Lexicographic Shellability of Partial Involutions

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Abstract

In this manuscript we study inclusion posets of Borel orbit closures on (symmetric) matrices. In particular, we show that the Bruhat poset of partial involutions is a lexicographiically shellable poset. Also, studying the embeddings of symmetric groups and involutions into rooks and partial involutions, respectively, we find new EL-labelings on permutations as well as on involutions.

1 Introduction.

Recall that a simplicial complex Δ is called *shellable* if there exits a linear ordering $F_1, F_2, \dots, F_k$ of the facets of Δ in such a way that, for each $j = 2, \dots, k$, the intersection of the sub-complex of F_j with the union of all sub-complexes of previous facets $F_1, \dots, F_{j-1}$ is a pure sub-complex of Δ of dimension $\dim F_j - 1$.

Although its definition is not illuminating, the notion of shellability has remarkable topological consequences. For example, if shellable, the simplicial complex Δ has the homotopy type of a wedge of spheres. See [4].

In this manuscript we are concerned with the shellability question of a simplicial complex arising from an action of the invertible upper triangular matrices on symmetric matrices. Let K denote an algebraically closed field of characteristic zero, $M = M_n(K)$ denote the affine variety of $n \times n$ matrices over K, and let Y denote an arbitrary subvariety of M. When a group G acts on Y, we denote by $B(Y; G)$ the set of G-orbit closures.

In this paper we focus on two main examples:

- $Y = Q$, the space of symmetric matrices in M, and $G = B_n$, the Borel group of invertible upper triangular matrices acting on Y via

$$x \cdot A = (x^{-1})^{\top} A x^{-1},$$

where $x^\top$ denotes the transpose of the matrix $x \in B_n$ and $A \in Q$.

- $Y = M$ and $G = B_n \times B_n$ acting on Y via

$$(x, y) \cdot A = xAy^{-1}, \tag{1}$$

where $x, y \in B_n$ and $A \in M$.

In the above cases, the set $B(Y; G)$ is finite, and furthermore, elements of $B(Y; G)$ are partially ordered with respect to set inclusion. Let $\Delta(Y)$ denote the set of all chains (sequences of nested orbit-closures) of the poset $(B(Y; G), \subseteq)$. Then $\Delta(Y)$ has the structure of a simplicial complex. Our first main result is that $\Delta(Q)$ is a shellable complex. In fact, we prove a much stronger statement; the poset $(B(Q; B_n), \subseteq)$ is "lexicographically shellable."

Introduced by Björner in [2] and advanced by Björner and Wachs in [3], the notion of lexicographic shellability is equivalent to finding a suitable labeling of the edges of the Hasse diagram of the poset under consideration. Thus, associated with each saturated chain is a sequence of labels, which provides an ordering of the faces of the simplicial complex $\Delta(Y)$.

Recall that the *rook monoid* R_n is the finite monoid of 0/1 matrices with at most one 1 in each row and each column. It is well known that the elements of R_n parametrize the orbits of the action (1) of $B_n \times B_n$ on M. See [11]. The elements of R_n are called *rooks*, or *rook matrices*. It is shown by Szechtman in [12] that each orbit closure in $B(Q; B_n)$ has a unique corresponding symmetric rook in R_n. Following [1], we call these rooks *partial involutions* as they satisfy the quadratic equation

$$x^2 = e,$$

where $e \in R_n$ is a diagonal matrix. We denote the set of all partial involutions in R_n by P_n.

The *Bruhat-Chevalley-Renner ordering* on rooks is defined by

$$r \leq t \iff B_n r B_n \subseteq \overline{B_n t B_n},\ r, t \in R_n.$$

Here, bar on the orbit $B_n t B_n$ stands for the Zariski closure in M. Corresponding partial order on P_n, which we denote by $\preceq$, is studied by Bagno and Cherniavsky [1]. If A and A' are two B_n-orbit closures in $B(Q; B_n)$, and, r and r' are two partial involutions representing A and A', respectively, then

$$r \preceq r' \iff A' \subseteq A.$$

Although $\preceq$ is more natural from geometric point of view, we prefer to work with its opposite, which we denote, by abuse of notation, by $\leq$, also. Our first main result is that

Theorem 1. *The poset $(P_n, \leq)$ is lexicographically shellable.*

Let S_n denote the symmetric group of permutations which is contained in R_n as the group of invertible rooks. In an increasing order of generality, the articles [7], [9], and [3] show that $(S_n, \leq)$ is a lexicographically shellable poset. Generalizing this result to R_n, [5] shows that R_n is a lexicographically shellable poset. See also [10].

Recall that a graded poset $(P, \leq)$ with the rank function $\rho : P \to \mathtt{N}$ is called *Eulerian*, if for all $x \leq y$ the equality

$$|\{z \in [x, y] : \ \rho(z) \text{ is odd}\}| = |\{z \in [x, y] : \ \rho(z) \text{ is even}\}|$$

holds.

Let $I_n \subset P_n$ denote the subset of involutions of maximal rank. Hence, I_n lies in S_n. It is shown by Incitti in [8] that I_n with respect to "opposite inclusion ordering" is not only lexicographically shellable but also Eulerian. Unfortunately, neither R_n nor P_n is Eulerian, so, we direct our attention to certain important subposets of them.

Let H denote the group of invertible elements of a monoid N. It is important for semigroup theorists to understand the structure of orbits of H on N for various actions. In this regard, we consider a two sided action of S_n on R_n:

$$(x, y) \cdot z = xzy^{-1}, \text{ for all } z \in R_n, \ x, y \in S_n. \tag{2}$$

There is natural restriction of this action to an S_n-action on partial involutions:

$$y \cdot t = (y^{-1})^{\top} t y^{-1}, \text{ for } t \in P_n, \ y \in S_n. \tag{3}$$

Let $R_{n,k} \subset R_n$ denote the rook matrices with k non-zero entries, and let $P_{n,k} = R_{n,k} \cap P_n$. Then any orbit of (2) is equal to one of $R_{n,k}$ for some k, and similarly, any orbit of (3) is equal to one of $P_{n,k}$ for some k.

Once k is fixed, the unions $\cup_{l \leq k} R_{n,l}$ and $\cup_{l \leq k} P_{n,l}$ parametrize Borel orbits in certain determinental varieties. Therefore, it is important to study the restriction of Bruhat-Chevalley-Renner ordering on these subposets.

Although they are significantly different from each other, $R_{n,k}$ and $P_{n,k}$ share many important properties. For example, both of them have the smallest and the largest elements. In fact, much more is true.

Given a positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Theorem 2. *For all $n \geq 1$ and $k \in [n]$, the subposets $R_{n,k} \subseteq R_n$ and $P_{n,k} \subseteq P_n$ are Eulerian if and only if $k = n$ or $k = n-1$.*

The proof of Theorem 2 relies on the following intriguing result:

Theorem 3. *For all $n \geq 1$,*

1. *$(R_{n,n-1} \cup R_{n,n}, \leq)$ is isomorphic to the poset $(S_{n+1}, \leq)$ and,*
2. *$(P_{n,n-1} \cup P_{n,n}, \leq)$ is isomorphic to the poset $(I_{n+1}, \leq)$.*

Let us point out that, as a remarkable corollary of Theorems 1 and 3 together with the main result of [5], we obtain new EL-labelings of S_{n+1} and I_{n+1} induced from their imbedding into R_n and P_n, respectively.

The organization of our paper is as follows. In Section 2 we introduce our notation and provide preliminaries. In Section 3 we study covering relations of the opposite order $\leq$ of $\preceq$. In Section 4 we prove Theorem 1, and finally, in Section 5 we prove Theorems 2 and 3.

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2 Background.

2.1 Lexicographic shellability.

We start with reviewing the notion of lexicographic shellability.

Let P be a finite poset with a maximum and a minimum element, denoted by $\hat{1}$ and $\hat{0}$, respectively. We assume that P is *graded* of *rank* n. In other words, all maximal chains of P have equal length n. Denote by $C(P)$ the set of covering relations

$$C(P) = \{(x, y) \in P \times P : \; y \text{ covers } x\}.$$

An *edge-labeling* on P is a map $f = f_{P,\Gamma} : C(P) \to \Gamma$ into some totally ordered set Γ. The *Jordan-Hölder sequence* (with respect to f) of a maximal chain $\mathfrak{c} : x_0 < x_1 < \cdots < x_{n-1} < x_n$ of P is the n-tuple

$$f(\mathfrak{c}) := (f((x_0, x_1)), f((x_1, x_2)), \ldots, f((x_{n-1}, x_n))) \in \Gamma^n.$$

Fix an edge labeling f, and a maximal chain $\mathfrak{c}: \ x_0 < x_1 < \cdots < x_n$. We call both the maximal chain $\mathfrak{c}$ and its image $f(\mathfrak{c})$ *increasing*, if

$$f((x_0, x_1)) \leq f((x_1, x_2)) \leq \cdots \leq f((x_{n-1}, x_n))$$

holds in Γ.

Let $k > 0$ be a positive integer and let Γ^k denote the $k-$fold cartesian product $\Gamma^k = \Gamma \times \cdots \times \Gamma$, totally ordered with respect to the lexicographic partial ordering. An edge labeling $f : C(P) \to \Gamma$ is called an *$EL-$labeling*, if

1. in every interval $[x, y] \subseteq P$ of rank $k > 0$ there exists a unique maximal chain $\mathfrak{c}$ such that $f(\mathfrak{c}) \in \Gamma^k$ is increasing,

2. the Jordan-Hölder sequence $f(\mathfrak{c}) \in \Gamma^k$ of the unique chain $\mathfrak{c}$ from (1) is the smallest among the Jordan-Hölder sequences of maximal chains $x = x_0 < x_1 < \cdots < x_k = y$.

A poset P is called *EL-shellable*, if it has an $EL-$labeling.

Remark 4. *Let P^{op} and Γ^{op} denote the opposites of $(P, \leq)$ and the total order Γ, respectively. Suppose $f : C(P) \to \Gamma$ is an EL-labeling for $(P, \leq)$. Then, the same underlying map $f : C(P^{op}) \to \Gamma^{op}$ gives an EL-labeling for P^{op}. Therefore, our first main result implies the lexicographic shellability of $(P_n, \preceq)$, also.*

2.2 Rooks and their enumeration.

We set up our notation for rook matrices and establish a preliminary enumerative result.

Let $x = (x_{ij}) \in R_n$ be a rook matrix of size n. Define the sequence $(a_1, \dots, a_n)$ by

$$a_j = \begin{cases} 0 & \text{if the } j\text{'th column consists of zeros,} \\ i & \text{if } x_{ij} = 1. \end{cases} \tag{4}$$

By abuse of notation, we denote both the matrix and the sequence $(a_1, \dots, a_n)$ by x. For example, the associated sequence of the partial permutation matrix

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is $x = (3, 0, 4, 0)$.

Once n is fixed, a rook matrix $x \in R_n$ with k-nonzero entries is called a *k-rook*. Observe that the number of k-rooks is given by the formula

$$|R_{n,k}| = k! \cdot \binom{n}{k}^2. \tag{5}$$

Indeed, to determine a k-rook, we first choose $n-k$ 0 zero rows and $n-k$ 0 columns. This is done in $\binom{n}{n-k}^2$ ways. Next we decide for the non-zero entires of the k-rook. Since deleting the zero rows and columns results in a permutation matrix of size k, there are $k!$ possibilities. Hence, the formula follows.

Let τ_n denote the number of invertible partial involutions. By default, we set $\tau_0 = 1$.

There is no closed formula for τ_n, however, there is a simple recurrence that it satisfies;

$$\tau_{n+1} = \tau_n + (n-1)\tau_{n-1} \ (n \geq 1) \tag{6}$$

There is a similar recurrence satisfied by the number of invertible n-rooks (permutations);

$$(n+1)! = n! + n^2 \cdot (n-1)! \ (n \geq 1). \tag{7}$$

It follows that

Lemma 5. *For all $n \geq 1$,*

1. $|R_{n,n-1} \cup R_{n,n}| = (n+1)!$,
2. $|P_{n,n-1} \cup P_{n,n}| = \tau_{n+1}$.

Proof. The first assertion follows from equations (7) and (5). The second assertion follows from equation (6) and the fact that $|P_{n,n-1}| = (n-1)\tau_{n-1}$. □

2.3 The poset $(P_n, \preceq)$.

We briefly review a combinatorial description of $\preceq$ as described by Bagno and Cherniavsky in [1].

Let $X = (x_{ij})$ be an $n \times m$ matrix. For each $1 \leq k \leq n$ and $1 \leq l \leq m$, denote by X_{kl} the upper-left $k \times l$ submatrix of X. Then the *rank-control matrix* of X is the $n \times m$ matrix $R(X) = (r_{kl})$ with entries given by

$$r_{kl} = rank(X_{kl}),$$

for $1 \leq k \leq n$ and $1 \leq l \leq m$. For example, for the partial involution $x = (1, 0, 3)$, the rank control matrix is

$$R(x) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{8}$$

Given matrices $A = (a_{kl})$ and $B = (b_{kl})$ of the same size with integer entries, we write $A \leq_R B$, if $a_{kl} \leq b_{kl}$ for all k and l.

Knowing the rank-control matrices of partial involutions $x, y \in P_n$ is enough to compare them with respect to $\preceq$:

$$x \preceq y \text{ if and only if } R(x) \leq_R R(y).$$

However, covering relations depend on a numerical invariant associated with the rank control matrices, which is defined as follows.

For any non-negative integer k, define $r_{0,k}$ to be 0. For a rank-control matrix $R(X) = (r_{ij})$, define

$$D(x) = \#\{(i, j) | 1 \leq i \leq j \leq n \text{ and } r_{ij} = r_{i-1,j-1}\}.$$

For example, if $R(x)$ is as in (8), then $D(x) = \#\{(2, 2), (2, 3)\} = 2$. In [1], Bagno and Cherniavsky prove that, in $(P_n, \preceq)$,

$$x \text{ covers } y \Leftrightarrow R(y) \leq_R R(x) \text{ and } D(x) = D(y) + 1.$$

We depict the Hasse diagram of the opposite partial order on partial involutions for $n = 3$ in Figure 3 below.

2.4 An EL-labeling of invertible involutions.

In [8], Incitti shows that the poset of invertible involutions is lexicographically shellable. Let us briefly recall his arguments.

For a permutation $\sigma \in S_n$, a *rise* of σ is a pair $(i, j) \in [n] \times [n]$ such that

$$i < j \text{ and } \sigma(i) < \sigma(j).$$

A rise (i, j) is called *free*, if there is no $k \in [n]$ such that

$$i < k < j \text{ and } \sigma(i) < \sigma(k) < \sigma(j).$$

For $\sigma \in S_n$, define its *fixed point set, its exceedance set* and its *defect set* to be

$$
\begin{aligned}
I_f(\sigma) &= Fix(\sigma) = \{i \in [n] : \sigma(i) = i\}, \\
I_e(\sigma) &= Exc(\sigma) = \{i \in [n] : \sigma(i) > i\}, \\
I_d(\sigma) &= Def(\sigma) = \{i \in [n] : \sigma(i) < i\},
\end{aligned}
$$

respectively.

Given a rise (i, j) of σ, its *type* is defined to be the pair (a, b), if $i \in I_a(\sigma)$ and $j \in I_b(\sigma)$, for some $a, b \in \{f, e, d\}$. We call a rise of type (a, b) an *ab-rise.* On the other hand, two kinds of *ee*-rises have to be distinguished from each other; an *ee*-rise is called *crossing*, if $i < \sigma(i) < j < \sigma(j)$, and it is called *non-crossing*, if $i < j < \sigma(i) < \sigma(j)$.

The rise (i, j) of an involution $\sigma \in I_n$ is called *suitable* if it is free and if its type is one of the following: $(f, f), (f, e), (e, f), (e, e), (e, d)$.

Finally, the *covering transformation*, denoted $ct_{(i,j)}(\sigma)$, of a suitable rise (i, j) of σ is the involution obtained from σ by moving the 1's from the black dots to the white dots as described in Table 1 of [8], depending on the type of (i, j).

It is proven in [8] that, if τ and σ are two involutions in I_n, then

$$
\tau \text{ covers } \sigma \text{ in } \leq \Leftrightarrow \tau = ct_{(i,j)}(\sigma), \text{ for some suitable rise } (i, j) \text{ of } \sigma. \tag{9}
$$

In particular, Incitti shows that the labeling

$$
F((\sigma, c_{(i,j)}(\sigma))) := (i, j) \in [n] \times [n]
$$

is an EL-labeling, hence, $(I_n, \leq)$ is a lexicographically shellable poset. In the next section we give a generalization of this result to the set all partial involutions.

3 Covering relations of partial involutions P_n.

The notion of suitable rise on involutions extends to the partial permutations $(P_n, \leq)$, verbatim. There are additional covering relations. In this section we exhibit all possible covering types.

Lemma 6. *Let x and y be two partial involutions. Then x covers y if and only if one of the following is true:*

1. *x and y have the same zero-rows and columns. Let $\widetilde{x}$ and $\widetilde{y}$ denote the full rank involutions obtained from x and y, respectively, by deleting common zero rows*

and columns. Then x covers y if and only if $\widetilde{x}$ covers $\widetilde{y}$. For example,

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

2. *Without removing a suitable rise, x is obtained from y by one of the following moves:*

   (a) *a 1 on the diagonal is moved down to the first available diagonal entry. It is possible for a 1 to be pushed out of the matrix. For example,*

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (b) *Two off-diagonal symmetric 1's are pushed right/down or down/ right to the first available entries at symmetric positions. There are two cases which we demonstrate by examples:*

      i. $y = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *is covered by* $x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

      ii. $y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *is covered by* $x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

   As a special case of ii., if there are no available entries at symmetric positions to push down and right, then the two 1's at positions (i, j) and (j, i) with $i > j$ are pushed to (i, i), and to the first available diagonal entry below (i, i). For example,

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   In this case, a single 1 is allowed to be pushed out of the matrix. For example, $y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *is covered by* $x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Before we start the proof, let us illustrate by an example, what it means to remove a suitable rise:

Example 7. *Let* $y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *and let* $x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. *Then x is obtained from y by a move as in 2.(b)i., however, it removes the suitable rise* $(1,3)$*. Therefore, it is not a covering relation.*

Proof. Comparing the rank-control matrices $R(\cdot)$ as well as the invariants $D(\cdot)$ of x and y, the "if" direction of the claim is straightforward to verify.

We prove the "only if" direction by contraposition. To this end let x denote a partial involution that covers $y \in P_n$, and x is not obtained by one the moves as in 1., 2.(a), or 2.(b).

Since 1. does not hold, x has a smallest row consisting of zeros such that the corresponding row of y contains a non-zero entry. Let i denote the index of this zero row of x. Notice that, if there is a zero row for both x and y with the same index, then removing this row and the corresponding column does not have any effect on the remaining entries of the rank-control matrices. Therefore, we assume that neither x nor y has a zero row before the i-th row.

There are two subcases;

1) the nonzero entry of y occurs before the i-th column,

2) it occurs on or after the i-th column.

We proceed with 1). Then y and x are as in

$$y = \left(\begin{array}{ccccccc} & & & 0 & & & \\ & A & & \vdots & & B & \\ & & & 1 & & & \\ & & & \vdots & & & \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ & C & & \vdots & & D & \\ & & & 0 & & & \end{array}\right) \text{ and } x = \left(\begin{array}{ccccccc} & & & 0 & & & \\ & A' & & \vdots & & B' & \\ & & & 0 & & & \\ & & & \vdots & & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ & C' & & \vdots & & D' & \\ & & & 0 & & & \end{array}\right),$$

where $A, A'B, B', \ldots$ stand for appropriate size matrices. Let $1 \leq k < i$ denote the index of the row of y with a 1 on its i-th entry.

Let Γ denote the set of coordinates of non-zero entries (r, s) of x satisfying $k \leq r \leq n$ and $i < s \leq n$. Since the upper $k \times n$ portions of both of y and x are rank k, $\Gamma \neq \emptyset$.

Let $(r,s) \in \Gamma$ denote the entry with smallest second coordinate. Unless $r = s$, we define $\tilde{x}$ to be the matrix obtained from x by moving the non-zero entries at the positions (r,s) and (s,r) (which exists, by symmetry) to the positions (r,i) and (i,r). If $r = s$, then $\tilde{x}$ is defined by moving the non-zero entry to the (i,i)-th position.

We claim that $y \leq \tilde{x} < x$. Indeed, since $\tilde{x}$ is obtained from x by reverse of the one of the moves 2.(a) or 2.(b), the second inequality is clear. The first inequality follows immediately from checking the corresponding rank-control matrices of x, $\tilde{x}$ and of y. Let us illustrate the procedure by two possible scenarios:

Example 8. *Let*

$$y = \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix} \text{ and } x = \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix}.$$

Then $i = 5$, $k = 1$, and the rank-control matrices of y and x are

$$R(y) = \begin{pmatrix} 0&0&0&0&1&1&1&1\\ 0&1&1&1&2&2&2&2\\ 0&1&2&2&3&3&3&3\\ 0&1&2&2&3&3&3&4\\ 1&2&3&3&4&4&4&5\\ 1&2&3&3&4&5&5&6\\ 1&2&3&3&4&5&5&6\\ 1&2&3&4&5&6&6&7 \end{pmatrix} \text{ and } R(x) = \begin{pmatrix} 0&0&0&0&0&1&1&1\\ 0&0&1&1&1&2&2&2\\ 0&1&2&2&2&3&3&3\\ 0&1&2&2&2&3&3&4\\ 0&1&2&2&2&3&3&4\\ 1&2&3&3&3&4&4&5\\ 1&2&3&3&3&4&5&6\\ 1&2&3&4&4&5&6&7 \end{pmatrix}.$$

In this case,

$$\tilde{x} = \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix} \text{ and } R(\tilde{x}) = \begin{pmatrix} 0&0&0&0&0&1&1&1\\ 0&0&1&1&1&2&2&2\\ 0&1&2&2&2&3&3&3\\ 0&1&2&2&2&3&3&4\\ 0&1&2&2&3&4&4&5\\ 1&2&3&3&4&5&5&6\\ 1&2&3&3&4&5&5&6\\ 1&2&3&4&5&6&6&7 \end{pmatrix}.$$

If

$$y = \begin{pmatrix} 0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} \text{ and } x = \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

then $i = 4$, $k = 1$ and the rank-control matrices are

$$R(y) = \begin{pmatrix} 0&0&0&0&1&1&1&1\\ 0&1&1&1&2&2&2&2\\ 0&1&2&2&3&3&3&3\\ 0&1&2&2&3&4&4&4\\ 1&2&3&4&4&5&5&5\\ 1&2&3&4&5&6&6&6\\ 1&2&3&4&5&6&7&7\\ 1&2&3&4&5&6&7&8 \end{pmatrix} \text{ and } R(x) = \begin{pmatrix} 0&0&0&0&0&1&1&1\\ 0&0&1&1&1&2&2&2\\ 0&1&2&2&2&3&3&3\\ 0&1&2&2&2&3&3&3\\ 0&1&2&2&2&3&4&4\\ 1&2&3&3&3&4&5&5\\ 1&2&3&3&4&5&6&6\\ 1&2&3&3&4&5&6&6 \end{pmatrix}.$$

In this case,

$$\tilde{x} = \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \text{ and } R(\tilde{x}) = \begin{pmatrix} 0&0&0&0&0&1&1&1\\ 0&0&1&1&1&2&2&2\\ 0&1&2&2&2&3&3&3\\ 0&1&2&2&2&3&4&4\\ 0&1&2&2&2&3&4&4\\ 1&2&3&3&3&4&5&5\\ 1&2&3&4&4&5&6&6\\ 1&2&3&4&4&5&6&6 \end{pmatrix}.$$

We proceed with the case 2) that the non-zero entry of y in its i-th column occurs at the k-th row, where $k \geq i$.

First of all, without loss of generality, we may assume that x has a non-zero entry in its $i+1$-st row, whose column index we denote by j_x.

Denote by $\tilde{y}$ the partial involution obtained from y by interchanging its i-th and $i+1$-st rows as well as interchanging its i-th and $i+1$-st columns. If exists, let j_y denote the column index of the non-zero entry of $\tilde{y}$ in its i-th row. If $j_y < k$, then,

$y < \tilde{y}$. Furthermore, in this case, because i-th row of x consists of 0's, $\tilde{y} < x$. In other words, we have $y < \tilde{y} < x$.

Therefore, we assume that $k < j_y$. In this case, if $k < j_x$, then let $\tilde{x}$ denote the partial involution obtained from x by interchanging its i-th and $i+1$-st rows as well as interchanging its i-th and $i+1$-st columns. Then we have $y < \tilde{x} < x$ and we are done. Therefore, we assume that $k > j_x$. But in this case $y < \tilde{y} < x$ holds. This finishes the proof of the case 2), and we conclude the result.

□

4 EL-labeling of P_n.

In this section we define an edge labeling of P_n and prove that it is an EL-labeling.

1. If the covering relation is derived from a regular covering of an involution, namely from a move that is as in Lemma 6, Part 1., then we use the labeling as defined in [8].

2. If the covering relation results from a move as in Lemma 6 Part 2.(a), namely from a diagonal push where the element that is pushed from is at the position (i, i), then we label it by (i, i).

3. Suppose that a covering relation is as in Lemma 6 (b). Observe that, in all of these covering relations, one of the 1's is pushed down and the other is pushed right. Let i denote the column index of the first 1 that is pushed to the right, and let j denote the index of the resulting column. Then we label the move by (i, j).

To illustrate the third labeling let us present a few examples. Also, see Figure 1 below for the labeling of P_3 which is depicted in one-line notation.

Example 9.

$$y = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding labeling here is $(3, 5)$.

Example 10.

$$y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \textit{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding labeling here is $(1, 3)$.

Example 11.

$$y = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \textit{ is covered by } x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The corresponding labeling here is $(2, 3)$.

If x covers y with label (i, j), then we refer to it as an $(i, j)-$*covering*. Alternatively, we call a covering relation a *c-cover*, if it is derived from an involution; a *d-cover*, if it is obtained by a shift of a diagonal element; an *r-cover*, if it is derived from a right/down or a down/right move.

Let Γ denote the lexicographic total order on the product $[n] \times [n]$. Then, for any $k > 0$, $\Gamma^k = \Gamma \times \cdots \times \Gamma$ is totally ordered with respect to lexicographic ordering. Finally, let $F : C(P_n) \to \Gamma$ denote the labeling function defined above.

For an interval $[x, y] \subseteq P_n$ and a maximal chain $\mathfrak{c} : x = x_0 < \cdots < x_k = y$, we denote by $F(\mathfrak{c})$ the Jordan-Hölder sequence of labels of $\mathfrak{c}$:

$$F(\mathfrak{c}) = (F((x_0, x_1)), \ldots, F((x_{k-1}, x_k))) \in \Gamma^k.$$

Proposition 12. *Let* $\mathfrak{c} : x = x_0 < \cdots < x_k = y$ *be a maximal chain in* $[x, y]$ *such that its Jordan-Hölder sequence* $F(\mathfrak{c})$ *is lexicographically smallest among all Jordan-Hölder sequences (of chains in* $[x, y]$*) in* Γ^k*. Then,*

$$F((x_0, x_1)) \leq F((x_1, x_2)) \leq \cdots \leq F((x_{k-1}, x_k)). \tag{10}$$

Proof. Assume that (10) is not true. Then, there exist three consecutive terms

$$x_{t-1} < x_t < x_{t+1}$$

in $\mathfrak{c}$, such that $F((x_{t-1}, x_t)) > F((x_t, x_{t+1}))$. We have 9 cases to consider.

Case 1: $type(x_{t-1}, x_t) = c$, and $type(x_t, x_{t+1}) = c$.
Case 2: $type(x_{t-1}, x_t) = d$, and $type(x_t, x_{t+1}) = d$.
Case 3: $type(x_{t-1}, x_t) = d$, and $type(x_t, x_{t+1}) = c$.
Case 4: $type(x_{t-1}, x_t) = c$, and $type(x_t, x_{t+1}) = d$.
Case 5: $type(x_{t-1}, x_t) = r$, and $type(x_t, x_{t+1}) = r$.
Case 6: $type(x_{t-1}, x_t) = d$, and $type(x_t, x_{t+1}) = r$.
Case 7: $type(x_{t-1}, x_t) = r$, and $type(x_t, x_{t+1}) = d$.
Case 8: $type(x_{t-1}, x_t) = r$, and $type(x_t, x_{t+1}) = c$.
Case 9: $type(x_{t-1}, x_t) = c$, and $type(x_t, x_{t+1}) = r$.

In each of these 9 cases, we either produce an immediate contradiction by showing that we can interchange the two moves, or we construct an element $z \in [x, y]$ which covers x_{t-1}, and such that $F((x_{t-1}, z)) < F((x_{t-1}, x_t))$. Since we assume that $F(\mathfrak{c})$ is the lexicographically first Jordan-Hölder sequence, the existence of z is a contradiction, too.

Case 1: Straightforward from the fact that type-c covering relations have identical labelings with Incitti's [8].

Case 2: Suppose that the first move is labeled (i, i) and the second one (j, j) with $j < i$. If the two moves are not interchangeable then (j, i) is a legal c- move in x_{t-1}. Since (j, i)-move is lexicographically smaller than (i, i)-move, we derive a contradiction.

Case 3: Let (i, i) be moved to (j, j) in the first step (type d-move), hence $i < j$. If the following c-move does not involve the entry at (j, j), then either the c- and the d-move commute with each other, or the rise for the c-move is not free in x_{t-1}. In that case there has to be an ef-rise involving the entry at the position (i, i). This ef-rise has a smaller label than (i, i), which is a contradiction.

Thus we may assume that the c-move involves the entry at the (j, j)- th position. Then the c-move has to come from either an ff-, an fe-, or an ef-rise.

ff is not possible, because if in $(a, b), a = j$, then $(a, b) > (i, i)$. If $b = j$, then $a < i$ if $(a, b) < (i, i)$. Therefore, the legal c-move (a, i) in x_{t-1} has a smaller label than (i, i). Contradiction..

fe is not possible since (j, b) is greater than (i, i).

Finally, ef is not possible: Let (k, j) be the label of the c-move. If (k, i) is a suitable rise in x_{t-1}, then $(k, i) > (i, i)$. If (k, i) is not a suitable rise in x_{t-1}, let $(j, j), (k, l)$, and (l, k) denote the entries involved in the c-move with $l < k$. Then $l < i < k$ and $(l, j) < (i, i)$. (l, j) is a legal r-move in x_{t-1} with a smaller label than (i, i). Contradiction.

Case 4: This is not possible since no c-move places a 1 on the diagonal such that moving this 1 gives rise to a smaller labeling than the c-move. Note that if there is a 1 on the diagonal before the c-move takes place, then moving this 1 first creates an element z with covering label lexicographically smaller that of the c-move. Thus we are done with this case.

Case 5: It is clear that moving the same 1 twice towards right gives an increasing sequence of labels hence this is not possible. If we move the element to the right which is already moved down via the first move, then switching the order of the moves give a lexicographically smaller sequence, hence a contradiction is obtained. If we move different elements to the right, then, if possible, we perform the second move first. If it is not possible to interchange the order of the r-moves, then by the first r-move a suitable rise is removed. But then the corresponding c-cover has a smaller label in x_{t-1} the r-move.

Case 6: We either perform the r-move first if possible, or perform the c-cover corresponding to the suitable rise removed by d-move which has a smaller label than the d-move in in x_{t-1}.

Case 7: Similar to *Case 6* so we omit the proof.

Case 8: The c-move has to include the elements moved by the previous r-move since otherwise a c-move can be performed first.

If the suitable rise is created by the r-move then the label of the r-move is smaller than the label of the c-move. Otherwise, there is a suitable rise in x_{t-1} involving the elements moved by the r-move. But the c-move corresponding to this suitable rise has a smaller label than the r-move.

Case 9: If the r-move does not involve an element moved by the c-move then perform the r-move first. If this is not possible then a suitable rise is removed by moving it. The c-move corresponding to this suitable rise has a smaller label than the other c-move.

If the r-move involves an element that is placed at this position by the preceding c-move, then we proceed to exhibit every c-move to exclude all of them:

ff: the label of c-move is (i, j). The smaller r-move involving a new element can only be (i, k) with $k < j$. But then (i, i) is possible in x_{t-1} and $(i, i) < (i, j)$.

fe: Similar to *ff* so we omit the proof.

ef: the label of c-move is (i, j). The smaller r-move involving a new element can only be (i, k) with $k < j$. Then (i, k) is possible in x_{t-1} and $(i, k) < (i, j)$.

non-crossing ee, crossing ee and ed are similar to ef so we omit the proof.

□

Proposition 13. *We use the notation of Proposition 12. There exists a unique maximal chain $x = x_0 < \cdots < x_k = y$ with $F((x_0, x_1)) \leq \cdots \leq F((x_{k-1}, x_k))$.*

Proof. We already know that the lexicographically first chain is increasing. Therefore, it is enough to show that there is no other increasing chain. We prove this by induction on the length of the interval $[x, y]$. Clearly, if y covers x, there is nothing to prove. So, we assume that for any interval of length k there exists a unique increasing maximal chain.

Let $[x, y] \subseteq P_n$ be an interval of length $k+1$, and let

$$\mathfrak{c} : x = x_0 < x_1 < \cdots < x_k < x_{k+1} = y$$

be the maximal chain such that $F(\mathfrak{c})$ is the lexicographically first Jordan-Hölder sequence in Γ^{k+1}.

Assume that there exists another increasing chain

$$\mathfrak{c}' : x = x_0 < x_1' < \cdots < x_k' < x_{k+1} = y.$$

Since the length of the chain

$$x_1' < \cdots < x_k' < x_{k+1} = y$$

is k, by the induction hypotheses, it is the lexicographically first chain between x_1' and y.

We are going to find contradictions to each of the following possibilities.

Case 1: $\mathit{type}(x_0, x_1) = c$, and $\mathit{type}(x_0, x_1') = c$,
Case 2: $\mathit{type}(x_0, x_1) = d$, and $\mathit{type}(x_0, x_1') = d$,
Case 3: $\mathit{type}(x_0, x_1) = d$, and $\mathit{type}(x_0, x_1') = c$,
Case 4: $\mathit{type}(x_0, x_1) = c$, and $\mathit{type}(x_0, x_1') = d$,
Case 5: $\mathit{type}(x_0, x_1) = r$, and $\mathit{type}(x_0, x_1') = r$,
Case 6: $\mathit{type}(x_0, x_1) = d$, and $\mathit{type}(x_0, x_1') = r$,
Case 7: $\mathit{type}(x_0, x_1) = r$, and $\mathit{type}(x_0, x_1') = d$,
Case 8: $\mathit{type}(x_0, x_1) = r$, and $\mathit{type}(x_0, x_1') = c$,
Case 9: $\mathit{type}(x_0, x_1) = c$, and $\mathit{type}(x_0, x_1') = r$,

In each of these cases we will construct a partial involution z such that z covers x_1' and $F((x_1', z)) < F((x_1', x_2'))$. Contradiction to the induction hypothesis.

Case 1: Proved in [8].

Case 2: $F(x_0, x_1) = (i, i) < F(x_0, x_1') = (j, j)$ with $i < j$. In x_1' (i, i) is a legal covering move. Hence we have our desired contradiction: $(j, j) \leq F((x_1', x_2')) \leq (i, i)$.

Case 3: $F(x_0, x_1) = (i, i) < F(x_0, x_1') = (j, k)$. There are two cases to consider: $i = j$ and $i < j$. If $i < j$ then we can reverse the order of the d and c move and get to the same contradiction as in *Case 2*. If $i = j$ then $k \neq i + 1$ since otherwise (i, i) wouldn't have been a possible move x_0. The d-cover moves (i, i) to (l, l) where $l < k$. But then (i, l) is a legal move of x'_1 and $(i, l) < (i, k)$ which is a contradiction.

Case 4: $F(x_0, x_1) = (i, j) < F(x_0, x'_1) = (k, k)$. There are two cases to be considered: $j = k$ and $j \neq k$. If $j \neq k$ then $k \notin [i, j]$ since otherwise (i, j) is not a suitable rise. Hence $k > j$. But this means the two covering moves are interchangeable. We get to the same contradiction as in the preceding cases.

If $j = k$ then (i, j) is a legal r-move of x'_1 with $(i, j) < (k, k)$. Contradiction.

Case 5: $F(x_0, x_1) = (i, j) < F(x_0, x'_1) = (k, l)$. $k > i$ since there is at most one legal r-move of each element. We also have $j < l$ since otherwise either (i, k) or (i, l) is a suitable rise with a label less than (i, j). We have two cases to consider:

(a) $i < j < k < l$

(b) $i < k < j < l$

In case (a) both moves are interchangeable.
In case (b) we have that (i, k) is a suitable rise in x_0 with $(i, k) < (i, j)$.

Case 6: $F(x_0, x_1) = (i, i) < F(x_0, x'_1) = (j, k)$. $j \neq i$ by construction. Hence $i < j < k$. Therefore (j, k) does not influence the move (i, i) and we derive a contradiction.

Case 7: $F(x_0, x_1) = (i, j) < F(x_0, x'_1) = (k, k)$. $k > j$ since otherwise a suitable rise is removed by (i, j). But then (i, j) is a legal move of x'_1. Contradiction.

Case 8: $F(x_0, x_1) = (i, j) < F(x_0, x'_1) = (k, l)$. Two cases need to be considered: $i < k$ and $i = k, i < j < l$. If $i < k$ then $j < k$ since otherwise a suitable rise would have been removed by (i, j). But this means that (i, j) is a legal move of x'_1. Contradiction.

If $i = k$ then the c-move corresponds to an ef, non-crossing ee, crossing ee or a ed rise. In each of these cases (i, j) is a legal move of x'_1.

Case 9: $F(x_0, x_1) = (i, j) < F(x_0, x'_1) = (k, l)$. We have two cases to consider:

(a) $i = k, i < j < l$

(b) $i < k$

(a) does not occur because then the r-move removes a suitable rise, hence, it is not a covering relation.
(b) Since we have $i < k < l$ and $i < j$, we consider $i < j < k < l, i < k < j < l$ and $i < k < l < j$. In all these cases the c- and the r-moves are interchangeable. This ends the proofs of our claims.

□

Combining previous two propositions we obtain our first main result.

Theorem 14. *The poset of partial permutations is lexicographically shellable.*

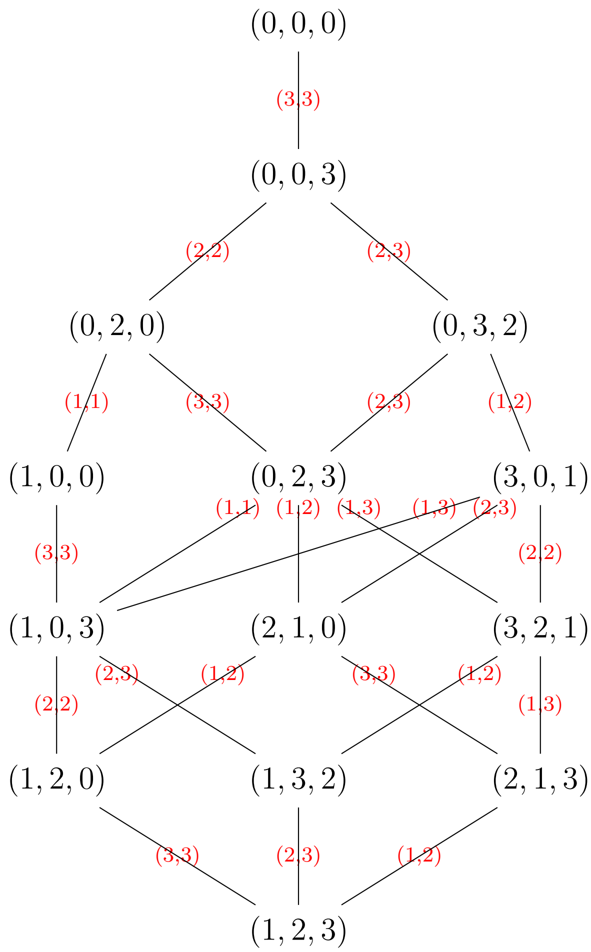

Figure 1: The EL-labeling of P_3.

5 Eulerian intervals.

In this section we prove Theorems 2 and 3.

There is a concrete way to compare two rooks given in one line notation in Bruhat-Chevalley-Renner ordering. For an integer valued vector $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, let

$\widetilde{a} = (a_{\alpha_1}, \ldots, a_{\alpha_n})$ be the rearrangement of the entries $a_1, \ldots, a_n$ of a in a non-increasing fashion;

$$a_{\alpha_1} \geq a_{\alpha_2} \geq \cdots \geq a_{\alpha_n}.$$

The *containment ordering*, "$\leq_c$," on $\mathbb{Z}^n$ is then defined by

$$a = (a_1, \ldots, a_n) \leq_c b = (b_1, \ldots, b_n) \iff a_{\alpha_j} \leq b_{\alpha_j} \text{ for all } j = 1, \ldots, n.$$

where $\widetilde{a} = (a_{\alpha_1}, \ldots, a_{\alpha_n})$, and $\widetilde{b} = (b_{\alpha_1}, \ldots, b_{\alpha_n})$.

Example 15. *Let $x = (4, 0, 2, 3, 1)$, and let $y = (4, 3, 0, 5, 1)$. Then $x \leq_c y$, because*

$$\widetilde{x} = (4, 3, 2, 1, 0) \quad \textit{and} \quad \widetilde{y} = (5, 4, 3, 1, 0).$$

For $k \in [n]$, the *k-th truncation* $a(k)$ of $a = (a_1, \ldots, a_n)$ is defined to be

$$a(k) = (a_1, a_2, \ldots, a_k).$$

Let $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ be two rooks in R_n. It is shown in [6] that

$$v \leq w \iff \widetilde{v(k)} \leq_c \widetilde{w(k)} \text{ for all } k = 1, \ldots, n.$$

Example 16. *Let $x = (0, 1, 2, 3, 4)$, and let $y = (4, 3, 2, 5, 1)$. Then $x \leq y$, because*

$$\begin{aligned}
\widetilde{x(1)} = (0) &\leq_c \widetilde{y(1)} = (4),\\
\widetilde{x(2)} = (1, 0) &\leq_c \widetilde{y(2)} = (4, 3),\\
\widetilde{x(3)} = (2, 1, 0) &\leq_c \widetilde{y(3)} = (4, 3, 2),\\
\widetilde{x(4)} = (3, 2, 1, 0) &\leq_c \widetilde{y(4)} = (5, 4, 3, 2),\\
\widetilde{x(5)} = (4, 3, 2, 1, 0) &\leq_c \widetilde{y(5)} = (5, 4, 3, 2, 1).
\end{aligned}$$

The next lemma, whose proof is omitted, shows that for two permutations x and y of S_n, the inequality $x \leq y$ can be decided in $n - 1$ steps.

Lemma 17. *Let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be two permutations in S_n. Then $x \leq y$ if and only if*

$$\widetilde{x(k)} \leq_c \widetilde{y(k)} \textit{ for } k = 1, \ldots, n - 1.$$

We are ready to prove the first half of Theorem 3.

Proposition 18. *The union $(R_{n,n-1} \cup R_{n,n}, \leq)$ is isomorphic to the poset $(S_{n+1}, \leq)$.*

We depict the isomorphism between S_4 and $R_{3,3} \cup R_{3,2}$ in Figure 2.

Proof. Let u and w denote the rooks $u = (0, 1, 2, \ldots, n)$ and $w = (n, n-1, \ldots, 2, 1)$. Then $R_{n,n-1} \cup R_n = [u, w]$.

We define a map ψ between $[v, w]$ and S_{n+1} as follows. If $x = (a_1, \ldots, a_n) \in [v, w]$, then

$$\psi(x) = (a_1 + 1, a_2 + 1, \ldots, a_n + 1, a_x), \tag{11}$$

where a_x is the unique element of the set

$$[n+1] \setminus \{a_1 + 1, a_2 + 1, \ldots, a_n + 1\}.$$

We have two immediate observations.

1. If x is already a permutation (in $R_{n,n}$), then $a_x = 1$.
2. ψ is injective, hence by Lemma 5, it is bijective as well.

Now, let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be two elements in $[v, w]$ such that $x \leq y$. For the sake of brevity, denote the "shifted" sequence $(a_1 + 1, \ldots, a_n + 1)$ associated with x by x'. Since increasing each entry of x and y by 1 does not change the relative sizes of the entries of x and y, we have

$$x' \leq y'.$$

Recall that this is equivalent to saying that $\widetilde{x'(k)} \leq_c \widetilde{y'(k)}$ for all $k = 1, \ldots, n$. Since, x' is the n-th truncation $\psi(x)(n)$ of the permutation $\psi(x)$, the proof of the theorem is complete by considering Lemma 17. The converse statement "$\psi(x) \leq \psi(y) \implies x \leq y$" follows from the same argument. Therefore, ψ is a poset isomorphism. $\square$

Unfortunately, the map ψ defined in (11) does not restrict to partial involutions nicely enough, therefore, we need another order preserving injection in $P_{n,n-1} \cup P_n$ onto I_{n+1}.

Let $u = (0, n, n-1, \ldots, 2)$ and let $\iota = (1, 2, \ldots, n)$. Observe that the rank-control matrix of u is the smallest, and that the rank-control matrix of ι is the largest among all elements of $P_{n,n-1} \cup P_{n,n}$. Therefore, the union $P_{n,n-1} \cup P_{n,n}$ is the underlying set of the interval $[\iota, u]$ of P_n.

Let $x = (a_1, \ldots, a_n) \in [\iota, u]$ be given in one-line notation. Then there are two cases:

1. there is an $i \in [n]$ such that $a_i = 0$,

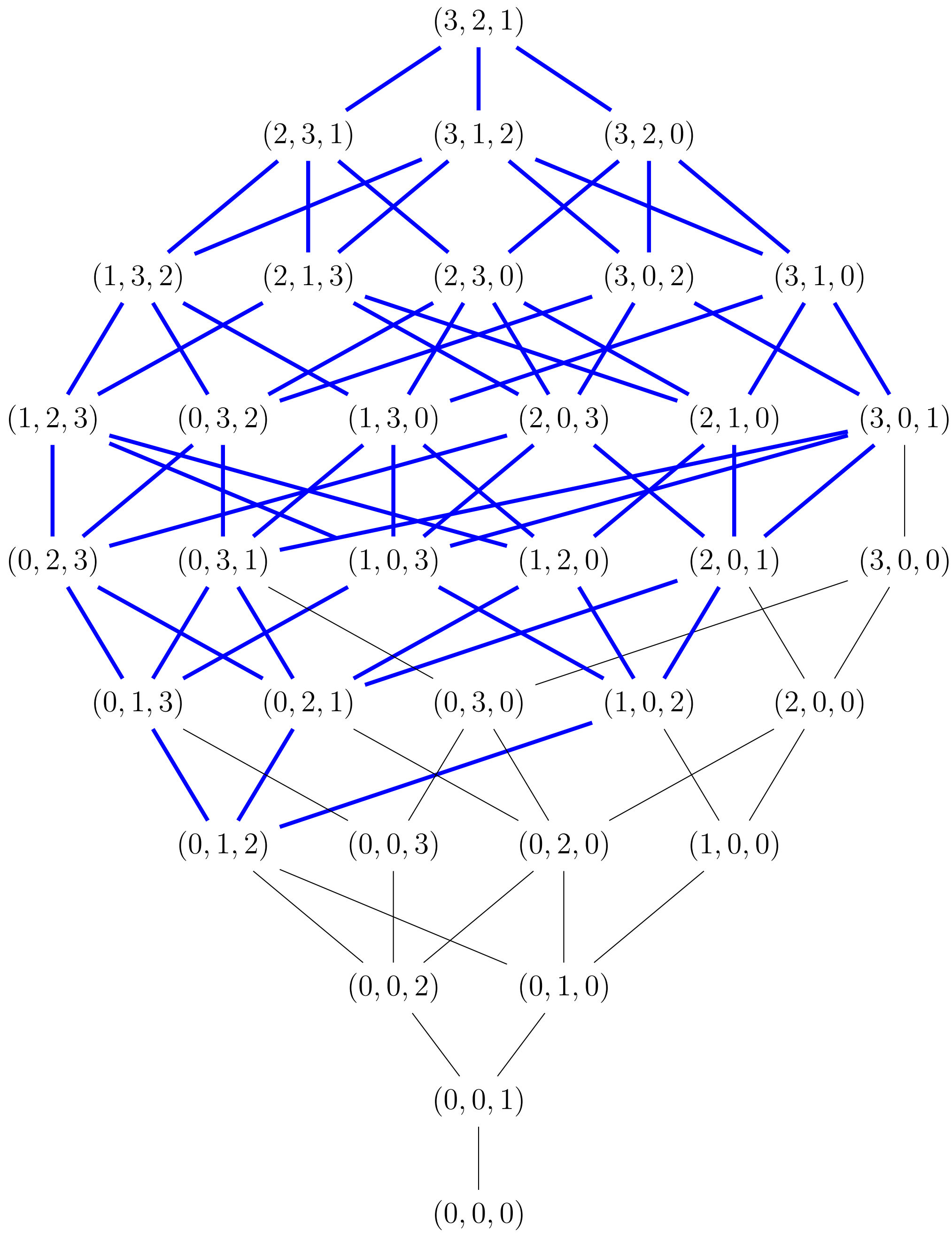

Figure 2: S_4 in $(R_3, \leq)$.

2. x is a permutation.

We start with the first case. If $a_i = 0$ for some $i \in [n]$, then we define $b_i = n+1$ and for $j \in [n] \setminus \{i\}$ we set $b_j = a_j$. In addition, in this case, we define b_{n+1} to be the unique element of the set $\{0, 1, \dots, n\} - \{a_1, \dots, a_n\}$. If latter case, we set $b_j = a_j$ for $j = 1, \dots, n$ and define $b_{n+1} = n+1$. Finally, we define $\phi : [\iota, u] \to I_{n+1}$ by

$$\phi(x) = (b_1, \dots, b_{n+1}). \tag{12}$$

For example,

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \phi(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We are ready to prove the second half of Theorem 3.

Proposition 19. *The union $(P_{n,n-1} \cup P_{n,n}, \leq)$ is isomorphic to the poset $(I_{n+1}, \leq)$.*

We depict the isomorphism between I_4 and $P_{3,3} \cup P_{3,2}$ in Figure 3.

Proof. Let ϕ be defined as in (12). By its construction, ϕ is injective. Therefore, by Lemma 5, Part 2., it is enough to show that ϕ is order preserving.

Let x and y be two elements in $[\iota, u]$ such that $x \leq y$. Then $R(y) \leq_R R(x)$.

Note that the upper-left $n \times n$ portion of the rank-control matrix of $\phi(x)$ is equal to $R(x)$. The same is true for $\phi(y)$ and $R(y)$.

Let $R^{\phi(x)}_{i,j}$ denote the (i,j)-th entry of $R(\phi(x))$. Then, since $\phi(x)$ is a permutation in I_{n+1}, we have

$$R^{\phi(x)}_{n+1,i} = i \text{ and } R^{\phi(x)}_{j,n+1} = j$$

for all $i, j \in [n+1]$. The same is true for $R(\phi(y))$. Therefore,

$$R(\phi(y)) \leq_R R(\phi(x))$$

and the proof is complete. □

It follows from Propositions 18 and 19 that Theorem 3 is true;

$$R_{n,n-1} \cup R_{n,n} \cong S_{n+1} \text{ and } P_{n,n-1} \cup P_{n,n} \cong I_{n+1}.$$

Next we prove Theorem 2, which states that $R_{n,k}$ and $P_{n,k}$ are Eulerian if and only if $k = n$ or $k = n-1$.

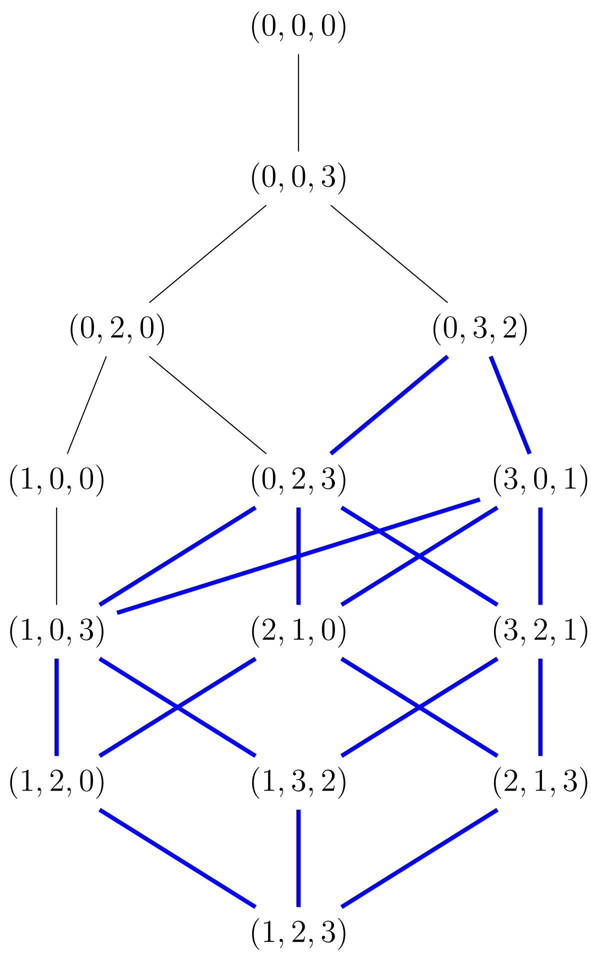

Figure 3: I_4 in $(P_3, \leq)$.

First of all, $R_{n,n} \cong S_n$, and by Theorem 3, $R_{n,n-1}$ is isomorphic to an interval in S_{n+1}. Thus, both $R_{n,n}$ and $R_{n,n-1}$ are Eulerian. The same argument is true for both of the posets $P_{n,n}$ and $P_{n,n-1}$. Therefore, to finish the proof of Theorem 2, it is enough to show that, for $k \neq n, n-1$, $R_{n,k}$ and $P_{n,k}$ are not Eulerian. To this end, for $k \leq n-2$, let v_k, v_k' and v_k'' denote the elements

$$\begin{aligned} v_k &= (0, \dots, 0, 0, 1, 2, \dots, k), \\ v_k' &= (0, \dots, 0, 1, 0, 2, \dots, k), \\ v_k'' &= (0, \dots, 1, 0, 0, 2, \dots, k) \end{aligned}$$

in $R_{n,k}$. Then the interval $[v_k, v_k''] \subset R_{n,k}$ has exactly three elements v_k, v_k', v_k'', hence it cannot be Eulerian.

Similarly, for $k \leq n-2$, let u_k, u_k' and u_k'' denote the elements

$$\begin{aligned} u_k &= (1, 2, \dots, k, 0, \dots, 0), \\ u_k' &= (1, 2, \dots, k-1, 0, k+1, 0, \dots, 0), \\ u_k'' &= (1, 2, \dots, k-1, 0, 0, k+2, 0, \dots, 0) \end{aligned}$$

in $I_{n,k}$. Then the interval $[u_k, u_k''] \subset P_{n,k}$ has exactly three elements u_k, u_k', u_k'', and therefore, it cannot be Eulerian. This finishes the proof of Theorem 2.

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