((A \otimes B) \otimes C) \\ {\scriptstyle T\alpha}\downarrow & & \downarrow{\scriptstyle T\alpha} \\ T(TA \otimes (TB \otimes TC)) & \underset{\mu\cdot T(\kappa\cdot(1\otimes\kappa))}{\overset{T(a\otimes(b\otimes c))}{\rightrightarrows}} & T(A \otimes (B \otimes C)) \end{array}$$

commute. By Corollary 2.5.3, $q_{A\boxtimes B,C}\cdot T(q_{A,B}\cdot\eta_{A\otimes B}\otimes 1_C)$ and $q_{A,B\boxtimes C}\cdot T(1_A\otimes q_{B,C}\cdot\eta_{B\otimes C})$ are respectively coequalizers of the top and bottom row, so there is a $\mathsf{C}^{\mathbb{T}}$-isomorphism $\overline{\alpha}_{A,B,C}:(A\boxtimes B)\boxtimes C\longrightarrow A\boxtimes(B\boxtimes C)$ induced by $T\alpha$:

$$\begin{array}{ccc} T((A\otimes B)\otimes C) & \xrightarrow{q\cdot T(q\cdot\eta\otimes 1)} & (A\boxtimes B)\boxtimes C \\ {\scriptstyle T\alpha}\downarrow & & \downarrow{\scriptstyle\overline{\alpha}} \\ T(A\otimes(B\otimes C)) & \xrightarrow{q\cdot T(1\otimes q\cdot\eta)} & A\boxtimes(B\boxtimes C)\ . \end{array}$$

Similarly, for $\mathsf{C}^{\mathbb{T}}$-morphisms $f:(A,a)\longrightarrow(A',a')$, $g:(B,b)\longrightarrow(B',b,)$, $h:(C,c)\longrightarrow(C',c')$, the inner and outer squares in

$$\begin{array}{ccc} T((TA\otimes TB)\otimes TC) & \underset{T((a\otimes b)\otimes c)}{\overset{\mu\cdot T(\kappa\cdot(\kappa\otimes 1))}{\rightrightarrows}} & T((A\otimes B)\otimes C) \\ {\scriptstyle T((Tf\otimes Tg)\otimes Th)}\downarrow & & \downarrow{\scriptstyle T((f\otimes g)\otimes h)} \\ T((TA'\otimes TB')\otimes TC') & \underset{T((a'\otimes b')\otimes c')}{\overset{\mu\cdot T(\kappa\cdot(\kappa\otimes 1))}{\rightrightarrows}} & T((A'\otimes B')\otimes C') \end{array}$$

commute, inducing a $\mathsf{C}^{\mathbb{T}}$-morphism $(f\boxtimes g)\boxtimes h$:

$$\begin{array}{ccc} T((A\otimes B)\otimes C) & \xrightarrow{q\cdot T(q\cdot\eta\otimes 1)} & (A\boxtimes B)\boxtimes C \\ {\scriptstyle T((f\otimes g)\otimes h)}\downarrow & & \downarrow{\scriptstyle (f\boxtimes g)\boxtimes h} \\ T((A'\otimes B')\otimes C') & \xrightarrow{q\cdot T(q\cdot\eta\otimes 1)} & (A'\boxtimes B')\boxtimes C'\ ; \end{array}$$

one reasons similarly for $f\boxtimes(g\boxtimes h)$. Naturality of $T\alpha$, commutativity of the diagrams for $(f\boxtimes g)\boxtimes h$, $f\boxtimes(g\boxtimes h)$ and $\overline{\alpha}_{A,B,C}$ together with the fact that $q_{A\boxtimes B,C}\cdot T(q_{A,B}\cdot\eta_{A\otimes B}\otimes 1_C)$ is epic yield naturality of $\overline{\alpha}$:

$$\begin{array}{ccc} (A\boxtimes B)\boxtimes C & \xrightarrow{\overline{\alpha}} & A\boxtimes(B\boxtimes C) \\ {\scriptstyle (f\boxtimes g)\boxtimes h}\downarrow & & \downarrow{\scriptstyle f\boxtimes(g\boxtimes h)} \\ (A'\boxtimes B')\boxtimes C' & \xrightarrow{\overline{\alpha}} & A'\boxtimes(B'\boxtimes C')\ . \end{array}$$

Commutativity of the coherence diagram

$$\begin{array}{ccccc} ((A\boxtimes B)\boxtimes C)\boxtimes D & \xrightarrow{\overline{\alpha}} & (A\boxtimes B)\boxtimes(C\boxtimes D) & \xrightarrow{\overline{\alpha}} & A\boxtimes(B\boxtimes(C\boxtimes D)) \\ {\scriptstyle \overline{\alpha}\boxtimes 1}\downarrow & & & & \uparrow{\scriptstyle 1\boxtimes\overline{\alpha}} \\ (A\boxtimes(B\boxtimes C))\boxtimes D & & \xrightarrow{\qquad\overline{\alpha}\qquad} & & A\boxtimes((B\boxtimes C)\boxtimes D) \end{array}$$

follows from a diagram chase involving the coherence diagram of $T(((A\otimes B)\otimes C)\otimes D)$ and the fact that the $\mathsf{C}^{\mathbb{T}}$-morphism of Corollary 2.5.4 is epic.

Unitariness. For a $\mathbb{T}$-algebra (A,a), the composite C-morphism

$$TE\otimes A\xrightarrow{1\otimes\eta}TE\otimes TA\xrightarrow{\kappa}T(E\otimes A)\xrightarrow{T\lambda}TA\xrightarrow{a}A$$

is a bimorphism: on one hand, we have

$$\begin{aligned}
&a \cdot T(a \cdot T\lambda_A \cdot \kappa_{E,A} \cdot (1_{TE} \otimes \eta_A)) \cdot \kappa_{TE,A} \\
&\qquad = a \cdot T\lambda_A \cdot \mu_{E\otimes A} \cdot T\kappa_{E,A} \cdot \kappa_{TE,TA} \cdot (1_{TTE} \otimes T\eta_A) \\
&\qquad = a \cdot T\lambda_A \cdot \kappa_{E,A} \cdot (\mu_E \otimes \mu_A) \cdot (1_{TTE} \otimes T\eta_A) \\
&\qquad = a \cdot T\lambda_A \cdot \kappa_{E,A} \cdot (\mu_E \otimes 1_{TA}) \ ,
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&a \cdot T\lambda_A \cdot \kappa_{E,A} \cdot (1_{TE} \otimes \eta_A) \cdot (\mu_E \otimes a) \\
&\qquad = a \cdot T\lambda_A \cdot \kappa_{E,A} \cdot (1_{TE} \otimes Ta \cdot \eta_{TA}) \cdot (\mu_E \otimes 1_{TA}) \\
&\qquad = a \cdot Ta \cdot T\lambda_{TA} \cdot \kappa_{E,TA} \cdot (1_{TE} \otimes \eta_{TA}) \cdot (\mu_E \otimes 1_{TA}) \\
&\qquad = a \cdot \mu_A \cdot T(T\lambda_A \cdot \kappa_{E,A} \cdot (\eta_E \otimes 1_{TA})) \cdot \kappa_{E,TA} \cdot (1_{TE} \otimes \eta_{TA}) \cdot (\mu_E \otimes 1_{TA}) \\
&\qquad = a \cdot T\lambda_A \cdot \mu_{E\otimes A} \cdot T\kappa_{E,A} \cdot \kappa_{TE,TA} \cdot (T\eta_E \otimes \eta_{TA}) \cdot (\mu_E \otimes 1_{TA}) \\
&\qquad = a \cdot T\lambda_A \cdot \kappa_{E,A} \cdot (\mu_E \otimes \mu_A) \cdot (T\eta_E \otimes \eta_{TA}) \cdot (\mu_E \otimes 1_{TA}) \\
&\qquad = a \cdot T\lambda_A \cdot \kappa_{E,A} \cdot (\mu_E \otimes 1_{TA}) \ .
\end{aligned}$$

By Lemma 2.3.3, there is therefore a unique $\mathsf{C}^{\mathbb{T}}$-morphism $\overline{\lambda}_A$ that makes the diagram

$$\begin{array}{ccc}
T(TE \otimes A) & \xrightarrow{T(a\cdot T\lambda\cdot\kappa\cdot(1\otimes\eta))} & TA \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle a} \\
TE \boxtimes A & \xrightarrow{\quad\overline{\lambda}\quad} & A
\end{array}$$

commute. Its inverse is the $\mathsf{C}^{\mathbb{T}}$-morphism induced by $l := q_{TE,A} \cdot T(\eta_E \otimes 1_A) \cdot T\lambda_A^{-1} : TA \longrightarrow TE \boxtimes A$. Indeed, since a is a coequalizer in $\mathsf{C}^{\mathbb{T}}$ of (Ta, μ_A), and

$$\begin{aligned}
&(q_{TE,A} \cdot T(\eta_E \otimes 1_A) \cdot T\lambda_A^{-1}) \cdot Ta \\
&\qquad = q_{TE,A} \cdot T(1_{TE} \otimes a) \cdot T(\eta_E \otimes 1_{TA}) \cdot T\lambda_{TA}^{-1} \\
&\qquad = q_{TE,A} \cdot T(\mu_E \otimes a) \cdot T(T\eta_E \otimes 1_{TA}) \cdot T(\eta_E \otimes 1_{TA}) \cdot T\lambda_{TA}^{-1} \\
&\qquad = q_{TE,A} \cdot \mu_{TE\otimes A} \cdot T\kappa_{TE,A} \cdot T(T\eta_E \otimes 1_A) \cdot T(\eta_E \otimes 1_{TA}) \cdot T\lambda_{TA}^{-1} \\
&\qquad = q_{TE,A} \cdot \mu_{TE\otimes A} \cdot TT(\eta_E \otimes 1_A) \cdot T\kappa_{E,A} \cdot T(\eta_E \otimes 1_{TA}) \cdot T\lambda_{TA}^{-1} \\
&\qquad = q_{TE,A} \cdot \mu_{TE\otimes A} \cdot TT(\eta_E \otimes 1_A) \cdot TT\lambda_A^{-1} \\
&\qquad = (q_{TE,A} \cdot T(\eta_E \otimes 1_{TA}) \cdot T\lambda_{TA}^{-1}) \cdot \mu_A \ ,
\end{aligned}$$

there is a unique induced $\mathsf{C}^{\mathbb{T}}$-morphism $A \longrightarrow TE \boxtimes A$ (given by $l \cdot \eta_A$). Using this computation, we can also compose l with $T(a \cdot T\lambda_A \cdot \kappa_{TE,A} \cdot (1_{TE} \otimes \eta_A))$ to obtain

$$\begin{aligned}
&q_{TE,A} \cdot T(\eta_E \otimes 1_A) \cdot T\lambda_A^{-1} \cdot T(a \cdot T\lambda_A \cdot \kappa_{TE,A} \cdot (1_{TE} \otimes \eta_A)) \\
&\qquad = q_{TE,A} \cdot T(\eta_E \otimes 1_A) \cdot T\lambda_A^{-1} \cdot T\lambda_A \cdot \mu_{TE\otimes A} \cdot T\kappa_{TE,A} \cdot T(1_{TE} \otimes \eta_A) \\
&\qquad = q_{TE,A} \cdot \mu_{TE\otimes A} \cdot T\kappa_{TE,A} \cdot T(T\eta_E \otimes \eta_A) \\
&\qquad = q_{TE,A} \cdot T(\mu_E \otimes a) \cdot T(T\eta_E \otimes \eta_A) \\
&\qquad = q_{TE,A} \ .
\end{aligned}$$

Since $q_{TE,A}$ is epic, we have $l \cdot \eta_A \cdot \overline{\lambda}_A = 1_{TE \boxtimes A}$. With

$$a \cdot (T(a \cdot T\lambda_A \cdot \kappa_{TE,A} \cdot (1_{TE} \otimes \eta_A))) \cdot (T(\eta_E \otimes 1_A) \cdot T\lambda_A^{-1}) = a \ ,$$

we obtain similarly $\overline{\lambda}_A \cdot l \cdot \eta_A = 1_A$. Hence, $\overline{\lambda}_A$ is an isomorphism in $\mathsf{C}^{\mathbb{T}}$. Naturality of $\overline{\lambda}$ follows from a standard diagram chase involving the defining diagrams of $\overline{\lambda}_A$, $\overline{\lambda}_B$, the facts that $1_{TE} \boxtimes f$ and f are $\mathsf{C}^{\mathbb{T}}$-morphisms, and that $q_{TE,A}$ is epic. The natural isomorphism $\overline{\rho}_A : A \boxtimes TE \longrightarrow A$ is obtained symmetrically via the defining diagram

$$\begin{array}{ccc} T(A \otimes TE) & \xrightarrow{T(a \cdot T\rho \cdot \kappa \cdot (\eta \otimes 1))} & TA \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle a} \\ A \boxtimes TE & \xrightarrow{\quad \overline{\rho} \quad} & A \ . \end{array}$$

Commutativity of

$$\begin{array}{ccc} (A \boxtimes TE) \boxtimes B & \xrightarrow{\quad \overline{\alpha} \quad} & A \boxtimes (TE \boxtimes B) \\ & {\scriptstyle \overline{\rho} \boxtimes 1} \searrow \quad \swarrow {\scriptstyle 1 \boxtimes \overline{\lambda}} & \\ & A \boxtimes B & \end{array}$$

follows again by a standard diagram chase involving the defining diagrams of the three given morphisms, and the fact that $q_{A \boxtimes TE,B} \cdot T(q_{A,TE} \cdot \eta_{A \otimes TE} \otimes 1_B)$ is epic.

Symmetry. By using symmetry of the monoidal monad $(\mathbb{T}, \kappa)$, one obtains the existence of a family of $\mathsf{C}^{\mathbb{T}}$-morphisms $\overline{\sigma}_{A,B} : A \boxtimes B \longrightarrow B \boxtimes A$ via the diagram

$$\begin{array}{ccc} T(A \otimes B) & \xrightarrow{T\sigma} & T(B \otimes A) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ A \boxtimes B & \xrightarrow{\quad \overline{\sigma} \quad} & B \boxtimes A \ . \end{array}$$

Naturality of $\overline{\sigma}$, as well as commutativity of the symmetry diagrams for a symmetric monoidal category then follows by straightforward diagrammatic arguments. □

In the case where $(\mathsf{C}, \otimes, E)$ is closed symmetric monoidal, Theorem 2.6.1 is dual to [8, Theorem 2.2]. The following result states this remark in a slightly more general form.

2.6.2 Corollary. *Let $(\mathbb{T}, \kappa)$ be a monoidal monad on a biclosed monoidal category $(\mathsf{C}, \otimes, E)$ with reflexive coequalizers. If T preserves reflexive coequalizers, then $(\mathsf{C}^{\mathbb{T}}, \boxtimes, TE)$ is a monoidal category whose structure morphisms are induced by those of $(\mathsf{C}, \otimes, E)$.*

Proof. By hypothesis, $X \otimes (-) : \mathsf{C} \longrightarrow \mathsf{C}$ and $(-) \otimes Y : \mathsf{C} \longrightarrow \mathsf{C}$ are left adjoints (for all $X, Y \in \operatorname{ob} \mathsf{C}$). Hence, the functors $T(X \otimes -), T(- \otimes Y) : \mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{C}^{\mathbb{T}}$ preserve reflexive coequalizers, and Theorem 2.6.1 applies. □

2.6.3 Examples.

(1) Theorem 2.6.1 does not apply *as is* to the identity monoidal monad $(\mathbb{I}, 1)$ on a monoidal category $(\mathsf{C}, \otimes, E)$. However, one can observe that the only reflexive coequalizers that are really needed are built upon the $\mathsf{C}^{\mathbb{T}}$-morphisms $q_{A,B}$ and $\mathbb{T}$-algebra structure morphisms—that in turn are defined via the components of the monad $\mathbb{T}$. By a finer analysis of the required coequalizers, one could obtain a version of the Theorem that would have Proposition 3.2.1 occur as the $\mathbb{T} = \mathbb{I}$ case.

(2) For the free abelian group cartesian monad $(\mathbb{A}, \kappa)$ on Set (Example 1.2.1(2)), Corollary 2.6.2 describes the usual tensor product over $\mathbb{Z}$ of the category $\mathsf{Set}^{\mathbb{A}} \cong \mathsf{AbGrp}$ of abelian groups (the hypotheses are verified by [10, Proposition 4] and the following discussion).

(3) For the monoidal powerset monad $(\mathbb{P}, \iota)$ on Set (Example 1.2.1(3)), Corollary 2.6.2 yields the classical tensor product on $\mathsf{Set}^{\mathbb{P}} \cong \mathsf{Sup}$ (see [16]).

(4) In the case of the filter monad $(\mathbb{F}, \kappa)$ on Set (Example 1.2.1(4)), Corollary 2.6.2 yields the tensor product on $\mathsf{Set}^{\mathbb{F}} \cong \mathsf{Cnt}$ appearing in [4, Proposition 3.19].

(5) For the monoidal category $(\mathsf{C}, +, \varnothing)$ of Example 1.2.1(5), we note that the functor $X + (-)$ preserves equalizers for all $X \in \operatorname{ob}\mathsf{C}$. In general, C is not monoidal closed (for example, if $\mathsf{C} = \mathsf{Set}$ and X is a non-empty set, the functor $X + (-)$ is not left adjoint, as it does not preserve coproducts), so Corollary 2.6.2 does not apply. Nevertheless, since any monad $\mathbb{T}$ on C is monoidal when equipped with the comparison natural transformation κ, it suffices that reflexive coequalizers exist in C and that T preserves them to obtain an induced monoidal structure on $\mathsf{C}^{\mathbb{T}}$ via Theorem 2.6.1. This monoidal structure turns out to be the coproduct in $\mathsf{C}^{\mathbb{T}}$ ([10, Proposition 2]).

2.7 Monoidal monad morphisms and Eilenberg–Moore categories. Once the monoidal structure on an Eilenberg–Moore category has been established, it is not surprising that monoidal monad morphisms induce monoidal functors, although these need not be strict as in Proposition 1.3.2.

2.7.1 Proposition. *Let $(\mathsf{C}, \otimes, E)$ be a monoidal category with reflexive coequalizers, and $(\mathbb{S}, \iota)$, $(\mathbb{T}, \kappa)$ monoidal monads on C such that $S(X \otimes -)$, $S(- \otimes Y)$, $T(X \otimes -)$ and $T(- \otimes Y)$ preserve reflexive coequalizers in C (for all $X, Y \in \operatorname{ob}\mathsf{C}$). If $\phi : (\mathbb{S}, \iota) \longrightarrow (\mathbb{T}, \kappa)$ is a monoidal monad morphism, then the induced functor*

$$\mathsf{C}^{\phi} : \mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{C}^{\mathbb{S}}$$

(that commutes with the forgetful functors to C) between the monoidal categories $(\mathsf{C}^{\mathbb{T}}, \boxtimes, TE)$ and $(\mathsf{C}^{\mathbb{S}}, \circledast, SE)$ is itself monoidal.

Proof. Set $\mathbb{S} = (S, \nu, \delta)$ and $\mathbb{T} = (T, \mu, \eta)$. Recall that the algebraic functor C^{ϕ} sends a $\mathbb{T}$-algebra (A, a) to the $\mathbb{S}$-algebra $(A, a \cdot \phi_A)$ and is identical on morphisms. For $\mathbb{T}$-algebras (A, a) and (B, b), one has $\phi_{A\otimes B} \cdot \nu_{A\otimes B} = \mu_{A\otimes B} \cdot \phi_{T(A\otimes B)} \cdot S\phi_{A\otimes B}$, that is, $\phi_{A\otimes B} : S(A \otimes B) \longrightarrow T(A \otimes B)$ is a

$\mathsf{C}^{\mathbb{S}}$-morphism. Hence, the inner- and outer-left squares in the diagram

$$\begin{array}{ccccc} S(SA\otimes SB) & \underset{S((a\cdot\phi)\otimes(b\cdot\phi))}{\overset{\nu\cdot S\iota}{\rightrightarrows}} & S(A\otimes B) & \xrightarrow{\quad p \quad} & A\circledast B \\ {\scriptstyle \phi\cdot S(\phi\otimes\phi)}\downarrow & & \downarrow{\scriptstyle\phi} & & \vdots\,{\scriptstyle\overline{\phi}} \\ T(TA\otimes TB) & \underset{\mu\cdot T\iota}{\overset{T(a\otimes b)}{\rightrightarrows}} & T(A\otimes B) & \xrightarrow{\quad q \quad} & A\boxtimes B \end{array}$$

commute in $\mathsf{C}^{\mathbb{S}}$. If $p_{A,B}$ denotes the coequalizer of the upper row, there is consequently a unique $\mathsf{C}^{\mathbb{S}}$-morphism $\overline{\phi}_{A,B} : (A\circledast B, (a\cdot\phi_A)\bowtie(b\cdot\phi_B)) \longrightarrow (A\boxtimes B, (a\bowtie b)\cdot\phi_{A\boxtimes B})$ that makes the square on the right commute. These $\overline{\phi}_{A,B}$ (for $A, B \in \operatorname{ob}\mathsf{C}^{\mathbb{T}}$) are easily seen to be natural in A and B.

A standard diagram chase involving defining diagrams of the induced structures yields commutativity of

$$\begin{array}{ccccc} (A\circledast B)\circledast C & \xrightarrow{\overline{\phi}\circledast 1} & (A\boxtimes B)\circledast C & \xrightarrow{\overline{\phi}} & (A\boxtimes B)\boxtimes C \\ {\scriptstyle\overline{\alpha}}\downarrow & & & & \downarrow{\scriptstyle\overline{\alpha}} \\ A\circledast(B\circledast C) & \xrightarrow{1\circledast\overline{\phi}} & A\circledast(B\boxtimes C) & \xrightarrow{\overline{\phi}} & A\boxtimes(B\boxtimes C)\ . \end{array}$$

Similarly, commutativity of

$$\begin{array}{ccccc} SE\circledast A & \xrightarrow{\phi\circledast 1} & TE\circledast A & \xrightarrow{\overline{\phi}} & TE\boxtimes A \\ & {\scriptstyle\overline{\lambda}}\searrow & \downarrow & \swarrow{\scriptstyle\overline{\lambda}} & \\ & & A & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} A\circledast SE & \xrightarrow{1\circledast\phi} & A\circledast TE & \xrightarrow{\overline{\phi}} & A\boxtimes TE \\ & {\scriptstyle\overline{\rho}}\searrow & & \swarrow{\scriptstyle\overline{\rho}} & \\ & & A & & \end{array}$$

is routinely verified. Hence, the functor C^{ϕ} with $\overline{\phi} : \mathsf{C}^{\phi}(-)\circledast\mathsf{C}^{\phi}(-) \longrightarrow \mathsf{C}^{\phi}(-\boxtimes-)$ and $\phi_E : SE \longrightarrow TE$ is monoidal. $\square$

2.7.2 Examples.

(1) The unit monad morphism $\eta : \mathbb{I} \longrightarrow \mathbb{T}$ of a monoidal monad $\mathbb{T}$ (Example 1.3.1(1)) induces the forgetful functor $\mathsf{C}^{\eta} : \mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{C}^{\mathbb{I}} \cong \mathsf{C}$ that is therefore monoidal.

(2) The principal filter monad morphism $\phi : \mathbb{P} \longrightarrow \mathbb{F}$ of Example 1.3.1(2) makes the forgetful functor $\mathsf{Cnt} \longrightarrow \mathsf{Sup}$ monoidal with respect to the tensors of Examples 2.6.3(3) and (4).

3 Actions

3.1 Monoids. A *monoid* in a monoidal category C is a C-object M together with two morphisms

$$m : M\otimes M \longrightarrow M\ , \quad e : E \longrightarrow M$$

such that the diagrams

$$
\begin{array}{ccccc}
(M\otimes M)\otimes M & \xrightarrow{\alpha} & M\otimes(M\otimes M) & \xrightarrow{1\otimes m} & M\otimes M \\
{\scriptstyle m\otimes 1}\downarrow & & & & \downarrow{\scriptstyle m} \\
M\otimes M & & \xrightarrow{\quad m \quad} & & M
\end{array}
\qquad
\begin{array}{ccccc}
E\otimes M & \xrightarrow{e\otimes 1} & M\otimes M & \xleftarrow{1\otimes e} & M\otimes E \\
 & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle\rho} & \\
 & & M & &
\end{array}
$$

commute. A *homomorphism of monoids* $f:(N,n,d)\longrightarrow(M,m,e)$ is a C-morphism $f:N\longrightarrow M$ such that the diagrams

$$
\begin{array}{ccc}
N\otimes N & \xrightarrow{f\otimes f} & M\otimes M \\
{\scriptstyle n}\downarrow & & \downarrow{\scriptstyle m} \\
M & \xrightarrow{\quad f \quad} & M
\end{array}
\qquad
\begin{array}{ccc}
 & E & \\
{\scriptstyle d}\swarrow & & \searrow{\scriptstyle e} \\
N & \xrightarrow{\quad f \quad} & M
\end{array}
$$

commute. The category of monoids in C with their homomorphisms is denoted by $\mathsf{Mon}(\mathsf{C})$.

3.1.1 Examples.

(1) For Set with its cartesian structure, $\mathsf{Mon}(\mathsf{Set})=\mathsf{Mon}$, the usual category of monoids with their homomorphisms.

(2) A unital ring R is an abelian group that is also a monoid in which the distributive laws hold, that is, the multiplication $R\times R\longrightarrow R$ is $\mathbb{Z}$-bilinear and is therefore equivalently described as a group homomorphism $R\otimes_{\mathbb{Z}}R\longrightarrow R$. Hence, unital rings are precisely the monoids in AbGrp (with its usual tensor product), and $\mathsf{Mon}(\mathsf{AbGrp})=\mathsf{Rng}$ is the category of unital rings and their homomorphisms.

(3) A quantale V is a complete lattice with a monoid operation $(-)\otimes(-):V\times V\longrightarrow V$ that preserves suprema in each variable; with the tensor product in Sup, the category of complete lattices and sup-preserving maps, the monoid operation may equivalently be considered a morphism $V\otimes V\longrightarrow V$ in Sup. Hence, one has $\mathsf{Mon}(\mathsf{Sup})=\mathsf{Qnt}$, the category of quantales and their homomorphisms.

(4) A monoid in the category Cnt is a continuous lattice V with a monoid operation $(-)\otimes(-):V\times V\longrightarrow V$ that preserves suprema and filtered infima in each variable. The unit interval $[0,1]$ with multiplication, or the extended positive reals $[0,\infty]$ with its opposite order and extended addition are both monoids in Cnt. In fact, they are isomorphic: the map that sends $x\in[0,1]$ to $-\ln(x)\in[0,\infty]$ defines an isomorphism of monoids $([0,1],\cdot,1)\cong([0,\infty]^{\mathrm{op}},+,0)$ in Cnt.

3.2 Actions in a monoidal category. Let C be a monoidal category, and $M=(M,m,e)$ a monoid in C. An *M-action* (more precisely, a *left M-action*) is an object A in C that comes with a C-morphism

$$a:M\otimes A\longrightarrow A$$

that makes the following diagrams commute:

$$\begin{array}{ccc} M\otimes(M\otimes A) & \xrightarrow{1\otimes a} & M\otimes A \\ {\scriptstyle (m\otimes 1)\cdot\alpha^{-1}}\downarrow & & \downarrow{\scriptstyle a} \\ M\otimes A & \xrightarrow{\quad a \quad} & A \end{array} \qquad\qquad \begin{array}{ccc} E\otimes A & \xrightarrow{e\otimes 1} & M\otimes A \\ & {\scriptstyle\lambda}\searrow & \downarrow{\scriptstyle a} \\ & & A\ . \end{array}$$

A C-morphism $f : A \longrightarrow B$ between M-actions (A, a) and (B, b) is *equivariant* if the diagram

$$\begin{array}{ccc} M\otimes A & \xrightarrow{1\otimes f} & M\otimes B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

commutes. The category of M-actions and equivariant C-morphisms is denoted by C^M, a notation that is motivated by the following result.

3.2.1 Proposition. *A monoid (M, m, e) in a monoidal category C gives rise to a monad $M = (M \otimes (-), \tilde{m}, \tilde{e})$ on C, where*

$$\tilde{m}_A = (m \otimes 1_A) \cdot \alpha^{-1}_{M,M,A} \qquad \textit{and} \qquad \tilde{e}_A = (e \otimes 1_A) \cdot \lambda^{-1}_A$$

for all $A \in \mathrm{ob}\,\mathsf{C}$. The Eilenberg–Moore category C^M of this monad is the category of M-actions and equivariant C-morphisms.

Proof. Direct verifications. □

3.2.2 Examples.

(1) If $\mathsf{C} = \mathsf{Set}$, and M is a monoid, the category Set^M is the usual category of M-actions and equivariant maps.

(2) The monoidal structure of AbGrp is given by its the tensor product over $\mathbb{Z}$, and a monoid R in AbGrp is a ring. Hence, AbGrp^R is the usual category R-Mod of left R-modules.

(3) Given a quantale $V = (V, \otimes, k)$, that is, a monoid is Sup, the category Sup^V is described as follows. A V-action X in Sup is a complete lattice X together with a bimorphism $(-) \cdot (-) : V \times X \longrightarrow X$ in Sup such that

$$(u \otimes v) \cdot x = u \cdot (v \cdot x) \quad , \qquad k \cdot x = x\ ,$$

for all $v \in V$, $x \in X$, and a sup-map $f : X \longrightarrow Y$ is equivariant whenever

$$f(v \cdot x) = v \cdot f(x)$$

for all $v \in V$, $x \in X$.

(4) The category $\mathsf{Cnt}^{[0,1]}$ of $[0, 1]$-actions in Cnt is isomorphic to the category $\mathsf{Cnt}^{[0,\infty]^{\mathrm{op}}}$ of $[0,\infty]^{\mathrm{op}}$-actions in Cnt studied in [4].

3.3 Monadic actions. In general, monadic functors do not compose. In the case of actions in $\mathsf{C}^{\mathbb{T}}$ however, they do (Theorem 3.3.3 below).

3.3.1 Proposition. *Let $(\mathbb{T},\kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C},\otimes,E)$ such that C has reflexive coequalizers, and $T(X\otimes -)$, $T(-\otimes Y)$ preserve reflexive coequalizers in C (for all $X,Y\in \mathrm{ob}\,\mathsf{C}$).*

A monoid (M,ξ) in $(\mathsf{C}^{\mathbb{T}},\boxtimes,TE)$ induces a monad $M\boxtimes\mathbb{T}$ on C whose functor is $M\boxtimes T(-):\mathsf{C}\longrightarrow\mathsf{C}$, and whose multiplication and unit are given by their components at $X\in\mathrm{ob}\,\mathsf{C}$ as follows:

$$\tilde{\mu}_X = (m\boxtimes 1_{TX})\cdot\overline{\alpha}^{-1}_{M,M,TX}\cdot(1_M\boxtimes(\xi\bowtie\mu_X))\ ,$$
$$\tilde{\eta}_X = (e\boxtimes 1_{TX})\cdot\overline{\lambda}^{-1}_{TX}\cdot\eta_X\ .$$

Proof. The composite of the adjunctions

$$(\mathsf{C}^{\mathbb{T}})^M \underset{\longrightarrow}{\overset{M\boxtimes(-)}{\underset{\perp}{\longleftarrow}}} \mathsf{C}^{\mathbb{T}} \underset{\longrightarrow}{\overset{T}{\underset{\perp}{\longleftarrow}}} \mathsf{C}$$

yield an adjunction $(\mathsf{C}^{\mathbb{T}})^M \underset{\longrightarrow}{\overset{M\boxtimes T(-)}{\underset{\perp}{\longleftarrow}}} \mathsf{C}$ that induces the described monad. $\square$

3.3.2 Lemma. *Let $(\mathbb{T},\kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C},\otimes,E)$ such that C has reflexive coequalizers, and $T(X\otimes -)$, $T(-\otimes Y)$ preserve reflexive coequalizers in C (for all $X,Y\in\mathrm{ob}\,\mathsf{C}$).*

The C-morphisms $(e\boxtimes 1_{TX})\cdot\overline{\lambda}^{-1}_{TX}:TX\longrightarrow M\boxtimes TX$ are the components of a monad morphism $\tau:\mathbb{T}\longrightarrow M\boxtimes\mathbb{T}$.

Proof. Naturality of the components $\tau_X := (e\boxtimes 1_{TX})\cdot\overline{\lambda}^{-1}_{TX}$ (for $X\in\mathrm{ob}\,\mathsf{C}$) is immediate. Agreement on the units is also easily verified:

$$\tilde{\eta}_X = (e\boxtimes 1_{TX})\cdot\overline{\lambda}^{-1}_{TX}\cdot\eta_X = \tau_X\cdot\eta_X\ .$$

For the multiplications, one has

$$\begin{aligned}\tilde{\mu}_X\cdot(1_M\boxtimes T\tau_X)\cdot\tau_{TX} &= (m\boxtimes 1_{TX})\cdot\overline{\alpha}^{-1}_{M,M,TX}\cdot(1_M\boxtimes((e\boxtimes 1_{TX})\cdot\overline{\lambda}^{-1}_{TX}\cdot\mu_X))\cdot\tau_{TX}\\ &= (\overline{\lambda}_M\boxtimes 1_{TX})\cdot((1_{TE}\boxtimes e)\boxtimes 1_{TX})\cdot\overline{\alpha}^{-1}_{TE,TE,TX}\cdot(1_{TE}\boxtimes(\overline{\lambda}^{-1}_{TX}\cdot\mu_X))\cdot\overline{\lambda}^{-1}_{TX}\\ &= (e\boxtimes 1_{TX})\cdot(\overline{\lambda}_{TE}\boxtimes 1_{TX})\cdot\overline{\alpha}^{-1}_{TE,TE,TX}\cdot\overline{\lambda}^{-1}_{TE\boxtimes TX}\cdot\overline{\lambda}^{-1}_{TX}\cdot\mu_X\\ &= (e\boxtimes 1_{TX})\cdot\overline{\lambda}^{-1}_{TX}\cdot\mu_X = \tau_X\cdot\mu_X\end{aligned}$$

(we use [5] for the penultimate equality), so τ is indeed a monad morphism. $\square$

3.3.3 Theorem. *Let $(\mathbb{T},\kappa)$ be a monoidal monad on a monoidal category $(\mathsf{C},\otimes,E)$ such that C has reflexive coequalizers, and $T(X\otimes -)$, $T(-\otimes Y)$ preserve reflexive coequalizers in C (for all $X,Y\in\mathrm{ob}\,\mathsf{C}$).*

For a monoid (M,ξ) in $(\mathsf{C}^{\mathbb{T}},\boxtimes,TE)$, there is an isomorphism between the category of algebras of the monad $M\boxtimes\mathbb{T}$ and the category of M-actions in $\mathsf{C}^{\mathbb{T}}$:

$$\mathsf{C}^{M\boxtimes\mathbb{T}}\cong(\mathsf{C}^{\mathbb{T}})^{M}\ .$$

In particular, the forgetful functor $(\mathsf{C}^{\mathbb{T}})^{M}\longrightarrow\mathsf{C}$ is strictly monadic.

Proof. The comparison functor $K:(\mathsf{C}^{\mathbb{T}})^{M}\longrightarrow\mathsf{C}^{M\boxtimes\mathbb{T}}$ sends a $\mathbb{T}$-algebra (A,a_1) with an action $a_2:M\boxtimes A\longrightarrow A$ in $\mathsf{C}^{\mathbb{T}}$ to the $(M\boxtimes\mathbb{T})$-algebra $(A,a_2\cdot(1_M\boxtimes a_1))$. We proceed to verify that K is an isomorphism.

The monad morphism $\tau:\mathbb{T}\longrightarrow M\boxtimes\mathbb{T}$ of Lemma 3.3.2 induces a functor $\mathsf{C}^{\tau}:\mathsf{C}^{M\boxtimes\mathbb{T}}\longrightarrow\mathsf{C}^{\mathbb{T}}$ that sends a $(M\boxtimes\mathbb{T})$-algebra (A,a) to the $\mathbb{T}$-algebra $(A,a\cdot\tau_A)$ and commutes with the forgetful functors to C. Set $a_1:=a\cdot\tau_A$, so that

$$T(TM\otimes TA)\ \underset{T(\xi\otimes a_1)}{\overset{\mu\cdot T\kappa}{\rightrightarrows}}\ T(M\otimes A)\xrightarrow{\ q\ }M\boxtimes A$$

is a coequalizer diagram. There is then a unique $\mathsf{C}^{\mathbb{T}}$-morphism $a_2:M\boxtimes A\longrightarrow A$ such that

$$\begin{array}{ccccc}
T(M\otimes A) & & \xrightarrow{\quad q\quad} & & M\boxtimes A\\
{\scriptstyle T(1\otimes\eta)}\downarrow & & & & \downarrow{\scriptstyle a_2}\\
T(M\otimes TA) & \xrightarrow{\ q\ } & M\boxtimes TA & \xrightarrow{\ a\ } & A
\end{array}$$

commutes. To see this, we use the universal property of $q_{M,A}$. Indeed, $\tau_A=(e\boxtimes 1_{TA})\cdot\overline{\lambda}_{TA}^{-1}:TA\longrightarrow M\boxtimes TA$ is a $\mathsf{C}^{\mathbb{T}}$-morphism, so the diagram

$$\begin{array}{ccccc}
T(TM\otimes TA) & \xrightarrow{T(\xi\otimes\eta)} & T(M\otimes TTA) & \xrightarrow{T(1\otimes T((e\boxtimes 1)\cdot\overline{\lambda}^{-1}))} & T(M\otimes T(M\boxtimes TA))\\
 & \searrow{\scriptstyle T(\xi\otimes 1)} & \downarrow{\scriptstyle T(1\otimes\mu)} & & \downarrow{\scriptstyle T(1\otimes(\xi\bowtie\mu))}\\
 & & T(M\otimes TA) & \xrightarrow{T(1\otimes((e\boxtimes 1)\cdot\overline{\lambda}^{-1}))} & T(M\otimes(M\boxtimes TA))
\end{array}$$

commutes; by definition of a $(M\boxtimes\mathbb{T})$-algebra (A,a), the diagram

$$\begin{array}{ccccccc}
T(M\otimes T(M\boxtimes TA)) & \xrightarrow{\ q\ } & M\boxtimes T(M\boxtimes TA) & \xrightarrow{1\boxtimes Ta} & M\boxtimes TA & & \\
{\scriptstyle T(1\otimes(\xi\bowtie\mu))}\downarrow & & \downarrow{\scriptstyle 1\boxtimes(\xi\bowtie\mu)} & & & \searrow{\scriptstyle a} & \\
T(M\otimes(M\boxtimes TA)) & \xrightarrow{\ q\ } & M\boxtimes(M\boxtimes TA) & \xrightarrow{(m\boxtimes 1)\cdot\overline{\alpha}^{-1}} & M\boxtimes TA & \xrightarrow{\ a\ } & A
\end{array}$$

also commutes. By glueing these diagrams together along $T(1_M\otimes(\xi\bowtie\mu_A))$, one obtains, with

$a_1 = a \cdot (e \boxtimes 1_{TA}) \cdot \overline{\lambda}_{TA}^{-1}$ and $q_{M,TA} \cdot T(1 \otimes Ta) = (1_M \boxtimes Ta) \cdot q_{M,T(M \boxtimes TA)}$,

$$\begin{aligned} & a \cdot q_{M,TA} \cdot T(1_M \otimes \eta_A) \cdot T(\xi \otimes a_1) \\ & \quad = a \cdot q_{M,TA} \cdot T(1_M \otimes Ta_1) \cdot T(\xi \otimes \eta_{TA}) \\ & \quad = a \cdot (1_M \boxtimes Ta) \cdot q_{M,T(M \boxtimes TA)} \cdot T(1_M \otimes T((e \boxtimes 1_{TA}) \cdot \overline{\lambda}_{TA}^{-1})) \cdot T(\xi \otimes \eta_{TA}) \\ & \quad = a \cdot (m \boxtimes 1_{TA}) \cdot \overline{\alpha}_{M,M,TA}^{-1} \cdot q_{M,M \boxtimes TA} \cdot T(1_M \otimes ((e \boxtimes 1_{TA}) \cdot \overline{\lambda}_{TA}^{-1})) \cdot T(\xi \otimes 1_{TA}) \\ & \quad = a \cdot (m \boxtimes 1_{TA}) \cdot \overline{\alpha}_{M,M,TA}^{-1} \cdot (1_M \boxtimes ((e \boxtimes 1_{TA}) \cdot \overline{\lambda}_{TA}^{-1})) \cdot q_{M,TA} \cdot T(\xi \otimes 1_{TA}) \\ & \quad = a \cdot (\overline{\rho}_M \boxtimes 1_{TA}) \cdot \overline{\alpha}_{M,TE,TA}^{-1} \cdot (1_M \boxtimes \overline{\lambda}_{TA}^{-1}) \cdot q_{M,TA} \cdot T(\xi \otimes 1_{TA}) \\ & \quad = a \cdot q_{M,TA} \cdot T(\xi \otimes 1_{TA}) \ . \end{aligned}$$

Since one also has

$$\begin{aligned} & a \cdot q_{M,TA} \cdot T(1_M \otimes \eta_A) \cdot \mu_{M \otimes A} \cdot T\kappa_{M,A} \\ & \quad = a \cdot q_{M,TA} \cdot \mu_{M \otimes TA} \cdot T\kappa_{M,TA} \cdot T(1_{TM} \otimes T\eta_A) \\ & \quad = a \cdot q_{M,TA} \cdot T(\xi \otimes 1_{TA}) \ , \end{aligned}$$

the existence and unicity of the required $\mathsf{C}^{\mathbb{T}}$-morphism a_2 follows. Moreover,

$$\begin{aligned} a \cdot q_{M,TA} &= a \cdot q_{M,TA} \cdot T(1_M \otimes \eta_A) \cdot T(1_M \otimes a_1) \\ &= a_2 \cdot q_{M,A} \cdot T(1_M \otimes a_1) \\ &= a_2 \cdot (1_M \boxtimes a_1) \cdot q_{M,TA} \ , \end{aligned}$$

so $a = a_2 \cdot (1_M \boxtimes a_1)$ because $q_{M,TA}$ is epic. Let us verify that $a_2 : M \boxtimes A \longrightarrow A$ defines an action. One has

$$\begin{aligned} a \cdot (1_M \boxtimes Ta) &= a_2 \cdot (1_M \boxtimes a_1) \cdot (1_M \boxtimes Ta) \\ &= a_2 \cdot (1_M \boxtimes a) \cdot (1_M \boxtimes (\xi \bowtie \mu_A)) \\ &= a_2 \cdot (1_M \boxtimes a_2) \cdot (1_M \boxtimes (1_M \boxtimes a_1)) \cdot (1_M \boxtimes (\xi \bowtie \mu_A)) \end{aligned}$$

and

$$\begin{aligned} a \cdot \tilde{\mu}_A &= a_2 \cdot (m \boxtimes 1_A) \cdot (1_{M \boxtimes M} \boxtimes a_1) \cdot \overline{\alpha}_{M,M,TA}^{-1} \cdot (1_M \boxtimes (\xi \bowtie \mu_A)) \\ &= a_2 \cdot (m \boxtimes 1_A) \cdot \overline{\alpha}_{M,M,A}^{-1} \cdot (1_M \boxtimes (1_M \boxtimes a_1)) \cdot (1_M \boxtimes (\xi \bowtie \mu_A)) \ . \end{aligned}$$

Notice that for any $\mathbb{T}$-algebra structure $b : TB \longrightarrow B$, the $\mathsf{C}^{\mathbb{T}}$-morphisms $T(1_M \otimes b)$ is an epimorphism, so that $q_{M,A} \cdot T(1_M \otimes b) = (1_M \boxtimes b) \cdot q_{M,TA}$ implies that $(1_M \boxtimes b)$ is epic. Since $a \cdot (1_M \boxtimes Ta) = a \cdot \tilde{\mu}_A$, and $(1_M \boxtimes (1_M \boxtimes a_1))$, $(1_M \boxtimes (\xi \bowtie \mu_A))$ are both epic, we obtain

$$a_2 \cdot (1_M \boxtimes a_2) = a_2 \cdot (m \boxtimes 1_A) \cdot \overline{\alpha}_{M,M,A}^{-1} \ ,$$

the first condition for a_2 to be an action. The second condition comes from

$$a \cdot \tilde{\eta}_A = a_2 \cdot (1_M \boxtimes a_1) \cdot (e \boxtimes 1_{TA}) \cdot \overline{\lambda}_{TA}^{-1} \cdot \eta_A = a_2 \cdot (e \boxtimes 1_A) \cdot \overline{\lambda}_A^{-1} \cdot a_1 \cdot \eta_A = a_2 \cdot (e \boxtimes 1_A) \cdot \overline{\lambda}_A^{-1}$$

with $a \cdot \tilde{\eta}_A = 1_A$. Finally, any $(M \boxtimes \mathbb{T})$-algebra homomorphism $f : (A, a) \longrightarrow (B, b)$ yields a $\mathbb{T}$-algebra homomorphism $f : (A, a \cdot \tau_A) \longrightarrow (B, b \cdot \tau_B)$ that is equivariant: one has

$$f \cdot a_2 \cdot (1_M \boxtimes a_1) = f \cdot a = b \cdot (1_M \boxtimes Tf) = b_2 \cdot (1_M \boxtimes b_1) \cdot (1_M \boxtimes Tf) = b_2 \cdot (1_M \boxtimes f) \cdot (1_M \boxtimes a_1) \ ,$$

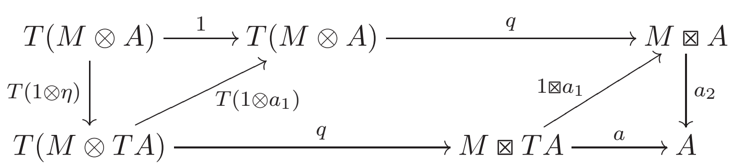
so that $f \cdot a_2 = b_2 \cdot (1_M \boxtimes f)$ because $(1_M \boxtimes a_1)$ is epic.

Hence, a $(M \boxtimes \mathbb{T})$-algebra (A, a) yields a $\mathbb{T}$-algebra (A, a_1) with an action $a_2 : M \boxtimes A \longrightarrow A$, and $K((A, a_1), a_2)$ returns the original $(M \boxtimes \mathbb{T})$-algebra (A, a), since $a = a_2 \cdot (1_M \boxtimes a_1)$.

Conversely, the K-image of a $\mathbb{T}$-algebra (A, a_1) with an action $a_2 : M \boxtimes A \longrightarrow A$ is a $(M \boxtimes \mathbb{T})$-algebra $(A, a_2 \cdot (1_M \boxtimes a_1))$. Since

$$a_2 \cdot (1_M \boxtimes a_1) \cdot \tau_A = a_2 \cdot (e \boxtimes 1_A) \cdot \overline{\lambda}_A^{-1} \cdot a_1 = a_1$$

and the diagram



commutes, one recuperates the original triplet $((A, a_1), a_2)$ from $(A, a_2 \cdot (1_M \boxtimes a_1))$; that is, K is an isomorphism. □

3.3.4 Examples.

(1) Even if a monoidal category $(\mathsf{C}, \otimes, E)$ has only trivial coequalizers, any monoid M in C yields an isomorphism $(\mathsf{C}^{\mathbb{I}})^M \cong \mathsf{C}^M \cong \mathsf{C}^{M \otimes \mathbb{I}}$ (see also Example 2.6.3(1)).

(2) If $\mathbb{T} = \mathbb{A}$ is the free abelian group monad and $M = R$ is a ring, the isomorphisms

$$R\text{-}\mathsf{Mod} \cong \mathsf{AbGrp}^R \cong (\mathsf{Set}^{\mathbb{A}})^R \cong \mathsf{Set}^{R \otimes_{\mathbb{Z}} \mathbb{A}}$$

recall the classical monadicity of R-modules over Set, and describe the free R-module over a set X as $R \otimes_{\mathbb{Z}} AbX$.

(3) For a quantale V, the category Sup^V of V-actions (Example 3.2.2(3)) is isomorphic to the category $\mathsf{Set}^{V \otimes \mathbb{P}}$. The classical description of the tensor in Sup (see for example [1]) yields isomorphisms

$$V \otimes PX \cong \mathsf{Sup}(\mathsf{Sup}(V, \mathsf{Sup}(PX, 2)), 2) \cong \mathsf{Set}(X, V) \ .$$

The case where V is integral was treated in [15], where it is proved that Sup^V is monadic over Set.

(4) The category $\mathsf{Cnt}^{[0,\infty]^{\mathrm{op}}}$ of Example 3.2.2(4) is known to be monadic ([4, Section 3.4]). Theorem 3.3.3 confirms that this result can be extended from $([0, \infty]^{\mathrm{op}}, +, 0)$ to any other monoid in Cnt.

3.4 Monad morphisms. Every pair of morphisms $f : N \longrightarrow M$ and $\phi : \mathbb{S} \longrightarrow \mathbb{T}$ induce a monad morphism $(f, \phi) : N \circledast \mathbb{S} \longrightarrow M \boxtimes \mathbb{T}$ (Proposition 3.4.2 below), and thus a functor $(\mathsf{C}^{\mathbb{T}})^M \longrightarrow (\mathsf{C}^{\mathbb{S}})^N$ between the respective categories of actions. The usual *restriction-of-scalars* functor between categories of modules then appears as the $\mathbb{S} = \mathbb{T} = \mathbb{A}$ instance of this result (Corollary 3.4.3).

3.4.1 Lemma. *Let $(\mathsf{C}, \otimes, E)$ be a monoidal category with reflexive coequalizers, and $(\mathbb{S}, \iota)$, $(\mathbb{T}, \kappa)$ monoidal monads on C such that $S(X \otimes -)$, $S(- \otimes Y)$, $T(X \otimes -)$ and $T(- \otimes Y)$ preserve reflexive coequalizers in C (for all $X, Y \in \operatorname{ob} \mathsf{C}$).*

If $\phi : \mathbb{S} \longrightarrow \mathbb{T}$ is a monoidal monad morphism and $((N, \zeta), n, d)$ is a monoid in $\mathsf{C}^{\mathbb{T}}$, then $(N, \zeta \cdot \phi_N)$ with multiplication $n \cdot \overline{\phi}_{N,N}$ and unit $d \cdot \phi_E$ is a monoid in $\mathsf{C}^{\mathbb{S}}$.

Proof. Since (N, ζ) is a $\mathbb{T}$-algebra, $(N, \zeta \cdot \phi_N)$ is an $\mathbb{S}$-algebra. The structures $d \cdot \phi_E : SE \longrightarrow N$ and $n \cdot \overline{\phi}_{N,N} : N \circledast N \longrightarrow N$ are $\mathsf{C}^{\mathbb{S}}$-morphisms (see the proof of Proposition 2.7.1). Commutativity of the corresponding monoid diagrams in $(\mathsf{C}^{\mathbb{S}}, \circledast, SE)$ follows from commutativity of the monoid diagrams of (N, n, d) in $(\mathsf{C}^{\mathbb{T}}, \boxtimes, TE)$ combined with commutativity of the diagrams showing that the functor $\mathsf{C}^{\phi} : \mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{C}^{\mathbb{S}}$ is monoidal with respect to $\overline{\phi}$ and ϕ_E (Proposition 2.7.1). □

3.4.2 Proposition. *Let $(\mathsf{C}, \otimes, E)$ be a monoidal category with reflexive coequalizers, and $(\mathbb{S}, \iota)$, $(\mathbb{T}, \kappa)$ monoidal monads on C such that $S(X \otimes -)$, $S(- \otimes Y)$, $T(X \otimes -)$ and $T(- \otimes Y)$ preserve reflexive coequalizers in C (for all $X, Y \in \operatorname{ob} \mathsf{C}$).*

If $\phi : \mathbb{S} \longrightarrow \mathbb{T}$ is a monoidal monad morphism, and $f : N \longrightarrow M$ a monoid homomorphism in $\mathsf{C}^{\mathbb{T}}$, then there is a monad morphism $(f, \phi) : N \circledast \mathbb{S} \longrightarrow M \boxtimes \mathbb{T}$ whose components at $X \in \operatorname{ob} \mathsf{C}$ are given by the $\mathsf{C}^{\mathbb{S}}$-morphism $\overline{\phi}_{M,TX} \cdot (f \circledast \phi_X) : N \circledast SX \longrightarrow M \boxtimes TX$.

Proof. By Lemma 3.4.1, N can be seen as a monoid in $(\mathsf{C}^{\mathbb{S}}, \circledast, SE)$, thus defining a monad $N \circledast \mathbb{S}$. Via the functor $\mathsf{C}^{\phi} : \mathsf{C}^{\mathbb{T}} \longrightarrow \mathsf{C}^{\mathbb{S}}$, the arrows f and ϕ_X are $\mathsf{C}^{\mathbb{S}}$-morphisms, so $f \circledast \phi_X$ is defined in $\mathsf{C}^{\mathbb{S}}$, and so is $\overline{\phi}_{M,TX}$ by Proposition 2.7.1. To verify that these components define a monad morphism, we use our usual notations $\mathbb{S} = (S, \nu, \delta)$, $\mathbb{T} = (T, \mu, \eta)$ for the monads, and (N, n, d), (M, m, e) for the monoids. Moreover, we let the context differentiate between the induced structure morphisms $\overline{\alpha}$, $\overline{\delta}$ and $\overline{\rho}$ of $\mathsf{C}^{\mathbb{T}}$ or $\mathsf{C}^{\mathbb{S}}$.

By using the definitions and properties of the involved morphisms, we compute

$$
\begin{aligned}
\overline{\phi}_{M,TX} \cdot (f \circledast \phi_X) \cdot \tilde{\delta}_X &= \overline{\phi}_{M,TX} \cdot (f \circledast \phi_X) \cdot (d \cdot \phi_E \circledast 1_{SX}) \cdot \overline{\lambda}_{SX}^{-1} \cdot \delta_X \\
&= \overline{\phi}_{M,TX} \cdot (e \circledast 1_{TX}) \cdot (\phi_E \circledast 1_{TX}) \cdot (1_{SE} \circledast \phi_X) \cdot \overline{\lambda}_{SX}^{-1} \cdot \delta_X \\
&= (e \boxtimes 1_{TX}) \cdot \overline{\phi}_{TE,TX} \cdot (\phi_E \circledast 1_{TX}) \cdot \overline{\lambda}_{TX}^{-1} \cdot \phi_X \cdot \delta_X \\
&= (e \boxtimes 1_{TX}) \cdot \overline{\lambda}_{TX}^{-1} \cdot \eta_X = \tilde{\eta}_X \ .
\end{aligned}
$$

For the multiplications, we use that

$$\begin{aligned}
&\overline{\phi}_{M,TX} \cdot (f \circledast \phi_X) \cdot (n \cdot \overline{\phi}_{N,N} \circledast 1_{SX}) \cdot \overline{\alpha}^{-1}_{N,N,SX} \\
&\qquad = (m \cdot (f \boxtimes f) \boxtimes 1_{TX}) \cdot \overline{\phi}_{N\boxtimes N,TX} \cdot (\overline{\phi}_{N,N} \circledast 1_{TX}) \cdot \overline{\alpha}^{-1}_{N,N,TX} \cdot (1_N \circledast (1_N \circledast \phi_X)) \\
&\qquad = (m \cdot (f \boxtimes f) \boxtimes 1_{TX}) \cdot \overline{\alpha}^{-1}_{N,N,TX} \cdot \overline{\phi}_{N,N\boxtimes TX} \cdot (1_N \circledast \overline{\phi}_{N,TX}) \cdot (1_N \circledast (1_N \circledast \phi_X)) \\
&\qquad = (m \boxtimes 1_{TX}) \cdot \overline{\alpha}^{-1}_{M,M,TX} \cdot \overline{\phi}_{M,M\boxtimes TX} \cdot (1_M \circledast \overline{\phi}_{M,TX}) \cdot (f \circledast (f \circledast \phi_X))
\end{aligned}$$

and since $\overline{\phi}_{M,TX} \cdot (f \circledast \phi_X) : N \circledast SX \longrightarrow M \boxtimes TX$ is a $\mathsf{C}^{\mathbb{S}}$-morphism,

$$\begin{aligned}
&\overline{\phi}_{M,M\boxtimes TX} \cdot (1_M \circledast \overline{\phi}_{M,TX}) \cdot (f \circledast (f \circledast \phi_X)) \cdot (1_N \circledast (\zeta \cdot \phi_N \bowtie \nu_X)) \\
&\qquad = \overline{\phi}_{M,M\boxtimes TX} \cdot (f \circledast ((\xi \bowtie \mu_X) \cdot \phi_{M\circledast TX})) \cdot (1_N \circledast S(\overline{\phi}_{M,TX} \cdot (f \circledast \phi_X))) \\
&\qquad = (1_M \boxtimes (\xi \bowtie \mu_X)) \cdot \overline{\phi}_{M,T(M\boxtimes TX)} \cdot (f \circledast \phi_{M\boxtimes TX}) \cdot (1_N \circledast S(\overline{\phi}_{M,TX} \cdot (f \circledast \phi_X))) \ .
\end{aligned}$$

Hence,

$$\begin{aligned}
\overline{\phi}_{M,TX} \cdot (f \circledast \phi_X) \cdot \tilde{\nu}_X &= \overline{\phi}_{M,TX} \cdot (f \circledast \phi_X) \cdot (n \cdot \overline{\phi}_{N,N} \circledast 1_{SX}) \cdot \overline{\alpha}^{-1}_{N,N,SX} \cdot (1_N \circledast (\zeta \cdot \phi_N \bowtie \nu_X)) \\
&= \tilde{\mu}_X \cdot \overline{\phi}_{M,T(M\boxtimes TX)} \cdot (f \circledast \phi_{M\boxtimes TX}) \cdot (1_N \circledast S(\overline{\phi}_{M,TX} \cdot (f \circledast \phi_X))) \ ,
\end{aligned}$$

so the components $\overline{\phi}_{M,TX} \cdot (f \circledast \phi_X)$ verify the two conditions for being a monad morphism. □

3.4.3 Corollary (Restriction of scalars)**.** *Let $(\mathsf{C}, \otimes, E)$ be a monoidal category with reflexive coequalizers, and $(\mathbb{T}, \kappa)$ a monoidal monad on C such that $T(X \otimes -)$ and $T(- \otimes Y)$ preserve reflexive coequalizers in C (for all $X, Y \in \operatorname{ob}\mathsf{C}$).*

If $f : N \longrightarrow M$ a monoid homomorphism in $\mathsf{C}^{\mathbb{T}}$, then the monad morphism $(f, 1_{\mathbb{T}}) : N \boxtimes \mathbb{T} \longrightarrow M \boxtimes \mathbb{T}$ induces a functor $\mathsf{C}^{(f,1_{\mathbb{T}})} : (\mathsf{C}^{\mathbb{T}})^M \longrightarrow (\mathsf{C}^{\mathbb{T}})^N$.

Proof. The monad morphism is the $\phi = 1_{\mathbb{T}}$ case of Proposiiton 3.4.2, and the result follows from the isomorphisms $(\mathsf{C}^{\mathbb{S}})^N \cong \mathsf{C}^{N\circledast\mathbb{S}}$ and $(\mathsf{C}^{\mathbb{T}})^M \cong \mathsf{C}^{M\boxtimes\mathbb{T}}$ (Theorem 3.3.3). □

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