TROPICALLY UNIRATIONAL VARIETIES

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Abstract. We introduce *tropically unirational varieties*, which are subvarieties of tori that admit dominant rational maps whose tropicalisation is surjective. The central (and unresolved) question is whether *all* unirational varieties are tropically unirational. We present several techniques for proving tropical unirationality, along with various examples.

1. Tropical Unirationality

Tropical geometry has proved useful for *implicitisation*, i.e., for determining equations for the image of a given polynomial or rational map [10, 11, 12]. The fundamental underlying observation is that tropicalising the map in a naive manner gives a piecewise linear map whose image is contained in the tropical variety of the image of the original map. Typically, this containment is strict, and for polynomial maps with *generic* coefficients the difference between the two sets was determined in [12]. Polynomial or rational maps arising from applications are typically highly non-generic, and yet it would be great if those maps could be tropicalised to determine the tropical variety of their image. Rather than realising that ambitious goal, this paper identifies a concrete research problem and presents several useful tools for attacking it.

Thus let K be an algebraically closed field with a non-Archimedean valuation $v : K \to \mathbb{R} \cup \{\infty\}$. We explicitly allow v to be trivial. Write $T = K^*$ for the one-dimensional torus over K and T^n for the n-dimensional torus. For a non-zero polynomial $f = \sum_\alpha c_\alpha x^\alpha \in K[x_1, \dots, x_m]$ we write $\mathrm{Trop}(f)$ for the *tropicalisation* of f, i.e., for the function $\mathbb{R}^m \to \mathbb{R}$ defined by

$$\mathrm{Trop}(f)(\xi) := \min_\alpha (v(c_\alpha) + x \cdot \alpha),\ \xi \in \mathbb{R}^m;$$

here $\cdot$ stands for the standard dot product on $\mathbb{R}^m$. Throughout this paper we will use greek letters to stand for tropical variables.

Strictly speaking, one should distinguish between a tropical polynomial and the function that it defines, but in this paper we will only need the latter. By Gauss's Lemma, we have $\mathrm{Trop}(fg) = \mathrm{Trop}(f) + \mathrm{Trop}(g)$ for non-zero polynomials f, g, and this implies that we can extend the operator Trop to *rational* functions by setting $\mathrm{Trop}(f/h) = \mathrm{Trop}(f) - \mathrm{Trop}(h)$. We further extend this definition to rational maps $\varphi = (f_1, \dots, f_n) : T^m \dashrightarrow T^n$ by setting $\mathrm{Trop}\,\varphi := (\mathrm{Trop}(f_1), \dots, \mathrm{Trop}(f_n)) : \mathbb{R}^m \to \mathbb{R}^n$.

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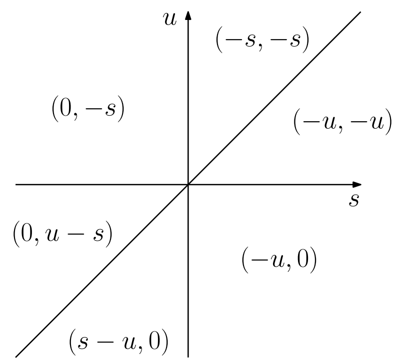

FIGURE 1. Outside the lines, $\mathrm{Trop}(\psi)$ is linear and of the indicated form.

If X is a subvariety of T^n, then we write $\mathrm{Trop}(X)$ for the tropicalisation of X, i.e., for the intersection of the corner loci of all $\mathrm{Trop}(f)$ as f ranges through the ideal of X in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Definition 1.1. A subvariety X of T^n is called *tropically unirational* if there exists a natural number p and a dominant rational map $\varphi : T^p \dashrightarrow X$ such that the image $\mathrm{im}\,\mathrm{Trop}(\varphi)$ equals $\mathrm{Trop}(X)$. The map φ is then called *tropically surjective*.

We recall that the inclusion $\mathrm{im}\,\mathrm{Trop}(\varphi) \subseteq \mathrm{Trop}(X)$ always holds [4]. The following example shows that this inclusion is typically strict, but that φ can sometimes be modified (at the expense of increasing p) so as to make the inclusion into an equality.

Example 1.2. Let $X \subseteq T^2$ be the line defined by $y = x + 1$, with the well-known tripod as its tropical variety. Then the rational map $\varphi : T^1 \dashrightarrow T^2,\ t \mapsto (t, t+1)$ is dominant, but the image of its tropicalisation only contains two of the rays of the tripod, so φ is not tropically surjective. However, the map $\psi : T^2 \dashrightarrow X \subseteq T^2$, $(s, u) \mapsto (\frac{1+s}{u-s}, \frac{1+u}{u-s})$ *is* tropically surjective. Indeed, see Figure 1: under $\mathrm{Trop}(\psi)$, the north-west and south-east quadrants cover the arms of the tripod in the north and east directions, respectively, and any of the two halfs of the north-east quadrant covers the arm of the tripod in the south-west direction. So X is tropically unirational. There is no tropically surjective rational map into X with $p = 1$.

The central question that interests us is the following.

Question 1.3. *Is every unirational variety tropically unirational?*

This paper is organised as follows. In Section 2 we review the known fact that (affine-)linear spaces and rational curves are tropically unirational. In Section 3 we prove that, at least for *rational* varieties, our central question above is equivalent to the apparently weaker question of whether $\mathrm{Trop}(X)$ is the union of *finitely many* images $\mathrm{im}\,\mathrm{Trop}(\varphi_i),\ i = 1, \ldots, N$ with each φ_i a rational map $T^{p_i} \dashrightarrow X$. This involves the concept of *reparameterisations*: precompositions $\varphi \circ \alpha$ of a dominant rational map φ into X with other rational maps α; since tropicalisation does not commute with composition, $\mathrm{Trop}(\varphi \circ \alpha)$ may hit points of $\mathrm{Trop}(X)$ that are not hit by $\mathrm{Trop}(\varphi)$. In Section 4 we introduce a somewhat ad-hoc technique for finding suitable (re)parameterisations. Together with tools from preceding sections this technique allows us, for example, to prove that the hypersurface of singular $n \times n$-matrices is tropically unirational for every n. In Section 5 we prove that for X unirational, every sufficiently generic point on $\mathrm{Trop}(X)$ has a $\dim(X)$-dimensional neighbourhood that is covered by $\mathrm{Trop}(\varphi)$ for suitable φ; here we require that K has characteristic zero. Combining reparameterisations, we find that there exist dominant maps into X whose tropicalisation hit full-dimensional subsets of all full-dimensional polyhedra in the polyhedral complex $\mathrm{Trop}(X)$. But more sophisticated methods, possibly from *geometric tropicalisation*, will probably be required to give a definitive answer to our central question.

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2. Linear spaces, toric varieties, homogenisation, curves

We start with some elementary constructions of tropically unirational subvarieties of tori.

Lemma 2.1. *If X is a tropically unirational subvariety of T^n, then so is its image $L_u\overline{\pi(X)}$ under any torus homomorphism $\pi : T^n \to T^q$ followed by left multiplication L_u with $u \in T^q$.*

Proof. If $\varphi : T^m \dashrightarrow X$ is tropically surjective, then we claim that so is $L_u \circ \pi \circ \varphi : T^m \dashrightarrow Y := \overline{\pi(X)}$. Indeed, since ϕ is a monomial map and L_u is just componentwise multiplication with non-zero scalars, we have $\mathrm{Trop}(L_u \circ \pi \circ \varphi) = \mathrm{Trop}(L_u) \circ \mathrm{Trop}(\pi) \circ \mathrm{Trop}(\varphi)$. Here the first map is a translation over the componentwise valuation $v(u)$ of u, and the second map is an ordinary linear map. The claim follows from the known fact that $\mathrm{Trop}(L_u)\,\mathrm{Trop}(\pi)\,\mathrm{Trop}(X) = \mathrm{Trop}(Y)$, which is a consequence of the main theorem of tropical geometry [2] $\square$

The following is a consequence of a theorem by Yu and Yuster [15].

Proposition 2.2. *If X is the intersection with T^n of a linear subspace of K^n, then X is tropically unirational.*

Proof. Let V be the closure of X in K^n, by assumption a linear subspace. The support of an element $v \in V$ is the set of i such that $v_i \neq 0$. Choose non-zero vectors $v_1, \ldots, v_p \in V$ such that the support of each vector in V contains the support of some v_i. Let $A \in K^{n\times p}$ be the matrix with columns $v_1, \ldots, v_p$, and let $v(A) \in (\mathbb{R} \cup \{\infty\})^{n\times p}$ be the image of A under coordinate-wise valuation. Then Yu-Yuster's theorem states that $\mathrm{Trop}(V) \subseteq (\mathbb{R} \cup \{\infty\})^n$ is equal to the image of $(\mathbb{R} \cup \{\infty\})^p$ under tropical matrix multiplication with $v(A)$. This implies that the rational map $T^p \dashrightarrow T^n,\ v \mapsto Av$ is tropically surjective. $\square$

Another argument for the tropical unirationality of linear spaces will be given in Section 4.

Lemma 2.3. *Let $X \subseteq T^n$ be a closed subvariety, and write $\tilde{X}$ for the the cone $\{(t, tp) \mid t \in K^*\}$ over X in T^{n+1}. Then $\tilde{X}$ is tropically unirational if and only if X is.*

Proof. If X is tropically unirational, then so is $T \times X \subseteq T^{n+1}$, and hence by Lemma 2.1 so is the image $\tilde{X}$ of the latter variety under the torus homomorphism $(t, p) \mapsto (t, tp)$. Conversely, if $\tilde{X}$ is tropically unirational, then so is its image X under the torus homomorphism $(t, p) \mapsto t^{-1}p$. □

Note that $\mathrm{Trop}(\tilde{X}) = \{0\} \times \mathrm{Trop}(X) + \mathbb{R}(1, \ldots, 1)$; we will use this in Section 3. We can now list a few classes of tropically unirational varieties.

Corollary 2.4. *Intersections with T^n of affine subspaces of K^n are tropically unirational.*

Proof. If X the intersection with T^n of an affine subspace of K^n, then the cone $\tilde{X}$ is the intersection with T^{n+1} of a linear subspace of K^{n+1}. Thus the corollary follows from Proposition 2.2 and Lemma 2.3. □

The following corollary has been known at least since Speyer's thesis [9].

Corollary 2.5. *Rational curves are unirational.*

Proof. Let $\varphi = (f_1, \ldots, f_n) : T \dashrightarrow T^n$ be a rational map, and let Y be the rational curve parameterised by it. Let $S \subseteq K$ be a finite set containing all roots and poles of the f_i, so that we can write

$$f_i(x) = c_i \prod_{s \in S} (x - s)^{e_{is}}.$$

where the c_i are non-zero elements of K and the e_{is} are integer exponents. Let $X \subseteq T^S$ be the image of the affine-linear linear map $T \dashrightarrow T^S$ given by $x \mapsto (x - s)_{s \in S}$. Then X is tropically unirational by Corollary 2.4. Let $\pi : T^S \to T^n$ be the torus homomorphism mapping $(z_s)_{s \in S}$ to $(\prod_{s \in S} z_s^{e_{is}})_i$, and let $u = (c_i)_i \in T^n$. Then the curve Y is the image of X under $L_u \circ \pi$, and the corollary follows from Lemma 2.1. □

Corollary 2.6. *The variety in $T^{m \times n}$ of $m \times n$-matrices of rank at most 2 is tropically unirational.*

Proof. Let $\varphi : T^m \times T^m \times T^n \times T^n \dashrightarrow T^{m \times n}$ be the rational map defined by

$$\varphi : (u, x, v, y) \mapsto \mathrm{diag}(u)(x\mathbf{1}^t + \mathbf{1}y^t)\,\mathrm{diag}(v),$$

where $\mathrm{diag}(u), \mathrm{diag}(v)$ are diagonal matrices with the entries of u, v along the diagonals; $\mathbf{1}^t, \mathbf{1}$ are the $1 \times n$ and the $m \times 1$ row vectors with all ones; and x, y are interpreted as column vectors. Elementary linear algebra shows that φ is dominant into the variety Y of rank-at-most-2 matrices. Moreover, φ is the composition of the linear map $(u, x, y, v) \mapsto (u, x\mathbf{1}^t + \mathbf{1}y^t, v)$ with the torus homomorphism $(u, z, v) \mapsto (\mathrm{diag}(u) z\,\mathrm{diag}(v))$. Hence Y is tropically unirational by Proposition 2.2 and Lemma 2.1. □

Corollary 2.7. *The affine cone over the Grassmannian of two-dimensional vector subspaces of an n-dimensional space (or more precisely its part in $T^{\binom{n}{2}}$ with non-zero Plücker coordinates) is tropically unirational.*

Proof. The proof is identical to the proof of Corollary 2.6, using the rational map

$$T^n \times T^n \mapsto T^{\binom{n}{2}},\ (u,x) \mapsto (u_i u_j (x_i - x_j))_{i<j}.$$

□

Interestingly, Grassmannians of two-spaces and varieties of rank-two matrices are among the few infinite families of varieties for which *tropical bases* are known [1]. It would be nice to have a direct link between this fact and the fact, used in the preceding proofs, that they are obtained by smearing around a linear space with a torus action.

Corollary 2.8. *The varieties defined by A-discriminants are tropically unirational.*

Proof. Like in the previous two cases, these varieties are obtained from a linear variety by smearing around with a torus action; this is the celebrated *Horn Uniformisation* [5, 6]. □

In fact, this linear-by-toric description of A-discriminants was used in [3] to give an efficient way to compute the Newton polytopes of these discriminants in the hypersurface case. A relatively expensive step in this computation is the computation of the tropicalisation of the kernel of A; the state of the art for this computation is [8].

3. Combining reparameterisations

A fundamental method for constructing tropically surjective maps into a unirational variety $X \subseteq T^n$ is precomposing one dominant map into X with suitable rational maps.

Definition 3.1. Given a rational dominant map $\varphi : T^m \dashrightarrow X \subseteq T^n$, a *reparameterisation* of φ is a rational map of the form $\varphi \circ \alpha$ where $\alpha : T^p \dashrightarrow T^m$ is a dominant rational map and p is some natural number.

The point is that, in general, $\mathrm{Trop}(\varphi \circ \alpha)$ is *not* equal to $\mathrm{Trop}(\varphi) \circ \mathrm{Trop}(\alpha)$. So the former tropical map has a chance of being surjective onto $\mathrm{Trop}(X)$ even if the latter is not.

Example 3.2. In Example 1.2, the map $\psi : (s,t) \mapsto (\frac{1+s}{u-s}, \frac{1+u}{u-s})$ is obtained from $\varphi : t \mapsto (t, t+1)$ by precomposing with the rational map α sending (s,u) to $\frac{1+s}{u-s}$. Hence ψ is a tropically surjective reparameterisation of the non-tropically surjective rational map φ.

This leads to the following sharpening of our Question 1.3.

Question 3.3. *Does every dominant rational map φ into a unirational variety $X \subseteq T^n$ have a tropically surjective reparameterisation?*

Note that if X is rational and $\varphi : T^m \dashrightarrow X \subseteq T^n$ is birational, then *every* dominant rational map $\psi : T^p \to X$ factors into the dominant rational map $(\varphi^{-1} \circ \psi) : T^p \dashrightarrow T^m$ and the map φ. So for such pairs (X, φ), the preceding question is *equivalent* to the question whether X is tropically unirational.

We will now show how to combine reparameterisations at the expense of enlarging the parameterising space T^p. For this we need a variant of Lemma 2.3. Let $\varphi = (\frac{f_1}{g}, \ldots, \frac{f_n}{g}) : T^m \dashrightarrow X \subseteq T^n$ be a dominant rational map where

$g, f_1, \ldots, f_n \in K[x_1, \ldots, x_m]$. Let $d > 0$ be a natural number greater than or equal to $\max\{\deg(g), \deg(f_1), \ldots, \deg(f_n)\}$, and define the homogenisations

$$\tilde{g} := x_0^d g(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \text{ and } \tilde{f}_i := x_0^d f_i(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}),\ i = 1, \ldots, n.$$

These are homogeneous polynomials of positive degree d in $n+1$ variables $x_0, \ldots, x_n$. The map $\tilde{\varphi} : T^{m+1} \dashrightarrow T^{n+1}$ with components $(\tilde{g}, \tilde{f}_1, \ldots, \tilde{f}_n)$ is called a *degree-d homogenisation of φ*. The components of one degree-d homogenisation of φ differ from those of another by a common factor, which is a rational function with numerator and denominator homogeneous of the same degree. Any degree-d homomgenisation of ϕ is dominant into the cone $\tilde{X}$ in T^{n+1} over X. Recall that $\operatorname{Trop}(\tilde{X}) = \{0\} \times \operatorname{Trop}(X) + \mathbb{R}(1, \ldots, 1)$. The following lemma is the analogue of this statement for $\operatorname{im}\operatorname{Trop}(\tilde{\varphi})$.

Lemma 3.4. *Let $\tilde{\varphi}$ be any degree-d homogenisation of φ. Then the image of* $\operatorname{Trop}(\tilde{\varphi})$ *equals* $\{0\} \times (\operatorname{im}\operatorname{Trop}(\varphi)) + \mathbb{R}(1, \ldots, 1)$.

Proof. For the inclusion $\supseteq$, let $\xi \in \mathbb{R}^m$ and let $\gamma \in \mathbb{R}$. Setting $\tilde{\xi} := (0, x) + \frac{\gamma}{d}(1, \ldots, 1) \in \mathbb{R}^{m+1}$ and using that $\tilde{g}$ is homogeneous of degree d we find that

$$\operatorname{Trop}(\tilde{g})(\tilde{\xi}) = \operatorname{Trop}(\tilde{g})(0, \xi) + \gamma = \operatorname{Trop}(g)(\xi) + \gamma;$$

and similarly for the $\tilde{f}_i$. This proves that

$$\operatorname{Trop}(\tilde{\varphi})(\tilde{\xi}) = \operatorname{Trop}(\varphi)(\xi) + \gamma(1, \ldots, 1),$$

from which the inclusion $\supseteq$ follows. For the inclusion $\subseteq$ let $\tilde{\xi} = (\tilde{\xi}_0, \ldots, \tilde{\xi}_m) \in \mathbb{R}^{m+1}$ and set $\xi_i := \tilde{\xi}_i - \tilde{\xi}_0,\ i = 1, \ldots, m$. Again by homogeneity we find

$$\operatorname{Trop}(\tilde{g})(\tilde{\xi}) = \operatorname{Trop}(\tilde{g})(0, \xi) + d\tilde{\xi}_0 = \operatorname{Trop}(g)(\xi) + d\tilde{\xi}_0$$

and similarly for the $\tilde{f}_i$. This implies

$$\operatorname{Trop}(\tilde{\varphi})(\tilde{\xi}) = \operatorname{Trop}(\varphi)(\xi) + d\tilde{\xi}_0(1, \ldots, 1),$$

which concludes the proof of $\subseteq$. □

Lemma 3.5 (Combination Lemma)**.** *Let $\varphi : T^m \dashrightarrow X \subseteq T^n$ and $\alpha_i : T^{p_i} \dashrightarrow T^m$ for $i = 1, 2$ be dominant rational maps. Then there exists a dominant rational map $\alpha : T^{p_1+p_2+1} \dashrightarrow T^m$ such that* $\operatorname{im}\operatorname{Trop}(\varphi \circ \alpha)$ *contains both* $\operatorname{im}\operatorname{Trop}(\varphi \circ \alpha_1)$ *and* $\operatorname{im}\operatorname{Trop}(\varphi \circ \alpha_2)$.

Proof. Consider a degree-d homogenisation $\tilde{\varphi} : T^{m+1} \dashrightarrow \tilde{X} \subseteq T^{n+1}$ and degree-e homogenisations $\tilde{\alpha}_1 : T^{p_1+1} \dashrightarrow T^{m+1}, \tilde{\alpha}_2 : T^{p_2+1} \dashrightarrow T^{m+1}$ of α_1, α_2, respectively. Define $\tilde{\alpha} : T^{p_1+1} \times T^{p_2+1} \dashrightarrow T^{m+1}$ by

$$\alpha(\tilde{u}, \tilde{v}) = \tilde{\alpha}_1(\tilde{u}) + \tilde{\alpha}_2(\tilde{v}).$$

We claim that

$$\operatorname{im}\operatorname{Trop}(\tilde{\varphi} \circ \tilde{\alpha}) \supseteq \operatorname{im}\operatorname{Trop}(\tilde{\varphi} \circ \tilde{\alpha}_i) \text{ for } i = 1, 2.$$

Indeed, since $\tilde{\varphi}$ has polynomial components and since $\tilde{\alpha}_2$ is homogeneous of positive degree e, we have

$$\tilde{\varphi}(\tilde{\alpha}_1(\tilde{u}) + \alpha_2(\tilde{v})) = \tilde{\varphi}(\tilde{\alpha}_1(\tilde{u})) + \text{ terms divisible by at least one variable } \tilde{v}_j.$$

As a consequence, for $(\tilde{\mu}, \tilde{\nu}) \in \mathbb{R}^{p_1+1} \times \mathbb{R}^{p_2+1}$ we have

$$\operatorname{Trop}(\tilde{\varphi} \circ \tilde{\alpha})(\tilde{\mu}, \tilde{\nu}) = \min\{\operatorname{Trop}(\tilde{\varphi} \circ \tilde{\alpha}_1)(\tilde{\mu}),\ \text{terms containing at least one } \tilde{\nu}_j\}.$$

Hence if $\tilde{\mu}$ is fixed first and $\tilde{\nu}$ is then chosen to have sufficiently large (positive) entries, then we find

$$\mathrm{Trop}(\tilde{\varphi} \circ \tilde{\alpha})(\tilde{\mu}, \tilde{\nu}) = \mathrm{Trop}(\tilde{\varphi} \circ \tilde{\alpha}_1)(\tilde{\nu}).$$

This proves that $\operatorname{im}\mathrm{Trop}(\tilde{\phi} \circ \tilde{\alpha}_1) \subseteq \operatorname{im}\mathrm{Trop}(\tilde{\phi} \circ \tilde{\alpha})$. Repeating the argument with the roles of 1 and 2 reversed proves the claim.

Now we carefully de-homogenise as follows. First, a straightforward computation shows that $\tilde{\varphi} \circ \tilde{\alpha}_i$ is a degree-de homogenisation of $\varphi \circ \alpha_i$ for $i = 1, 2$, hence by Lemma 3.4 we have

$$\operatorname{im}(\mathrm{Trop}(\tilde{\varphi} \circ \tilde{\alpha}_i)) = \{0\} \times \operatorname{im}\mathrm{Trop}(\varphi \circ \alpha_i) + \mathbb{R}(1, \ldots, 1).$$

Similarly, writing $\tilde{\alpha} = (a_0, \ldots, a_m) : T^{p_1+1+p_2+1} \dashrightarrow T^m$ for the components of $\tilde{\alpha}$ we define $\alpha : T^{p_1+p_2+1} \dashrightarrow T^m$ as the de-homogenisation of $\tilde{\alpha}$ given by

$$\alpha(u, \tilde{v}) = \left(\frac{a_1(1, u, \tilde{v})}{a_0(1, u, \tilde{v})}, \ldots, \frac{a_m(1, u, \tilde{v})}{a_0(1, u, \tilde{v})} \right).$$

A straightforward computation shows that $\tilde{\phi} \circ \tilde{\alpha}$ is a degree-de homogenisation of $\phi \circ \alpha$. Hence by Lemma 3.4 we have

$$\operatorname{im}(\mathrm{Trop}(\tilde{\varphi} \circ \tilde{\alpha})) = \{0\} \times \operatorname{im}\mathrm{Trop}(\varphi \circ \alpha) + \mathbb{R}(1, \ldots, 1).$$

Now the desired containment

$$\operatorname{im}\mathrm{Trop}(\phi \circ \alpha) \supseteq \operatorname{im}\mathrm{Trop}(\phi \circ \alpha_i) \text{ for } i = 1, 2$$

follows from

$$\begin{aligned} \{0\} \times \operatorname{im}\mathrm{Trop}(\phi \circ \alpha) &= (\{0\} \times \mathbb{R}^n) \cap \operatorname{im}\mathrm{Trop}(\tilde{\phi} \circ \tilde{\alpha}) \\ &\supseteq (\{0\} \times \mathbb{R}^n) \cap \operatorname{im}\mathrm{Trop}(\tilde{\phi} \circ \tilde{\alpha}_i) \\ &= \{0\} \times \operatorname{im}\mathrm{Trop}(\phi \circ \alpha_i) \end{aligned}$$

□

4. Birational projections

In this section we show that rational subvarieties of T^n that have sufficiently many birational toric projections are tropically unirational. Here is a first observation.

Lemma 4.1. *Let $X \subseteq T^n$ be an algebraic variety and $\pi : T^n \to T^d$ a torus homomorphism whose restriction to X is birational, with rational inverse φ. Then $\mathrm{Trop}(\pi) \circ \mathrm{Trop}(\varphi)$ is the identify on $\mathbb{R}^d$.*

Proof. Let $\eta \in \mathbb{R}^d$ be a point where $\mathrm{Trop}(\varphi)$ is (affine-)linear. Such points form the complement of a codimension-1 subset and are therefore dense in $\mathbb{R}^d$. Hence it suffices to prove that $\mathrm{Trop}(\varphi)(\eta)$ maps to η under $\mathrm{Trop}(\pi)$. Let $y \in T^d$ be a point with $v(y) = \eta$ where φ is defined and such that $x := \varphi(y) \in X$ satisfies $\pi(x) = y$. Such points exist because $v^{-1}(\eta)$ is Zariski-dense in T^d. Now $\xi := v(x)$ equals $\mathrm{Trop}(\varphi)(\eta)$ by linearity of $\mathrm{Trop}(\varphi)$ at η and $\mathrm{Trop}(\pi)\xi = \eta$ by linearity of $\mathrm{Trop}(\pi)$. □

For our criterion we need the following terminology.

Definition 4.2. Let $P \subseteq \mathbb{R}^n$ be a d-dimensional polyhedron and let $A : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map. Then P is called *A-horizontal* if $\dim AP = d$.

Proposition 4.3. *Let $X \subseteq T^n$ be an algebraic variety and $\pi : T^n \to T^d$ a torus homomorphism whose restriction to X is birational, with rational inverse φ. Using the Bieri-Groves theorem, write $\mathrm{Trop}(X) = \bigcup_i P_i$ where the P_i are finitely many d-dimensional polyhedra. Then $\mathrm{im}\,\mathrm{Trop}(\varphi)$ is the union of all $\mathrm{Trop}(\pi)$-horizontal polyhedra P_i.*

Proof. Let P_i be a $\mathrm{Trop}(\pi)$-horizontal polyhedron. We want to prove that $\mathrm{Trop}(\varphi) \circ \mathrm{Trop}(\pi)$ is the identity on P_i. To this end, let $\xi \in P_i$ be such that $\mathrm{Trop}(\varphi)$ is affine-linear at $\eta := \mathrm{Trop}(\pi)\xi$. The fact that P_i is $\mathrm{Trop}(\pi)$-horizontal implies that such ξ are dense in P_i. To prove that $\mathrm{Trop}(\varphi)(\eta)$ equals ξ let $x \in X$ be a point with $v(x) = \xi$ such that φ is defined at $y := \pi(x)$ and satisfies $\varphi(y) = x$. The existence of such a point follows from birationality and the density of fibers in X of the valuation map [7]. Now η equals $v(y)$ by linearity of $\mathrm{Trop}(\pi)$ and $\xi = v(x) = v(\varphi(y))$ equals $\mathrm{Trop}(\varphi)(\eta)$ by linearity of $\mathrm{Trop}(\varphi)$ at η. Hence $\mathrm{Trop}(\varphi) \circ \mathrm{Trop}(\pi)$ is the identity on P_i, as claimed. Thus $\mathrm{im}\,\mathrm{Trop}(\varphi)$ contains P_i. Since the projections of $\mathrm{Trop}(\pi)$-horizontal polyhedra P_i together form all of $\mathbb{R}^d$, we also find that $\mathrm{im}\,\mathrm{Trop}(\varphi)$ does not contain any points outside those polyhedra. □

Corollary 4.4. *Let $X \subseteq T^n$ be a birational variety and write $\mathrm{Trop}(X) = \bigcup_i P_i$ as in Proposition 4.3. If for each P_i there exists a torus homomorphism $\pi : T^n \to T^d$ that is birational on X and for which P_i is $\mathrm{Trop}(\pi)$-horizontal, then X is tropically unirational.*

Proof. In that case, there exist finitely many homomorphisms $\pi_1, \ldots, \pi_N : T^n \to T^d$, birational when restricted to X, such that each P_i is $\mathrm{Trop}(\pi_j)$-horizontal for at least one j. Then Proposition 4.3 shows that the rational inverse φ_j of π_j satisfies $P_i \subseteq \mathrm{im}\,\mathrm{Trop}(\varphi_j)$. Now use the Combination Lemma 3.5. □

In particular, when all coordinate projections to tori of dimension $\dim X$ are birational the variety X is tropically unirational. This is the case in the following statement.

Corollary 4.5. *For any natural number n the variety of singular $n \times n$-matrices is tropically unirational.*

Proof. A matrix entry m_{ij} of a singular matrix can be expressed as a rational function of all other $n^2 - 1$ entries (with denominator equal to the corresponding $(n-1)\times(n-1)$-subdeterminant). This shows that the map $T^{n^2} \to T^{n^2-1}$ forgetting m_{ij} is birational. Any (m^2-1)-dimensional polyhedron in $\mathbb{R}^{m^2}$ is horizontal with respect to some coordinate projection $\mathbb{R}^{n^2} \to \mathbb{R}^{n^2-1}$, and this holds *a fortiori* for the cones of tropically singular matrices. Now apply Corollary 4.4. □

Corollary 4.4 also gives an alternative proof of Corollary 2.4 stating that tropicalisations of affine-linear spaces are tropically unirational.

Second proof of Corollary 2.4. Let X be the intersection with T^n of a d-dimensional linear space in K^n. For each polyhedron P_i of $\mathrm{Trop}(X)$ there exists a coordinate projection $\pi : T^n \to T^I$, with I some cardinality-d subset of the coordinates, such that P_i is $\mathrm{Trop}(\pi_I)$-horizontal. Here we have not yet used that X is affine-linear. Then the restriction $\pi_I : X \to T^I$ is dominant, and since X is affine-linear, it is also birational. Now apply Corollary 4.4. □

We continue with an example of a determinantal variety of codimension larger than one whose unirationality is a consequence of Corollary 4.4.

Example 4.6. Let $V \subseteq M_{4\times 5}(K)$ be the variety of matrices of rank smaller than or equal to 3. The ideal of V contains all maximal minors and the dimension of V equals 18. One way to see the latter statement is to write a matrix in V in the following form,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad A \in M_{3\times 3}, \quad B \in M_{3\times 2}, \quad C \in M_{1\times 3}, \quad D \in M_{1\times 2}.$$

There are no conditions on A, B and C, while D is uniquely determined by the choice of A, B and C. The dimension thus equals $3\cdot 3 + 3\cdot 2 + 1\cdot 3 = 18$.

Let (m_{ij}) denote the standard coordinate functions on $M_{4\times 5}$. We aim to show that the projection into any subset of X of size 18 is birational. Let $z_1 = m_{i,j}$ and $z_2 = m_{l,k}$ be the indices of the coordinate functions left out of the projection. Note that if z_1 appears in a maximal minor, in which z_2 doesn't, then z_1 is a rational function of the coordinate functions in the maximal minor. In particular, if z_1 and z_2 are in different columns, there exist such maximal minors for z_1 and z_2 and hence both are rational in the remaining 18 coordinate function.

The case that z_1 and z_2 are in the same column requires some calculation. Suppose without loss of generality that $z_1 = m_{3,4}$ and $z_2 = m_{4,5}$. Then,

$$\begin{aligned} 0 &= m_{35}\det M_{124,234} + m_{45}\det M_{123,234} + m_{15}\det M_{234,234} + m_{25}\det M_{124,234}, \\ 0 &= m_{35}\det M_{124,134} + m_{45}\det M_{123,134} + m_{15}\det M_{234,134} + m_{25}\det M_{134,134} \end{aligned}$$

by cofactor expansion of the determinants of the matrices $M_{1234,2345}$ and $M_{1234,1345}$. The set of equations has a unique solution for m_{35} and m_{45} when

$$\det M_{124,134}\det M_{123,134} \neq \det M_{124,234}\det M_{123,134},$$

showing that the projection is birational.

We conclude this section with a beautiful example, suggested to us by Filip Cools and Bernd Sturmfels, and worked out in collaboration with Wouter Castryck and Filip Cools.

Example 4.7. Let $Y \subseteq T^5$ be parameterised by $(s^4, s^3t, \ldots, t^4)$, $(s,t) \in T^2$, the affine cone over the rational normal quartic. Write $X := \overline{Y+Y} \subseteq T^5$, the first secant variety. Writing $z_0, \ldots, z_4$ for the coordinates on T^5, X is the hyperplane defined by

$$\begin{aligned} &\det \begin{bmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{bmatrix} \\ &= z_0z_2z_4 + 2z_1z_2z_3 - z_1^2z_4 - z_0z_3^2 - z_2^3 \\ &= a + 2b - c - d - e. \end{aligned}$$

This polynomial is homogeneous both with respect to the ordinary grading of $K[z_0, \ldots, z_4]$ and with respect to the grading where z_i gets degree i. Hence its Newton polygon is three-dimensional; see Figure 2. Modulo its two-dimensional lineality space the tropical variety $\mathrm{Trop}(X)$ is two-dimensional. Intersecting with a sphere yields Figure 3.

Now set $I := \{1,2,3,4\}$ and $J := \{0,1,2,3\}$. Then the coordinate projections $\pi_I : T^5 \to T^I$, $\pi_J := T^5 \to T^I$ are birational, since z_0 and z_4 appear only linearly

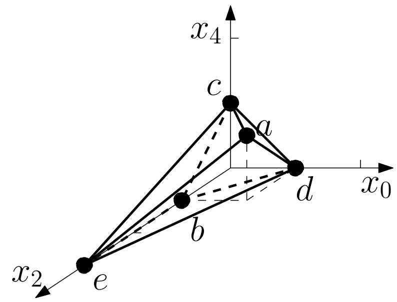

FIGURE 2. The Newton polytope of the Hankel determinant. Only the exponents of z_0, z_2, z_4 have been drawn.

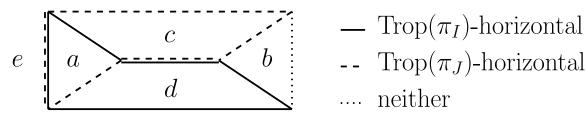

FIGURE 3. The tropical variety of the Hankel determinant. Two-dimensional regions correspond to the monomials a, b, c, d, e.

in the Hankel determinant. Let P be a full-dimensional cone in $\mathrm{Trop}(X)$, and let $\{\alpha, \beta\}$ be the corresponding edge in the Newton polygon. Then P is $\mathrm{Trop}(\pi_I)$-horizontal if and only if $\alpha_0 \neq \beta_0$ and $\mathrm{Trop}(\pi_J)$-horizontal if and only if $\alpha_4 \neq \beta_4$. Figure 3 shows that all but one of the cones are, indeed, horizontal with respect to one of these projections. Let P denote the cone corresponding to the edge between the monomials $b = z_1 z_2 z_3$ and $e = z_2^3$. By Proposition 4.3 and the Combination Lemma 3.5, there exists a rational parameterisation of X whose tropicalisation covers all cones of $\mathrm{Trop}(X)$ except, possibly, P. We now set out to find a parameterisation whose tropicalisation covers P.

Let $\zeta \in P$. By Lemma 3.4 we may assume that ζ is of the form $(\zeta_0, \zeta_1, 2\zeta_1, 3\zeta_1, \zeta_2)$. We aim to show that there exist two reparametrisations of $\phi : T^4 \to X$, where

$$\phi(u_0, u_1, v_0, v_1) = (u_0 + v_1, u_0 v_0 + u_1 v_1, u_0 v_0^2 + u_1 v_1^2, u_0 v_0^3 + u_1 v_1^3, u_0 v_0^4 + u_1 v_1^4).$$

such that ζ is in the image of at least one of them. Note that from the defining inequalities of P it follows that $\zeta_0 \geq 0$ and $\zeta_2 \geq 4\zeta_1$. Let $i \in K$ be a fourth root of unity and consider the map $\psi : T^3 \to T^4$ defined by

$$\psi(x_0, x_1, x_2) = \left(1 + x_0, -1, i x_1, -x_1(1 + x_2 x_1^{-4})\right).$$

A short computation shows that the restriction of the tropicalisation of $\phi \circ \psi$ to the cone defined by $\xi_0 \geq 0$, $\xi_2 \geq 4\xi_1$ and $\xi_2 \leq \xi_0 + 4\xi_1$ is the linear function $(\xi_0, \xi_1, \xi_2) \mapsto (\xi_0, \xi_1, 2\xi_2, 3\xi_1, \xi_2)$. If ζ satisfies $\zeta_2 \leq \zeta_0 + 4\zeta_1$ then the image of $(\zeta_0, \zeta_1, \zeta_2)$ under this tropicalisation is exactly ζ.

If ζ satisfies $\zeta_2 \geq \zeta_0 + 4\zeta_1$ it is in the image of the tropicalisation of $\phi \circ \psi \circ \iota$, where

$$\iota(x_0, x_1, x_2) = (x_0^{-1}, x_1^{-1}, x_2^{-1}).$$

The tropicalisation is linear on the cone $0 \geq \xi_0$, $4\xi_1 \geq \xi_2$ and $\xi_2 \geq \xi_0 + \xi_1$ and maps $-(\zeta_0, \zeta_1, \zeta_2)$ to ζ.

5. Very local reparameterisations

Let $X \subseteq \mathrm{T}^n$ be a d-dimensional rational variety that is the closure of the image of a rational map $\varphi : T^m \dashrightarrow T^n$. Suppose without loss of generality that X is defined over a valued field (K, v) such that $v(K^*)$ is a finite dimensional vector space over $\mathbb{Q}$. Write $\bar{\xi}$ for the image of $\xi \in \mathbb{R}$ under the canonical projection $\mathbb{R} \to \mathbb{R}/v(K^*)$. We can now state the main result of this section and its corollary.

Theorem 5.1. *Assume that the field K has characteristic zero. Let $\xi \in \mathrm{Trop}(X)$ and set d to be the dimension of the $\mathbb{Q}$-vectorspace spanned by $\bar{\xi}_1, \ldots, \bar{\xi}_n$. There exists a rational map $\alpha : T^d \dashrightarrow T^m$ and an open subset $U \subseteq \mathbb{R}^d$ such that the restriction of $\mathrm{Trop}(\varphi \circ \alpha)$ to U is an injective affine linear map, whose image contains ξ.*

Corollary 5.2. *Assume that K has characteristic zero. Let $\{C_1, \ldots, C_k\}$ be a finite set of $v(K^*)$-rational polyhedra of dimension $\dim X$ such that*

$$\mathrm{Trop}(X) = \bigcup_{i=1}^{k} C_i.$$

There exist a natural number p and a rational map $\alpha : T^p \dashrightarrow T^m$ such that the image of $\mathrm{Trop}(\varphi \circ \alpha)$ intersects each C_i in a $\dim X$-dimensional subset.

Proof. By the theorem, for each cone C_i there exists a reparametrisation α_i such that the tropicalization of $\varphi \circ \alpha_i$ hits C_i in a full dimensional subset. They can be combined using the Combination Lemma. □

The main step in the proof of the theorem is Proposition 5.5, which is a valuation theoretic result.

Let ξ be a point of $\mathbb{R}^n$. Such a point defines a valuation v_ξ on the field of rational functions $L = K(y_1, \ldots, y_n)$ of T^n by

$$v_\xi(h) = \mathrm{Trop}(h)(\xi), \qquad h \in L$$

Let L_ξ denote the completion of L with respect to v_ξ and denote its algebraic closure by $\overline{L_\xi}$. That closure is equipped with the unique valuation whose restriction to L_ξ equals v_ξ [13, §144]. Denote by $K[y_1^{\mathbb{Q}}, \ldots, y_n^{\mathbb{Q}}]$ the subring of $\overline{L_\xi}$ generated by all roots of the elements $y_1, \ldots, y_n$.

The next lemmata deal with the case $n = 1$. They allow us to prove Proposition 5.5 below by means of induction on the number of variables.

Lemma 5.3. *Let $\xi \in \mathbb{R}$ such that ξ is not in the $\mathbb{Q}$-vectorspace spanned by $v(K^*)$. Then $K[t^{-1}, t]$ is dense in $K(t)_\xi$*

Proof. Let $p/q \in K(t)$. If q is a monomial we are done. Suppose it isn't. The valuation on q is of the form $v_\xi(q) = \min_i v(p_i) + i\xi$. Moreover, the minimum is attained exactly once since otherwise ξ would be a $\mathbb{Q}$-multiple of some element of

$v(K^*)$. Say it is attained at j. Compute,

$$\begin{aligned}\frac{p}{q} &= \frac{p}{a_j t^j + (q - a_j t^j)} \\ &= a_j t^j \frac{p}{1 + (q - a_j t^j)/(a_j t^j)} \\ &= a_j t^j p \sum_{n=0}^{\infty} \Big(\frac{(q - a_j t^j)}{a_j t^j} \Big)^n\end{aligned}$$

The convergence of the power series with respect to w is a consequence of $w(q - a_j t^j) > w(a_j t^j)$. The limit is easily seen to to coincide with p/q. This completes the proof. □

Lemma 5.4. *Suppose K is algebraically closed of characteristic 0. Then $K[t^{\mathbb{Q}}]$ is dense in $\overline{K(t)_\xi}$,*

Proof. Denote the residue field of $K(t)_\xi$ by k. Note that it is also the residue field of $K(t)$ under ξ, by the conditions on ξ.

We prove by induction on d that all zeroes in $\overline{K(t)_\xi}$ of a polynomial of degree d over $K(t)_\xi$ can be approximated arbitrarily well with elements of $K[t^{\mathbb{Q}}]$. For $d = 1$ this is the content of Lemma 5.3. Assume that the statement is true for all degrees lower than d. We follow the proof of [14, §14, Satz].

Let $P(S) = S^d + a_{d-1}S^{d-1} + \ldots + a_0 \in K(t)_\xi[S]$. After a coordinate change replacing S by $S - \frac{1}{d}a_{d-1}$ we may assume that $a_{d-1} = 0$. Indeed, a root s of the original polynomial can be approximated well by elements of $K[t^{\mathbb{Q}}]$ if and only if the root $s + \frac{1}{d}a_{d-1}$ can be approximated well, since a_{d-1} itself can be approximated well.

If now all a_i are zero, then we are done. Otherwise, let the minimum among the numbers $v(a_{d-i})/i$ be $\omega + q\tau$, where $\omega \in v(K^*)$ and $q \in \mathbb{Q}$, and let c be a constant in K with valuation ω. Setting $S = ct^qU$ transforms P into

$$c^d t^{dq}(U^d + b_{d-2}U^{d-2} + \ldots + b_0),$$

where each b_i is an element of $K(t^{1/p})_\xi$ of valuation at least zero, with p the denominator of q. Moreover, some b_i has valuation zero. Let $Q(U)$ denote the polynomial in the brackets. The image of $Q(U)$ in the polynomial ring $k[U]$ over the residue field is neither U^d as b_i has non-zero image in L, nor a d-th power of an other linear form as the coefficient of U^{d-1} is zero. Hence the image of $Q(U)$ in $k[U]$ has at least two distinct roots in the algebraically closed residue field k, and therefore factors over k into two relatively prime polynomials. By Hensel's lemma [14, §144], Q itself factors over $K(t^{1/p})_\xi$ into two polynomials of positive degree. By induction the roots of these polynomials can be approximated arbitrarily well by elements of $K[t^{\mathbb{Q}}]$, hence so can the roots of Q and of P. □

Proposition 5.5. *Let (K, v) be an algebraically closed field of characteristic 0 with valuation v and $\xi \in \mathbb{R}^n$ whose entries are $\mathbb{Q}$-linearly independent over $\mathbb{R}/v(K^*)$. Then $K[y_1^{\mathbb{Q}}, \ldots, y_n^{\mathbb{Q}}]$ is dense in $\overline{L_\xi}$.*

Proof. Follows from Lemma 5.4 by induction on the number of variables. □

We are now ready to prove the main result.

Proof of Theorem 5.1. Choose $\tau_1, \dots, \tau_d \in \mathbb{R}$ such that their projections in $\mathbb{R}/v(K^*)$ form a basis of the $\mathbb{Q}$-vectorspace spanned by $\bar{\xi_1}, \dots, \bar{\xi_n}$. Let $t_1, \dots, t_n$ be variables and denote by L the field $K(t_1, \dots, t_n)$ equipped with the unique valuation w that extends v and satisfies $w(t_i) = \tau_i$.

There exists a point $x' \in T^m(\overline{L_\xi})$ such that $w(\varphi(x')) = \xi$. By Proposition 5.5 there exists an approximation $x \in T^m(K[t_1^{\mathbb{Q}}, \dots, t_d^{\mathbb{Q}}])$ of x' that satisfies $w(\varphi(x)) = \xi$. Choose $e \in \mathbb{N}$ such that every coefficient of x is already in $K[t_1^{\pm 1/e}, \dots, t_d^{\pm 1/e}]$ and set $s_i = t_i^{1/e}$. Thus, $x \in T^m(K[s_1^{\pm}, \dots, s_d^{\pm}])$, and hence defines a rational map $T^d \to T^m$. Denote this map α. We show that there exists a neighbourhood of $\sigma = \frac{1}{e}(\tau_1, \dots, \tau_d)$ such that the restriction of $\mathrm{Trop}(\varphi \circ \alpha)$ to U satisfies the conclusions of the theorem.

First, note that, by construction of α, $\mathrm{Trop}(\varphi \circ \alpha)(\sigma) = \xi$. Every component $\varphi_i(x) = \varphi_i(\alpha(s_1, \dots, s_d))$ of φ is a Laurent polynomial over K in the s_j with a unique term $z_i s_1^{b_{i,1}} \cdots s_d^{b_{i,d}}$ of minimal valuation $\xi_i = v(z_i) + \sum_{j=1}^d b_{i,j}\sigma_j$. If we let σ vary in a small neighbourhood and change the valuations of the s_i accordingly, then for each i the same term of φ_i has the minimal valuation. Hence $\mathcal{T}(\varphi \circ \alpha)$ is linear at σ with differential the matrix (b_{ij}). Finally, as the numbers $\eta_1, \dots, \eta_n$ span the same $\mathbb{Q}$-space as $\sigma_1, \dots, \sigma_d$ modulo $v(K^*)$, the matrix (b_{ij}) has full rank d. This completes the proof. □

6. Concluding remarks

The concept of tropical surjectivity of a rational map seems natural and concrete, and, as far as we know, not to have been studied before. This paper presented some methods of determining whether a rational map is tropically surjective, and aims to be a starting point for further study. In particular, the question whether every unirational variety is tropically unirational is still open. It seems likely that tehniques from geometric tropicalisation will prove useful in making further progress on this question.

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