On radiative damping in plasma-based accelerators

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Radiative damping in plasma-based electron accelerators is analyzed. The electron dynamics under combined influence of the constant accelerating force and the classical radiation reaction force is studied. It is shown that electron acceleration cannot be limited by radiation reaction. If initially the accelerating force was stronger than the radiation reaction force then the electron acceleration is unlimited. Otherwise the electron is decelerated by radiative damping up to a certain instant of time and then accelerated without limits. Regardless of the initial conditions the infinite-time asymptotic behavior of an electron is governed by self-similar solution providing unlimited acceleration. The relative energy spread induced by the radiative damping decreases with time in the infinite-time limit.

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The plasma-based methods of electron acceleration demonstrate an impressive progress in the last ten years. The quasimonoenergetic electron bunches are generated in laser-plasma acceleration experiments [1]. The electron energy in laser wakefield acceleration experiments exceeds 1 GeV for cm-scale acceleration length [2] and energy doubling of 42 GeV electrons in a meter-scale plasma wakefield accelerator is demonstrated [3]. Recently the physics of linear colliders based on laser-plasma accelerators have been discussed [4, 5].

The accelerating structure in the plasma-based methods is a plasma wave generated behind the driver which can be the laser pulse or the electron bunch. There is a number of effects which limit the energy gain in the plasma-based accelerators [6]. One of the main limitations comes from the dephasing. The velocity of the relativistic electrons becomes slightly higher than the plasma wave phase velocity, which is determined by the driver velocity. The accelerated electrons slowly outrun the plasma wave and leave the accelerating phase. This problem can be partially solved by the use of proper longitudinal gradient of plasma density [7, 8]. Another limitation is caused by the driver depletion as the driver energy converts into the energy of the plasma wave. The driver evolution during acceleration (e. g. laser pulse diffraction or electron bunch expansion) also imposes certain restrictions on the electron energy gain. In the case of laser-plasma accelerators the laser pulse can be guided over long distances in the preformed plasma density channel [9] or with relativistic optical guiding when diffraction is compensated by relativistic self-focusing [10]. In general, in order to accelerate electrons far beyond the energy limited by these effects the multistage schemes can be used.

The electron acceleration in the plasma wave is accompanied with the transverse betatron oscillations caused by the action of the focusing force on the electron from the plasma wakefield. The accelerating force and the focusing force acting on the relativistic electron near the driver axis can be approximated as follows $F_{acc} = fmc\omega_p$ and $F_\perp \simeq -m\kappa^2\omega_p^2 r$, respectively, where r is the transverse displacement of the electron from the driver axis, f and κ are the numerical factor and the focusing constant, respectively, determined by the parameters of the driver and the plasma, $\omega_p = \left(4\pi e^2 n/m\right)^{1/2}$ is the plasma frequency, n is the density of the background plasma, m and $e = -|e|$ are the electron mass and the electron charge, respectively [6]. For example, if the driver is the linearly polarized Gaussian laser pulse with resonant pulse duration then $f = 0.35a_0^2 \simeq 0.7$ and $\kappa^2 \simeq 0.11$, where $a_0 = eE_L/(mc\omega_L) = 2^{1/2}$ is chosen, E_L is the laser field amplitude, ω_L is the laser frequency [5]. The period of the betatron oscillations is $\omega_\beta = \omega_p\kappa\gamma^{-1/2}$, where γ is the relativistic gamma-factor of the electron.

The electrons undergoing betatron oscillations emit synchrotron radiation [11, 12]. The radiated power can be estimated as follows $P_{rad} \simeq 2r_e\gamma^2F_\perp^2/(3mc)$, where $r_e = e^2/(mc^2) \simeq 3\cdot 10^{-13}$ cm is the classical electron radius, c is the speed of light. Since the power is proportional to the square of the electron energy, the radiation losses can stop electron acceleration at some threshold value of the electron energy. The threshold energy can be estimated by balancing the accelerating force and the radiation reaction force, $F_{rrf} \simeq P_{rad}/c$, so that $\gamma_{th}^2 \simeq f/(\epsilon\kappa^4R_\beta^2)$, where $R_\beta = k_p r$ is the normalized amplitude of betatron oscillations, $\epsilon = 2r_e\omega_p/(3c)$ and $k_p = \omega_p/c$. The threshold energy is ~ 100 GeV for $f = 0.7$, $n = 10^{19}\,\mathrm{cm}^{-3}$ and $R_\beta = 1$ and $\kappa^2 = 0.11$. Therefore the radiative damping may be a serious limitation of electron acceleration.

The electron acceleration in plasma with the radiation reaction effect has been studied theoretically [4, 5, 13, 14]. The radiation reaction has been treated as a perturbation [13]. The first-order radiative correction to the energy gain of the accelerated electron bunch and the energy spread induced by radiation emission have been derived for the constant accelerating force. The dependence of the electron energy on time has been calculated in the plasma channel without the accelerating force and with the radiation reaction force [14]. Here we study the electron acceleration treating the radiation damping unperturbatively and analyzing the infinite-time limit.

We start from the relativistic equation for electron motion in an electromagnetic field with the radiative reaction force in Landau-Lifshits form [15]

$$\gamma \frac{du^i}{dt} = \frac{cr_e}{e} F^{ik} u_k + \frac{2r_e^2}{3mc} F^i_{rad}, \quad (1)$$

where $F^i_{rad} = F^i_1 + F^i_2 + F^i_3$, $F^i_1 = (e/r_e)\left(\partial F^{ik}/\partial x^l\right) u_k u^l$, $F^i_2 = -F^{il} F_{kl} u^k$, $F^i_3 = \left(F_{kl} u^l\right)\left(F^{km} u_m\right) u^i$, F_{ik} is the electromagnetic field tensor, u_k is the 4-velocity of the electron. The first term in Eq. (1) corresponds to the Lorentz force and the last term corresponds to the radiation reaction force. We assume that the ultrarelativistic electrons ($\gamma \gg 1$) are accelerated along x-axis by the force $F_{acc} \gg F_\perp v_\perp / c$ and undergo betatron oscillations driven by the focusing force $F_\perp \simeq -m\kappa^2 \omega_p^2 y$. Under our assumptions, $F_3 \gg F_1$, F_2 and the focusing forces make a major contribution to the energy losses through radiation. It is convenient to introduce new variables $P = (p_y/mc)\epsilon^{1/2} f^{1/2}$, $Y = y k_p f^{3/2} \epsilon^{1/2}$, $T = \omega_p t \kappa^2 / f$, $G = \gamma \kappa^2 f^{-2}$. Then Eq. (1) can be reduced to the form

$$\frac{dP}{dT} = -Y - Y^2 PG, \quad (2)$$

$$\frac{dY}{dT} = \frac{P}{G}, \quad (3)$$

$$\frac{dG}{dT} = 1 - Y^2 G^2, \quad (4)$$

The obtained equations describe the betatron oscillations with the radiative damping. The first term on the right-hand side of Eq. (4) describes the action of the accelerating force, while the second term describes the radiative damping.

When the number of the betatron oscillations is large, we can use the averaging method [16]. To do this let us introduce a new variable, S, so that $2S = |U|^2 = Y^2 + P^2/G = R_\beta^2 f^3 \epsilon \simeq 2\langle Y^2 \rangle$ and $U \exp\left(i \int G^{-1/2} dT\right) = Y - iG^{-1/2} P$. After averaging over the fast time related to the betatron oscillations the averaged equations are

$$\frac{dS}{dT} = -\frac{1}{2}\frac{S}{G} - \frac{1}{4} G S^2, \quad (5)$$

$$\frac{dG}{dT} = 1 - SG^2. \quad (6)$$

As $G > 0$ and $S > 0$ then $dS/dt < 0$ and the amplitude of the betatron oscillations always decreases with time. This means that for arbitrary electron energy the betatron oscillation amplitude will be small enough at certain instance of time to be radiation reaction force less than the accelerating force.

At the absence of the accelerating force ($f = 0$), it follows from Eqs. (5) and (6) that $SG^{-1/4} =$ const and $\gamma = \gamma_0 \left(1 + 5\epsilon R^2_{\beta,0} \gamma_0 \omega_p t / 16\right)^{-4/5}$, which is in agreement

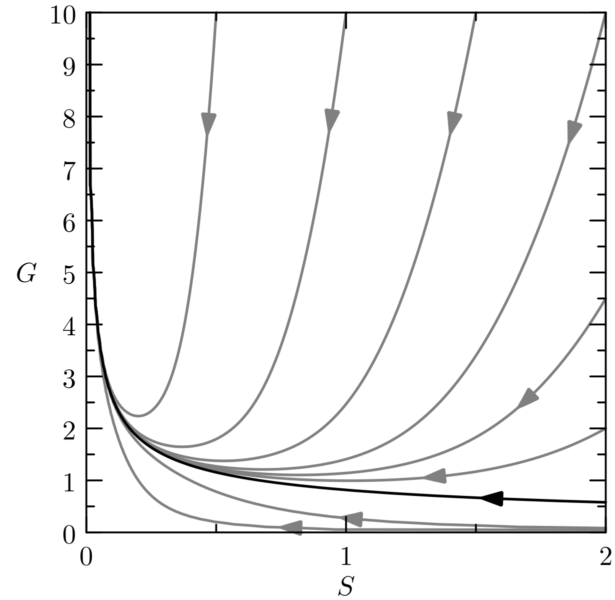

FIG. 1: The phase portrait of the system governed by Eqs. (5) and (6).

with the solution calculated in Ref. [14], where $R_{\beta,0} = R_\beta(t = 0)$. At the absence of the radiation reaction (the last terms in RHS of Eqs. (5) and (6) are absent) we get $G = G_0 + T$, $\sqrt{G} S =$ const. The radiation reaction effect can be treated as a perturbation. To the first order in the radiation reaction force the normalized electron energy is $G = G_0 + T - (2/5)\left[1 - (G_0 + T)^{5/2}\right]$, which is in agreement with the result obtained in Ref. [13].

The system of Eqs. (5) and (6) has integral of motion

$$I = \frac{1 - 3SG^2/2}{S^{9/4}\left(SG^2\right)^{3/4}} = \text{const.} \quad (7)$$

The electron trajectories in the phase space $S - G$ are the integral lines determined by Eq. (7). The phase portrait of the system governed by Eqs. (5) and (6) is shown in Fig. 1. It is seen from Fig. 1 that if initially the accelerating force is stronger than the radiation reaction force ($SG^2 < 1$) then the electron energy monotonically increases with time. Otherwise the electron energy decays up to the time instance when $F_{acc} = F_{rrf}$ (that corresponds to $SG^2 = 1$) and then it monotonically increases with time. It is also seen from Fig. 1 that all electron trajectories merge in the the limit $t \to \infty$ so that $G \to \infty$ and $S \to 0$. It follows from Eq. (7) that $S = 2G^{-2}/3$ in this limit. We will call the electron acceleration in this limit as an asymptotic acceleration regime (AAR).

We verify our analytical results by numerical simulations. The exact equation (1) and the averaged equations of motions (5) and (6) are integrated numerically for test electrons for $f = 0.1$ and $n = 10^{15}\,\text{cm}^{-3}$. For simplicity,

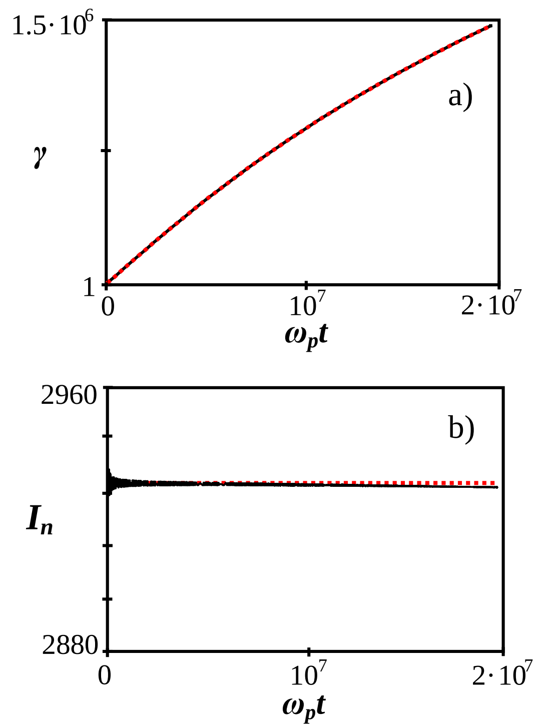

FIG. 2: The dependence of a) γ and b) I_n on $\omega_p t$ calculated by solving of the exact Eq. (1) (black solid lines) and by solving of the approximate Eqs. (5)-(6) (red dashed lines) for $f = 0.1$, $\kappa^2 = 0.5$, $n = 10^{15}$ cm^{-3} and for initial conditions $\gamma_0 = 2000$, $R_{\beta,0} = 0.8$, $p_{y,0} = 0$.

we consider the structure of the transverse electromagnetic field similar to the bubble regime: $\kappa^2 = 0.5$ and $E_\perp \approx H_\perp$. The dependence of the normalized integral of motion $I_n = I^{-1}(\epsilon\kappa f^2)^{-3}$, and γ on $\omega_p t$ for initial condition $\gamma_0 = 2000$ and $R_{\beta,0} = 0.8$, $p_{y,0} = 0$ is shown on Fig. 2. It is seen from Fig. 2 that the solution of the exact equations and that of the approximate averaged are in a good agreement. Moreover, the integral I is almost constant for the exact equations (1) (see Fig. 2c).

We can introduce new variables $g = G/G_{tr}$, $\tau = T/T_{tr}$ and $s = (S/S_{tr})(G/G_{tr})^{-1/4}$, where $G_{tr} = T_{tr} = S_{tr}^{-2} = I^{2/9}$. Then Eqs. (5), (6) and (7) are reduced to the form which does not depend on any parameters. Therefore the characteristic time of transition to AAR is $\sim T_{tr}$. The solution of the equations can be written in term of hypergeometric function, ${}_2F_1(a, b; c; z)$, [18] as follows $\varphi(s) - \varphi(s_0) = \tau$, $\varphi(s) = 2^{4/9}\left(3 + 2s^2\right)^{5/9} s^{-4/9} - 2^{13/9}3^{5/9}s^{14/9}\,{}_2F_1\left(7/9, 4/9; 16/9; -2s^2/3\right)$, where $s_0 = s(\tau = 0)$. The asymptotic expansions of function $\varphi(s)$ are $\varphi(s) \approx 3(3s/2)^{-4/9}$ for $s \ll 1$, $\varphi(s) \approx \delta + s^{-4/3}$ for $s \gg 1$, where $\delta \approx 1.85$. Thus in the limit $\tau \gg 1$ $s \sim \tau^{-9/4} \ll 1$ and $g \sim \tau \gg 1$.

To derive the asymptotic solution the initial condition should be applied. We assume that $S_0 G_0^2 \ll 1$ (so that $s_0 \ll 1$ and $I \simeq S^{-3}G^{-3/2}$) which is typical for the initial parameters of the electron beam. For example, this condition is fulfilled for the initial parameters $\gamma_0 mc^2 < 0.1$ TeV, $n < 10^{18}$ cm^{-3}, $R_{\beta,0} = 1$, $f = 0.7$, $\kappa^2 = 0.11$. Making of use the asymptotic expansion for $s \ll 1$ and $s_0 \gg 1$ we have $(9/4)s^{-4/9} \approx \tau + \delta$. Therefore the normalized electron energy and the square of the normalized betatron amplitude are in the limit $T \gg T_{tr}$

$$G = \frac{\delta}{3}G_{tr} + \frac{1}{3}T, \quad S = \frac{2}{3}G^{-2}. \tag{8}$$

We can conclude that in AAR $F_{rrf} = 2F_{acc}/3$ so that the electron energy increases linearly with time while the betatron amplitude is reversely proportional to the time.

The averaged equations of motions (5) and (6) are integrated numerically for the test electrons with the same parameters as for Fig. 2 for three values of the initial betatron amplitude $R_{\beta,0} = 0.8$, 0.2, 0.1. It is seen from Fig. 3 that the asymptotic solution (8) is in a good agreement with the result of numerical integration.

The radiation damping rate varies for the electrons with different betatron oscillation amplitudes. This causes the energy spread in the electron bunch accelerated in the plasma wave. We assume that the amplitude of the betatron oscillations of the electrons in the accelerated bunch is uniformly distributed in the range $R_{min} < R_{\beta,0} < R_{max}$ and $R_{max} \gg R_{min}$. We also again assume that $S_0 G_0^2 \ll 1$. Then the normalized mean energy and the normalized square of the relative energy spread are in AAR

$$\langle G \rangle \simeq \frac{2}{R_{max}^2}\int_{R_{min}}^{R_{max}} G R_{\beta,0} dR_{\beta,0} \simeq G_{max}\delta + \frac{T}{3}, \tag{9}$$

$$\sigma_G^2 = \left\langle G^2 \right\rangle - \left\langle G \right\rangle^2 \simeq G_{max}^2 \frac{\delta^2}{3}\left(\frac{R_{max}}{R_{min}}\right)^{2/3}, \tag{10}$$

where $G_{max} = G_{tr}(R_{\beta,0} = R_{max})$. It follows from Eqs. (9) and (10) that the relative energy spread, $\sigma_G/\langle G \rangle$, decreases with time in AAR.

Eqs. (2)-(4) are derived under conditions that $F_\perp$ gives the main contribution to the radiative damping and $F_3 \gg F_1$, F_2. However $F_\perp$ goes to zero in the limit $t \to \infty$. Therefore we should check: should the accelerating force and terms F_1, F_2 be taken into account in the radiation reaction force in this limit? First it is significant that the radiation reaction force remains constant in AAR because $F_\perp \sim R_\beta \to 0$ and $\gamma \to \infty$ for $t \to \infty$ in such way that $R_\beta^2\gamma^2 = $ const. Making of use Eq. (8) and relation $v_y \sim \omega_\beta y$ we get $F_2/F_3 \sim f\epsilon \ll 1$ and $F_1/F_3 \sim (3/4)\kappa^2 f \gamma^{-1/2}\epsilon^{1/2} \ll 1$, where we assume that $\kappa \sim f \sim 1$. The contribution from the accelerating force (or from E_x) to F_3 is of the order $F_2/F_3 \ll 1$. Therefore our model defined by Eqs. (5) and (6) is valid in AAR. For

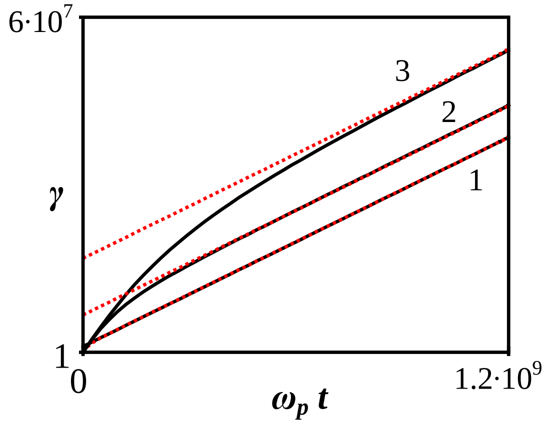

FIG. 3: The dependence of γ on $\omega_p t$ in AAR: analytic solution (red dashed lines) and numerical solution (black solid line) for $R_{\beta,0} = 0.8$ (lines 1), $R_{\beta,0} = 0.2$ (lines 2) and $R_{\beta,0} = 0.1$ (lines 3). The other parameters are the same as in Fig. 2.

high energy electrons quantum electrodynamics (QED) effects can be important. The energy of the photon emitted by the accelerated electron can be so high that the quantum recoil becomes strong. The photon emission can be treated in classical approach if QED parameter $\chi = \left[(mc\gamma\mathbf{E} + \mathbf{p}\times\mathbf{H})^2 - (\mathbf{p}\cdot\mathbf{E})^2\right]^{1/2} / (mcE_{cr}) \simeq \gamma F_\perp/(eE_{cr})$ is much less than unity, where $E_{cr} = m^2c^3/(e\hbar) \approx 1.32\times 10^{16}$ V/cm is the QED critical field [17]. χ can be estimated in AAR as follows $\chi \approx \left[(2f/\alpha)\left(\hbar\omega_p/mc^2\right)\right]^{1/2} \ll 1$, where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant. Therefore the classical approach for the radiation reaction force is valid in the limit $t\to\infty$ because, like for the corrections to the radiation reaction force, the growth of γ in χ is compensated by decreasing of $F_\perp$.

The distance passed by the electron before reaching AAR is $k_p l_{tr} \simeq \left(f/\kappa^2\right) T_{tr} \simeq 1.6\left(\epsilon^2\gamma_0 R^4_{\beta,0} f\kappa^8\right)^{-1/3}$. For the initial parameters $n = 10^{18}$ cm^{-3}, $R_{\beta,0} = 1$, $\gamma_0 = 2\cdot 10^3$, $f = 0.7$, $\kappa^2 = 0.11$ the electron comes into AAR after passing 7800 laser-driven acceleration stages with total distance $l_{tr} \simeq 73$ m, achieving the energy $\gamma mc^2 \simeq 5$ TeV and $R_\beta \simeq 0.008$, where the stage distance is chosen to be equal to the half dephasing length [5] and the distance between the acceleration stages is neglected. For the rarefied plasma $n = 10^{15}$ cm^{-3}, AAR is achieved in 78 stages with $l_{tr} \simeq 23$ km, $\gamma mc^2 \simeq 48$ TeV and $R_\beta \simeq 0.005$. AAR may be achieved within one acceleration stage in the proton-driven acceleration schemes because of very large dephasing length [19].

In conclusions, we have shown that the electron acceleration is not limited by the radiative damping in plasma-based accelerators. Even if the radiation reaction force is stronger than the accelerating force at the beginning, then acceleration eventually succeeds deceleration with time. The damping of the betatron oscillations leads to the transition to the self-similar asymptotic acceleration regime in the infinite-time limit when the radiation reaction force becomes equal to two thirds of the accelerating force. The relative energy spread induced by the radiative damping in the accelerated electron bunch decreases with time in this regime. This opens possibility to use high density plasma at the late stages of multi-stage plasma-based accelerators despite the fact that the radiative damping is enhanced as density increases. The high density plasma can be favorable because it provides high accelerating gradient and, thus, reduces the length of the acceleration stages. The obtained results can be also applied to any other accelerating systems with the linear focusing forces.

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