COLLAPSING AND ESSENTIAL COVERINGS

TAKAO YAMAGUCHI

Abstract. In the present paper, we consider the family of all compact Alexandrov spaces with curvature bound below having a definite upper diameter bound of a fixed dimension. We introduce the notion of essential coverings by contractible metric balls, and provide a uniform bound on the numbers of contractible metric balls forming essential coverings of the spaces in the family. In particular, this gives another view for Gromov's Betti number theorem.

1. Introduction

It is an important problem to find the relation between curvature and topology of Riemannian manifolds. In the study of the finiteness of compact Riemannian manifolds with uniformly bounded curvature, originated by Cheeger [3] and Weinstein [16], it was a crucial idea to cover a manifold in a certain class by convex metric balls whose number is uniformly bounded depending only on the class. The minimal number of such metric balls covering a manifold represents the complexity of the manifold. For a certain class of compact Riemannian manifolds with a lower curvature bound, Grove, Petersen and Wu [9] used contractible balls in place of convex balls to get a topological finiteness of the manifolds in the class. Those are results in the non-collapsing cases.

In the present paper, we consider the collapsing case for compact Alexandrov spaces M with curvature bounded below. The Perelman stability theorem [14] shows that a small metric ball around a given point of M is homeomorphic to the tangent cone, and hence contractible.

If M is collapsed, the sizes of contractible metric balls must be very small, and therefore the minimal number of contractible metric balls covering M becomes large. In other words, it is not efficient to cover the whole M by contractible metric balls. In the present paper, to overcome this difficulty, we introduce the notion of an *essential covering* of M in place of a usual covering.

To illustrate the notion of essential covering, let us take a flat torus $T^2_\epsilon = S^1(1) \times S^1(\epsilon)$ for a small $\epsilon > 0$, where $S^1(\epsilon)$ is the circle of length

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ϵ. The torus T^2_ϵ can be covered by two thin metric balls B_α, $\alpha \in \{1, 2\}$. Each ball B_α is isotopic to a much smaller concentric metric ball $\hat{B}_\alpha$ of radius, say 2ϵ. If one tries to cover B_α by contractible metric balls, we need too many, about $[1/\epsilon]$-pieces of such balls. In stead, we take a covering of $\hat{B}_\alpha$. It is possible to cover $\hat{B}_\alpha$ by two contractible metric balls $\{B_{\alpha\beta}\}_{\beta=1}^2$. Thus we have a collection of four contractible metric balls $\{B_{\alpha\beta}\}$, which is an essential covering of T^2_ϵ. Although it is not a usual covering of T^2_ϵ, deforming and enlarging $B_{\alpha\beta}$ by isotopies, we obtain a covering $\{\tilde{B}_{\alpha\beta}\}$ of T^2_ϵ by contractible open subsets $\tilde{B}_{\alpha\beta}$. In that sense, the essential covering seems to contain an essential property of T^2_ϵ.

In the general collapsing case with a lower curvature bound, it is believed that a collapsed space has a certain fiber structure in a generalized sense such that the fibers shrink to points (see [15], [17]). Although it is not established yet, a fiber which is not visible yet may shrink to a point with different scales in different directions, in general. This suggests that we have to repeat the above process of taking a smaller concentric metric ball and of covering it by much smaller metric balls at most n-times, $n = \dim M$, to finally reach contractible metric balls (see Examples 4.2 and 4.3). In this way, we come to the notion of an *essential covering* of M with *depth* $\le n$.

As illustrated above, an essential covering is not a usual covering of M, but it contains an essential feature on the complexity of the space M. Actually by deforming and enlarging the balls in the essential covering by isotopies of M in a systematic way, we obtain a real open covering of M.

We define the geometric invariant $\tau_n(M)$ as the minimal number of contractible metric balls forming essential coverings of M with depth $\le n$. See Section 4 for a more refined formulation of $\tau_n(M)$. In particular, if M is a Riemannian manifold, we can replace contractible metric balls by metric balls homeomorphic to an n-disk in the definition of $\tau_n(M)$.

For a positive integer n and $D > 0$, we denote by $\mathcal{A}(n, D)$ the isometry classes of n-dimensional compact Alexandrov spaces with curvature ≥ -1 and diameter $\le D$. In this paper, we shall prove

Theorem 1.1. *For given n and D, there is a positive integer $N(n, D)$ such that $\tau_n(M) \le N(n, D)$ for all M in $\mathcal{A}(n, D)$.*

This gives a new geometric restriction on the spaces in $\mathcal{A}(n, D)$ even in the case of Riemannian manifolds. A more detailed information on the essential covering of M is given in Theorem 4.4.

The minimal number cell(M) of open cells needed to cover a compact manifold M is an interesting topological invariant. It is known however that cell$(M) \le \dim M + 1$ (see [4]). This suggests that to have better understanding of the complexity of a compact Riemannian manifold or a compact Alexandrov space concerning the minimal number of some basic subsets needed to cover it, we have to consider a *metric* invariant rather than a topological invariant. This is the reason why we mainly consider coverings by metric balls.

Working with concentric coverings, Theorem 1.1 yields the following uniform bound on the total Betti number:

Corollary 1.2 ([5],[13]). *For given n and D, there is a positive integer $C(n, D)$ such that if M is in $\mathcal{A}(n, D)$, then*

$$\sum_{i=0}^{n} b_i(M; F) \le C(n, D),$$

where F is any field.

In the original work [5], Gromov developed the critical point theory for distance functions to obtain an explicit bound on the total Betti numbers for Riemannian manifolds. The argument in [13] is a natural extension of that in [5] to Alexandrov spaces. Unfortunately our bound is not explicit. However our approach provides a conceptually clear view of what the essence of Corollary 1.2 is like.

For the proof of Theorem 1.1, we use the convergence and collapsing methods. If a space M in $\mathcal{A}(n, D)$ does not collapse, the stability theorem immediately yields the consequence. If M collapses to a lower dimensional space, we use the rescaling method, which was used in [15] and [17] in some special cases. We first generalize those results to the general case. Using this rescaling method, we can grasp the proper size of a collapsed "fiber" although it is not visible. This enables us to have a covering $\{B_{\alpha_1}\}$ of M such that each ball B_{α_1} is, under some rescaling of metric with the fiber size, close to a complete noncompact Alexandrov space Y_1 of nonnegative curvature with $\dim Y_1 > \dim X$ for the pointed Gromov-Hausdorff topology. If the "fiber" uniformly shrinks to a point, the new convergence $B_{\alpha_1} \to Y_1$ does not collapse. In the other collapsing case of $\dim Y_1 < n$, we again grasp the size of a new "fiber" in the collapsing $B_{\alpha_1} \to Y_1$ with the help of the rescaling method. From this, we see that a much smaller concentric subball $\hat{B}_{\alpha_1}$ of B_{α_1}, which is isotopic to B_{α_1}, can be covered by small metric balls $\{B_{\alpha_1\alpha_2}\}_{\alpha_2}$ whose number is uniformly bounded such that each $B_{\alpha_1\alpha_2}$ is, under some rescaling with the size of a new "fiber", close to a complete noncompact Alexandrov space Y_2 of nonnegative curvature with $\dim Y_2 > \dim Y_1$ for the pointed Gromov-Hausdorff topology. Repeating this process at most $n - \dim X$ times and using the stability theorem, we finally get an essential covering of M by contractible metric balls, as required.

Corollary 1.2 follows almost directly from Theorem 1.1 and the topological lemma of [5]. Actually we formulate and prove a more general result for every subset of an n-dimensional complete Alexandrov space in terms of δ-content (see Theorem 5.2).

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2. Preliminaries

We refer to Burago, Gromov and Perelman [2] for the basic materials on Alexandrov spaces with curvature bounded below.

Let M be an Alexandrov space with curvature bounded below, say $\geq \kappa$. For two points x and y in M, a minimal geodesic joining x to y is denoted by xy for simplicity. For any geodesic triangle Δxyz in M with vertices x, y and z, we denote by $\tilde{\Delta} xyz$ a *comparison triangle* in the κ-plane M^2_κ, the simply connected complete surface with constant curvature κ. The angle between the geodesics xy and yz in M is denoted by $\angle xyz$, and the corresponding angle of $\tilde{\Delta} xyz$ by $\tilde{\angle} xyz$. It holds that

$$\angle xyz \geq \tilde{\angle} xyz.$$

Let $\Sigma_p = \Sigma_p(M)$ denote the space of directions at $p \in M$. Let $K_p = K_p(M)$ be the tangent cone at p with vertex o_p, the Euclidean cone over Σ_p. For a closed set $A \subset M$ and $p \in M - A$, we denote by $A' = A'_p$ the subset of Σ_p consisting of all the directions of minimal geodesics from p to A.

From now on, we assume that M is finite-dimensional. It is known that Σ_p (resp. K_p) is a $(n-1)$-dimensional compact (resp. n-dimensional complete noncompact) Alexandrov space with curvature ≥ 1 (resp. curvature ≥ 0), where $n = \dim M$.

It is well-known that as $r \to 0$, $(\frac{1}{r}M, p)$ converges to (K_p, o_p) with respect to the pointed Gromov-Hausdorff topology, where $\frac{1}{r}M$ denotes the rescaling of the original distance of M multiplied by $\frac{1}{r}$.

We denote by $\mathcal{A}_p(n)$ the isometry classes of n-dimensional complete pointed Alexandrov spaces (M, p) with curvature ≥ -1.

The following results play crucial roles in this paper.

Theorem 2.1 ([7],[8]). *$\mathcal{A}(n, D)$ (resp. $\mathcal{A}_p(n)$) is relatively compact with respect to the Gromov-Hausdorff distance (resp. the pointed Gromov-Hausdorff topology).*

Consider the distance function $d_p(x) = d(p, x)$ from a point $p \in M$. A point $q \neq p$ is a *critical point* of d_p if $\tilde{\angle} pqx \leq \pi/2$ for all $x \in M$.

For $0 < r < R$, $A(p; r, R)$ denotes the closed annulus $\bar{B}(p, R) - B(p, r)$, where $B(p, r)$ is the open metric ball around p of radius r.

Lemma 2.2 ([10],[5],[14]). *If d_p has no critical points on $A(p; r, R)$, then $A(p; r, R)$ is homeomorphic to $\partial B(p, r) \times [0, 1]$.*

Theorem 2.3 ([14], cf. [11]). *Let an infinite sequence (M_i, p_i) in $\mathcal{A}_p(n)$ converge to a space (M, p) in $\mathcal{A}_p(n)$ with respect to the pointed Gromov-Hausdorff topology. Take an $r > 0$ such that there are no critical points* *of d_p on $B(p,r)-\{p\}$. Then $B(p_i,r)$ is homeomorphic to both $B(p,r)$ and K_p for large i.*

3. Rescaling metrics

Let a sequence (M_i,p_i) in $\mathcal{A}_p(n)$ converge to a pointed Alexandrov space (X,p) with curvature ≥ -1 with respect to the pointed Gromov-Hausdorff topology. It is a fundamental problem to find topological relation between $B(p_i,r)$ and $B(p,r)$ for a small but fixed positive number r and large i.

In the case when $\dim X = n$, take $r>0$ so that the distance function d_p has no critical points on $B(p,r)-\{p\}$. Then Theorem 2.3 shows that $B(p_i,r)$ is homeomorphic to $B(p,r)$ for large i.

In this section, from now on, we consider the collapsing case when $1 \leq \dim X \leq n-1$. Since we are concerned with the topology of a neighborhood of p_i, we may assume

Assumption 3.1. $B(\tilde{p}_i,r)$ is not homeomorphic to an n-disk for any $\tilde{p}_i$ with $d(p_i,\tilde{p}_i)\to 0$ and for any sufficiently large i.

The following is a generalization of the Key lemma 3.6 in [15] and Theorem 4.1 in [17].

Theorem 3.2. *Under Assumption 3.1, there exist $\hat{p}_i \in B(p_i,r)$ and a sequences $\delta_i \to 0$ such that*

1. *$d(\hat{p}_i,p_i)\to 0$;*
2. *$d_{\hat{p}_i}$ has no critical points on $A(\hat{p}_i;R\delta_i,r)$ for every $R\geq 1$ and large i compared to R. In particular, $B(\hat{p}_i,r)$ is homeomorphic to $B(\hat{p}_i,R\delta_i)$;*
3. *for any limit (Y,y_0) of $(\frac{1}{\delta_i}M_i,\hat{p}_i)$, we have $\dim Y \geq \dim X+1$.*

The essential idea of the proof of Theorem 3.2 is the same as in [15]. In [15] however, we had to suppose that the function $\hat{f}: K_p \to \mathbb{R}$ constructed there takes a strict local maximum at the vertex o_p of K_p. Since this does not hold in general, we must modify the construction. Some simplification of the proof is also made here.

For positive numbers θ and ϵ with $\epsilon \ll \theta \leq \pi/100$, take a positive number $r = r(p,\theta,\epsilon)$ such that

1. $\angle xpy - \tilde{\angle} xpy < \epsilon$ for every $x,y\in\partial B(p,2r)$;
2. $\{y'_p\}_{y\in\partial B(p,2r)}$ is ϵ-dense in Σ_p.

Note that the above (2) implies that there are no critical points of d_p on $B(p,r)-\{p\}$. Let $\{x_\alpha\}_\alpha$ be a θr-discrete maximal system in $\partial B(p,r)$. For a small positive number ϵ, take an ϵr-discrete maximal system $\{x_{\alpha\beta}\}_\beta$, $1\leq\beta\leq N_\alpha$, in $B(x_\alpha,\theta r)\cap\partial B(p,r)$. Let $\xi_{\alpha\beta}\in\Sigma_p$ be the direction of geodesic $px_{\alpha\beta}$. Note that $\{\xi_{\alpha\beta}\}_\beta$ is $\epsilon/2$-discrete. A standard covering argument implies that

$$N_\alpha \geq \mathrm{const}\left(\frac{\theta}{\epsilon}\right)^{\dim X-1}. \tag{3.1}$$

We consider the following functions f_α and f on M:

$$f_\alpha(x)=\frac{1}{N_\alpha}\sum_{\beta=1}^{N_\alpha} d(x_{\alpha\beta},x), \qquad f(x)=\min_\alpha f_\alpha(x).$$

A similar construction was made in [12] to define a strictly concave function on a neighborhood of a given point of an Alexandrov space. The effectiveness of the use of those functions was suggested to the author by Vitali Kapovitch.

Lemma 3.3. *For every $x \in B(p,r/2)$, we have $f(x) \leq r - d(p,x)/2$. In particular, the restriction of f to $B(p,r/2)$ has a strict maximum at p.*

Proof. Take $y \in \partial B(p,r)$ and x_α with $\angle xpy < \epsilon$ and $d(y,x_\alpha) < \theta r$. It follows that $\angle xpx_{\alpha\beta} < 5\theta$. Let $\gamma : [0,d] \to X$ be a minimal geodesic joining p to x. By the curvature assumption with trigonometry, we see that $\angle x\gamma(t)x_{\alpha\beta} < \pi/4$. The first variation formula then implies that $d(x_{\alpha\beta},x) \leq r - d(p,x)/2$, and therefore $f(x) \leq f_\alpha(x) \leq r - d(p,x)/2$. □

Proof of Theorem 3.2. Take a μ_i-approximation

$$\phi_i : B(p,1/\mu_i) \to B(p_i,1/\mu_i),$$

with $\phi_i(p)=p_i$, where $\mu_i \to 0$ as $i \to \infty$. Let $x^i_{\alpha\beta} := \phi_i(x_{\alpha\beta})$, and define the functions f^i_α and f^i on M_i by

$$f^i_\alpha(x)=\frac{1}{N_\alpha}\sum_{\beta=1}^{N_\alpha} d(x^i_{\alpha\beta},x), \qquad f^i(x)=\min_\alpha f^i_\alpha(x).$$

Note that $f^i_\alpha \circ \phi_i \to f_\alpha$ and $f^i \circ \phi_i \to f$. By Lemma 3.3, there is a point $\hat{p}_i \in B(p_i,r/2)$ such that

(1) $(p_i,\hat{p}_i) \to 0$;
(2) the restriction of f^i to $B(p_i,r/3)$ takes a maximum at $\hat{p}_i$.

Consider the distance function $d_{\hat{p}_i}$. By Assumption 3.1, there is a critical point of $d_{\hat{p}_i}$ in $B(\hat{p}_i,r)$. Let δ_i be the maximum distance between $\hat{p}_i$ and the critical point set of $d_{\hat{p}_i}$ within $B(\hat{p}_i,r)$. Note that $\delta_i \to 0$. Let $\hat{q}_i$ be a critical point of $d_{\hat{p}_i}$ within $B(\hat{p}_i,r)$ realizing δ_i. We may assume that $(\frac{1}{\delta_i}M_i,\hat{p}_i)$ converges to a complete noncompact pointed Alexandrov space (Y,y_0) with nonnegative curvature. Let $z_0 \in Y$ be the limit of $\hat{q}_i$ under this convergence. We denote by $\hat{d}=\frac{1}{\delta_i}d$ the distance of $\frac{1}{\delta_i}M_i$. Consider the function

$$h^i_{\alpha\beta}(x) := \hat{d}(x^i_{\alpha\beta},x) - \hat{d}(x^i_{\alpha\beta},\hat{p}_i),$$

which is 1-Lipschitz, and bounded on every bounded set. Therefore passing to a subsequence, we may assume that $h_{\alpha\beta}$ converges to a 1-Lipschitz function $h_{\alpha\beta}$ on Y.

Let

$$h_\alpha = \frac{1}{N_\alpha}\sum_{\beta=1}^{N_\alpha} h_{\alpha\beta}, \qquad h = \min_\alpha h_\alpha,$$
$$h^i_\alpha = \frac{1}{N_\alpha}\sum_{\beta=1}^{N_\alpha} h^i_{\alpha\beta}, \qquad h^i = \min_\alpha h^i_\alpha.$$

Since $h^i = [f^i - f^i(\hat{p}_i)]/\delta_i$, h takes a maximum at y_0. Let $x_{\alpha\beta}(\infty)$ denote the the element of the ideal boundary $Y(\infty)$ of Y defined by the limit ray, say $\gamma_{\alpha\beta}$, from y_0 of the geodesic $\hat{p}_i x^i_{\alpha\beta}$ under the convergence $(\frac{1}{\delta_i}M_i, \hat{p}_i) \to (Y, y_0)$. Let $v_{\alpha\beta} \in \Sigma_{y_0}$ and $v \in \Sigma_{y_0}$ denote the direction of $\gamma_{\alpha\beta}$ and yz respectively. Since z_0 is a critical point of d_{y_0}, we have $\tilde{\angle} y_0 z_0 x_{\alpha\beta}(\infty) \le \pi/2$. Since Y has nonnegative curvature, it follows that $\angle(v, v_{\alpha\beta}) \ge \tilde{\angle} z_0 y_0 x_{\alpha\beta}(\infty) \ge \pi/2$, for every α and β. Choosing α with $h'_{y_0}(v) = (h_\alpha)'_{y_0}(v)$, we obtain

$$0 \ge h'_{y_0}(v) = \frac{1}{N_\alpha}\sum_{\beta=1}^{N_\alpha} -\cos\angle(v, v_{\alpha\beta}),$$

and therefore $\angle(v, v_{\alpha\beta}) = \pi/2$. Since

$$\begin{aligned}\angle(v_{\alpha\beta}, v_{\alpha\beta'}) &= \lim_{t\to 0} \tilde{\angle}\gamma_{\alpha\beta}(t) y_0 \gamma_{\alpha\beta'}(t)\\ &\ge \tilde{\angle} x_{\alpha\beta} p x_{\alpha\beta'}\\ &\ge \epsilon/2,\end{aligned}$$

for every $1 \le \beta \ne \beta' \le N_\alpha$, $\{v_{\alpha\beta}\}_{\beta=1}^{N_\alpha}$ is $\epsilon/2$-discrete in $\partial B(v, \pi/2) \subset \Sigma_{y_0}$. Since Σ_{y_0} has curvature ≥ 1, there is an expanding map from $\partial B(v, \pi/2)$ to the unit sphere $S^{\dim Y - 2}(1)$. It follows that

$$N_\alpha \le \text{const}\, \epsilon^{-(\dim Y - 2)}. \tag{3.2}$$

Since this holds for any sufficiently small ϵ, from (3.1) and (3.2) we can conclude $\dim Y \ge \dim X + 1$. This completes the proof of Theorem 3.2. □

4. Isotopy covering systems and essential coverings

Let M be a compact n-dimensional Alexandrov space with curvature bounded below. For an open metric ball B of M, we denote by λB the concentric ball of radius λr. We call a concentric ball $\hat{B} \subset B$ an *isotopic subball* of B if there is a homeomorphism $M \to M$ sending $\hat{B}$ onto B

and leaving the outside of a neighborhood of $\bar{B}$ fixed. For instance, this is the case when d_p has no critical points on $\bar{B} - \hat{B}$ (Lemma2.2).

Consider the following system $\mathcal{B} = \{B_{\alpha_1 \cdots \alpha_k}\}$ consisting of open metric balls $B_{\alpha_1 \cdots \alpha_k}$ of M, where the indices $\alpha_1, \dots, \alpha_k$ range over

$$1 \le \alpha_1 \le N_1, \quad 1 \le \alpha_2 \le N_2(\alpha_1),$$
$$1 \le \alpha_k \le N_k(\alpha_1 \cdots \alpha_{k-1}),$$

and $1 \le k \le \ell$ for some ℓ depending on the choice of the indices $\alpha_1, \alpha_2, \dots$. Note that the range of α_k also depends on $\alpha_1 \cdots \alpha_{k-1}$. Let A be the set of all multi-indices $\alpha = \alpha_1 \cdots \alpha_k$ such that $B_{\alpha_1 \cdots \alpha_k} \in \mathcal{B}$. For each $\alpha = \alpha_1 \cdots \alpha_k \in A$, put $|\alpha| := k$ and call it the length of α.

Let X be a subset of M. We call $\mathcal{B}$ an *isotopy covering system* of X if it satisfies the following:

(1) $\{B_{\alpha_1}\}_{\alpha_1=1}^{N_1}$ covers X;
(2) $B_{\alpha_1 \cdots \alpha_{k-1}} \supset B_{\alpha_1 \cdots \alpha_k}$;
(3) $\{B_{\alpha_1 \cdots \alpha_k}\}_{\alpha_k=1}^{N_k(\alpha_1 \cdots \alpha_{k-1})}$ is a covering of an isotopic subball $\hat{B}_{\alpha_1 \cdots \alpha_{k-1}}$ of $B_{\alpha_1 \cdots \alpha_{k-1}}$;
(4) there is a uniform bound d such that $|\alpha| \le d$ for all $\alpha \in A$.

We call N_1 the *first degree* of the system $\mathcal{B}$, and $N_k(\alpha_1 \cdots \alpha_{k-1})$ the *k-th degree* of $\mathcal{B}$ with respect to $\alpha_1 \cdots \alpha_{k-1}$.

Let $\hat{A}$ be the set of all maximal multi-indices $\alpha_1 \cdots \alpha_\ell$ in A in the sense that there are no $\alpha_{\ell+1}$ with $B_{\alpha_1 \cdots \alpha_\ell \alpha_{\ell+1}} \in \mathcal{B}$. Then $\mathcal{U} := \{B_\alpha\}_{\alpha \in \hat{A}}$ is called an *essential covering* of B. In other words, $\mathcal{U}$ is the collection of the metric balls lying on the bottom of the system $\mathcal{B}$.

We show that the essential covering $\mathcal{U} = \{B_\alpha\}_{\alpha \in \hat{A}}$ produces a covering $\tilde{\mathcal{U}} = \{\tilde{B}_\alpha\}_{\alpha \in \hat{A}}$ of X such that $\tilde{B}_\alpha$ is homeomorphic (actually isotopic) to B_α. Let $h_{\alpha_1 \cdots \alpha_{k-1}} : M \to M$ be a homeomorphism sending $\hat{B}_{\alpha_1 \cdots \alpha_{k-1}}$ onto $B_{\alpha_1 \cdots \alpha_{k-1}}$ and leaving the outside of a neighborhood of $\bar{B}_{\alpha_1 \cdots \alpha_{k-1}}$ fixed. For each $\alpha = \alpha_1 \cdots \alpha_\ell \in \hat{A}$, consider the open set

$$\tilde{B}_\alpha := h_{\alpha_1} \circ h_{\alpha_1 \alpha_2} \circ \cdots \circ h_{\alpha_1 \cdots \alpha_{\ell-1}}(B_\alpha).$$

For each $1 \le \alpha_1 \le N_1$, let $A(\alpha_1)$ be the set of all multi-indices $\alpha \in A$ of the forms $\alpha = \alpha_1 \cdots \alpha_k$ whose leading term is equal to α_1 and $k \ge 2$. From construction, we have

$$B_{\alpha_1} \subset \bigcup_{\alpha \in A(\alpha_1)} \tilde{B}_\alpha$$

and therefore $\tilde{\mathcal{U}} = \{\tilde{B}_\alpha\}_{\alpha \in \hat{A}}$ provides a covering of X.

We call

$$d_0 := \max_{\alpha \in \hat{A}} |\alpha|$$

the *depth* of both $\mathcal{B}$ and $\mathcal{U}$. Note that if $d_0 = 1$, then $\mathcal{B} = \mathcal{U}$ is a usual covering of X.

Let $\mathcal{C}(n)$ be the set of all isometry classes of the Euclidean cone over $(n-1)$-dimensional compact Alexandrov spaces with curvature ≥ 1. We say that $\mathcal{B}$ and $\mathcal{U}$ are *modeled* on $\mathcal{C}(n)$ if each B_α in $\mathcal{U}$ is homeomorphic to a space in $\mathcal{C}(n)$.

For any positive integer d, we denote by $\tau_d(X)$ the minimal number of metric balls forming an essential covering $\mathcal{U}$ of X with depth $\leq d$ modeled on $\mathcal{C}(n)$. Note that $\tau_{d_1}(B) \geq \tau_{d_2}(B)$ if $d_1 \leq d_2$.

For open metric ball B of M having a proper isotopic subball, we set

$$\tau_d^*(B) = \min_{\hat{B}} \tau_d(\hat{B}),$$

where $\hat{B}$ runs over all isotopic subballs of B. If B itself is homeomorphic to a space in $\mathcal{C}(n)$, we define

$$\tau_0(B) = \tau_0^*(B) = 1.$$

From definition, we immediately have

Lemma 4.1. *Suppose that X is covered by metric balls $\{B_{\alpha_1}\}_{\alpha_1=1}^{N_1}$ having proper isotopic subballs. Then we have*

$$\tau_{d+1}(X) \leq \sum_{\alpha_1=1}^{N_1} \tau_d^*(B_{\alpha_1}).$$

Example 4.2. For a positive number ϵ, let us consider the flat torus

$$T_\epsilon^n = S^1(1) \times S^1(\epsilon) \times S^1(\epsilon^2) \times \cdots \times S^1(\epsilon^{n-1}).$$

An obvious observation similar to that in the introduction shows $\tau_n(T^n) \leq 2^n$. Note that $\lim_{\epsilon\to 0} \tau_d(T_\epsilon^n) = \infty$ for every $1 \leq d \leq n-1$.

Example 4.3. Let N be an n-dimensional simply connected Lie group, and $\mathfrak{n}$ its Lie algebra. Take a triangular basis $x_1, \ldots, x_n$ of $\mathfrak{n}$ in the sense that $[x_i, x] \in \mathfrak{n}_{i-1}$ for every $x \in \ell$, where $\mathfrak{n}_{i-1}$ is spanned by $x_1, \ldots, x_{i-1}$. For $\epsilon > 0$, put $\epsilon_i := \epsilon^{n^{n-i}}$, and define the inner product on $\mathfrak{n}$ by

$$||x||_\epsilon^2 = \epsilon_1^2 a_1^2 + \cdots + \epsilon_n^2 a_n^2,$$

for $x = \sum a_i x_i$. We equip N the corresponding left invariant metric g_ϵ. For a given uniform discrete subgroup Γ of N, consider the quotient $M_\epsilon := (\Gamma\backslash N, g_\epsilon)$. Note that the sectional curvature of M_ϵ is uniformly bounded and $\delta_\epsilon^1 := \mathrm{diam}(M_\epsilon) \to 0$ as $\epsilon \to 0$ (see [6]). Now under the rescaling of metric $\frac{1}{\delta_\epsilon^1} M_\epsilon$ collapses to a circle. We then have a fibration

$$\Gamma_1\backslash N_1 \to M_\epsilon \to S^1,$$

with a nilmanifold $\Gamma_1\backslash N_1$ as fiber. Thus M_ϵ can be covered by two thin metric balls B_{α_1}, $\alpha_1 \in \{1, 2\}$, each of which is homeomorphic to $\Gamma_1\backslash N_1 \times [0,1]$. Let $\delta_\epsilon^2 := \mathrm{diam}(\Gamma_1\backslash N_1)$. Under the rescaling of metric $\frac{1}{\delta_\epsilon^1} B_{\alpha_1}$ collapses to $S^1 \times \mathbb{R}$. Now an isotopic subball $\hat{B}_{\alpha_1}$ of B_{α_1} has a fibration

$$\Gamma_2\backslash N_2 \to \hat{B}_{\alpha_1} \to S^1 \times [0,1],$$

with a nilmanifold $\Gamma_2\backslash N_2$ as fiber. Thus $\hat{B}_{\alpha_1}$ can be covered by two metric balls $B_{\alpha_1\alpha_2}$, $\alpha_2 \in \{1,2\}$, each of which is homeomorphic to $\Gamma_2\backslash N_2 \times [0,1]^2$. Repeating this, we finally have $\tau_n(M_\epsilon) \le 2^n$. Note that $\lim_{\epsilon\to 0} \tau_d(M_\epsilon) = \infty$ for every $1 \le d \le n-1$.

Let $A(n)$ denote the set of all isometry classes of n-dimensional complete Alexandrov spaces with curvature ≥ -1. Theorem 1.1 is an immediate consequence of the following

Theorem 4.4. *For given n and $D > 0$, there are constants C_n and $C_n(D)$ such that for every metric ball B of radius $\le D$ in $M \in \mathcal{A}(n)$, there is an isotopy covering system $\mathcal{B} = \{B_{\alpha_1\cdots\alpha_k}\}$ of B with depth $\le n$ modeled on $\mathcal{C}(n)$ such that*

(1) *the first degree $\le C_n(D)$;*

(2) *any other higher degree $\le C_n$.*

In particular $\tau_n(B) \le C_n(D)(C_n)^{n-1}$.

We first prove the local version of Theorem 4.4.

Lemma 4.5. *There is a positive number C_n satisfying the following: For a given infinite sequence (M_i, p_i) in $\mathcal{A}_p(n)$ with $\inf \operatorname{diam}(M_i) > 0$, there is a subsequence (M_j, p_j) for which we have a positive number $r > 0$ and $\hat{p}_j \in M_j$ with $d(p_j, \hat{p}_j) \to 0$ such that $\tau^*_{n-1}(B(\hat{p}_j, r)) \le C_n$.*

Proof. We prove it by contradiction. If the conclusion does not hold, we would have an infinite sequence (M_i, p_i) in $\mathcal{A}_p(n)$ such that for every $r > 0$ and every $\hat{p}_i \in M_i$ with $d(p_i, \hat{p}_i) \to 0$, we have $\tau^*_{n-1}(B(\hat{p}_j, r)) \to \infty$ for any subsequence $\{j\}$ of $\{i\}$. By Theorem 2.1, we have a subsequence $\{j\}$ such that (M_j, p_j) converges to a pointed space (X, p). Set $k = \dim X$.

We claim that $\tau^*_{n-k}(B(\hat{p}_j, r)) \le C$ for some $r > 0$ and constant C independent of j, where $\hat{p}_j$ is a point of M_j with $d(p_j, \hat{p}_j) \to 0$. Since this is a contradiction, this will complete the proof.

We prove the claim by the reverse induction on k. If $k = n$, then Theorem 2.3 shows that there is an $r > 0$ such that $B(p_j, r)$ is homeomorphic to K_p, yielding $\tau^*_0(B(p_j, r)) = 1$. Therefore together with the diameter assumption, we only have to investigate the case $1 \le \dim X \le n-1$. Suppose the claim holds for $\dim X = k+1, \dots, n$, and consider the case of $\dim X = k$. Take $r = r_p > 0$, $\hat{p}_j$ and $\delta_j \to 0$ as in Theorem 3.2. Namely passing to a subsequence, we may assume that $(\frac{1}{\delta_j} M_j, \hat{p}_j)$ converges to a pointed complete noncompact nonnegatively curved space (Y, y_0) with $\dim Y \ge \dim X + 1$ such that $B(\hat{p}_j, R\delta_j)$ is an isotopic subball of $B(\hat{p}_j, r)$ for every $R \ge 1$ and large j compared to R. Applying the induction hypothesis to the convergence $(\frac{1}{\delta_j} M_j, \hat{p}_j) \to (Y, y_0)$, we have the following: For each $z \in B(y_0, 2)$, there are $z^j \in (\frac{1}{\delta_j} M_j, \hat{p}_j)$ and $r_z > 0$ such that $\tau^*_{n-k-1}(B(z^j, r_z; \frac{1}{\delta_j} M_j)) \le C$ for some constant

C independent of j. By compactness, there are finitely many points $z_\alpha \in B(y_0, 2)$ and $z_\alpha^j \in M_j$ converging to z_α together with $r_\alpha > 0$ such that

$$\bigcup B(z_\alpha, r_\alpha/2) \supset B(y_0, 2), \qquad \tau^*_{n-k-1}(B(z_\alpha^j, r_\alpha; \frac{1}{\delta_j} M_j)) \le C_\alpha.$$

Note that $\cup B(z_\alpha^j, r_\alpha; \frac{1}{\delta_j} M_j) \supset B(\hat{p}_j, 2; \frac{1}{\delta_j} M_j)$ for large i. Thus we can conclude

$$\begin{aligned}\tau^*_{n-k}(B(\hat{p}_j, r)) &\le \tau_{n-k}(B(\hat{p}_j, 2\delta_j)) \\ &\le \sum_\alpha \tau^*_{n-k-1}(B(\hat{z}_\alpha^j, r_\alpha)) \\ &\le \sum C_\alpha < \infty.\end{aligned}$$

□

Proof of Theorem 4.4. The proof is by contradiction. If the conclusion does not hold, we would have an infinite sequence of metric balls B_i of spaces $M_i \in \mathcal{A}(n)$ such that for every essential covering system $\mathcal{B}^i$ of B_i with depth $\le n$ modeled on $\mathcal{C}(n)$, either $\liminf N_1^i = \infty$ or $\liminf N_k^i > C_n$, where N_1^i, N_k^i are the degrees of $\mathcal{B}^i$, and C_n is the positive constant given in Lemma 4.5. Let p_i be the center of B_i. By Theorem 2.1, we may assume that (M_i, p_i) converges to a pointed complete Alexandrov space (X, p) with curvature ≥ -1 with respect to the pointed Gromov-Hausdorff topology. We may also assume that B_i converges to a metric ball B around p under this convergence. If X is a point, we rescale the metric of M_i so that the new diameter is equal to 1. Thus we may assume that $1 \le \dim X \le n$. Applying Lemma 4.5 to the convergence $B_i \to B$, we obtain finitely many points $\{x_\alpha\}_{\alpha=1}^N$ of B and positive numbers r_α with $B \subset \cup B(x_\alpha, r_\alpha/2)$ such that for a subsequence $\{j\}$ of $\{i\}$, we get $p_\alpha^j \in M_j$ converging to x_α with $\tau^*_{n-1}(B(p_\alpha^j, r_\alpha)) \le C_n$ for every $1 \le \alpha \le N$. Together with the covering $\{B(p_\alpha^j, r_\alpha)\}_\alpha$ of M_j, this enables us to obtain an essential covering system $\mathcal{B}^j$ of B_j with depth $\le n$ modeled on $\mathcal{C}(n)$ such that $N_1^j \le N$ and $N_k^j \le C_n$. This is a contradiction. □

Remark 4.6. Let $\mathcal{M}(n)$ denote the subfamily of $\mathcal{A}(n)$ consisting of Riemannian manifolds. By Theorem 4.4, each metric ball of radius $\le D$ in $M \in \mathcal{M}(n)$ has an essential covering with depth $\le n$ modeled on $\mathcal{C}(n)$ whose number is uniformly bounded. In this case, one can easily check from the proof that each metric ball in the essential covering is homeomorphic to an n-disk. Namely, for $\mathcal{M}(n)$, we can take the single n-dimensional Euclidean space $\mathbb{R}^n$ as the model family in stead of $\mathcal{C}(n)$.

Remark 4.7. Let $\delta > 0$ be given. Under the situation of Theorem 4.4, if we restrict ourselves to metric balls of radii $< \delta$, we can construct an isotopy covering system $\mathcal{B} = \{B_{\alpha_1\cdots\alpha_k}\}$ of B with depth $\le n$ modeled on $\mathcal{C}(n)$ such that

(1) the radius of B_{α_1} is less than δ for every $1 \le \alpha_1 \le N_1$;
(2) the first degree $N_1 \le C_n(D,\delta)$ for some uniform constant $C_n(D,\delta)$;
(3) any other higher degree $\le C_n$.

In particular $\tau_n(B) \le C_n(D,\delta)(C_n)^{n-1}$.

Parhaps Examples 4.2 and 4.3 will be ones of maximal case.

Conjecture 4.8. Let M be an n-dimensional compact Alexandrov space with nonnegative curvature. Then $\tau_n(M) \le 2^n$.

5. Betti numbers

In this section, we apply Theorem 4.4 to prove Corollary 1.2. We consider homology groups with any coefficient field F. Let $\beta(\quad)$ denote the total Betti number for simplicity.

We make use of the following machinery in [5], whose proof is based on Leray's spectral sequence.

Lemma 5.1 (Topological lemma ([5])). *Let B^i_α, $1 \le \alpha \le N$, $0 \le i \le n+1$, be open subsets of an n-dimensional space M, with*

$$B^0_\alpha \subset B^1_\alpha \subset \cdots B^{n+1}_\alpha,$$

and set $A^i := \cup_{i=1}^N B^i_\alpha$. Let I_+ denote the set of all multi-indices $(\alpha_1,\dots,\alpha_m)$ with $1 \le \alpha_1 < \dots < \alpha_m \le N$ and with non-empty intersection $\cap_{j=1}^m B^{n+1}_{\alpha_j}$. For each $\mu = (\alpha_1,\dots,\alpha_m) \in I_+$, let $f^i_\mu : H_*(B^i_{\alpha_1} \cap \cdots \cap B^i_{\alpha_m}) \to H_*(B^{i+1}_{\alpha_1} \cap \cdots \cap B^{i+1}_{\alpha_m})$ be the inclusion homomorphism. Then the rank of the inclusion homomorphism $H_*(A^0) \to H_*(A^{n+1})$ is bounded above by the sum

$$\sum_{0 \le i \le n, \mu \in I_+} \operatorname{rank} f^i_\mu$$

For any subset $X \subset M$ and $\delta \ge 0$, let $U_\delta(X) := \{x \mid d(x,X) \le \delta\}$. We define *$\delta$-content*, denoted by δ-cont(X) of X as the rank of the inclusion homomorphism $H_*(X) \to H_*(U_\delta(X))$. Observe that 0-cont$(X) = \beta(X)$ may be infinite. However we have

Theorem 5.2. *For given n, $D > 0$ and $\delta > 0$, there is a positive integer $C(n,D,\delta)$ such that if X is a subset of diameter $\le D$ in an n-dimensional complete Alexandrov space M with curvature ≥ -1, then*

$$\delta\text{-cont}(X) \le C(n,D,\delta).$$

Corollary 1.2 is a direct consequence of Theorem 5.2. Although it is not explicitly stated in [5] or [1], Theorem 5.2 also follows from the methods there. Below we give the proof of Theorem 5.2 based on Theorem 4.4.

For a subset X of diameter less than D in a space $M \in \mathcal{A}(n)$, let B an open metric D-ball in M containing X. For $\delta > 0$, take an isotopy covering system $\mathcal{B} = \{B_{\alpha_1\cdots\alpha_k}\}$ of B with depth $\le n$ modeled on $\mathcal{C}(n)$ satisfying the conclusion of Theorem 4.4 and Remark 4.7 such that the radii of B_{α_1} are less than $10^{-(n+2)}\delta$ for all $1 \le \alpha_1 \le N_1$. To apply Lemma 5.1, we let $\lambda_i := 10^i$ for $0 \le i \le n+1$, and put

$$B^i_{\alpha_1\cdots\alpha_k} := \lambda_i B_{\alpha_1\cdots\alpha_k}.$$

In view of the conclusion (2) of Theorem 3.2, we may assume that

(1) $B^{n+1}_{\alpha_1\cdots\alpha_k} \subset B_{\alpha_1\cdots\alpha_{k-1}}$;
(2) $B^i_{\alpha_1\cdots\alpha_k}$ is an isotopic subball of $B^{i+1}_{\alpha_1\cdots\alpha_k}$,

for each $1 \le \alpha_k \le N_k(\alpha_1 \cdots \alpha_{k-1})$ and $0 \le i \le n+1$.

Let $\mathcal{U} = \{B_\alpha\}_{\alpha \in \hat{A}}$ be the essential covering of B associated with $\mathcal{B}$.

Lemma 5.3. *For every $\alpha = \alpha_1 \cdots \alpha_\ell \in \hat{A}$ and every $1 \le k \le \ell$ we have*

$$\beta(B_{\alpha_1\cdots\alpha_k}) \le C_n.$$

Proof. We prove it by the reverse induction on k. The case $k = \ell$ is clear since $B_{\alpha_1\cdots\alpha_\ell}$ is contractible. Suppose the conclusion $\beta(B_{\alpha_1\cdots\alpha_{k+1}}) \le C_n$ for all $1 \le \alpha_{k+1} \le N_{k+1}$. Let $\hat{B}_{\alpha_1\cdots\alpha_k}$ be the isotopic subball of $B_{\alpha_1\cdots\alpha_k}$ such that

$$\hat{B}_{\alpha_1\cdots\alpha_k} \subset \bigcup_{\alpha_{k+1}=1}^{N_{k+1}} B_{\alpha_1\cdots\alpha_{k+1}}.$$

Since $(\alpha_1, \dots, \alpha_k)$ is fixed, we put

$$\hat{B} := \hat{B}_{\alpha_1\cdots\alpha_k}, \qquad B := B_{\alpha_1\cdots\alpha_k},$$
$$B_\alpha := B_{\alpha_1\cdots\alpha_k\alpha}, \qquad B^i_\alpha := \lambda_i B_\alpha,$$

for each $1 \le \alpha \le N_{k+1}$. Let $A^i = \bigcup_{\alpha=1}^{N_{k+1}} B^i_\alpha$. From the inclusions $\hat{B} \subset A^0 \subset A^{n+1} \subset B$, we have $\beta(\hat{B}) = \beta(B) \le \text{rank of}\,[H_*(A^0) \to H_*(A^{n+1})]$. Let I_+ denote the set of multi-indices of intersection for the covering $\{B^{n+1}_\alpha\}_\alpha$. For each $\mu = (\gamma_1, \dots, \gamma_m) \in I_+$, let B_{γ_s} have minimal radius among $\{B_{\gamma_j}\}_{j=1}^m$. Let $f^i_\mu : H_*(B^i_{\gamma_1} \cap \cdots \cap B^i_{\gamma_m})) \to H_*(B^{i+1}_{\gamma_1} \cap \cdots \cap B^{i+1}_{\gamma_m})$ be the inclusion homomorphism. From the inclusions

$$B^i_{\gamma_1} \cap \cdots \cap B^i_{\gamma_m} \subset B^i_{\gamma_s} \subset \frac{1}{2} B^{i+1}_{\gamma_s} \subset B^{i+1}_{\gamma_1} \cap \cdots \cap B^{i+1}_{\gamma_m},$$

we have

$$\begin{aligned}\text{rank}(f^i_\mu) &\le \text{rank of}\,[H_*(B^i_{\gamma_s}) \to H_*(\frac{1}{2}B^{i+1}_{\gamma_s})] \\ &= \beta(B_{\gamma_s}) \\ &\le C_n.\end{aligned}$$

Lemma 5.1 then shows $\beta(B) \le (n+1)2^{C_n} C_n$. □

Proof of Theorem 5.2. Without loss of generality, we may assume that $\{B_{\alpha_1}\}_{\alpha_1=1}^{N(X)}$ is a covering of X for some $N(X)$ with $N(X) \le N_1$. By Lemma 5.3, we have in particular $\beta(B_{\alpha_1}) = \beta(B^i_{\alpha_1}) \le C_n$ for all $1 \le \alpha_1 \le N(X)$ and $0 \le i \le n+1$. Therefore applying Lemma 5.1 to the concentric coverings $\{B^i_{\alpha_1}\}_{\alpha_1=1}^{N(X)}$ of X together with

$$X \subset \bigcup_{\alpha_1=1}^{N(X)} B_{\alpha_1} \subset \bigcup_{\alpha_1=1}^{N(X)} B^{n+1}_{\alpha_1} \subset U_\delta(X),$$

we have

$$\delta\text{-cont}(X) \le (n+1)2^{C_n(D,\delta)} C_n.$$

This completes the proof of Theorem 5.2. □

For a subset X of a metric space, we define the *homological injectivity radius* of X, denoted by $\mathrm{hom.inj}(X)$, as the supremum of $\delta \ge 0$ such that the inclusion homomorphism $H_*(X) \to H_*(U_\delta(X))$ is injective for any coefficient field.

The following is an immediate consequence of Theorem 5.2.

Corollary 5.4. *For a space M in $\mathcal{A}(n)$, let X_i be a sequence of subsets of M with $\lim \beta(X_i) = \infty$. Then one of the following must occur:*

(1) $\liminf \mathrm{hom.inj}(X_i) = 0$;
(2) $\limsup \mathrm{diam}(X_i) = \infty$.

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Institute of Mathematics, University of Tsukuba, Tsukuba 305-8571, JAPAN
E-mail address: takao@math.tsukuba.ac.jp