DIOPHANTINE APPROXIMATION WITH RESTRICTED NUMERATORS AND DENOMINATORS ON SEMISIMPLE GROUPS

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ABSTRACT. We consider the problem of Diophantine approximation on semisimple algebraic groups by rational points with restricted numerators and denominators and establish a quantitative approximation result for all real points in the group by rational points with any prescribed denominator and an almost prime numerator.

1. INTRODUCTION

In this paper we are interested in the problem of Diophantine approximation of real points x by rational points $\frac{u}{v}$, where the numerator u and the denominator v are restricted to interesting arithmetic sets; for instance, when u, v are primes or r-primes. Recall that an integer is called r-prime if it is a product of at most r prime factors counted with multiplicities.

Starting from the work of Vinogradov [V], the question for which $\alpha > 1$ the inequality

$$\left| x - \frac{u}{v} \right| \leq v^{-\alpha}, \quad x \in \mathbb{R}, \tag{1.1}$$

has infinitely many solutions with integral u and prime v attracted significant attention (see [V77, H83, B86, J93, H96, HBJ02, M09]). When u is r-prime and v is prime, this question has been investigated in [V76, H84], but it still seems open when both u and v are assumed to be prime (see [R77, S82]).

Here we consider an analogous question for the Diophantine approximation on semisimple algebraic groups. For instance, let us consider a special linear group $\mathrm{SL}_N = \{x \in \mathrm{Mat}_N(\mathbb{C}) : \det(x) = 1\}$. It is well known that $\mathrm{SL}_N(\mathbb{Q})$ is dense in $\mathrm{SL}_N(\mathbb{R})$, and explicit quantitative density estimates have been established in [GGN]. Now it is natural to ask whether we can approximate any $x \in \mathrm{SL}_N(\mathbb{R})$ by rational points $z \in \mathrm{SL}_N(\mathbb{Q})$ whose coordinates have prescribed arithmetic properties. In particular,

Question 1.1. *Is the set of points in $\mathrm{SL}_N(\mathbb{Q})$ with prime denominators and r-prime numerators dense in $\mathrm{SL}_N(\mathbb{R})$?*

As we shall show, this is indeed the case, and moreover, a quantitative estimate similar to (1.1) holds.

In full generality, our result deals with a simply connected semisimple algebraic $\mathbb{Q}$-group $\mathrm{G} \subset \mathrm{GL}_N$ which is isotropic over $\mathbb{Q}$ and $\mathbb{Q}$-simple. It is known that $\mathrm{G}(\mathbb{Q})$ is dense in $\mathrm{G}(\mathbb{R})$. Moreover, it follows from the strong approximation property [PR, §7.4] that for every $n \geq 2$, the set $\mathrm{G}(\mathbb{Z}[1/n])$ is dense in $\mathrm{G}(\mathbb{R})$. Every $z \in \mathrm{G}(\mathbb{Q})$ can be uniquely written as $z = v^{-1} u$ with $v \in \mathbb{N}$ and $u \in \mathrm{Mat}_N(\mathbb{Z})$ such that $\gcd(u_{11}, \ldots, u_{NN}, v) = 1$. We use the denominator $\mathrm{den}(z) := v$ to measure complexity of rational points and quantify their density in $\mathrm{G}(\mathbb{R})$ with respect to a right invariant Riemannian metric d on $\mathrm{G}(\mathbb{R})$.

Let $f_1, \ldots, f_t$ be a collection of polynomials on $\mathrm{Mat}_N(\mathbb{C})$ with integral coefficients. We assume that f_i's considered as elements of the coordinate ring $\mathbb{Q}[\mathrm{G}]$ are nonzero, distinct, and absolutely irreducible. We say that an element $z \in \mathrm{Mat}_N(\mathbb{Z}[1/n])$ is *r-prime*, with respect to the family of polynomials $f_1, \ldots, f_t$, if $f_1(z) \cdots f_t(z)$ is a product of at most r prime factors in the ring $\mathbb{Z}[1/n]$.

Our main result is the following

Theorem 1.2. *Given a simply connected semisimple algebraic group* $\mathrm{G} \subset \mathrm{GL}_\mathrm{N}$ *defined over* $\mathbb{Q}$*, which is isotropic over* $\mathbb{Q}$ *and* $\mathbb{Q}$*-simple, and a collection of polynomials* $f_1, \ldots, f_t$ *as above, there exist explicit* $\alpha > 0$ *and* $r \in \mathbb{N}$ *such that for every* $x \in \mathrm{G}(\mathbb{R})$ *and* $n \geq n_0(x)$*, one can find* $z \in \mathrm{G}(\mathbb{Q})$ *satisfying*

$$d(x, z) \leq n^{-\alpha},$$
$$\mathrm{den}(z) = n,$$

and r-prime in $\mathrm{Mat}_n(\mathbb{Z}[1/n])$*. Moreover, the constant* $n_0(x)$ *is uniform over* x *in bounded subsets of* $\mathrm{G}(\mathbb{R})$*.*

Remark 1.3. Let $\|\cdot\|_\infty$ be the Euclidean norm on $\mathrm{Mat}_N(\mathbb{R})$. Using that $\mathrm{G}(\mathbb{R})$ is a submanifold of $\mathrm{Mat}_N(\mathbb{R})$, one can check that

$$\|x_1 - x_2\|_\infty \ll d(x_1, x_2), \quad x_1, x_2 \in \mathrm{G}(\mathbb{R}),$$

where the implied constant is uniform over x_1, x_2 in bounded subset of $\mathrm{G}(\mathbb{R})$. Hence, Theorem 1.2 also implies a Diophantine approximation result with respect to the Euclidean norm.

Next we consider the case when G is not necessarily isotropic over $\mathbb{Q}$. Then the set $\mathrm{G}(\mathbb{Z}[1/p])$, where p is prime, might be discrete in $\mathrm{G}(\mathbb{R})$. In fact, $\mathrm{G}(\mathbb{Z}[1/p])$ is dense in $\mathrm{G}(\mathbb{R})$ if and only if the group G is isotropic over the p-adic field $\mathbb{Q}_p$ (see [PR, Th. 7.12]). Under this assumption we prove a weaker version of Theorem 1.2 for $\mathrm{G}(\mathbb{Z}[1/p]) \subset \mathrm{G}(\mathbb{R})$ where the parameters $\alpha, r, n_0(x)$ might depend on p.

Theorem 1.4. *Given a simply connected* $\mathbb{Q}$*-simple algebraic group* $\mathrm{G} \subset \mathrm{GL}_\mathrm{N}$ *defined over* $\mathbb{Q}$*, a collection of polynomials* $f_1, \ldots, f_t$ *as above, and a finite collection* $\mathcal{P}$ *of primes such that* G *is isotropic over* $\mathbb{Q}_p$ *for all* $p \in \mathcal{P}$*,* *there exist explicit* $\alpha > 0$ *and* $r \in \mathbb{N}$ *such that for every* $x \in \mathrm{G}(\mathbb{R})$ *and* $n \geq n_0(x)$ *whose prime divisors are in* $\mathcal{P}$*, one can find* $z \in \mathrm{G}(\mathbb{Q})$ *satisfying*

$$d(x, z) \leq n^{-\alpha},$$
$$\mathrm{den}(z) = n,$$

and r*-prime in* $\mathrm{Mat}(\mathbb{Z}[1/n])$*. Moreover, the constant* $n_0(x)$ *is uniform over* x *in bounded subsets of* $\mathrm{G}(\mathbb{R})$*.*

More explicit statements of Theorems 1.2 and 1.4 are given in Section 6 below.

The proof of the main theorems is based on the uniform spectral gap property for the automorphic unitary representations and the asymptotic analysis of suitable averaging operators combined with standard number-theoretic sieving arguments. In the following section we introduce essential notation and outline the strategy of the proof in more details.

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2. Initial set-up

Throughout the paper, p always denotes a prime number.

Let $\mathrm{G} \subset \mathrm{GL}_N$ be a simply connected $\mathbb{Q}$-simple algebraic group defined over $\mathbb{Q}$. We also use the same notation for the corresponding integral model of G defined by the embedding of G into GL_N.

For $n \in \mathbb{N}$, we set

$$G_n^{\mathrm{f}} := \prod_{p|n} \mathrm{G}(\mathbb{Q}_p) \quad \text{and} \quad G_n := \mathrm{G}(\mathbb{R}) \times G_n^{\mathrm{f}}.$$

Let

$$\Gamma_n := \mathrm{G}(\mathbb{Z}[1/n]).$$

We consider Γ_n as a subgroup of G_n embedded in G_n diagonally. Then Γ_n is a discrete subgroup with finite covolume (see [PR, §5.4]).

For every prime p, we fix a special maximal compact open subgroup U_p of $\mathrm{G}(\mathbb{Q}_p)$ (as defined in [BrT72, T79]), so that $U_p = \mathrm{G}(\mathbb{Z}_p)$ for almost all p.

We denote by $m_{\mathrm{G}(\mathbb{Q}_p)}$ the Haar measure on $\mathrm{G}(\mathbb{Q}_p)$ normalised so that

$$m_{\mathrm{G}(\mathbb{Q}_p)}(U_p) = 1.$$

The Haar measure $m_{G_n^{\mathrm{f}}}$ on G_n^{f} is the product of the measures $m_{\mathrm{G}(\mathbb{Q}_p)}$ over p dividing n. The Haar measure m_{G_n} on G_n is the product of a Haar measure $m_{\mathrm{G}(\mathbb{R})}$ on $\mathrm{G}(\mathbb{R})$ and the measure $m_{G_n^{\mathrm{f}}}$.

For $q \in \mathbb{N}$, coprime to n, we define the congruence subgroups

$$\Gamma_n(q) := \{\gamma \in \Gamma_n : \gamma = \mathrm{id} \bmod q\}.$$

Clearly, $\Gamma_n(q)$ is a finite index normal subgroup of Γ_n, and the space

$$Y_{n,q} := G_n/\Gamma_n(q)$$

has finite volume. For simplicity, we also set $Y_n := G_n/\Gamma_n$. We denote by $m_{Y_{n,q}}$ the invariant measure on $Y_{n,q}$ induced by m_{G_n} and by $\mu_{Y_{n,q}}$ the probability invariant measure on $Y_{n,q}$, so that $\mu_{Y_{n,q}} = \frac{m_{Y_{n,q}}}{m_{Y_{n,q}}(Y_{n,q})}$.

We denote by $\|\cdot\|_p$ the maximum norm on $\mathrm{Mat}_N(\mathbb{Q}_p)$. Given $n \in \mathbb{N}$ with prime decomposition $n = \prod_p p^{\alpha_p}$, we set

$$B_{n,p} := \{g \in \mathrm{G}(\mathbb{Q}_p) : \|g\|_p = p^{\alpha_p}\} \quad \text{and} \quad B_n^{\mathrm{f}} := \prod_{p|n} B_{n,p}. \tag{2.1}$$

Note that $B_{n,p}$ is a compact subset of $\mathrm{G}(\mathbb{Q}_p)$, which is invariant under the compact open subgroup $\mathrm{G}(\mathbb{Z}_p)$. In particular, $B_{n,p}$ is invariant under U_p for almost all p. We fix a right-invariant Riemannian metric on $\mathrm{G}(\mathbb{R})$. For $x \in \mathrm{G}(\mathbb{R})$ and $\epsilon > 0$, we set

$$\begin{aligned} B^\infty(x,\epsilon) &:= \{g \in \mathrm{G}(\mathbb{R}) : d(g,x) \le \epsilon\}, \\ B_n(x,\epsilon) &:= B^\infty(x,\epsilon) \times B_n^{\mathrm{f}}. \end{aligned}$$

We denote by $L^2(Y_{n,q}) = L^2(Y_{n,q}, \mu_{Y_{n,q}})$ the Hilbert space of square-integrable functions on $Y_{n,q}$, and by $L^2_0(Y_{n,q})$ the subspace of functions with zero integral. For $p = \infty$ and a prime p dividing n, the group $\mathrm{G}(\mathbb{Q}_p)$ naturally acts on $Y_{n,q}$, and we denote by $\pi_{Y_{n,q},p}$ the corresponding unitary representation of $\mathrm{G}(\mathbb{Q}_p)$ on $L^2(Y_{n,q})$. We denote by $\pi_{Y_{n,q}}$ and $\pi_{Y_{n,q},\mathrm{f}}$ the unitary representations of G_n and G_n^{f} on $L^2(Y_{n,q})$ respectively. It would be also convenient to denote by $\pi^0_{Y_{n,q},p}$, $\pi^0_{Y_{n,q}}$, $\pi^0_{Y_{n,q},\mathrm{f}}$ the restrictions of the above representations to $L^2_0(Y_{n,q})$.

Given a unitary representation π of a locally compact group G on a Hilbert space $\mathcal{H}$ and a finite Borel measure β on G, we denote by $\pi(\beta) : \mathcal{H} \to \mathcal{H}$ the corresponding averaging operator defined by

$$\pi(\beta)v = \int_G \pi(g)v\, d\beta(g), \quad v \in \mathcal{H}.$$

We note that if β is a probability measure, then $\|\pi(\beta)\| \le 1$.

The crucial ingredient of our argument is the study of suitable averaging operators on the space $L^2(Y_{n,q})$. Namely, we denote by $\beta_{n,x,\epsilon}$ the uniform probability measure on G_n supported on the set $B_n(x,\epsilon)$. This defines an averaging operator

$$\pi_{Y_{n,q}}(\beta_{n,x,\epsilon}) : L^2(Y_{n,q}) \to L^2(Y_{n,q}) : \qquad (2.2)$$
$$f \quad \mapsto \frac{1}{m_{G_n}(B_n(x,\epsilon))} \int_{B_n(x,\epsilon)} \pi_{Y_{n,q}}(g) f \, dm_{G_n}(g).$$

A unitary representation π of a locally compact group G on a Hilbert space $\mathcal{H}$ is called L^{r+} *integrable* if there exists a dense family of vectors $v_1, v_2 \in \mathcal{H}$ such that the function

$$g \mapsto \langle \pi(g)v_1, v_2 \rangle , \quad g \in G,$$

is in $L^{r+\delta}(G)$ for every $\delta > 0$. In our setting, it follows from the property (τ), established in [C03], that the representations $\pi_{Y_{n,q},p}$, restricted to $L^2_0(Y_{n,q})$, are uniformly integrable. Namely, there exists $r \geq 2$, independent of n, q, p, such that all the representations $\pi_{Y_{n,q},p}$, restricted to $L^2_0(Y_{n,q})$, are L^{r+} integrable. We denote by $r(\mathrm{G})$ the least real number with this property. Let $\iota(\mathrm{G})$ be the least even integer greater than or equal to $r(\mathrm{G})/2$ if $r(\mathrm{G}) > 2$, and $\iota(\mathrm{G}) = 1$ if $r(\mathrm{G}) = 2$.

Outline of the proof. In Section 3, we analyse the asymptotic behaviour of the averaging operators $\pi_{Y_{n,q}}(\beta_{n,x,\epsilon})$ and establish a quantitative mean ergodic theorem for them, namely, an estimate of the form

$$\left\| \pi_{Y_{n,q}}(\beta_{n,x,\epsilon})|_{L^2_0(Y_{n,q})} \right\| \ll m_{G^{\mathrm{f}}_n}(B^{\mathrm{f}}_n)^{-\theta}, \qquad (2.3)$$

where $\theta > 0$ is determined by the integrability exponent $r(\mathrm{G})$. This argument is based on the techniques developed in [GN, GN12a], but it is crucial for our application that the implied constant in (2.3) is uniform on n, and this requires additional considerations.

Section 4 plays auxiliary role. In this section we establish several volume estimates which might be of independent interest. We use these estimates in the later sections to guarantee uniformity in the parameter n.

In Section 5, we use (2.3) to estimate the cardinality of elements in $\bar{\gamma}\Gamma_n(q)$ lying in the regions $B_n(x,\epsilon)$. Typically, such a counting estimate requires that the regions are well-rounded in the sense of [GN12a, Def. 1.1], but the regions $B_n(x,\epsilon)$ are not well-rounded as $\epsilon \to 0^+$. Nonetheless, we shall establish a quantitative estimate for $|B_n(x,\epsilon) \cap \bar{\gamma}\Gamma_n(q)|$ as $n \to \infty$.

Finally, in Section 6, we use a combinatorial sieving argument as in [HR, NS10, GN12b] to estimate the cardinality of almost prime points lying in the regions $B_n(x,\epsilon)$. This leads to the proof of the main theorems.

3. Averaging operators

In this section, we study the asymptotic behaviour of the averaging operators $\pi_{Y_{n,q}}(\beta_{n,x,\epsilon})$ defined in (2.2). Similar problem has been previously investigated in [GN, GN12a]. In particular, the following theorem can be proved by adopting the methods of [GN12a, Theorem 4.5].

Theorem 3.1. *For every $\eta > 0$ and $f \in L^2(Y_{n,q})$,*

$$\left\| \pi_{Y_{n,q}}(\beta_{n,x,\epsilon})f - \int_{Y_{n,q}} f\, d\mu_{Y_{n,q}} \right\|_2 \ll_{n,\eta} m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-(2\iota(\mathrm{G}))^{-1}+\eta} \|f\|_2,$$

where the implied constant depends only on the set of prime divisors of n.

Proof. The statement of the theorem is equivalent to the estimate

$$\left\| \pi^0_{Y_{n,q}}(\beta_{n,x,\epsilon}) \right\| \ll_{n,\eta} m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-(2\iota(\mathrm{G}))^{-1}+\eta}, \quad \eta > 0.$$

We observe that the probability measure $\beta_{n,x,\epsilon}$ decomposes as a product

$$\beta_{n,x,\epsilon} = \beta^\infty_{x,\epsilon} \otimes \left(\bigotimes_{p|n} \beta_{n,p} \right),$$

where $\beta^\infty_{x,\epsilon}$ is the uniform probability measure on $\mathrm{G}(\mathbb{R})$ supported on $B^\infty(x,\epsilon)$, and $\beta_{n,p}$'s are the uniform probability measures on $\mathrm{G}(\mathbb{Q}_p)$ supported on $B_{n,p}$. This implies that $\pi^0_{Y_{n,q}}(\beta_{n,x,\epsilon})$ can be written as a product of commuting operators

$$\pi^0_{Y_{n,q}}(\beta_{n,x,\epsilon}) = \pi^0_{Y_{n,q},\infty}(\beta^\infty_{x,\epsilon}) \prod_{p|n} \pi^0_{Y_{n,q},p}(\beta_{n,p}).$$

Since $\|\pi^0_{Y_{n,q},\infty}(\beta^\infty_{x,\epsilon})\| \le 1$, we obtain

$$\left\| \pi^0_{Y_{n,q}}(\beta_{n,x,\epsilon}) \right\| \le \prod_{p|n} \left\| \pi^0_{Y_{n,q},p}(\beta_{n,p}) \right\|. \tag{3.1}$$

The argument as in [GN12a, Theorem 4.5] gives the estimate

$$\left\| \pi^0_{Y_{n,q},p}(\beta_{n,p}) \right\| \ll_{p,\eta} m_{\mathrm{G}(\mathbb{Q}_p)}(B_{n,p})^{-(2\iota(\mathrm{G}))^{-1}+\eta}, \quad \eta > 0, \tag{3.2}$$

which completes the proof of the theorem. □

We note that the crucial estimate (3.2) is based on Nevo's transfer principle (see [N98]) and the spherical Kunze–Stein inequality (see, for instance, [C97]). The implied constant in (3.2) can be estimated explicitly, but unfortunately it blows up as $p \to \infty$.

For our purposes we need the following uniform version of Theorem 3.1.

Theorem 3.2. *For every $\eta > 0$ and $f \in L^2(Y_{n,q})$,*

$$\left\| \pi_{Y_{n,q}}(\beta_{n,x,\epsilon})f - \int_{Y_{n,q}} f\, d\mu_{Y_{n,q}} \right\|_2 \ll_{\eta} m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-(4\iota(\mathrm{G}))^{-1}+\eta} \|f\|_2.$$

Proof. As in the above proof, we need to estimate $\|\pi^0_{Y_{n,q}}(\beta_{n,x,\epsilon})\|$, and because of (3.1), it is sufficient to give a bound on the norms of $\pi^0_{Y_{n,q},p}(\beta_{n,p})$. We claim that

$$\left\|\pi^0_{Y_{n,q},p}(\beta_{n,p})\right\| \le c_{p,\eta}\, m_{\mathrm{G}(\mathbb{Q}_p)}(B_{n,p})^{-(4\iota(\mathrm{G}))^{-1}+\eta}, \quad \eta > 0, \tag{3.3}$$

where the constant $c_{p,\eta} \ge 1$ satisfies

$$\prod_p c_{p,\eta} < \infty.$$

Clearly, (3.1) combined with (3.3) implies the theorem.

To prove (3.3) we consider the tensor-power representation $(\pi^0_{Y_{n,q},p})^{\otimes\iota(\mathrm{G})}$. It follows from Nevo's spectral transfer principle [N98, Th. 1] that

$$\left\|\pi^0_{Y_{n,q},p}(\beta_{n,p})\right\| \le \left\|(\pi^0_{Y_{n,q},p})^{\otimes\iota(\mathrm{G})}(\beta_{n,p})\right\|^{1/\iota(\mathrm{G})}. \tag{3.4}$$

Since $\pi^0_{Y_{n,q},p}$ is $L^{r(\mathrm{G})+}$ integrable, the representation $(\pi^0_{Y_{n,q},p})^{\otimes\iota(\mathrm{G})}$ is L^{2+} integrable. Therefore, by [CHH88], the representation $(\pi^0_{Y_{n,q},p})^{\otimes\iota(\mathrm{G})}$ is weakly contained in the regular representation $\lambda_{\mathrm{G}(\mathbb{Q}_p)}$ of $\mathrm{G}(\mathbb{Q}_p)$ on $L^2(\mathrm{G}(\mathbb{Q}_p))$. This implies the estimate

$$\left\|(\pi^0_{Y_{n,q},p})^{\otimes\iota(\mathrm{G})}(\beta_{n,p})\right\| \le \left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\beta_{n,p})\right\|. \tag{3.5}$$

Let $\tilde{B}_{n,p} := U_p B_{n,p} U_p$ and $\tilde{\beta}_{n,p}$ be the uniform probability measure supported on $\tilde{B}_{n,p}$. Recall that for almost all p, we have $U_p = \mathrm{G}(\mathbb{Z}_p)$. For those p, we have $\tilde{B}_{n,p} = B_{n,p}$ and

$$\left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\beta_{n,p})\right\| = \left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\tilde{\beta}_{n,p})\right\|. \tag{3.6}$$

To deal with the remaining finite set of primes, we observe that $B_{n,p} \subset \tilde{B}_{n,p}$, and hence for every $f \in L^2(\mathrm{G}(\mathbb{Q}_p))$,

$$\begin{aligned}\left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\beta_{n,p})f\right\| &\le \left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\beta_{n,p})|f|\right\| \\ &\le \frac{m_{\mathrm{G}(\mathbb{Q}_p)}(\tilde{B}_{n,p})}{m_{\mathrm{G}(\mathbb{Q}_p)}(B_{n,p})} \cdot \left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\tilde{\beta}_{n,p})|f|\right\|.\end{aligned}$$

Hence,

$$\left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\beta_{n,p})\right\| \le \frac{m_{\mathrm{G}(\mathbb{Q}_p)}(\tilde{B}_{n,p})}{m_{\mathrm{G}(\mathbb{Q}_p)}(B_{n,p})} \cdot \left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\tilde{\beta}_{n,p})\right\|. \tag{3.7}$$

Since the group U_p is compact, there exists $\kappa_p \in \mathbb{N}$ such that

$$\tilde{B}_{n,p} \subset \{g \in \mathrm{G}(\mathbb{Q}_p) :\, p^{\alpha_p-\kappa_p} \le \|g\|_p \le p^{\alpha_p+\kappa_p}\}. \tag{3.8}$$

Consider the function

$$v(\ell) := m_{\mathrm{G}(\mathbb{Q}_p)}\left(\{g \in \mathrm{G}(\mathbb{Q}_p) :\, \|g\|_p = p^\ell\}\right).$$

By [D84, Th. 7.4], the sum $\sum_{\ell\ge 0} v(\ell)p^{-\ell s}$ is a rational function of p^{-s}. Therefore,

$$v(\ell) = \sum_{i=1}^{i_0} r_i(\ell)p^{a_i\ell}$$

for some polynomials r_i and $a_i \in \mathbb{Z}$. In view of (3.8), this implies that for some $c_p \ge 1$,

$$m_{\mathrm{G}(\mathbb{Q}_p)}(\tilde{B}_{n,p}) \le c_p\, m_{\mathrm{G}(\mathbb{Q}_p)}(B_{n,p}).$$

Hence, it follows from (3.6) and (3.7) that

$$\left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\beta_{n,p})\right\| \le c_p \left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\tilde{\beta}_{n,p})\right\|, \tag{3.9}$$

where $c_p = 1$ for almost all p.

Since $\tilde{\beta}_{n,p}$ is bi-invariant under U_p, we can estimate the norm $\left\|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\tilde{\beta}_{n,p})\right\|$ using Herz' majoration argument, as explained in [C97]. Indeed, since U_p's are special subgroups, it follows from the structure theory [T79, 3.3.2] that there exists a closed amenable subgroup Q_p such that

$$\mathrm{G}(\mathbb{Q}_p) = U_pQ_p, \tag{3.10}$$

i.e., $\mathrm{G}(\mathbb{Q}_p)$ is an Iwasawa group in the sense of [GN, Def. 5.1(1)]. The Herz' majoration argument [C97] can be applied to any Iwasawa group. It gives that for every $f \in L^2(\mathrm{G}(\mathbb{Q}_p))$ and $s \in [1,2)$,

$$\begin{aligned} \|\lambda_{\mathrm{G}(\mathbb{Q}_p)}(\tilde{\beta}_{n,p})f\|_2 &\le a_{p,s}\, \|\tilde{\beta}_{n,p}\|_s \|f\|_2 = a_{p,s}\, m_{\mathrm{G}(\mathbb{Q}_p)}(\tilde{B}_{n,p})^{-1+1/s}\|f\|_2 \\ &\le a_{p,s}\, m_{\mathrm{G}(\mathbb{Q}_p)}(B_{n,p})^{-1+1/s}\|f\|_2. \end{aligned} \tag{3.11}$$

The constant $a_{p,s}$ in the above estimate is explicit and computed in terms of the Harish-Chandra function Ξ_p (see [GN, Def. 5.1(2)]), which we now recall.

Let Δ_p denote the modular function on the group Q_p. For $g \in \mathrm{G}(\mathbb{Q}_p)$, we denote by $q(g)$ the Q_p-component of g with respect to the decomposition (3.10). The Harish-Chandra function on $\mathrm{G}(\mathbb{Q}_p)$ is defined by

$$\Xi_p(g) = \int_{U_p} \Delta_p(q(gu))^{-1/2}\, dm_{\mathrm{G}(\mathbb{Q}_p)}(u).$$

The constant $a_{p,s}$ in (3.11) is given by

$$a_{p,s} = \|\Xi_p\|_t$$

with $t = (1-1/s)^{-1}$. We have

$$a_{p,s} \ge \Xi_p(e) m_{\mathrm{G}(\mathbb{Q}_p)}(U_p)^{1/t} = 1,$$

and by [GN12a, Prop. 6.3], when $t > 4$ (that is, when $s < 4/3$),

$$\prod_p a_{p,s} < \infty.$$

Combining (3.4), (3.5), (3.9), and (3.11), we conclude that

$$\left\|\pi^0_{Y_{n,q},p}(\beta_{n,p})\right\| \le (a_{p,s}c_p)^{1/\iota(\mathrm{G})}\, m_{\mathrm{G}(\mathbb{Q}_p)}(B_{n,p})^{(-1+1/s)/\iota(\mathrm{G})},$$

where

$$\prod_p (a_{p,s}c_p)^{1/\iota(\mathrm{G})} < \infty$$

for $s < 4/3$. This implies (3.3) and completes the proof of the theorem. □

4. Volume estimates

This section plays auxiliary role and can be skipped for the first reading. Here we prove uniform estimates for the volumes of the sets Y_n and the sets B_n^{f}.

Proposition 4.1.

$$\inf_{n\in\mathbb{N}} m_{Y_n}(Y_n) > 0 \quad \textit{and} \quad \sup_{n\in\mathbb{N}} m_{Y_n}(Y_n) < \infty.$$

Proof. For $n \in \mathbb{N}$ we define

$$\mathcal{O}_n^{\mathrm{f}} := \prod_{p|n} \mathrm{G}(\mathbb{Z}_p), \tag{4.1}$$

which is a compact open subgroup of G_n^{f}.

To prove the first claim, we fix a sufficiently small open subset $\mathcal{O}^\infty$ of $\mathrm{G}(\mathbb{R})$ and set $\mathcal{O}_n := \mathcal{O}^\infty \times \mathcal{O}_n^{\mathrm{f}}$. We claim that $\mathcal{O}_n$ injects into $Y_n = G_n/\Gamma_n$ under the projection map $G_n \to G_n/\Gamma_n$. Indeed, if for some $\gamma \in \Gamma_n$, we have $\mathcal{O}_n\gamma \cap \mathcal{O}_n \ne \emptyset$, then it follows that $\gamma \in \mathrm{G}(\mathbb{Z})$. Therefore, it is sufficient to take $\mathcal{O}^\infty$ which injects into $\mathrm{G}(\mathbb{R})/\mathrm{G}(\mathbb{Z})$. Then

$$m_{Y_n}(Y_n) \ge m_{\mathrm{G}(\mathbb{R})}(\mathcal{O}^\infty) \prod_{p|n} m_{\mathrm{G}(\mathbb{Q}_p)}(\mathrm{G}(\mathbb{Z}_p)).$$

Since $m_{\mathrm{G}(\mathbb{Q}_p)}(\mathrm{G}(\mathbb{Z}_p)) = m_{\mathrm{G}(\mathbb{Q}_p)}(U_p) = 1$ for almost all p, this proves the first claim.

Now we turn to the proof of the second claim. We fix a prime p_0 such that G is isotropic over $\mathbb{Q}_p$ and $U_p = \mathrm{G}(\mathbb{Z}_p)$ (such a prime exists by [PR, Th. 6.7]). We consider the two separate cases depending on whether p_0 divides n or not.

Suppose that p_0 divides n and write $n = p_0^\alpha n_0$ with n_0 coprime to p. We identify G_{p_0} and $G_{n_0}^{\mathrm{f}}$ with the corresponding subgroups of G_n, so that

$$G_n = G_{p_0} G_{n_0}^{\mathrm{f}}. \tag{4.2}$$

Since G is isotropic over $\mathbb{Q}_p$, it follows that G_{p_0} is not compact, and by the strong approximation property [PR, §7.4], $G_{p_0}\Gamma_n$ is dense in G_n. In particular, it follows that for every $g\in G_n$,

$$G_{p_0}\Gamma_n\cap O^{\rm f}_{n_0}g\neq\emptyset.$$

This proves that

$$G_n=G_{p_0}O^{\rm f}_{n_0}\Gamma_n.$$

Let Ω_{p_0} be a measurable fundamental set in G_{p_0} for the right action of Γ_{p_0}. Then $G_{p_0}=\Omega_{p_0}\Gamma_{p_0}$, and every element $g\in G_n$ can be written with respect to the decomposition (4.2) as

$$g=(\omega\delta,o)\cdot(\gamma,\gamma)=(\omega,o\delta^{-1})\cdot(\delta\gamma,\delta\gamma),$$

where $\omega\in\Omega_{p_0}$, $\delta\in\Gamma_{p_0}$, $o\in O^{\rm f}_{n_0}$, and $\gamma\in\Gamma_n$. Since $o\delta^{-1}\in O^{\rm f}_{n_0}$ and $\delta\gamma\in\Gamma_n$, this shows that

$$G_n=\Omega_{p_0}O^{\rm f}_{n_0}\Gamma_n.$$

Therefore,

$$m_{Y_n}(Y_n)\le m_{G_n}(\Omega_{p_0}O^{\rm f}_{n_0})=m_{G_{p_0}}(\Omega_{p_0})\cdot m_{G^{\rm f}_{n_0}}(O^{\rm f}_{n_0}). \tag{4.3}$$

We observe that

$$m_{G^{\rm f}_{n_0}}(O^{\rm f}_{n_0})=\prod_{p|n_0}m_{{\rm G}(\mathbb{Q}_p)}({\rm G}(\mathbb{Z}_p)).$$

Since ${\rm G}(\mathbb{Z}_p)$ is compact, $m_{{\rm G}(\mathbb{Q}_p)}({\rm G}(\mathbb{Z}_p))<\infty$. Moreover, for almost p, we have ${\rm G}(\mathbb{Z}_p)=U_p$ and $m_{{\rm G}(\mathbb{Q}_p)}({\rm G}(\mathbb{Z}_p))=1$. Hence, the second claim of the lemma follows from (4.3).

Now suppose that p_0 does not divide n. In this case, we identify G_n and ${\rm G}(\mathbb{Q}_{p_0})$ with the corresponding subgroups of G_{np_0}, so that

$$G_{np_0}=G_n{\rm G}(\mathbb{Q}_{p_0}). \tag{4.4}$$

Let Ω_n be a measurable fundamental set in G_n for the right action of Γ_n. We claim that the natural projection map

$$\Omega_n\times{\rm G}(\mathbb{Z}_{p_0})\to Y_{np_0}=G_{np_0}/\Gamma_{np_0} \tag{4.5}$$

defined by the decomposition (4.4) is one-to-one. Indeed, suppose that for some $g_1,g_2\in\Omega_n$ and $u_1,u_2\in{\rm G}(\mathbb{Z}_{p_0})$, there exists $\gamma\in\Gamma_{np_0}$ such that $g_1\gamma=g_2$ and $u_1\gamma=u_2$. Then

$$\gamma=u_1^{-1}u_2\in\Gamma_{np_0}\cap{\rm G}(\mathbb{Z}_{p_0})=\Gamma_n$$

and since Ω_n is a fundamental set, it follows that $\gamma=e$. This proves the claim. It is clear the map (4.5) sends the product measure $m_{G_n}\otimes m_{{\rm G}(\mathbb{Q}_{p_0})}$ to the measure $m_{Y_{np_0}}$. Hence, we obtain

$$m_{G_n}(\Omega_n)m_{{\rm G}(\mathbb{Q}_{p_0})}({\rm G}(\mathbb{Z}_{p_0}))\le m_{Y_{np_0}}(Y_{np_0}).$$

Since it follows from the previous paragraph that the right hand side is uniformly bounded, we conclude that $m_{Y_n}(Y_n)=m_{G_n}(\Omega_n)$ is uniformly bounded as well. This completes the proof of the proposition. $\square$

We say that a number n is *isotropic* if for every prime divisor p of n, the group G is isotropic over $\mathbb{Q}_p$. In particular, if G is isotropic over $\mathbb{Q}$, then every number is isotropic.

Proposition 4.2. *There exists $a > 0$ such that for every isotropic number n,*

$$m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}}) \ge c(n)\, n^a,$$

where $c(n) > 0$ depends only on the set of prime divisors of n.

Moreover, if G *is isotropic over $\mathbb{Q}$, then $c(n)$ is independent of n.*

Proof. Let $n = \prod_p p^{\alpha_p}$ be the prime decomposition of n. Since B_n^{f} is the product of the sets $B_{n,p}$ (see (2.1)), it is sufficient to prove that there exists $c_p > 0$ such that

$$m_{\mathrm{G}(\mathbb{Q}_p)}(B_{n,p}) \ge c_p\, (p^{\alpha_p})^a,$$

When G is isotropic over $\mathbb{Q}$, we show that $c_p = 1$ for almost all p.

Recall that U_p is a special maximal compact subgroup. Therefore, by [T79, 3.3.3], we have a Cartan decomposition

$$\mathrm{G}(\mathbb{Q}_p) = U_p \mathrm{Z}_p(\mathbb{Q}_p) U_p,$$

where Z_p is the centraliser of a maximal $\mathbb{Q}_p$-split torus S_p in G. We fix a set Π_p of (restricted) simple roots for G with respect to S_p, and set

$$S_p^+ = \{s \in \mathrm{S}(\mathbb{Q}_p) :\, |\chi(s)|_p \ge 1 \quad \text{for } \chi \in \Pi_p\}.$$

Then the Cartan decomposition takes form

$$\mathrm{G}(\mathbb{Q}_p) = U_p S_p^+ \Omega_p U_p, \tag{4.6}$$

where Ω_p is a finite subset of $\mathrm{Z}_p(\mathbb{Q}_p)$. Since U_p is compact and Ω_p is finite, there exists $c_p' > 0$ such that for $g = u_1 s\omega u_2 \in U_p S_p^+ \Omega_p U_p$, we have

$$\|g\| \le c_p' \|s\|_p. \tag{4.7}$$

Consider the representation $\rho : \mathrm{G} \to \mathrm{GL}_N$ corresponding to the embedding $\mathrm{G} \subset \mathrm{GL}_N$ and denote by Φ_p the set of weights of this representation with respect to S_p. Since S_p is split over $\mathbb{Q}_p$, the action of $\mathrm{S}_p(\mathbb{Q}_p)$ on $\mathbb{Q}_p^N$ is completely reducible, and

$$\|s\|_p \le c_p'' \cdot \max_{\xi \in \Phi_p} |\xi(s)|_p \tag{4.8}$$

for some $c_p'' > 0$.

Let $\Pi_p^\vee$ denote the set of fundamental weights corresponding to Π_p. A weight ξ is called dominant if

$$\xi = \prod_{\psi \in \Pi_p^\vee} \psi^{n_\psi}$$

for some nonnegative integers n_ψ. It follows from [OV, Ch. 3, §1.9] that there exists $k_0 \in \mathbb{N}$ such that every weight ξ can be written as

$$\xi^{k_0} = \prod_{\chi \in \Pi_p} \chi^{m_\chi} \tag{4.9}$$

with $m_\chi \in \mathbb{Z}$. Moreover, if ξ is dominant, then the integers m_χ are nonnegative.

By [BoT65, Th. 7.2], there exists a semisimple subgroup $\tilde{\mathrm{G}}_p$ of G which is split over $\mathbb{Q}_p$, contains S_p as a maximal torus, and the set Π_p forms the set of simple roots for S_p in $\tilde{\mathrm{G}}_p$. The linear representations of $\tilde{\mathrm{G}}_p$ defined over $\mathbb{Q}_p$ are described by the theory of highest weights. In particular, it follows from the description of possible weights (see [OV, Ch. 3, §2.2]) that the maximum in (4.8) can be taken over the dominant weights in Φ_p. Moreover, it follows from the classification of semisimple groups and their representations that the set of all possible weights appearing in $\rho|_{\mathrm{S}_p}$ for some p is finite. Let Δ_p be the product of all positive roots of S_p in G counted with multiplicities. Then we deduce from (4.9) that there exists $\ell \in \mathbb{N}$, independent of p, such that

$$\xi \le \Delta_p^\ell \tag{4.10}$$

for all dominant weights ξ appearing in $\rho|_{\mathrm{S}_p}$.

Now combining (4.7), (4.8) and (4.10), we deduce that for all $g = u_1 s\omega u_2 \in U_p S_p^+ \Omega_p U_p$, we have

$$\|g\|_p \le (c_p' c_p'')\, |\Delta_p(s)|^\ell,$$

and when $g \in B_{n,p}$, we obtain

$$|\Delta_p(s)| \ge (c_p' c_p'')^{-1/\ell}\, (p^{\alpha_p})^{1/\ell}. \tag{4.11}$$

By [M71, 3.2.15],

$$m_{\mathrm{G}(\mathbb{Q}_p)}(U_p s U_p) \ge |\Delta_p(s)|. \tag{4.12}$$

Since both $\mathrm{G}(\mathbb{Z}_p)$ and U_p are compact open subgroups, $\mathrm{G}(\mathbb{Z}_p) \cap U_p$ has finite index in U_p, and there exists an open normal subgroup V_p of U_p contained in $\mathrm{G}(\mathbb{Z}_p)$. Then for $g = u_1 s\omega u_2 \in U_p S_p^+ \Omega_p U_p$, we obtain

$$\begin{aligned} m_{\mathrm{G}(\mathbb{Q}_p)}(V_p g V_p) &= m_{\mathrm{G}(\mathbb{Q}_p)}(u_1 V_p s(\omega V_p \omega^{-1})\omega u_2) \\ &= m_{\mathrm{G}(\mathbb{Q}_p)}(V_p s(\omega V_p \omega^{-1})). \end{aligned} \tag{4.13}$$

By compactness, both V_p and $V_p \cap \omega V_p \omega^{-1}$ have finite index in U_p. Therefore, it follows that

$$m_{\mathrm{G}(\mathbb{Q}_p)}(U_p s U_p) \le c_p'''\, m_{\mathrm{G}(\mathbb{Q}_p)}(V_p s(\omega V_p \omega^{-1})) \tag{4.14}$$

for some $c_p''' > 0$. Hence, when $g \in B_{n,p}$, we deduce from (4.11)–(4.12) combined with (4.13)–(4.14) that

$$m_{\mathrm{G}(\mathbb{Q}_p)}(V_p g V_p) \ge (c_p' c_p'')^{-1/\ell} (c_p''')^{-1}\, (p^{\alpha_p})^{1/\ell}.$$

Since $V_p \subset \mathrm{G}(\mathbb{Z}_p)$, we have $V_p g V_p \subset B_{n,p}$, and this proves the first claim of the proposition.

To prove the second claim, let us assume first that G is split over $\mathbb{Q}$. Then the Cartan decomposition (4.6) is of the form $\mathrm{G}(\mathbb{Q}_p) = U_p S_p^+ U_p$. For almost all p, we can take S_p to be a maximal $\mathbb{Q}$-split torus S. Then the action of $\mathrm{S}(\mathbb{Q})$ on $\mathbb{Q}^N$ is completely reducible. Since for almost all p, we have $U_p = \mathrm{G}(\mathbb{Z}_p)$, the estimate (4.7) holds with $c'_p = 1$ for almost all p. Since the action of S is completely reducible over $\mathbb{Q}$, for $s \in \mathrm{S}$,

$$\rho(s) = \sum_{\xi \in \Phi_p} \xi(s) v_\xi$$

for some $v_\xi \in \mathrm{Mat}_N(\mathbb{Q})$. Hence, in the estimate (4.8),

$$c''_p = \max_\xi \|v_\xi\|_p,$$

and it is clear that for almost all p, $c''_p = 1$. Finally, in the argument (4.13)–(4.14), we can take $V_p = U_p = \mathrm{G}(\mathbb{Z}_p)$ for almost all p. Therefore, for almost all p, we obtain

$$m_{\mathrm{G}(\mathbb{Q}_p)}(B_{n,p}) \geq (p^{\alpha_p})^{1/\ell},$$

which completes the proof of the proposition when G is split.

Now let G be isotropic over $\mathbb{Q}$. Then by [BoT65, Th. 7.2], there exists a semisimple $\mathbb{Q}$-subgroup H of G which is split over $\mathbb{Q}$. Moreover, by [BoT72, Cor. 4.6], the group H is simply connected. It follows from the previous paragraph that for almost all p,

$$m_{\mathrm{H}(\mathbb{Q}_p)}(B_{n,p} \cap \mathrm{H}(\mathbb{Q}_p)) \geq (p^{\alpha_p})^{1/\ell}. \tag{4.15}$$

Here $m_{\mathrm{H}(\mathbb{Q}_p)}$ is the Haar measure on $\mathrm{H}(\mathbb{Q}_p)$ which is normalised so that $m_{\mathrm{H}(\mathbb{Q}_p)}(U_p^{\mathrm{H}}) = 1$, where U_p^{H} is a special maximal compact subgroup of $\mathrm{H}(\mathbb{Q}_p)$ which is chosen so that $U_p^{\mathrm{H}} = \mathrm{H}(\mathbb{Z}_p)$ for almost all p. It follows from the uniqueness of invariant measures that the measure $m_{\mathrm{G}(\mathbb{Q}_p)}$ restricted to the open set $U_p\mathrm{H}(\mathbb{Q}_p)$ is given by

$$f \mapsto \frac{1}{m_{\mathrm{H}(\mathbb{Q}_p)}(U_p \cap \mathrm{H}(\mathbb{Q}_p))} \int_{U_p \times \mathrm{H}(\mathbb{Q}_p)} f(uh)\, dm_{\mathrm{G}(\mathbb{Q}_p)}(u) dm_{\mathrm{H}(\mathbb{Q}_p)}(h) \tag{4.16}$$

for $f \in C_c(U_p\mathrm{H}(\mathbb{Q}_p))$. Indeed, the measure m, defined by (4.16), is invariant under the transitive action

$$x \mapsto uxh^{-1}, \quad x \in U_p\mathrm{H}(\mathbb{Q}_p),\ (u,h) \in U_p \times \mathrm{H}(\mathbb{Q}_p),$$

and satisfies the normalisation $m(U_p) = 1$, so that it has to be equal to the measure $m_{\mathrm{G}(\mathbb{Q}_p)}$ restricted to $U_p\mathrm{H}(\mathbb{Q}_p)$. It follows from (4.16) that

$$m_{\mathrm{G}(\mathbb{Q}_p)}(U_p(B_{n,p} \cap \mathrm{H}(\mathbb{Q}_p))) \geq \frac{m_{\mathrm{H}(\mathbb{Q}_p)}(B_{n,p} \cap \mathrm{H}(\mathbb{Q}_p))}{m_{\mathrm{H}(\mathbb{Q}_p)}(U_p \cap \mathrm{H}(\mathbb{Q}_p))}. \tag{4.17}$$

For almost all p, we have

$$U_p \cap \mathrm{H}(\mathbb{Q}_p) = \mathrm{G}(\mathbb{Z}_p) \cap \mathrm{H}(\mathbb{Q}_p) = \mathrm{H}(\mathbb{Z}_p),$$

so that

$$m_{\mathrm{H}(\mathbb{Q}_p)}(U_p \cap \mathrm{H}(\mathbb{Q}_p)) = 1. \tag{4.18}$$

Since the set $B_{n,p}$ is U_p-invariant for almost all p, it follows that for almost all p,

$$m_{\mathrm{G}(\mathbb{Q}_p)}(B_{n,p}) \geq m_{\mathrm{G}(\mathbb{Q}_p)}(U_p(B_{n,p} \cap \mathrm{H}(\mathbb{Q}_p))). \tag{4.19}$$

Therefore, it follows from (4.15), combined with (4.17)–(4.19), that

$$m_{\mathrm{H}(\mathbb{Q}_p)}(B_{n,p}) \geq (p^{\alpha_p})^{1/\ell}$$

for almost all p. This completes the proof of the proposition. □

5. Counting estimates

In this section we prove an estimate on the number of lattice points in the regions $B_n(x,\epsilon)$.

Theorem 5.1. *For every coprime $n, q \in \mathbb{N}$, $\bar\gamma \in \Gamma_n$, $x \in \mathrm{G}(\mathbb{R})$, $\kappa, \eta > 0$, and $\epsilon \in (0, \epsilon_0(\kappa, x)]$, the following estimate holds*

$$\begin{aligned}|B_n(x,\epsilon) \cap \bar\gamma\Gamma_n(q)| =& \frac{m_{G_n}(B_n(x,\epsilon))}{m_{Y_{n,q}}(Y_{n,q})} + O_x\left(\epsilon^{\kappa+\dim(\mathrm{G})} m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})\right) \\ &+ O_{x,\eta}\left(\epsilon^{-\kappa\dim(\mathrm{G})} m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-(4\iota(\mathrm{G}))^{-1}+\eta}\right),\end{aligned}$$

where $\epsilon_0(\kappa, x)$ and the implied constants are uniform over x in bounded sets.

This result will be deduced from Theorem 3.2 following the strategy of [GN12a].

We set

$$\mathcal{O}_n(\epsilon) := B^\infty(e,\epsilon) \times \prod_{p|n} \mathrm{G}(\mathbb{Z}_p), \tag{5.1}$$

where $B^\infty(e,\epsilon)$ is the ϵ-neighbourhood of identity in $\mathrm{G}(\mathbb{R})$ with respect to the Riemannian metric d on $\mathrm{G}(\mathbb{R})$. Since the metric d is right invariant, $\mathcal{O}_n(\epsilon)$ is a symmetric neighbourhood of identity in G_n.

Lemma 5.2. *For every $n \in \mathbb{N}$ and $\epsilon \in (0,1]$,*

$$\epsilon^{\dim(\mathrm{G})} \ll m_{G_n}(\mathcal{O}_n(\epsilon)) \ll \epsilon^{\dim(\mathrm{G})}.$$

Proof. Writing $B^\infty(e,\epsilon)$ in the exponential coordinates for the Riemannian metric, we deduce that

$$\epsilon^{\dim(\mathrm{G})} \ll m_{\mathrm{G}(\mathbb{R})}(B^\infty(e,\epsilon)) \ll \epsilon^{\dim(\mathrm{G})}. \tag{5.2}$$

We recall that the measures $m_{\mathrm{G}(\mathbb{Q}_p)}$ are normalised so that $m_{\mathrm{G}(\mathbb{Q}_p)}(U_p) = 1$, and $U_p = \mathrm{G}(\mathbb{Z}_p)$ for all but finitely many primes p. For the remaining primes, we observe that since U_p and $\mathrm{G}(\mathbb{Z}_p)$ are compact open subgroups in

$\mathrm{G}(\mathbb{Q}_p)$, it follows that $U_p \cap \mathrm{G}(\mathbb{Z}_p)$ has finite index in both U_p and $\mathrm{G}(\mathbb{Z}_p)$. Therefore,

$$1 \ll \prod_{p|n} m_{\mathrm{G}(\mathbb{Q}_p)}(\mathrm{G}(\mathbb{Z}_p)) \ll 1.$$

Combining this estimate with (5.2), we deduce the lemma. □

Lemma 5.3. (i) *For every $n \in \mathbb{N}$, $x \in \mathrm{G}(\mathbb{R})$, and $\epsilon, \epsilon' \in (0,1]$,*

$$\mathcal{O}_n(\epsilon')B_n(x,\epsilon)\mathcal{O}_n(\epsilon') \subset B_n(x,\epsilon + c_1(x)\epsilon'),$$

where $c_1(x)$ is uniform over x in bounded sets.

(ii) *For every $n \in \mathbb{N}$, $x \in \mathrm{G}(\mathbb{R})$, and $\epsilon, \epsilon' \in (0, \epsilon_0(x)]$,*

$$m_{G_n}(B_n(x,\epsilon+\epsilon')) \le m_{G_n}(B_n(x,\epsilon)) + c_2(x)\epsilon'\epsilon^{\dim(\mathrm{G})-1}\, m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}}),$$

where $\epsilon_0(x)$ and $c_2(x)$ are uniform over x in bounded sets.

Proof. To prove (i), we observe that $B_n(x,\epsilon) = B^\infty(x,\epsilon) \times \prod_{p|n} B_{n,p}$ and the sets $B_{n,p}$ are invariant under $\mathrm{G}(\mathbb{Z}_p)$. Thus, it suffices to prove that for every $u_1, u_2 \in B^\infty(e,\epsilon')$ and $b \in B^\infty(x,\epsilon)$, we have

$$d(x, u_1 b u_2) \le \epsilon + c_1(x)\epsilon'. \tag{5.3}$$

Using the right invariance of the Riemannian metric on $\mathrm{G}(\mathbb{R})$, we obtain

$$\begin{aligned} d(x,u_1bu_2) &= d(x(bu_2)^{-1},u_1) \le d(x(bu_2)^{-1},e) + d(e,u_1) \\ &\le d(x,bu_2) + \epsilon' \le d(xb^{-1}, bu_2b^{-1}) + \epsilon' \\ &\le d(xb^{-1},e) + d(e,bu_2b^{-1}) + \epsilon' \le \epsilon + d(e,bu_2b^{-1}) + \epsilon'. \end{aligned}$$

Since $d(e,bu_2b^{-1}) \ll_b d(e,u_2) \le \epsilon'$ where the implied constant is uniform over b in bounded sets, we deduce (5.3).

To prove (ii), it is sufficient to show that

$$m_{\mathrm{G}(\mathbb{R})}(B^\infty(x,\epsilon+\epsilon')) - m_{\mathrm{G}(\mathbb{R})}(B^\infty(x,\epsilon)) \le c_2(x)\epsilon'\epsilon^{\dim(\mathrm{G})-1}.$$

This follows from the disintegration formula for the measure $m_{\mathrm{G}(\mathbb{R})}$ as in [S, p. 66]. □

Let $\chi_{n,\epsilon}$ be the constant multiple of the characteristic function of the set $\mathcal{O}_n(\epsilon)$ which is normalised so that $\int_{G_n} \chi_{n,\epsilon}\, dm_{G_n} = 1$. We also define a function $\phi^{\bar\gamma}_{n,q,\epsilon}$ on $Y_{n,q} = G_n/\Gamma_n(q)$ by

$$\phi^{\bar\gamma}_{n,q,\epsilon}(g\Gamma_n(q)) := \sum_{\gamma\in\Gamma_n(q)} \chi_{n,\epsilon}(g\gamma\bar\gamma).$$

Clearly, $\phi^{\bar\gamma}_{n,q,\epsilon}$ is a bounded measurable function on $Y_{n,q}$ with compact support. The following lemma shows that averages of this function can be used to approximate the cardinality $|B_n(x,\epsilon) \cap \bar\gamma\Gamma_n(q)|$. Note that $\bar\gamma\Gamma_n(q) = \Gamma_n(q)\bar\gamma$ because $\Gamma_n(q)$ is normal in Γ_n.

Lemma 5.4. *For every coprime* $n, q \in \mathbb{N}$, $\bar{\gamma} \in \Gamma_n$, $x \in \mathrm{G}(\mathbb{R})$, $\epsilon, \epsilon' \in (0,1]$, *and* $h \in \mathcal{O}_n(\epsilon')$,

$$|B_n(x,\epsilon) \cap \bar{\gamma}\Gamma_n(q)| \le \int_{B_n(x,\epsilon+c_1(x)\epsilon')} \phi^{\bar{\gamma}}_{n,q,\epsilon'}(g^{-1}h\Gamma_n(q))\, dm_{G_n}(g),$$
$$|B_n(x,\epsilon) \cap \bar{\gamma}\Gamma_n(q)| \ge \int_{B_n(x,\epsilon-c_1(x)\epsilon')} \phi^{\bar{\gamma}}_{n,q,\epsilon'}(g^{-1}h\Gamma_n(q))\, dm_{G_n}(g).$$

With the help of Lemma 5.3(i), the proof of Lemma 5.4 is the same as the proof of Lemma 2.1 in [GN12a].

Proof of Theorem 5.1. We first show that there exists $\theta_0 > 0$ such that for any distinct $\gamma_1, \gamma_2 \in \Gamma_n$,

$$\mathcal{O}_n(\theta_0)\gamma_1 \cap \mathcal{O}_n(\theta_0)\gamma_2 = \emptyset. \tag{5.4}$$

Indeed, suppose that for some

$$(g_\infty, g_f), (h_\infty, h_f) \in \mathcal{O}_n(\theta) = B^\infty(e,\theta) \times \prod_{p|n} \mathrm{G}(\mathbb{Z}_p) \quad \text{and} \quad \gamma \in \Gamma_n,$$

we have

$$(g_\infty, g_f) = (h_\infty, h_f)(\gamma,\gamma).$$

Then

$$(\gamma,\gamma) = (g_\infty h_\infty^{-1}, g_f h_f^{-1}) \in B^\infty(e,\theta)^2 \times \prod_{p|n} \mathrm{G}(\mathbb{Z}_p).$$

In particular, we conclude that $\gamma \in \mathrm{G}(\mathbb{Z})$, and hence $\gamma = e$ if θ is sufficiently small.

It follows from (5.4) for every $\theta \in (0, \theta_0]$,

$$\mu_{Y_{n,q}}(\mathcal{O}_n(\theta)\Gamma_n(q)) = \frac{m_{G_n}(\mathcal{O}_n(\theta))}{m_{Y_{n,q}}(Y_{n,q})}. \tag{5.5}$$

Moreover,

$$\int_{Y_{n,q}} \phi^{\bar{\gamma}}_{n,q,\theta}\, d\mu_{Y_{n,q}} = \int_{G_n} \chi_{n,\theta}(g)\, \frac{dm_{G_n}(g)}{m_{Y_{n,q}}(Y_{n,q})} = \frac{1}{m_{Y_{n,q}}(Y_{n,q})},$$

and similarly,

$$\|\phi^{\bar{\gamma}}_{n,q,\theta}\|_2^2 = \int_{G_n} \chi^2_{n,\theta}(g)\, \frac{dm_{G_n}(g)}{m_{Y_{n,q}}(Y_{n,q})} = \frac{m_{G_n}(\mathcal{O}_n(\theta))^{-1}}{m_{Y_{n,q}}(Y_{n,q})}.$$

By Theorem 3.2, for every $\rho, \eta > 0$, there exists $c_\eta > 0$ such that

$$\left\| \pi_{Y_{n,q}}(\beta_{n,x,\rho})\phi^{\bar{\gamma}}_{n,q,\theta} - \int_{Y_{n,q}} \phi^{\bar{\gamma}}_{n,q,\theta}\, d\mu_{Y_{n,q}} \right\|_2$$
$$\le c_\eta\, m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-(4\iota(\mathrm{G}))^{-1}+\eta} \|\phi^{\bar{\gamma}}_{n,q,\theta}\|_2.$$

Therefore, we deduce that for every $\delta > 0$,

$$\mu_{Y_{n,q}}\left(\left\{h\Gamma_n(q) : \left|\pi_{Y_{n,q}}(\beta_{n,x,\rho})\phi^{\bar{\gamma}}_{n,q,\theta}(h\Gamma_n(q)) - \frac{1}{m_{Y_{n,q}}(Y_{n,q})}\right| > \delta\right\}\right)$$
$$\leq c_\eta^2\delta^{-2}\frac{m_{G_n}(\mathcal{O}_n(\theta))^{-1}}{m_{Y_{n,q}}(Y_{n,q})}m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-(2\iota(\mathrm{G}))^{-1}+2\eta}.$$

Let us take $\delta > 0$ such that

$$\mu_{Y_{n,q}}(\mathcal{O}_n(\theta)\Gamma_n(q)) > c_\eta^2\delta^{-2}\frac{m_{G_n}(\mathcal{O}_n(\theta))^{-1}}{m_{Y_{n,q}}(Y_{n,q})}m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-(2\iota(\mathrm{G}))^{-1}+2\eta}. \tag{5.6}$$

Note that it follows from (5.5) that we may choose δ so that

$$\delta = O_\eta\left(m_{G_n}(\mathcal{O}_n(\theta))^{-1}m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-(4\iota(\mathrm{G}))^{-1}+\eta}\right)$$
$$= O_\eta\left(\theta^{-\dim(\mathrm{G})}m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-(4\iota(\mathrm{G}))^{-1}+\eta}\right),$$

where we used Lemma 5.2. Then we deduce from (5.6) that there exists $h \in \mathcal{O}_n(\theta)$ satisfying

$$\left|\pi_{Y_{n,q}}(\beta_{n,x,\rho})\phi^{\bar{\gamma}}_{n,q,\theta}(h\Gamma_n(q)) - \frac{1}{m_{Y_{n,q}}(Y_{n,q})}\right| \leq \delta,$$

which gives

$$\left|\int_{B_n(x,\rho)}\phi^{\bar{\gamma}}_{n,q,\theta}(g^{-1}h\Gamma_n(q))\,dm_{G_n}(g) - \frac{m_{G_n}(B_n(x,\rho))}{m_{Y_{n,q}}(Y_{n,q})}\right| \leq \delta\, m_{G_n}(B_n(x,\rho)).$$

Therefore,

$$\int_{B_n(x,\rho)}\phi^{\bar{\gamma}}_{n,q,\theta}(g^{-1}h\Gamma_n(q))\,dm_{G_n}(g) \tag{5.7}$$
$$=\frac{m_{G_n}(B_n(x,\rho))}{m_{Y_{n,q}}(Y_{n,q})} + O_\eta\left(\theta^{-\dim(\mathrm{G})}m_{G_n}(B_n(x,\rho))m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-(4\iota(\mathrm{G}))^{-1}+\eta}\right).$$

Now to finish the proof of the theorem, we observe that according to Lemma 5.4,

$$|B_n(x,\epsilon)\cap\bar{\gamma}\Gamma_n(q)| \leq \int_{B_n(x,\epsilon+c_1(x)\epsilon^{\kappa+1})}\phi^{\bar{\gamma}}_{n,q,\epsilon^{\kappa+1}}(g^{-1}h\Gamma_n(q))\,dm_{G_n}(g).$$

Combining this estimate with (5.7), we obtain

$$|B_n(x,\epsilon)\cap\bar{\gamma}\Gamma_n(q)| \leq \frac{m_{G_n}(B_n(x,\epsilon+c_1(x)\epsilon^{\kappa+1}))}{m_{Y_{n,q}}(Y_{n,q})} \tag{5.8}$$
$$+ O_\eta\left(\epsilon^{-(\kappa+1)\dim(\mathrm{G})}m_{G_n}(B_n(x,\epsilon+c_1(x)\epsilon^{\kappa+1}))m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-(4\iota(\mathrm{G}))^{-1}+\eta}\right).$$

By Lemma 5.3(ii), for sufficiently small $\epsilon > 0$,

$$m_{G_n}(B_n(x,\epsilon+c_1(x)\epsilon^{\kappa+1})) = m_{G_n}(B_n(x,\epsilon)) + O_x\left(\epsilon^{\kappa+\dim(\mathrm{G})}m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})\right),$$

where the implied constants are uniform over x in bounded sets. Also, we have

$$m_{G_n}(B_n(x,\epsilon+c_1(x)\epsilon^{\kappa+1}))=O_x\left(\epsilon^{\dim(\mathrm{G})}m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})\right).$$

Therefore, (5.8) implies that

$$\begin{aligned}|B_n(x,\epsilon)\cap\bar\gamma\Gamma_n(q)|\le&\frac{m_{G_n}(B_n(x,\epsilon))}{m_{Y_{n,q}}(Y_{n,q})}+O_x\left(\epsilon^{\kappa+\dim(\mathrm{G})}m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})\right)\\&+O_{x,\eta}\left(\epsilon^{-\kappa\dim(\mathrm{G})}m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-(4\iota(\mathrm{G}))^{-1}+\eta}\right).\end{aligned}$$

Here we used that $m_{Y_{n,q}}(Y_{n,q})\gg 1$ which follows from Proposition 4.1. This proves the required upper bound for $|B_n(x,\epsilon)\cap\bar\gamma\Gamma_n(q)|$.

The proof of the lower bound is similar, and we use the second estimate from Lemma 5.4. Note that in this proof we need to arrange that $\epsilon-c_1(x)\epsilon^{\kappa+1}>0$ which holds for sufficiently small ϵ, depending on κ,x. □

6. Completion of the proof

In this section, we finish the proof of our main results stated in the Introduction. It would be convenient to introduce a parameter

$$a(\mathrm{G}):=\limsup_{n\to\infty}\frac{\log m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})}{\log n}. \tag{6.1}$$

Note that according to Theorem 5.1, the quantity $a(\mathrm{G})$ measures the polynomial growth rate of the number of rational points in G with given denominator and lying in a bounded subset of $\mathrm{G}(\mathbb{R})$. By Proposition 4.2, $a(\mathrm{G})>0$ if G is isotropic over $\mathbb{Q}$. Given a finite set $\mathcal{P}$ of prime numbers, we also set

$$a(\mathrm{G},\mathcal{P}):=\limsup_{n\to\infty}{}'\ \frac{\log m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})}{\log n},$$

where the lim sup is taken over all integers n with prime divisors in $\mathcal{P}$. If the group G is isotropic over $\mathbb{Q}_p$ for every $p\in\mathcal{P}$, then $a(\mathrm{G},\mathcal{P})>0$ by Proposition 4.2.

Throughout this section we use the following simplified notation:

$$d:=\dim(\mathrm{G}),\quad a:=a(\mathrm{G}),\quad a_{\mathcal{P}}:=a(\mathrm{G},\mathcal{P}),\quad \iota:=\iota(\mathrm{G}).$$

Recall that $\iota(\mathrm{G})$ is computed in terms of the integrability exponent of automorphic representations (see Section 2).

For an integral polynomial f, we denote by $\Delta_n(f)$ the positive integer, coprime to n, representing the greatest common divisor of $f(\gamma)$, $\gamma\in\Gamma_n$, in the ring $\mathbb{Z}[1/n]$. We denote by $\delta_n(f)$ the number of prime factors of $\Delta_n(f)$. Note that $\Delta_n(f)$ divides $\Delta_1(f)$ and $\delta_n(f)\le\delta_1(f)$.

The following result is a more precise version of Theorem 1.2.

Theorem 6.1. *Let* $\mathrm{G} \subset \mathrm{GL_N}$ *be a simply connected* $\mathbb{Q}$*-simple algebraic group defined over* $\mathbb{Q}$*, which is isotropic over* $\mathbb{Q}$ *and* $f_1, \dots, f_t$ *a collection of polynomials as in the Introduction. Then for every* $x \in \mathrm{G}(\mathbb{R})$*,* $\alpha \in (0, \alpha_0)$ *with* $\alpha_0 := d^{-1}a(4\iota)^{-1}$ *and* $n \geq n_0(\alpha, x)$*, there exists* $z \in \mathrm{G}(\mathbb{Q})$ *satisfying*

$$d(x, z) \leq n^{-\alpha},$$
$$\mathrm{den}(z) = n,$$

and r*-prime in* $\mathrm{Mat}_N(\mathbb{Z}[1/n])$*, where*

$$r = \delta_n(f_1 \cdots f_t) + \left\lceil \frac{9t \deg(f_1 \cdots f_t)(d+1)^2}{a(4\iota)^{-1} - \alpha d} \right\rceil.$$

The constant $n_0(\alpha, x)$ *is uniform over* x *in bounded subsets of* $\mathrm{G}(\mathbb{R})$*.*

Proof. By Theorem 5.1, for every coprime $n, q \in \mathbb{N}$, $\bar{\gamma} \in \Gamma_n$, $x \in \mathrm{G}(\mathbb{R})$, $\kappa, \eta > 0$, and $\epsilon \in (0, \epsilon_0(\kappa, x)]$,

$$\begin{aligned} |B_n(x, \epsilon) \cap \bar{\gamma}\Gamma_n(q)| =& \frac{m_{G_n}(B_n(x, \epsilon))}{m_{Y_{n,q}}(Y_{n,q})} + O_x\left(\epsilon^{\kappa+d} m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})\right) \\ &+ O_{x,\eta}\left(\epsilon^{-\kappa d} m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-(4\iota(\mathrm{G}))^{-1}+\eta}\right), \end{aligned} \tag{6.2}$$

where the implied constants are uniform over x in bounded sets. We apply this estimate with $\epsilon_n = m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-\alpha'}$ where n is sufficiently large, $\alpha' \in (0, \alpha_0')$, and $\alpha_0' := \alpha_0/a = d^{-1}(4\iota)^{-1}$. To optimise the error term in (6.2), we choose

$$\kappa = \frac{(4\iota)^{-1} - \alpha' d}{\alpha'(d+1)}. \tag{6.3}$$

Note that the parameter α_0' is chosen to guarantee that $\kappa > 0$. Then (6.2) becomes

$$|B_n(x, \epsilon_n) \cap \bar{\gamma}\Gamma_n(q)| = \frac{m_{G_n}(B_n(x, \epsilon_n))}{m_{Y_{n,q}}(Y_{n,q})} + O_{x,\eta}\left(m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-\alpha'(\kappa+d)+\eta}\right). \tag{6.4}$$

It would be convenient to set

$$T_n(x) := |B_n(x, \epsilon_n) \cap \Gamma_n|.$$

Since

$$m_{\mathrm{G}(\mathbb{R})}(B^\infty(x, \epsilon)) = m_{\mathrm{G}(\mathbb{R})}(B^\infty(e, \epsilon)) \gg \epsilon^d,$$

we have

$$m_{G_n}(B_n(x, \epsilon_n)) \gg m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-\alpha' d}. \tag{6.5}$$

Using (6.5) and Proposition 4.1, we deduce from (6.4) with sufficiently small $\eta > 0$ that

$$\begin{aligned} T_n(x) &\gg \frac{m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-\alpha' d}}{m_{Y_n}(Y_n)} + O_{x,\eta}\left(m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-\alpha'(\kappa+d)+\eta}\right) \\ &\gg m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-\alpha' d} \end{aligned} \tag{6.6}$$

when n is sufficiently large. In particular, it follows from the definition of $a = a(\mathrm{G})$ that for every $b \in (0, a)$ and sufficiently large n,

$$T_n(x) \gg n^{b(1-\alpha' d)}. \tag{6.7}$$

Since

$$m_{Y_{n,q}}(Y_{n,q}) = m_{Y_n}(Y_n) \cdot [\Gamma_n : \Gamma_n(q)],$$

it follows from (6.4) that

$$|B_n(x, \epsilon_n) \cap \bar{\gamma}\Gamma_n(q)| = \frac{T_n(x)}{[\Gamma_n : \Gamma_n(q)]} + O_{x,\eta}\left(m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-\alpha'(\kappa+d)+\eta} \right). \tag{6.8}$$

Every $w \in \mathbb{Z}[1/n]$ can be uniquely written as $w = u \cdot [w]_n$ where u is a unit in $\mathbb{Z}[1/n]$ and $[w]_n \in \mathbb{Z}_{\geq 0}$ is coprime to n. Let $f = f_1 \cdots f_t$ and $\mathcal{P}_{n,z}$ be the set of prime numbers which are coprime to $\Delta_n(f)n$ and bounded by z. We denote by $S_{n,z}(x)$ the cardinality of the set of $\gamma \in B_n(x, \epsilon_n) \cap \Gamma_n$ such that $[f(\gamma)]_n$ is coprime to $\mathcal{P}_{n,z}$ (equivalently, $f(\gamma)$ is coprime to $\mathcal{P}_{n,z}$ in the ring $\mathbb{Z}[1/n]$).

We will apply the combinatorial sieve as in [HR, Th. 7.4] (see also [NS10, Sec. 2]) to estimate the quantity $S_{n,z}(x)$. For this we let

$$a_k := |\{\gamma \in B_n(x, \epsilon_n) \cap \Gamma_n : \ [f(\gamma)]_n = k\}|.$$

Then $T_n(x) = \sum_{k \geq 0} a_k$. To apply the combinatorial sieve we need to verify the following three conditions:

(A_0) For every square-free q divisible only by primes in $\mathcal{P}_{n,z}$,

$$\sum_{k=0 \bmod q} a_k = \frac{\rho(q)}{q} T_n(x) + R_q, \tag{6.9}$$

where $\rho(q)$ is a nonnegative multiplicative function such that for primes $p \in \mathcal{P}_{n,z}$, there exists $c_1 < 1$ satisfying

$$\frac{\rho(p)}{p} \leq c_1. \tag{6.10}$$

(A_1) Summing over square-free q divisible only by primes in $\mathcal{P}_{n,z}$,

$$\sum_{q \leq T_n(x)^\tau}{}' |R_q| \leq c_2 T_n(x)^{1-\zeta}$$

for some $c_2, \tau, \zeta > 0$.

(A_2) For some $w \in [2, z]$,

$$-l \leq \sum_{p \in \mathcal{P}_{n,z} : w \leq p < z} \frac{\rho(p) \log p}{p} - t \log \frac{z}{w} \leq c_3 \tag{6.11}$$

for some $c_3, l, t > 0$.

Once the conditions $(A_0), (A_1)$, and (A_2) are verified, by [HR, Th. 7.4], for $z = T_n(x)^{\tau/s}$ with $s > 9t$, we have the following estimate

$$S_{n,z}(x) \geq T_n(x)W(z)\left(C_1 - C_2 l \frac{(\log\log 3T_n(x))^{3t+2}}{\log T_n(x)}\right), \tag{6.12}$$

where

$$W(z) = \prod_{p\in\mathcal{P}_{n,z}:p\leq z}\left(1 - \frac{\rho(p)}{p}\right),$$

and the constants $C_1, C_2 > 0$ are determined by $c_1, c_2, c_3, \tau, \zeta, t$.

We denote by $\Gamma_n^{(q)}$ the image of Γ_n in $\mathrm{GL}_N(\mathbb{Z}[1/n]/q\mathbb{Z}[1/n])$. Note that

$$\Gamma_n^{(q)} \simeq \Gamma_n/\Gamma_n(q).$$

To verify (A_0), we observe that (6.8) implies that

$$\begin{aligned}
\sum_{k=0 \bmod q} a_k &= |\{\gamma \in B_n(x,\epsilon_n)\cap\Gamma_n :\ f(\gamma) = 0 \bmod q\}| \\
&= \sum_{\bar\gamma\in\Gamma_n^{(q)}:f(\bar\gamma)=0 \bmod q} |B_n(x,\epsilon_n)\cap\bar\gamma\Gamma_n(q)| \\
&= |\Gamma_n^{(q)}\cap\{f=0\}|\left(\frac{T_n(x)}{[\Gamma_n:\Gamma_n(q)]} + O_{x,\eta}\left(m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-\alpha'(\kappa+d)+\eta}\right)\right) \\
&= \frac{\rho(q)}{q}T_n(x) + O_{x,\eta}\left(|\Gamma_n^{(q)}\cap\{f=0\}|\cdot m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-\alpha'(\kappa+d)+\eta}\right),
\end{aligned}$$

where

$$\rho(q) := \frac{q|\Gamma_n^{(q)}\cap\{f=0\}|}{[\Gamma_n:\Gamma_n(q)]}.$$

Since q is coprime to $\Delta_n(f)$, the polynomial f is not identically zero on $\Gamma_n^{(q)}$ and $\rho(q) < q$. Moreover, it follows from the strong approximation property that

$$\Gamma_n^{(q)} \simeq \prod_{p|q}\Gamma_n^{(p)},$$

and for almost all primes p,

$$\Gamma_n^{(p)} \simeq \mathrm{G}^{(p)}(\mathbb{F}_p),$$

where $\mathrm{G}^{(p)}$ denotes the reduction of G modulo p. Therefore, arguing as in [NS10, Sec. 4.1], we deduce that the function ρ is multiplicative, (6.10) holds, and for every prime $p\in\mathcal{P}_{n,z}$,

$$\rho(p) = t + O(p^{-1/2}). \tag{6.13}$$

This proves (A_0) with

$$R_q = O_{x,\eta}\left(|\Gamma_n^{(q)}\cap\{f=0\}|\cdot m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-\alpha'(\kappa+d)+\eta}\right).$$

It follows from (6.6) that

$$\sum_{q\le T_n(x)^\tau}{}' |R_q| \ll_{x,\eta} \sum_{q\le T_n(x)^\tau} q^d m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{1-\alpha'(\kappa+d)+\eta}$$
$$\ll_{x,\eta} (T_n(x)^\tau)^{d+1} T_n(x)^{(1-\alpha'(\kappa+d)+\eta)/(1-\alpha' d)}.$$

Since η can be taken to be arbitrary positive number, we conclude that

$$\sum_{q\le T_n(x)^\tau}{}' |R_q| \ll_{x,\eta} T_n(x)^{1-\zeta}$$

with some $\zeta > 0$, when

$$\tau < \tau_0 := (1-(1-\alpha'(\kappa+d))/(1-\alpha' d))/(d+1). \tag{6.14}$$

Note that since $\kappa > 0$, it follows that $\tau_0 > 0$. This proves (A_1). From (6.3),

$$\tau_0 = \frac{(4\iota)^{-1} - \alpha' d}{(d+1)^2(1-\alpha' d)}.$$

Using (6.13), we can also establish condition (A_2) with

$$l = O\left(\log\log(\Delta_n(f)n)\right) = O\left(\log\log n\right).$$

This argument is exactly the same as in the proof of [GN12b, Th. 5.1].

Now we are in position to apply the main sieving argument (6.12). Note that for $z = T_n(x)^{\tau/s}$, it follows from (6.13) that

$$W(z) \gg (\log z)^{-t}.$$

Therefore, (6.12) gives

$$S_{n,T_n(x)^{\tau/s}}(x) \ge \frac{T_n(x)}{(\log T_n(x))^t}\left(C_1 - C_2'(\log\log n)\frac{(\log\log 3T_n(x))^{3t+2}}{\log T_n(x)}\right),$$

Using (6.7), we deduce that for sufficiently large n,

$$S_{n,T_n(x)^{\tau/s}}(x) \gg_x \frac{T_n(x)}{(\log T_n(x))^t}.$$

Every γ counted in $S_{n,T_n(x)^{\tau/s}}(x)$ satisfies

$$d(x,\gamma) \le \epsilon_n \quad \text{and} \quad \gamma \in B_n^{\mathrm{f}}. \tag{6.15}$$

This implies that $\mathrm{den}(\gamma) = n$, and the numerator of γ is $O_x(n)$. In particular,

$$[f(\gamma)]_n \ll_x n^{\deg(f)}.$$

On the other hand, for any γ counted in $S_{n,T_n(x)^{\tau/s}}(x)$, all prime numbers p which are coprime to $\Delta_n(f)$ and divide $[f(\gamma)]_n$ must satisfy

$$p > z = T_n(x)^{\tau/s}.$$

Thus, using (6.7), we deduce that the number of such prime factors is bounded from above by

$$\frac{\log(n^{\deg(f)}) + O_x(1)}{\log(T_n(x)^{\tau/s})} = \frac{s\deg(f)}{\tau b(1-\alpha' d)} + o_x(1).$$

Since this estimate holds for all $s > 9t$, $b < a$ and $\tau < \tau_0$, the number of such prime factors is at most

$$\left\lceil \frac{9t\deg(f)}{\tau_0 a(1-\alpha' d)} \right\rceil = \left\lceil \frac{9t\deg(f)(d+1)^2}{a((4\iota)^{-1}-\alpha' d)} \right\rceil$$

provided that n is sufficiently large. Therefore, we conclude that the element γ is r-prime in $\mathrm{Mat}_N(\mathbb{Z}[1/n])$ with

$$r = \delta_n(f) + \left\lceil \frac{9t\deg(f)(d+1)^2}{a((4\iota)^{-1}-\alpha' d)} \right\rceil.$$

For every $b \in (0,a)$ and sufficiently large n,

$$\epsilon_n = m_{G_n^{\mathrm{f}}}(B_n^{\mathrm{f}})^{-\alpha'} \le n^{-b\alpha'}.$$

Therefore, it follows from (6.15) that for every $\alpha < a\alpha_0' = \alpha_0$ and sufficiently large n,

$$d(x,\gamma) \le n^{-\alpha}.$$

This completes the proof of the theorem. □

When G is not assumed to be isotropic over $\mathbb{Q}$, we have the following version of Theorem 6.1.

Theorem 6.2. *Let* $\mathrm{G} \subset \mathrm{GL_N}$ *be a simply connected* $\mathbb{Q}$*-simple algebraic group defined over* $\mathbb{Q}$*,* $f_1, \ldots, f_t$ *a collection of polynomials as in the Introduction, and* $\mathcal{P}$ *a finite collection of prime number such that* G *is isotropic over* $\mathbb{Q}_p$ *for all* $p \in \mathcal{P}$*. Then for every* $x \in \mathrm{G}(\mathbb{R})$*,* $\alpha \in (0, \alpha_0)$ *with* $\alpha_0 := d^{-1} a_{\mathcal{P}}(2\iota)^{-1}$ *and* $n \ge n_0(\alpha, x)$ *whose prime divisors are in* $\mathcal{P}$*, there exists* $z \in \mathrm{G}(\mathbb{Q})$ *satisfying*

$$d(x,z) \le n^{-\alpha},$$
$$\mathrm{den}(z) = n,$$

and r*-prime in* $\mathrm{Mat}_N(\mathbb{Z}[1/n])$*, where*

$$r = \delta_n(f_1\cdots f_t) + \left\lceil \frac{9t\deg(f_1\cdots f_t)(d+1)^2}{a_{\mathcal{P}}(2\iota)^{-1}-\alpha d} \right\rceil.$$

The constant $n_0(\alpha, x)$ *is uniform over* x *in bounded subsets of* $\mathrm{G}(\mathbb{R})$*.*

The proof of Theorem 6.2 goes along the same lines as the proof of Theorem 6.1, but instead of the estimate on the averaging operators given by Theorem 3.2, we use Theorem 3.1. This leads to slightly better estimates for the parameters α and r, but the parameter $n_0(\alpha, x)$ now might depend on the set $\mathcal{P}$.

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