C-SORTABLE WORDS AS GREEN MUTATION SEQUENCES

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Abstract. Let Q be an acyclic quiver and $\mathbf{s}$ be a sequence $\mathbf{s}$ with elements in the vertex set Q_0. We describe a sequence of simple (backward) tilting in the bounded derived category $\mathcal{D}(Q)$, starting from the standard heart $\mathcal{H}_Q = \operatorname{mod} \mathbf{k}Q$ and ending at the heart $\mathcal{H}_{\mathbf{s}}$ in $\mathcal{D}(Q)$. Then we interpret Keller's green mutation via King-Qiu's Ext-quiver of hearts, which provides a proof of Keller's theorem, that $\mathbf{s}$ is a green mutation sequence if and only if every heart in the simple tilting sequence is greater than or equal to $\mathcal{H}_Q[-1]$; it is maximal if and only if $\mathcal{H}_{\mathbf{s}} = Q[-1]$. Further, fix a Coxeter element c in the Coxeter group W_Q of Q, which is admissible with respect to the orientation of Q. We show that the sequence $\widetilde{\mathbf{w}}$ induced by a c-sortable word $\mathbf{w}$ is a green mutation sequence. As a consequence, we obtain a bijection between the set of c-sortable words and finite torsion class in $\mathcal{H}_Q$, which was first proved by Thomas and was also obtained by Amiot-Iyama-Reiten-Todorov. As byproducts, the interpretations of inversions, descents and cover reflections of a c-sortable word $\mathbf{w}$, and thus noncrossing partitions, as well as the wide subcategories associated to $\mathcal{H}_{\mathbf{w}}$, are given in terms of non-green vertices.

Key words: Coxeter group, quiver mutation, cluster theory, exchange graph

Introduction

Cluster algebra was invented by Fomin-Zelevinsky in 2000, attempting to understand total positivity in algebraic groups and canonical bases in quantum groups. It has been heavily studied during the last decant due to its wide connection to many areas in mathematics, (for more details, see the introduction survey [6]). The combinatorial ingredient in the cluster theory is quiver mutation, which leads to the categorification of cluster algebra via quiver representation theory due to Buan-Marsh-Reineke-Reiten-Todorov in 2005.

Recently, Keller spotted a remarkable special case of quiver mutation by adding certain restrictions, known as the green quiver mutation (Definition 2.2); using which, he obtained results concerning Kontsevich-Soibelman's noncommutative Donaldson-Thomas invariant via quantum cluster algebras. Inspired by Keller [5] and Nagao [7], King-Qiu [8] studied the exchange graphs of hearts and clusters in various categories associated to cluster categories, with applications to stability conditions and quantum dilogarithm identities in [9].

Our first aim in this paper is to explain Keller's green mutation result in terms of tilting. More precisely, a green sequence $\mathbf{s}$ induces a path $\mathrm{P}(\mathbf{s})$ in the exchange graph Eg_Q (cf. Definition 1.2), that is, a sequence of simple (backward) tilting. Thus

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$\mathbf{s}$ corresponds to a heart $\mathcal{H}_{\mathbf{s}}$, which provides a proof of Keller's theorem. Here is a summarization of the results in Section 2.

Theorem 0.1. *Let Q be an acyclic quiver.*

- *A sequence $\mathbf{s}$ is a green mutation sequence if and only if $\mathcal{H} \geq \mathcal{H}_Q[-1]$ for any $\mathcal{H}$ in the path $\mathrm{P}(\mathbf{s})$.*
- *A vertex $j \in Q_0$ is a green vertex for some green mutation sequence $\mathbf{s}$ if and only if the corresponding simple in $\mathcal{H}_{\mathbf{s}}$ is in $\mathcal{H}_Q$.*
- *A green sequence $\mathbf{s}$ is maximal if and only if $\mathcal{H}_{\mathbf{s}} = \mathcal{H}_Q[-1]$. Hence the mutated quivers associated to two maximal green mutation sequences are isomorphic.*
- *The simples of the wide subcategory $\mathcal{W}\mathbf{s}$ associated to the torsion class $\mathcal{T}_{\mathbf{s}}$ are precisely the non-green simples in of $\mathcal{H}_{\mathbf{s}}$ shifting by minus one.*

Our second focus is on c-sortable words (c for Coxeter element), defined by Reading [10], who showed bijections between c-sortable words, c-clusters and noncrossing partitions in finite case (Dynkin case). Ingalls-Thomas extended Reading's result in the direction of representation theory and gave bijections between many sets (see [4, p. 1534]). The bijection between c-sortable words and finite torsion classes was first generalized by Thomas [11] and also obtained by Amiot-Iyama-Reiten-Todorov [1] via layers for preprojective algebras. We will interpret a c-sortable word as a green mutation sequence (Theorem 3.1) and obtain many consequences, summarized by the following theorem.

Theorem 0.2. *For an acyclic quiver Q with an admissible Coxeter element c. Then any c-sortable word $\mathbf{w}$ induces a green mutation sequence $\widetilde{\mathbf{w}}$ and we have the following bijections.*

- $\{$*the c-sortable word* $\mathbf{w}\} \overset{1-1}{\longleftrightarrow} \{$*the finite torsion class* $\mathcal{T}_{\mathbf{w}}$ *in* $\mathcal{H}_Q = \operatorname{mod} \mathbf{k}Q\}$.
- $\{$*the inversion* t_T *for* $\mathbf{w}\} \overset{1-1}{\longleftrightarrow} \{$*the indecomposable* T *in* $\mathcal{T}_{\mathbf{w}}\}$.
- $\{$*the descent* s_j *for* $\mathbf{w}\} \overset{1-1}{\longleftrightarrow} \{$*the non green vertex* j *for* $\mathbf{w}\}$.
- $\{$*the cover reflection* t_T *for* $\mathbf{w}\} \overset{1-1}{\longleftrightarrow} \{$*the non-green simple* T *in* $\mathcal{H}_{\mathbf{w}}\}$.

Further, if Q is of Dynkin type, the noncrossing partition $\mathrm{nc}_c(\mathbf{w})$ associated to $\mathbf{w}$ can be calculated as

$$\mathrm{nc}_c(\mathbf{w}) = \prod_{j \in \mathrm{V}^{\text{-}}(\mathbf{w})} s_j^{\mathbf{w}},$$

with $\operatorname{rank} \mathrm{nc}_c(\mathbf{w}) = \# \mathrm{V}^{\text{-}}(\mathbf{w})$, where $\mathrm{V}^{\text{-}}(\mathbf{w})$ is the set of the non-green vertices and $s_j^{\mathbf{w}}$ is the reflection corresponding the j-th simple in $\mathcal{H}_{\mathbf{w}}$. Also, the tree of c-sortable words (with respect to the week order) is isomorphic to a supporting tree of the exchange graph Eg_Q.

These results give a deeper understanding of the results of Ingalls-Thomas [4]. Note that all our bijections are consistent with theirs, cf. Table 1 and [4, Table 1]. Also, the construction from c-sortable words to the green mutation sequences should be the 'dual' construction of Amiot-Iyama-Reiten-Todorov [1] and provides a combinatorial perspective to attack their problems at end of their paper.

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1. Preliminaries

Fixed an algebraically-closed field $\mathbf{k}$. Throughout this paper, Q will be a finite acyclic quiver with vertex set $Q_0 = \{1, \ldots, n\}$ (unless otherwise states). The path algebra is denoted by $\mathbf{k}Q$. Let $\mathcal{H}_Q := \operatorname{mod} \mathbf{k}Q$ be the category of finite dimensional $\mathbf{k}Q$-modules, which is an abelian category, and $\mathcal{D}(Q) := \mathcal{D}^b(\mathcal{H}_Q)$ be its bounded derived category, which is a triangulated category. We denote by $\operatorname{Sim} \mathcal{A}$ a complete set of non-isomorphic simples in an abelian category $\mathcal{A}$ and let

$$\operatorname{Sim} \mathcal{H}_Q = \{S_1, \ldots, S_n\},$$

where S_i is the simple $\mathbf{k}Q$-module corresponding to vertex $i \in Q_0$.

1.1. **Coxeter group and words.** Recall that the *Euler form*

$$\langle -, - \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

associated to the quiver Q is defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i \in Q_0} a_i b_i - \sum_{(i \to j) \in Q_1} a_i b_j.$$

Denote by $(-, -)$ the symmetrized Euler form, i.e. $(\mathbf{a}, \mathbf{b}) = \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{a} \rangle$. Moreover for $M, L \in \operatorname{mod} \mathbf{k}Q$, we have

$$\langle \underline{\dim} M, \underline{\dim} L \rangle = \underline{\dim} \operatorname{Hom}(M, L) - \underline{\dim} \operatorname{Ext}^1(M, L), \tag{1.1}$$

where $\underline{\dim} E \in \mathbb{N}^{Q_0}$ is the *dimension vector* of any $E \in \operatorname{mod} \mathbf{k}Q$. Let $V = K(\mathbf{k}Q) \otimes \mathbb{R}$. For any non-zero $v \in V$, define a *reflection*

$$s_v(u) = u - \frac{2(v, u)}{(u, u)} v.$$

We will write $s_M = s_{\underline{\dim} M}$ for $M \in \mathcal{H}_Q \sqcup \mathcal{H}_Q[-1]$.

The *Coxeter group* $W = W_Q$ is the group of transformations generated by the *simple reflections* $s_i = s_{\underline{\dim} S_i}, i \in Q_0$. The (real) *roots* in W are $\{w(e_i) \mid w \in W, i \in Q_0\}$; the *positive roots* are those root which are a non-negative (integral) combination of the e_i. Note that, the reflection of a positive root is in W. Denote by T the set of all the reflections of W, that is, the set of all conjugates of the simple reflections of W. A *Coxeter element* for W is the product of the simple reflections in some order. For a Coxeter element $c = s_{\sigma_1} \ldots s_{\sigma_n}$, we say it is *admissible* with respect to the orientation of Q, if there is no arrow from σ_i to σ_j in Q for any $i > j$.

A word $\mathbf{w}$ in W is an expression in the free monoid generated by $s_i, i \in Q_0$. For $w \in W$, denote by $l(w)$ its *length*, that is, the length of the shortest word for w as a product of simple reflections. A *reduced* word $\mathbf{w}$ for an element $w \in W$ is a word such that $\mathbf{w} = w$ with minimal length. The notion of reduced word leads to the *weak order* $\leq$ on W, i.e. $x \leq y$ if and only if x has a reduced expression which is a prefix of some reduced word for y.

For a word $\mathbf{w}$ in W_Q, we have the following notions.

- An *inversion* of $\mathbf{w}$ is a reflection t such that $l(t\mathbf{w}) \leq l(\mathbf{w})$. The set of inversions of $\mathbf{w}$ is denoted by $\mathrm{Inv}(w)$.
- A *descent* of $\mathbf{w}$ is a simple reflection s such that $l(\mathbf{w}s) \leq l(\mathbf{w})$. The set of descent of $\mathbf{w}$ is denoted by $\mathrm{Des}(w)$.
- A *cover reflection* of w is a reflection t such that $t\mathbf{w} = \mathbf{w}s$ for some descent s of $\mathbf{w}$. The set of cover reflections of $\mathbf{w}$ is denoted by $\mathrm{Cov}(w)$.

Definition 1.1 (Reading, [10])**.** For a word $a = a_1 \ldots a_k$, define the support $\mathrm{supp}(a)$ to be $\{a_1, \ldots, a_k\}$. Fix a Coxeter element $c = s_{\sigma_1} \ldots s_{\sigma_n}$. A word $\mathbf{w}$ is call *c-sortable* if it has the form $\mathbf{w} = c^{(0)}c^{(1)} \ldots c^{(m)}$, where $c^{(i)}$ are subwords of c satisfying

$$\mathrm{supp}(c^{(0)}) \subseteq \mathrm{supp}(c^{(1)}) \subseteq \cdots \subseteq \mathrm{supp}(c^{(m)}) \subseteq Q_0.$$

Similarly to normal words, a T-word is an expression in the free monoid generated by elements in the set T of all reflections. Denote by by $l_T(w)$ its *absolute length*, that is, the length of the shortest word for w as a product of arbitrary reflections. So we have the notion of reduced T-words, which induces the *absolute order* $\leq_{\mathrm{T}}$ on W.

The *noncrossing partitions*, with respect to a Coxeter element c, for W are elements between the identity and c, with respect to the absolute order. The *rank* of a noncrossing partition is its absolute length.

1.2. **Hearts and tilting.** Recall that, a *t-structure* on a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{P} \subset \mathcal{D}$, satisfying $\mathcal{P}[1] \subset \mathcal{P}$ and being the torsion part of some torsion pair (with respect to triangles) $\langle \mathcal{P}, \mathcal{P}^{\perp} \rangle$ in $\mathcal{D}$. A t-structure $\mathcal{P}$ is *bounded* if, for every object M, the shifts $M[k]$ are in $\mathcal{P}$ for $k \gg 0$ and in $\mathcal{P}^{\perp}$ for $k \ll 0$. The *heart* of a t-structure $\mathcal{P}$ is the full subcategory $\mathcal{H} = \mathcal{P}^{\perp}[1] \cap \mathcal{P}$ and any bounded t-structure is determined by its heart.

By the tilting theory in the sense of Happel-Reiten-Smalø, for any heart $\mathcal{H}$ (in a triangulated category) with torsion pair $\langle \mathcal{F}, \mathcal{T} \rangle$, there exists the following two hearts with torsion pairs

$$\mathcal{H}^{\sharp} = \langle \mathcal{T}, \mathcal{F}[1] \rangle, \quad \mathcal{H}^{\flat} = \langle \mathcal{T}[-1], \mathcal{F} \rangle.$$

We call $\mathcal{H}^{\sharp}$ the *forward tilt* of $\mathcal{H}$ with respect to the torsion pair $\langle \mathcal{F}, \mathcal{T} \rangle$, and $\mathcal{H}^{\flat}$ the *backward tilt* of $\mathcal{H}$. Clearly $\mathcal{H}^{\flat} = \mathcal{H}^{\sharp}[-1]$.

We say a forward tilt is *simple*, if the corresponding torsion free part is generated by a single rigid simple object S. We denote the new heart by $\mathcal{H}^{\sharp}_S$. Similarly, a backward tilt is simple if the corresponding torsion part is generated by such a simple and the new heart is denoted by $\mathcal{H}^{\flat}_S$. The simple tilting leads to the notation of exchange graphs.

Definition 1.2. [8] The *exchange graph* $\mathrm{Eg}\,\mathcal{D}(Q)$ of a triangulated category $\mathcal{D}$ to be the oriented graph, whose vertices are all hearts in $\mathcal{D}$ and whose edges correspond to the simple ***backward*** titling between them. We denote by $\mathrm{Eg}^{\circ}\,\mathcal{D}(Q)$ the 'principal' component of $\mathrm{Eg}\,\mathcal{D}(Q)$, that is, the connected component containing the heart $\mathcal{H}_Q$. Further more, denote by Eg_Q full subgraph of $\mathrm{Eg}^{\circ}\,\mathcal{D}(Q)$ consisting of those hearts which are backward tilts of $\mathcal{H}_Q$.

By [8], every heart in $\mathrm{Eg}^{\circ}\,\mathcal{D}(Q)$ is finite and rigid (i.e. has finite many simple, each of which is rigid) and

$$\mathrm{Eg}_Q = \{\mathcal{H} \in \mathrm{Eg}^{\circ}(Q) \mid \mathcal{H}_Q[-1] \leq \mathcal{H} \leq \mathcal{H}_Q\}.$$

Remark 1.3. Unfortunately, we take a different convention to King-Qiu [8] (backward tilting instead of forward). Thus a exchange graph Eg in this paper has the opposite orientation of the exchange graph EG there.

1.3. **Simple (backward) tilting sequence.** Let $\mathbf{s} = i_1 \dots i_m$ be a sequence with $i_j \in Q_0$ and we have a sequence of hearts $\mathcal{H}_{\mathbf{s},j}$ with simples

$$\operatorname{Sim} \mathcal{H}_{\mathbf{s},j} = \{S_i^{\mathbf{s},j} \mid i \in Q_0\}, \quad 0 \le j \le m,$$

inductively defined as follows.

- $\mathcal{H}_{\mathbf{s},0} = \mathcal{H}_Q$ with $S_i^{\mathbf{s},0} = S_i$ for any $i \in Q_0$.
- For $0 \le j \le m-1$, we have

$$\mathcal{H}_{\mathbf{s},j+1} = (\mathcal{H}_{\mathbf{s},j})^\flat_{S_j^{\mathbf{s},j}}.$$

Note that $\operatorname{Sim} \mathcal{H}_{\mathbf{s},j+1}$ is given by formula [8, Proposition 5.2 (5.2)] in terms of $\operatorname{Sim} \mathcal{H}_{\mathbf{s},j}$, which inherited a labeling by Q_0. Define

$$\mathcal{H}_{\mathbf{s}} = \mathcal{H}_{\mathbf{s}}(Q) \colon = \mathcal{H}_{\mathbf{s},m}$$

and $\mathrm{P}(\mathbf{s})$ to be the path $\mathrm{P}(\mathbf{s}) = T_m^{\mathbf{s}} \cdots T_1^{\mathbf{s}}$ as follow

$$\mathrm{P}(\mathbf{s}) =\colon \mathcal{H}_Q \, \mathcal{H}_{\mathbf{s},0} \xrightarrow{T_1^{\mathbf{s}}} \mathcal{H}_{\mathbf{s},1} \xrightarrow{T_2^{\mathbf{s}}} \dots \xrightarrow{T_m^{\mathbf{s}}} \mathcal{H}_{\mathbf{s},m} = \mathcal{H}_{\mathbf{s}},$$

in $\operatorname{Eg}^\circ \mathcal{D}(Q)$, where $T_j^{\mathbf{s}} = S_j^{\mathbf{s},j-1}$ is the j-th simple in $\mathcal{H}_{\mathbf{s}_{j-1}}$. As usual, the support $\operatorname{supp} \mathrm{P}(\mathbf{s})$ of $\mathrm{P}(\mathbf{s})$ is the set $\{T_1^{\mathbf{s}}, \dots, T_m^{\mathbf{s}}\}$.

Remark 1.4. Note that, for a heart $\mathcal{H}$ in $\operatorname{Eg}^\circ \mathcal{D}(Q)$ other than $\mathcal{H}_Q$, there is no prior labeling for its simples by Q_0. Different sequences of simple tilting might induce different labeling of simples in $\operatorname{Sim} \mathcal{H}$ (cf. Figure 1).

2. Green mutation

In this section, the interpretation of the green mutation via King-Qiu's Ext-quiver of heart is given, which provides a proof of Keller's theorem.

2.1. **Green quiver mutation.**

Definition 2.1 (Fomin-Zelevinsky)**.** Let R be a finite quiver without loops or 2-cycles. The *mutation* μ_k on R at vertex k is a quiver $R' = \mu_k(R)$ obtaining from R as follows

- adding an arrow $i \to j$ for any pair of arrows $i \to k$ and $k \to j$ in R;
- reversing all arrows incident with k;
- deleting as many 2-cycles as possible.

It is straightforward to see that the mutation is an involution, i.e. $\mu_k^2 = id$. A *mutation sequence* $\mathbf{s} = i_1 \dots i_m$ on R is a sequence with $i_j \in R_0$ and define

$$R_{\mathbf{s}} \colon = \mu_{\mathbf{s}}(R) = \mu_{i_m}(\mu_{i_{m-1}}(\dots \mu_{i_1}(R) \dots)).$$

As in Section 1.3, a (green) mutation sequence $\mathbf{s}$ induces a sequence of simple (backward) tilting and a heart $\mathcal{H}_{\mathbf{s}}$.

Let $\widetilde{Q}$ be the *principal extension* of Q, i.e. the quiver obtained from Q by adding a new frozen vertex i' and a new arrow $i' \to i$ for each vertex $i \in Q_0$. Note that we will never mutate a quiver at a frozen vertex and so mutation sequences $\mathbf{s}$ for $\widetilde{Q}$ are precisely mutation sequences of Q.

Definition 2.2 (Keller [5])**.** Let $\mathbf{s}$ be a mutation sequence of $\widetilde{Q}$. A vertex j in the quiver $\widetilde{Q}_{\mathbf{s}}$ is called *green* if there is no arrows from j to a frozen vertex i'. Let $\mathrm{V}^{\text{-}}(\mathbf{s})$ be the set of non-green vertices in $\widetilde{Q}_{\mathbf{s}}$ for $\mathbf{s}$. A *green mutation sequence* $\mathbf{s}$ on Q (or $\widetilde{Q}$) is a mutation sequence on Q such that every mutation in the sequence is at a green vertex in the corresponding quiver. Such a green mutation sequence $\mathbf{s}$ is *maximal* if $\mathrm{V}^{\text{-}}(\mathbf{s}) = \emptyset$.

2.2. **Principal extension of Ext-quivers.** Following [8], we will also use Ext-quivers of hearts interpret green mutation.

Definition 2.3 (King-Qiu)**.** Let $\mathcal{H}$ be a finite heart in a triangulated category $\mathcal{D}$ with $\mathbf{S}_{\mathcal{H}} = \bigoplus_{S \in \operatorname{Sim} \mathcal{H}} S$. The Ext-quiver $\mathcal{Q}(\mathcal{H})$ is the (positively) graded quiver whose vertices are the simples of $\mathcal{H}$ and whose graded edges correspond to a basis of $\operatorname{End}^{\bullet}(\mathbf{S}_{\mathcal{H}}, \mathbf{S}_{\mathcal{H}})$. Further, define the *CY-N double* of a graded quiver $\mathcal{Q}$, denoted by $\mathrm{CY}^N(\mathcal{Q})$, to be the quiver obtained from $\mathcal{Q}$ by adding an arrow $T \to S$ of degree $N-k$ for each arrow $S \to T$ of degree k and adding a loop of degree N at each vertex.

As the principal extension of a quiver Q, we consider the analogue for $\mathcal{H}_Q$. Since Q is a subquiver of its extension $\widetilde{Q}$, $\mathcal{H}_Q$ and $\mathcal{D}(Q)$ are subcategories of $\mathcal{H}_{\widetilde{Q}}$ and $\mathcal{D}(\widetilde{Q})$ respectively. For a sequence $\mathbf{s}$, it also induces a simple tilting sequence in $\mathcal{D}(\widetilde{Q})$ (starting at $\mathcal{H}_{\widetilde{Q}}$) and corresponds to a heart, denoted by $\widetilde{\mathcal{H}_{\mathbf{s}}}$.

Let the set of simples in $\operatorname{Sim} \mathcal{H}_{\widetilde{Q}} - \operatorname{Sim} \mathcal{H}_Q$ be

$$\operatorname{Sim} \mathcal{H}_{Q'} :\ = \{S_i' \mid i \in Q_0\}.$$

A straightforward calculation gives

$$\operatorname{Hom}^k(S_i', S_j) = \delta_{ij}\delta_{1k}, \quad \forall i, j \in Q_0, k \in \mathbb{Z}.$$

Hence, for any $M \in \mathcal{H}_Q$, we have

$$\operatorname{Hom}^k(\bigoplus_{i \in Q_0} S_i', M) \neq 0 \quad \iff \quad k = 1. \tag{2.1}$$

We have the following lemma.

Lemma 2.4. *For any sequence* $\mathbf{s}$*, we have* $\operatorname{Sim} \widetilde{\mathcal{H}_{\mathbf{s}}} = \operatorname{Sim} \mathcal{H}_{\mathbf{s}} \cup \operatorname{Sim} \mathcal{H}_{Q'}$.

Proof. Use induction on the length of $\mathbf{s}$ starting from the trivial case when $\mathbf{s} = \emptyset$. Suppose that $\mathbf{s} = \mathbf{t}j$ with $\operatorname{Sim} \widetilde{\mathcal{H}_{\mathbf{t}}} = \operatorname{Sim} \mathcal{H}_{\mathbf{t}} \cup \operatorname{Sim} \mathcal{H}_{Q'}$. By [8, Lemma 3.4], we have $\mathcal{H}_{\mathbf{t}} \leq \mathcal{H}_Q$ and hence the homology of any object in $\mathcal{H}_t$, with respect to $\mathcal{H}_Q$, lives in non-positive degrees. Thus, any $M \in \mathcal{H}_{\mathbf{t}}$ admits a filtration with factors $S_i[k], i \in Q_0, k \leq 0$. As s' is a source in $\widetilde{Q}$ for any $s \in Q_0$, S_s' is an injective in $\mathcal{H}_{\widetilde{Q}}$ which implies that $\operatorname{Ext}^1(S_i[k], S_s') = 0$ for any $i \in Q_0$ and $k \leq 0$. Therefore, we have $\operatorname{Ext}^1(M, S_s') = 0$ for any $M \in \mathcal{H}_{\mathbf{t}}$, in particular, for $M = S_j^{\mathbf{t}}$. Then applying [8, formula (5.2)] to the backward tilt ${\mathcal{H}_{\mathbf{t}}}^{\flat}_{S_j^{\mathbf{t}}}$ and $(\widetilde{\mathcal{H}_{\mathbf{t}}})^{\flat}_{S_j^{\mathbf{t}}}$, gives $\operatorname{Sim} \widetilde{\mathcal{H}_{\mathbf{s}}} = \operatorname{Sim} \mathcal{H}_{\mathbf{s}} \cup \operatorname{Sim} \mathcal{H}_{Q'}$. $\square$

By the lemma, we know that $\mathcal{Q}(\mathcal{H}_{\mathbf{s}})$ is a subquiver of $\mathcal{Q}(\widetilde{\mathcal{H}_{\mathbf{s}}})$.

Definition 2.5. Given a sequence $\mathbf{s}$, define the *principal extension* of the Ext-quiver $\mathcal{Q}(\mathcal{H}_{\mathbf{s}})$ to be the Ext-quiver $\mathcal{Q}(\widetilde{\mathcal{H}_{\mathbf{s}}})$ while the vertices in $\operatorname{Sim}\mathcal{H}_{Q'}$ are the frozen vertices.

From the proof of Lemma 2.4, it is straightforward to see the following.

Lemma 2.6. *Every frozen vertices S'_i is a source in $\mathcal{Q}(\widetilde{\mathcal{H}_{\mathbf{s}}})$.*

2.3. **Keller's theorem.** Before we proof Keller's observations for green mutation, we import a result from [8] concerning the relation between quivers (for clusters) and Ext-quivers.

Lemma 2.7 (King-Qiu)**.** *If $\widetilde{\mathcal{H}_{\mathbf{s}}} \in \operatorname{Eg}_{\widetilde{Q}}$ for some sequence $\mathbf{s}$, then $\widetilde{Q}_{\mathbf{s}}$ is canonically isomorphic to the degree one part of $\mathrm{CY}^3(\mathcal{Q}(\widetilde{\mathcal{H}_{\mathbf{s}}}))$.*

Proof. First, $\operatorname{Eg}_{\widetilde{Q}}$ is isomorphic to the cluster exchange graph for $\widetilde{Q}$ and hence a heart $\widetilde{\mathcal{H}_{\mathbf{s}}} \in \operatorname{Eg}_Q$ corresponding to a cluster $\widetilde{C_{\mathbf{s}}}$ (cf. [4] and [8]). By [3], $\widetilde{Q}_{\mathbf{s}}$ is the degree zero part of the colored quiver $\mathcal{Q}(\widetilde{C_{\mathbf{s}}})$ associated to $C_{\mathbf{s}}$ while $\mathcal{Q}(\widetilde{C_{\mathbf{s}}})$ is the CY-2 completion of $\widetilde{Q}_{\mathbf{s}}$. Moreover, by [8, Proposition 6.7], after shift grading by one, this CY-2 completion is isomorphic to the CY-3 completion of $\mathcal{Q}(\widetilde{\mathcal{H}_{\mathbf{s}}})$. Thus the lemma follows. $\square$

Now we proceed to prove our first theorem.

Theorem 2.8. *Let Q be an acyclic quiver and $\mathbf{s}$ be a green mutation sequence for Q. Then we have the following.*

1°. $\mathcal{H}_Q[-1] \le \mathcal{H}_{\mathbf{s}} \le \mathcal{H}_Q$ and hence $\mathcal{H}_{\mathbf{s}} \in \operatorname{Eg}_Q$.
2°. $\widetilde{Q}_{\mathbf{s}}$ is canonically isomorphic to the degree one part of $\mathrm{CY}^3(\mathcal{Q}(\widetilde{\mathcal{H}_{\mathbf{s}}}))$.
3°. A vertex j in $\widetilde{Q}_{\mathbf{s}}$ is green if and only if the corresponding simple $S^{\mathbf{s}}_j$ is in $\mathcal{H}_Q$.
4°. A vertex j in $\widetilde{Q}_{\mathbf{s}}$ is not green if and only if the corresponding simple $S^{\mathbf{s}}_j$ is in $\mathcal{H}_Q[-1]$. In such case, there is no arrow from j to any of the frozen vertices i'.

Proof. We use induction on the length of $\mathbf{s}$ starting with trivial case when $l(\mathbf{s}) = 0$. Now suppose that the theorem holds for any green mutation sequence of length less than m and consider the case when $l(\mathbf{s}) = m$. Let $\mathbf{s} = \mathbf{t}j$ where $l(\mathbf{t}) = m - 1$ and j is a green vertex in $\widetilde{Q}_{\mathbf{t}}$.

First, the simple $S^{\mathbf{t}}_j$ corresponding to j is in $\mathcal{H}_Q$, by 3° of the induction step, which implies 1° by [8, Lemma 5.4, 1°].

Second, as $\operatorname{Sim}\widetilde{\mathcal{H}_{\mathbf{s}}} = \operatorname{Sim}\mathcal{H}_{\mathbf{s}} \cup \operatorname{Sim}\mathcal{H}_{Q'}$ by Lemma 2.4, $\mathcal{H}_{\mathbf{s}} \in \operatorname{Eg}_Q$ is equivalent to $\widetilde{\mathcal{H}_{\mathbf{s}}} \in \operatorname{Eg}_{\widetilde{Q}}$. Then 2° follows from Lemma 2.7.

Third, since $\mathcal{H}_Q$ is hereditary, 1° implies that any simple $S^{\mathbf{s}}_j \in \operatorname{Sim}\mathcal{H}_{\mathbf{s}}$ is in either $\mathcal{H}_Q$ or $\mathcal{H}_Q[-1]$. If $S^{\mathbf{s}}_j$ is in $\mathcal{H}_Q$, by (2.1), there are arrows $S'_i \to S^{\mathbf{s}}_j$ in $\mathcal{Q}(\widetilde{\mathcal{H}_{\mathbf{s}}})$ and each of which has degree one. Further, by 2°, any such degree one arrow corresponds to an arrow $i' \to j$ in $\widetilde{Q}_{\mathbf{s}}$. Therefore 3° follows. Similarly, if $S^{\mathbf{s}}_j$ is in $\mathcal{H}_Q[-1]$, there are arrows $S'_i \to S^{\mathbf{s}}_j$ in $\mathcal{Q}(\widetilde{\mathcal{H}_{\mathbf{s}}})$, each of which has degree two and corresponds to an arrow $i' \leftarrow j$ in $\widetilde{Q}_{\mathbf{s}}$. Thus 4° follows too. $\square$

The consequences of theorem include a criterion for a sequence being green mutation sequence and one of Keller's original statement about maximal green mutation sequences.

Corollary 2.9. *A sequence* $\mathbf{s}$ *is a green mutation sequence if and only if* $\mathcal{H} \geq \mathcal{H}_Q[-1]$ *for any* $\mathcal{H} \in \operatorname{supp} \mathrm{P}(\mathbf{s})$*. Further, a green mutation sequence* $\mathbf{s}$ *is maximal if and only if* $\mathcal{H}_{\mathbf{s}} = \mathcal{H}_Q[-1]$*. Thus a maximal green mutation sequence* $\mathbf{s}$ *satisfies* $\widetilde{Q}_{\mathbf{s}} \cong \widetilde{Q}$*.*

Proof. The necessity of first statement follows from 1° of Theorem 2.8. For the sufficiency, we only need to show that if $\mathbf{t}$ is a green mutation sequence and $\mathbf{s} = \mathbf{t}j$ satisfies $\mathcal{H}_{\mathbf{s}} \geq \mathcal{H}_Q[-1]$, for some $j \in Q_0$, then $\mathbf{s}$ is also a green mutation sequence. Since $\mathcal{H}_{\mathbf{t}} \geq \mathcal{H}_Q[-1]$, by [8, Lemma 5.4, 1°] we know that $\mathcal{H}_{\mathbf{s}} \geq \mathcal{H}_Q[-1]$ implies $S_j^{\mathbf{t}}$ is in $\mathcal{H}_Q$. But this means j is a green vertex for $\mathbf{t}$, by 3° of Theorem 2.8, as required.

For the second statement, $\mathbf{s}$ is a maximal, if and only if $S_i^{\mathbf{s}} \in \mathcal{H}_Q[-1]$ for any $i \in Q_0$, or equivalently, $\mathcal{H}_{\mathbf{s}} = \mathcal{H}_Q[-1]$, as required. $\square$

Example 2.10. We borrow an example of A_2 type green mutations from Keller [5] (but the orientation slightly differs). Figure 1 gives two different maximal green mutation sequences (121 and 21) which end up being isomorphic to each other.

2.4. **Wide subcategory via non-green vertices.** In this section, we aim to show the non-green simples are precisely the simples in the wide subcategory $\mathcal{W}_{\mathbf{s}}$ corresponds to the torsion class $\mathcal{T}_{\mathbf{s}}$ in the sense of Ingalls-Thomas.

Recall that a wide subcategory is an exact abelian categories closed under extensions. Further, given a finite generated torsion class $\mathcal{T}$ in $\mathcal{H}_Q$, define the corresponding wide

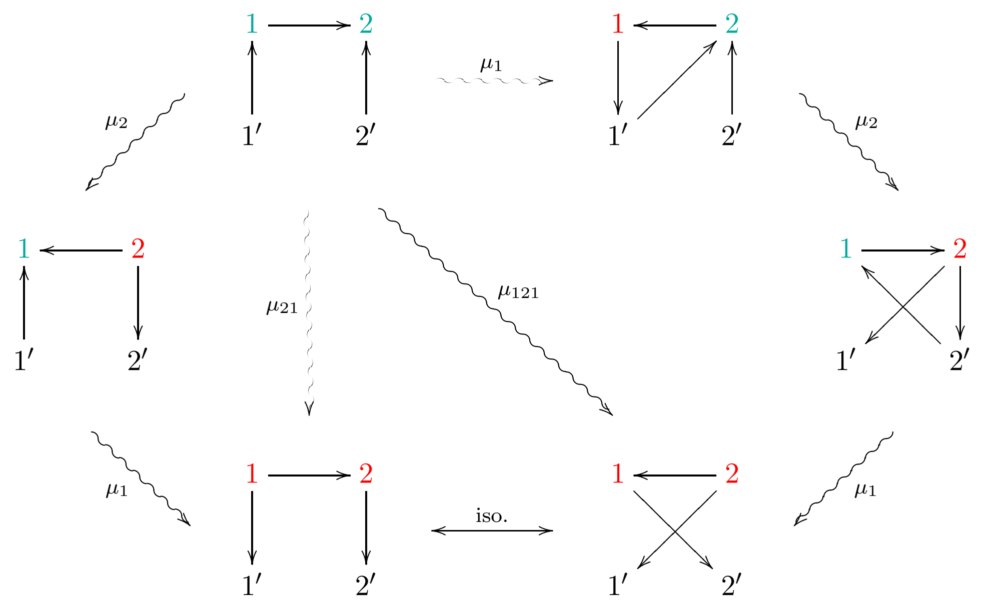

FIGURE 1. Two maximal green mutation sequences for an A_2 quiver

subcategory $\mathcal{W}(\mathcal{T})$ to be

$$\{M \in \mathcal{T} \mid \forall (f; X \to M) \in \mathcal{T}, \ker(f) \in \mathcal{T}\}. \tag{2.2}$$

First, we give another characterization for $\mathcal{W}(\mathcal{T})$.

Proposition 2.11. *Let $\langle \mathcal{F}, \mathcal{T} \rangle$ be a finite generated torsion pairs in $\mathcal{H}_Q$ and $\mathcal{H}^\sharp$ be the corresponding backward tilt. Then we have*

$$\operatorname{Sim} \mathcal{W}(\mathcal{T}) = \mathcal{T} \cap \operatorname{Sim} \mathcal{H}^\sharp. \tag{2.3}$$

Proof. By [4] and [8], such torsion pair corresponds to a cluster tilting object (in the cluster category of $\mathcal{D}(Q)$) and thus the heart $\mathcal{H}^\sharp$ is in Eg_Q and hence finite. Noticing that $\mathcal{H}^\sharp$ admits a torsion pair $\langle \mathcal{T}, \mathcal{F}[1] \rangle$, any its simple is either in $\mathcal{T}$ or $\mathcal{F}[1]$. Let $\mathcal{W}$ be the wide subcategory of $\mathcal{H}^\sharp$ generated by simples in $\mathcal{T} \cap \operatorname{Sim} \mathcal{H}^\sharp$.

First, for any $S \in \mathcal{T} \cap \operatorname{Sim} \mathcal{H}^\sharp$ and

$$(f : X \to S) \in \mathcal{T} \subset \mathcal{H}^\sharp,$$

f is surjective (in $\mathcal{H}^\sharp$) since S is a simple. Thus $\ker(f)$ is in $\mathcal{T}$ since $\mathcal{T}$ is a torsion free class in $\mathcal{H}^\sharp$, which implies $S \in \mathcal{W}(\mathcal{T})$. Therefore $\mathcal{W} \subset \mathcal{W}(\mathcal{T})$ and we claim that they are equal.

If not, let M in $\mathcal{W}(\mathcal{T}) - \mathcal{W}$ whose simple filtration in $\mathcal{H}^\sharp$ (with factors in $\operatorname{Sim} \mathcal{H}^\sharp$) has minimal number of factors. Let S be a simple top of M and then $X = \ker(M \twoheadrightarrow S)$ is in $\mathcal{T}$. If S is in $\mathcal{T} \cap \operatorname{Sim} \mathcal{H}^\sharp$, then X is in $\mathcal{W}(\mathcal{T}) - \mathcal{W}$ with less simple factors, contradicting to the choice of M. Hence $S \in \mathcal{F}[1] \cap \operatorname{Sim} \mathcal{H}^\sharp$. Then we obtain a short exact sequence

$$0 \to X \hookrightarrow M \twoheadrightarrow S \to 0$$

in $\mathcal{H}^\sharp$ which became a short exact sequence

$$0 \to S[-1] \hookrightarrow X \overset{f}{\twoheadrightarrow} M \to 0$$

in $\mathcal{H}_Q$. But $\ker(f) = S[-1] \in \mathcal{F}$, which contradicts to the fact that M is in $\mathcal{W}(\mathcal{T})$ (cf. (2.2)). Therefore $\mathcal{W}(\mathcal{T}) = \mathcal{W}$ or (2.3). $\square$

An immediate consequence of this corollary is as follows, noticing that a function Z (known as the *central charge*) from $\operatorname{Sim} \mathcal{H}^\sharp$ to the upper half plane (of the complex plane $\mathbb{C}$) gives a stability condition, in the sense of Bridgeland (cf. [9]), on the triangulated category $\mathcal{D}(Q)$.

Corollary 2.12. *A finite generated wide subcategories in $\mathcal{H}_Q$ is a semistable subcategory of some Bridgeland stability condition on $\mathcal{D}(Q)$.*

Remark 2.13. Note that Corollary 2.12 implies Ingalls-Thomas' result, that every wide subcategory in $\mathcal{H}_Q$ is a semistable subcategory for some θ-stability condition, in the sense of King, on $\mathcal{H}_Q$.

We end this section by showing that the simples of the wide subcategory associated to a green mutation sequence. Let $\mathbf{s}$ be a green mutation sequence. We will write $\mathcal{W}_\mathbf{s}$ for the wide subcategory $\mathcal{W}(\mathcal{T}_\mathbf{s})$ of the torsion pair

$$\mathcal{T}_\mathbf{s} = \mathcal{H}_Q \cap \mathcal{H}_\mathbf{s}[1]. \tag{2.4}$$

Recall that $\mathrm{V}^{-}(\mathbf{s})$ is the set of non-green vertices of a green mutation sequence $\mathbf{s}$. Denote by $\mathrm{V}^{-}(\mathcal{H}_{\mathbf{s}})$ the set of non-green simples, that is, the simples corresponds to vertices in $\mathrm{V}^{-}(\mathbf{s})$.

Corollary 2.14. *Let s be a green mutation sequence. Then* $\operatorname{Sim}\mathcal{W}_{\mathbf{s}}[1] = \mathrm{V}^{-}(\mathcal{H}_{\mathbf{s}})$.

Proof. By 3° and 4° of Theorem 2.8, we have

$$\mathrm{V}^{-}(\mathcal{H}_{\mathbf{s}}) = \mathcal{H}_Q[-1] \cap \operatorname{Sim}\mathcal{H}_{\mathbf{s}} = \mathcal{T}[-1] \cap \operatorname{Sim}\mathcal{H}_{\mathbf{s}}\,.$$

Noticing that $\mathcal{H}_{\mathbf{s}}[1]$ is the forward tilt of $\mathcal{H}_Q$ with respect to the torsion class $\mathcal{T}_{\mathbf{s}}$, the corollary follows from Proposition 2.11. □

3. C-sortable words

In this section, we will show that it is natural to interpret a c-sortable word as a green mutation sequence, which produces many consequences.

3.1. **Main results.** Denote by $\widetilde{\mathbf{w}} = i_1 \dots i_k$ the sequence induced from a c-sortable word $\mathbf{w} = s_{i_1} \dots s_{i_k}$. Note that $\mathbf{w}$ induces a path $\mathrm{P}(\widetilde{\mathbf{w}})$ and a heart $\mathcal{H}_{\widetilde{\mathbf{w}}}$ as in Section 1.3. We will drop the tilde of $\widetilde{\mathbf{w}}$ later when it appears in the subscript or superscript.

Theorem 3.1. *Let Q be an acyclic quiver and c be an admissible Coxeter element with respect to the orientation of Q. Let $\mathbf{w}$ be a c-sortable word and we have the following.*

1°. $\widetilde{\mathbf{w}}$ is a green mutation sequence.

2°. For any $i \in Q_0$, let $s_i^{\mathbf{w}}$ be the reflection of $S_i^{\mathbf{w}}$, the i-th simple of $\mathcal{H}_{\mathbf{w}}$. Then

$$s_i^{\mathbf{w}} \cdot \mathbf{w} = \mathbf{w} \cdot s_i. \tag{3.1}$$

3°. Let $\mathcal{T}_{\mathbf{w}}$ is defined as in (2.4) and we have $\operatorname{Ind}\mathcal{T}_{\mathbf{w}} = \operatorname{supp}\mathrm{P}(\mathbf{w})$.

Proof. We use induction on $l(\mathbf{w}) + \#Q_0$ staring with the trivial case $l(\mathbf{w}) = 0$. Suppose that the theorem holds for any $(Q, c, \mathbf{w})$ with $l(\mathbf{w}) + \#Q_0 < m$. Now we consider the case when $l(\mathbf{w}) + \#Q_0 = m$. Let $c = s_1 c_-$ without lose of generality.

If s_1 is not the initial of $\mathbf{w}$, then the theorem reduces to the case for $(Q_-, c_-, \mathbf{w})$, where Q_- is the full subquiver with vertex set $Q_0 - \{1\}$, which is true by the inductive assumption.

Next, suppose that s_1 is the initial of $\mathbf{w}$, so $\mathbf{w} = s_1\mathbf{v}$ for some $\mathbf{v}$. Denote by $\widetilde{\mathbf{v}}$ the sequence induced by $\mathbf{v}$. Let $Q_+ = \mu_1(Q)$, $c_+ = s_1 c s_1$ and we identify

$$\mathcal{H}_{Q_+} = \operatorname{mod}\mathbf{k}Q_+ \quad \text{with} \quad \mathcal{H}_{s_1} = (\mathcal{H}_Q)^{\flat}_{S_1}$$

via a so-called APR-tilting (reflecting the source 1 of Q).

For 2°, consider the influence of the APR-tilting on the dimension vectors and Coxeter group. we know that for any $M \in \mathcal{H}_Q - \{S_1\}$, the $\underline{\dim}_+ M$ with respect to Q_+ equals $s_1(\underline{\dim}\, M)$. Thus the reflection t_M of M for Q_+ equals $s_1 s_M s_1$. In particular, the reflection $t_i^{\mathbf{v}}$ of $S_i^{\mathbf{w}}$ for Q_+ equals $s_1 s_i^{\mathbf{w}} s_1$. Then formula (3.1) gives $t_i^{\mathbf{v}} \cdot \mathbf{v} = \mathbf{v} \cdot s_i$ or $s_i^{\mathbf{w}} \cdot \mathbf{w} = \mathbf{w} \cdot s_i$, as required.

Further more, by [10, Lemma 2.5], $\mathbf{v}$ is c_+-sortable and hence the theorem holds for $(Q_+, c_+, \mathbf{v})$. Let $\mathbf{v} = \mathbf{u}s_j$, then the theorem also holds for $(Q, c, s_1\mathbf{u})$. Let $T = S_j^{\mathbf{w}}$

the j-th simple of $\mathcal{H}_{\mathbf{w}}$. Use the criterion in Corollary 2.9 for being a green mutation sequence, we know that

$$\begin{cases} (\mathcal{H}_{\mathbf{w}})^{\sharp}_{T} = \mathcal{H}_{s_1\mathbf{u}} \geq \mathcal{H}_Q[-1], \\ \mathcal{H}_{\mathbf{w}} = \mathcal{H}_{\mathbf{w}}(Q) = \mathcal{H}_{\mathbf{v}}(Q_+) \geq \mathcal{H}_{Q_+}[-1]. \end{cases} \tag{3.2}$$

If $\mathcal{H}_{\mathbf{w}} \geq \mathcal{H}_Q[-1]$ fails, comparing (3.2) with

$$\operatorname{Ind}\mathcal{H}_{Q_+}[-1] = \operatorname{Ind}\mathcal{H}_Q[-1] - \{S_1[-1]\} \cup \{S_1[-2]\},$$

we must have $T = S_1[-2]$. However, by formula (3.1) for $(Q, c, s_1\mathbf{u})$ and $j \in Q_0$, noticing that the j-th simple of $\mathcal{H}_{s_1\mathbf{u}}$ is $T[1]$, we have

$$s_{T[1]} \cdot (s_1\mathbf{u}) = (s_1\mathbf{u}) \cdot s_j.$$

The RHS is $\mathbf{w}$ while the LHS equals to $s_1^2\mathbf{u} = \mathbf{u}$, which is a contradiction to the fact that the c-sortable word $\mathbf{w}$ is reduced. So $\mathcal{H}_{\mathbf{w}} \geq \mathcal{H}_Q[-1]$, and thus $\widetilde{\mathbf{w}}$ is a green mutation sequence, by Corollary 2.9, as required.

Finally, we have $\operatorname{Ind}\mathcal{T}(Q)_{\mathbf{w}} = \operatorname{Ind}\mathcal{T}_{\mathbf{v}}(Q_+) \cup \{S_1\}$ which implies 3°. □

3.2. **Consequences.** In this subsection, we discuss various corollaries of Theorem 3.1. First, we prove the bijection between c-sortable words and finite torsion classed in $\mathcal{H}_Q$.

Corollary 3.2 (Thomas, Amiot-Iyama-Reiten-Todorov)**.** *There is a bijection between the set of c-sortable words and the set of finite torsion classes in $\mathcal{H}_Q$, sending such a word $\mathbf{w}$ to $\mathcal{T}_{\mathbf{w}}$.*

Proof. Clearly, every torsion class $\mathcal{T}_{\mathbf{w}}$ induced by a c-sortable word $\mathbf{w}$ is finite. To see two different c-sortable words $\mathbf{w}_1$ and $\mathbf{w}_2$ induce different finite torsion classes, we use the induction on $l(\mathbf{w})$. Then it is reduced to the case when the initials of $\mathbf{w}_1$ and $\mathbf{w}_2$ are different. Without lose of generality, let the initial s_1 of $\mathbf{w}_1$ is on the left of the initial s_2 of $\mathbf{w}_2$ in expression

$$c = \cdots s_1 \cdots s_2 c'$$

of the Coxeter element c. Now, the sequence of simple tilting $\widetilde{\mathbf{w}}_2$ takes place in the full subcategory

$$\mathcal{D}(Q_{\mathrm{res}}) \subset \mathcal{D}(Q),$$

where Q_{res} is the full subquiver of Q restricted to $\operatorname{supp}(s_2 c')$. Thus the simple S_1 will never appear in the path $\mathrm{P}(\mathbf{w}_2)$ which implies $\mathcal{T}_{\widetilde{\mathbf{w}}_1} \neq \mathcal{T}_{\widetilde{\mathbf{w}}_2}$ by 3° of Theorem 3.1. Therefore, we have an injection from the set of c-sortable words to the set of finite torsion classes in $\mathcal{H}_Q$.

To finish, we need to show the surjectivity, i.e. any finite torsion class $\mathcal{T}$ is equal to $\mathcal{T}_{\mathbf{w}}$ for some c-sortable words. This is again by induction for $(Q, c, \mathcal{T})$, on $\#\operatorname{Ind}\mathcal{T} + \#Q_0$, starting with the trivial case when $\#\operatorname{Ind}\mathcal{T} = 0$. Suppose that the surjectivity hold for any $(Q, c, \mathcal{T})$ with $\#\operatorname{Ind}\mathcal{T} + \#Q_0 < m$ and consider the case when $\#\operatorname{Ind}\mathcal{T} + \#Q_0 = m$. Let $c = s_1 c_-$ without lose of generality.

If the simple injective S_1 of $\mathcal{H}_Q$ is not in $\mathcal{T}$, we claim that $\mathcal{T} \subset \mathcal{H}_{Q_-} \subset \mathcal{H}_Q$, where Q_- is the full subquiver with vertex set $Q_0 - \{1\}$. If so, the theorem reduces to the case for $(Q_-, c_-, \mathbf{s})$, which holds by the inductive assumption. To see the claim, choose any $M \in \mathcal{H}_Q - \mathcal{H}_{Q_-}$. Then S_1 is a simple factor of M in its canonical filtration and hence the top, since S_1 is injective. Thus $\mathrm{Hom}(M,S)\neq 0$. But $S_1\notin\mathcal{T}$ implies S_1 is in the torsion free class corresponds to $\mathcal{T}$. So $M\notin\mathcal{T}$, which implies $\mathcal{T}\subset\mathcal{H}_{Q_-}$ as required.

If the simple injective S_1 of $\mathcal{H}_Q$ is in $\mathcal{T}$, then consider the quiver $Q_+=\mu_1(Q)$ and the torsion class

$$\mathcal{T}_+=\mathrm{add}\left(\mathrm{Ind}\,\mathcal{T}-\{S_1\}\right).$$

Similar to the proof of Theorem 3.1, we know that the claim holds for $(Q_+,c+,\mathcal{T}_+)$, where $c_+=s_1cs_1$. i.e. $\mathcal{T}_+=\mathcal{T}_{\mathbf{v}}$ for some c_+-sortable word $\mathbf{v}$. But $\mathbf{w}=s_1\mathbf{v}$ is a c-sortable word by [10, Lemma 2.5] and we have

$$\mathrm{Ind}\,\mathcal{T}_{\mathbf{w}}=\{S_1\}\cup\mathrm{Ind}\,\mathcal{T}_{\mathbf{v}}=\mathrm{Ind}\,\mathcal{T},$$

or $\mathcal{T}=\mathcal{T}_{\mathbf{w}}$, as required. □

Second, we claim that the path $\mathrm{P}(\mathbf{w})$ has maximal length.

Corollary 3.3. *Let* $\mathbf{w}$ *be a* c*-sortable word. Then* $\mathrm{P}(\mathbf{w})$ *is the longest directed path in* Eg_Q *connecting* $\mathcal{H}$ *and* $\mathcal{H}_{\mathbf{w}}$*.*

Proof. By 4° of Theorem 3.1, the number of indecomposables in $\mathcal{T}_{\mathbf{w}}$ is exactly the length of $\mathrm{P}(\mathbf{w})$. Then the corollary follows from the fact that, each time we do a backward tilt in the sequence $\widetilde{\mathbf{w}}$, the torsion class adds at least a new indecomposable, i.e. the simple where the tilting is at. □

Third, we describe the properties of a c-sortable word $\mathbf{w}$ in terms of non-green vertices of the corresponding green mutation sequence $\widetilde{\mathbf{w}}$. Recall that $\mathrm{V}^{\text{-}}(\mathbf{w})$ is the set of non-green vertices of a green mutation sequence $\widetilde{\mathbf{w}}$ and $\mathrm{V}^{\text{-}}(\mathcal{H}_{\mathbf{w}})$ the set of (non-green) simples in $\mathcal{H}_{\mathbf{w}}$.

Corollary 3.4. *Let* Q *be an acyclic quiver and* c *be an admissible Coxeter element with respect to the orientation of* Q*. For a* c*-sortable word* $\mathbf{w}$*, the set of its inversions, descents and cover reflections are given as follows*

$$\mathrm{Inv}(\mathbf{w})=\{s_T\mid T\in\mathrm{supp}\,\mathrm{P}(\mathbf{w})\},\tag{3.3}$$

$$\mathrm{Des}(\mathbf{w})=\{s_i\mid i\in\mathrm{V}^{\text{-}}(\mathbf{w})\},\tag{3.4}$$

$$\mathrm{Cov}(\mathbf{w})=\{s_T\mid T\in\mathrm{V}^{\text{-}}(\mathcal{H}_{\mathbf{w}})\},\tag{3.5}$$

where $\mathbf{s}_T$ *is the reflection of* T*.*

Proof. First of all, as the proof of Theorem 3.1 or [4, Theorem 4.3], we have (3.3) by inducting on $l(\mathbf{w})+\#Q_0$.

For any $j\in\mathrm{V}^{\text{-}}(\mathbf{s})$, by 4° of Theorem 2.8, we have the corresponding simple $S_j^{\mathbf{w}}$ is in $\mathcal{H}_Q[-1]$ and hence $S_j^{\mathbf{w}}[1]$ the torsion class $\mathcal{T}_{\mathbf{w}}$. By formula (3.3), we know that $s_i^{\mathbf{w}}$ is in $\mathrm{Inv}(\mathbf{w})$ and hence s_i is in $\mathrm{Des}(\mathbf{w})$ by (3.1).

For any $j\notin\mathrm{V}^{\text{-}}(\mathbf{s})$, by 3° of Theorem 2.8, and hence the corresponding simple $S_j^{\mathbf{w}}$ is in $\mathcal{H}_Q$ but not in the torsion class $\mathcal{T}_{\mathbf{w}}$. Then $\underline{\dim}\,S_j^{\mathbf{w}}$ is not equal to any $\underline{\dim}\,T, T\in\mathcal{T}_{\mathbf{w}}$ since $\mathcal{T}_{\mathbf{w}}$ is a simple in $\mathcal{H}_{\mathbf{w}}\supset\mathcal{T}$. Again, by formula (3.3), we know that $s_i^{\mathbf{w}}$ is not in $\mathrm{Inv}(\mathbf{w})$ and hence s_i is not in $\mathrm{Des}(\mathbf{w})$ by (3.1).

Therefore, (3.4) and (3.5) both follow. □

In the finite case, there are two more consequences. The first one is about the supporting trees of the (cluster) exchange graphs.

Corollary 3.5. *Let Q be a Dynkin quiver. For any $\mathcal{H} \in \mathrm{Eg}_Q$, there is a unique c-sortable word* $\mathbf{w}$ *such that $\mathcal{H} = \mathcal{H}_{\mathbf{w}}$. Equivalently, the tree of c-sortable word* $\mathbf{w}$ *(with respect to the week order) is isomorphic to a supporting tree of the exchange graph* Eg_Q.

Proof. First, notice that all c-sortable words forms a tree with respect to the week order. Then the corollary follows from 3° of Theorem 3.1 and the fact that any torsion class in $\mathcal{H}_Q$ is finite. □

We finish this section by showing a formula of a T-reduced expression for noncrossing partitions via non-green vertices. Let nc_c be Reading's map from c-sortable words to noncrossing partitions. We have the following formula.

Corollary 3.6. *Let Q be a Dynkin quiver. Keep the notation of Theorem 3.1, we have the following formula*

$$\mathrm{nc}_c(\mathbf{w}) = \prod_{j \in \mathrm{V}^{\text{-}}(\mathbf{w})} s_j^{\mathbf{w}},$$

with $\operatorname{rank} \mathrm{nc}_c(\mathbf{w}) = \# \mathrm{V}^{\text{-}}(\mathbf{w})$.

Proof. The corollary follows from (3.5) and Reading's map ([10, Section 6]). □

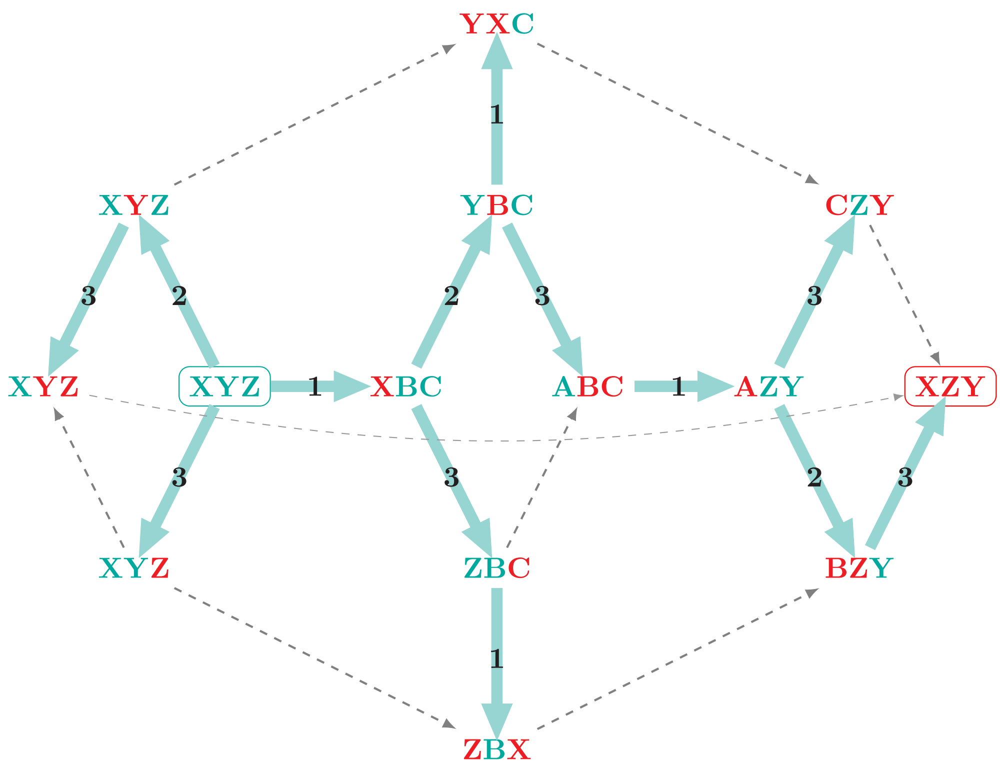

FIGURE 2. The supporting tree of Eg_Q with respect to $c = s_1 s_2 s_3$

4. Example: Associahedron

Example 4.1. Consider an A_3 type quiver $Q\colon 2 \leftarrow 1 \rightarrow 3$ with $c = s_1 s_2 s_3$. We have the tree of c-sortable words is given below.

$$\begin{tikzcd}
 & s_2 \arrow[r] & s_2s_3 & s_1s_2|s_1 & & & \\
e \arrow[ur] \arrow[r] \arrow[dr] & s_1 \arrow[r] \arrow[dr] & s_1s_2 \arrow[ur] \arrow[r] & s_1s_2s_3 \arrow[r] & s_1s_2s_3|s_1 \arrow[r] \arrow[dr] & s_1s_2s_3|s_1s_2 \arrow[r] & s_1s_2s_3|s_1s_2s_3 \\
 & s_3 & s_1|s_3 \arrow[r] & s_1s_3|s_1 & & s_1s_2s_3|s_1s_3 &
\end{tikzcd}$$

Moreover, a piece of AR-quiver of $\mathcal{D}(Q)$ is as follows

$$\begin{tikzcd}
\mathrm{Z} \arrow[dr] & & \mathrm{B} \arrow[dr] & & \mathrm{Y} \arrow[dr] & & \mathrm{C} \arrow[dr] & \\
 & \mathrm{A} \arrow[ur] \arrow[dr] & & \mathrm{X} \arrow[ur] \arrow[dr] & & \mathrm{A} \arrow[ur] \arrow[dr] & & \mathrm{X} \\
\mathrm{Y} \arrow[ur] & & \mathrm{C} \arrow[ur] & & \mathrm{Z} \arrow[ur] & & \mathrm{B} \arrow[ur] &
\end{tikzcd}$$

where the green vertices are the indecomposables in $\mathcal{H}_Q$ and the red ones are there shift minus one. Note that $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ are the simples S_1, S_2, S_3 in $\mathcal{H}_Q$ respectively.

Figure 2 is the exchange graph Eg_Q (cf. [8, Figure 1 and 4]). where we denote a heart $\mathcal{H}_{\mathbf{w}}$ by the set of its simples $S_1^{\mathbf{w}} S_2^{\mathbf{w}} S_3^{\mathbf{w}}$ (in order). The green edges are the green mutations in some green mutation sequences induced from c-sortable words. The number on a green edges indicates where the mutation is at. Note that the underlying graph of Figure 2 is the associahedron (of dimension 3).

Further, Table 1 is a list of correspondences between c-sortable words, hearts (as in the Figure 2), descents, cover reflection, inversions and (finite) torsion classes. Note that this table is consistent with [4, Table 1], in the sense that the objects in the j-th row here are precisely objects in the j-th row there.

TABLE 1. Example: A_3

c-sortable word $\mathbf{w}$	Heart $\mathcal{H}_{\mathbf{w}}$	Descent $\mathrm{Des}(\mathbf{w})$	Cover ref. $\mathrm{Cov}(\mathbf{w})$	Torsion class $\mathcal{T}_{\mathbf{w}}$
$s_1s_2s_3\vert s_1s_2s_3$	XZY	s_1, s_2, s_3	t_X, t_Y, t_Z	$\underline{\mathrm{XBCAZY}}$
$s_1s_2s_3\vert s_1s_2$	BZY	s_1, s_2	t_B, t_Z	$\mathrm{X}\underline{\mathrm{B}}\mathrm{C}\underline{\mathrm{AZ}}$
$s_1s_2s_3\vert s_1s_3$	CZY	s_1, s_3	t_C, t_Y	$\mathrm{XB}\underline{\mathrm{CAY}}$
s_2s_3	XYZ	s_2, s_3	t_Y, t_Z	$\underline{\mathrm{YZ}}$
$s_1s_2s_3$	ABC	s_2, s_3	t_B, t_C	$\mathrm{X}\underline{\mathrm{BC}}$
$s_1s_3\vert s_1$	ZBX	s_1, s_3	t_Z, t_X	$\underline{\mathrm{XCZ}}$
$s_1s_2\vert s_1$	YXC	s_1, s_2	t_Y, t_X	$\underline{\mathrm{XBY}}$
s_2	XYZ	s_2	t_Y	$\underline{\mathrm{Y}}$
s_3	XYZ	s_3	t_Z	$\underline{\mathrm{Z}}$
$s_1s_2s_3\vert s_1$	AZY	s_1	t_A	$\mathrm{XBC}\underline{\mathrm{A}}$
s_1s_2	YBC	s_2	t_B	$\mathrm{X}\underline{\mathrm{B}}$
s_1s_3	ZBC	s_3	t_C	$\mathrm{X}\underline{\mathrm{C}}$
s_1	XBC	s_1	t_X	$\underline{\mathrm{X}}$
e	XYZ	$\emptyset$	$\emptyset$	$\emptyset$

N.B.1: $\{t_X, t_Y, t_Z, t_A, t_B, t_C\} = \{s_1, s_2, s_3, s_2s_3s_1s_3s_2, s_1s_2s_1, s_1s_3s_1\}$.
N.B.2: The underlines objects in $\mathcal{T}_{\mathbf{w}}$ form the wide subcategory $\mathcal{W}_{\mathbf{w}}$.

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