Absolutely symmetric trees and complexity of natural number

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Abstract. We consider the rooted trees which not have isomorphic representation and introduce a conception of complexity a natural number also. The connection between quantity such trees with n edges and a complexity of natural number n is established. The recurrent ratio for complexity of a natural number is founded. An expression for calculation of difference complexities of two adjacent natural numbers is constructed. It is proved that this difference equal 1 if and only if a natural number is simple. From proved theorems it follows corollaries.

Almost every book on graph theory contains some parts devoted to trees (see, e.g., [1, 2, 3, 4]). The concept of a tree has been entered for the first time by Kirchhoff [5] in connection with research of electric chains. Later this concept also was independent is entered by Cayley [6] and it had been received the first basic results in the theory of trees. Trees have wide appendices in various area of a science. The analysis of publications [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16] allows to assume that a class of rooted trees which not have isomorphic trees distinct from them in the scientific literature is disregarded. The connection between quantity such trees with n edges and a complexity of natural number n is established in this article.

Definition 1. A flat geometrical realization of a rooted tree is called absolutely symmetric rooted tree (a.s.r.t.) if it not have isomorphic representative distinct from them.

Evidently at $n = 1, 2$ all flat geometrical realization of rooted trees are a.s.r.t.. Since $n = 3$ among realization of trees with n edges there are isomorphic trees.

It is possible to make other inductive definition a.s.r.t. equivalent given above.

Induction basis. The edge with the allocated end a (fig.1) is a.s.r.t. with a root a.

Induction step. Let A - be a a.s.r.t. with a root a (fig.2). Then the tree C (fig.3a) obtained from A by "connecting" a new edge to the root a, become a a.s.r.t. with the root c, where c is the free end of the connected edge. Further, the tree D (fig.3b), obtained from $k, k > 1, A$ by joining their roots, become a a.s.r.t. with the root a.

Let $A = (a_1, a_2, \ldots, a_k)$ a vector with k natural components $a_i \geq 1$. For A we will define functional $f(A)$ recurrently: $f(a_1) = a_1, f(a_1, a_2, \ldots, a_{i+1}) = (f(a_1, a_2, \ldots, a_i) + 1)a_{i+1}$.

Definition 2. The quantity $T(n)$ of various vectors A with $f(A) = n$ is called a complexity of number n.

Theorem 1. The quantity of a.s.r.t. with n edges equal $T(n)$.

Proof. We show at first to any vector $A = (a_1, a_2, \ldots, a_k), a_i \geq 1$, it is possible to put in conformity a.s.r.t. with $f(A)$ edges. We prove it an induction on k. For $k = 1$ $A = (a_1), f(A) = a_1$. We put to vector A in conformity in this case a rooted tree $A_1^{'}$ (fig.4) which is a.s.r.t. with a_1 edges and with a root a. We assume now, that for any $l \leq i$ to vector $A = (a_1, a_2, \ldots, a_l)$ corresponds a.s.r.t. $A_l^{'}$ with number of edges equal $f(A)$ and with a root a. We will show how, that to vector $A = (a_1, a_2, \ldots, a_{i+1})$ can put in conformity a.s.r.t. $A_{i+1}^{'}$ with number of edges $f(A)$.Really, under the assumption, to vector $A = (a_1, a_2, \ldots, a_i)$ corresponds a.s.r.t. $A_i^{'}$ (fig.5a) with number of edges equal $f(A)$ and with a root a. Then to vector $A = (a_1, a_2, \ldots, a_i, a_{i+1})$ we will put in conformity rooted tree $A_{i+1}^{'}$ with a root c (fig.5b) obtained from a_{i+1} trees $A_i^{''}$ (fig.5c) by joining their roots c. Evidently, $A_{i+1}^{'}$ is a.s.r.t. with number of edges equal $f(A) = (f(a_1, a_2, \ldots, a_i) + 1)a_{i+1}$, as was to be shown. Thus to any vector $A = (a_1, a_2, \ldots, a_k), a_i \geq 1$ it is possible to put in conformity a.s.r.t. with $f(A)$ edges.

We show now, that to any a.s.r.t. with number of edges n there correspond some vector $A = (a_1, a_2, \ldots, a_k)$ wits $f(A) = n$. We prove it an induction on n. If $n = 1$, a.s.r.t. with number of edges 1 (fig.1) correspond vector $A = (1)$ with $f(A) = 1$ We assume now, that for any $l \leq k$ a.s.r.t. with number of edges l there correspond some vector $A = (a_1, a_2, \ldots, a_m)$ with $f(A) = l$. Let now it is given a.s.r.t. with number of edges $k+1$. To inductive definition a.s.r.t. two cases are possible: the tree looks like, presented on fig.6a or 6b, where A and B a.s.r.t. such, that vectors corresponding to them $A^{'} = (a_1, a_2, \ldots, a_s), B^{'} = (a_1^{'}, a_2^{'}, \ldots, a_r^{'})$ have $f(A^{'}) = k, f(B^{'}) = m$.

Then a.s.r.t. C(fig.6a) and D(fig.6b) there will correspond vectors $A_C = (a_1, a_2, \ldots, a_s, 1), A_D = (a_1^{'}, a_2^{'}, \ldots, a_r^{'}, t)$. Evidently, $f(A_C) = k+1, f(A_D) = (f(B^{'})+1)t = k+1$. Thus, to any a.s.r.t. with number of edges n there correspond some vector $A = (a_1, a_2, \ldots, a_k)$ with $f(A) = n$.

Corollary 1. If n is simple then all vectors $A = (a_1, a_2, \ldots, a_k), k > 1$, for which $f(A) = n$ have $a_k = 1$.

Corollary 2. $T(n)$ is the quantity of presentations of n in look $a_1 a_2 \cdots a_k + a_2 a_3 \cdots a_k + \cdots + a_k$.

Theorem 2. $T(n) = \sum_r T(r-1)$, where $r \geq 1$ divide $n, T(0) = 1$.

Proof. We prove it an induction on n. $T(1) = 1$ (fig.1). Really, $T(1) = \sum_r T(r-1) = T(0) = 1$.We assume now, that the statement is true for all $l \leq k$, i.e. $T(l) = \sum_r T(r-1)$. Let now $l = k+1$. A.s.r.t. with $k+1$ edges can be one of two kinds (fig.6a) or(fig.6b). Obviously, the number of a.s.r.t a kind (fig.6a) will be $T(k)$, and the number of a.s.r.t. a kind (fig.6b) will be $\sum_{r'} T(r'-1)$, where r' divide $k+1$ and $r' < k+1$. Thus, the general number of a.s.r.t. with $k+1$ edges will be $T(k+1) = T(k) + \sum_{r'} T(r'-1) = \sum_r T(r-1)$. From here for any n it is had

$T(n) = \sum_r T(r-1)$ (1)

For example, if $n = 12$ then $T(12) = T(0) + T(1) + T(2) + T(3) + T(5) + T(11) = 40$.

Corollary 1. Number n is simple if and only if $T(n) = 1 + T(n-1)$

Corollary 2. If $n = p^r$, where p is simple then $T(n) = \sum_{k=0}^{r} T(p^k - 1)$.

Corollary 3. If n is compound then $T(n) < 1 + T(n-1) + (2\lfloor\sqrt{n}\rfloor - 2)T\left(\frac{n}{\min\limits_i p_i} - 1\right)$, where p_i is simple divider of n.

Corollary 4. $T(n) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. It statement is proved on induction. For $n = 1$ it is true. We assume now that it is true for all $l \leq k$, i.e. $T(l) \leq \binom{l}{\lfloor \frac{l}{2} \rfloor}$. Then from corollary 2 we receive $T(k+1) < 1 + T(k) + (2\lfloor\sqrt{k+1}\rfloor - 2)T\left(\frac{k+1}{\min\limits_i p_i} - 1\right) \leq \binom{k+1}{\lfloor \frac{k+1}{2} \rfloor}$. Thus, for any n we have $T(n) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Let $T^*(n) = T(n) - T(n-1), n \geq 1$. Evidently, for $n \geq 2$ $T^*(n)$ represent a difference on complexity of two adjacent natural numbers, i.e. on how many complexity of natural number n is more than complexity of $n-1$.

Theorem 3. $T^*(n) = \sum_{p_i} T(\frac{n}{p_i}) - \sum_{p_i, p_j} T(\frac{n}{p_i p_j}) + \cdots + (-1)^{m-1} T(\frac{n}{p_1 p_2 \cdots p_m})$, where $p_1, p_2, \ldots, p_m$ runs all in pairs various simple dividers of number $n \geq 2$

Proof. According to (1) $T(n) = \sum_r T(r-1) = \sum_{r'} T(r'-1) + T(n-1)$ From here $T^*(n) = \sum_{r'} T(r'-1)$ It is obvious, if $n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$, where

$p_1, p_2, \dots, p_m, m \geq 2$, runs all in pairs various simple dividers of number n then $\sum_{i=1}^{m} T(\frac{n}{p_i}) \geq \sum_{r'} T(r'-1)$ as any divider r' of number n distinct from n is a divider at least one of numbers $\frac{n}{p_i}, i = \overline{1,m}$. At numbers $\frac{n}{p_i}, i = \overline{1,m}$, can be the general dividers. There fore, using a method of inclusions and exeption [16] we receive $T^*(n) = \sum_{p_i} T(\frac{n}{p_i}) - \sum_{p_i, p_[j} T(\frac{n}{p_i p_j}) + \cdots + (-1^{m-1} T(\frac{n}{p_1 p_2 \cdots p_m})$.

Corollary. $T^*(n) = 1$ if and only if n is simple.

Proof. Let n is simple. Then $T^* = T(n) - T(n-1) = T(0) + T(n-1) - T(n-1) = T(0) = 1$. Let now $T^* = 1$. Then as $T^* = T(n) - T(n-1) = 1$, $T(n) = 1 + T(n-1)$. But $T(n) = \sum_r T(r-1) = T(0) + T(n-1) + \sum_{r''} T(r''-1)$, where $r'' > 1$ divide n and $r'' \neq n$. From here $\sum_{r''} T(r''-1) = 0$ Hence n is simple.

From definition $T^*(n), n \geq 1$, we have $T(n) = 1 + \sum_{k=1}^{n} T^*(k)$ The first 40 values of numbers $T^*(n), n = 1, 2, \dots, 40$, are that 0,1,1,2,1,4,1,5,3,7,1,13,1,12, 8,16,1,26,1,29,13,28,1,51,6,42,19,56,1,87,1,77,29,79,16,124,1,106,43,145.

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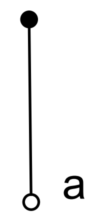
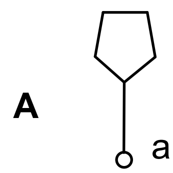


Figure 1:



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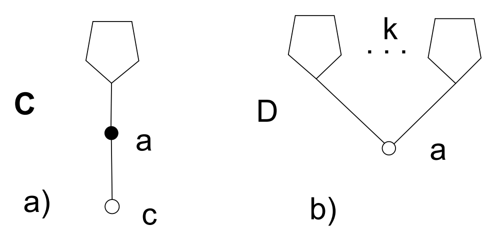

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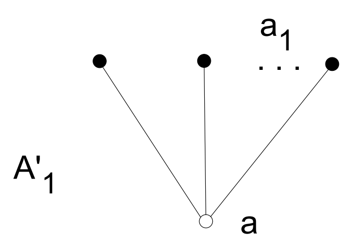

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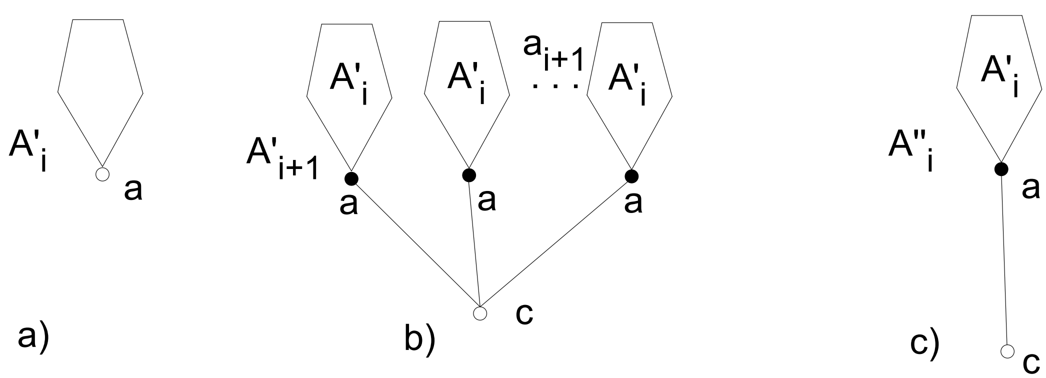

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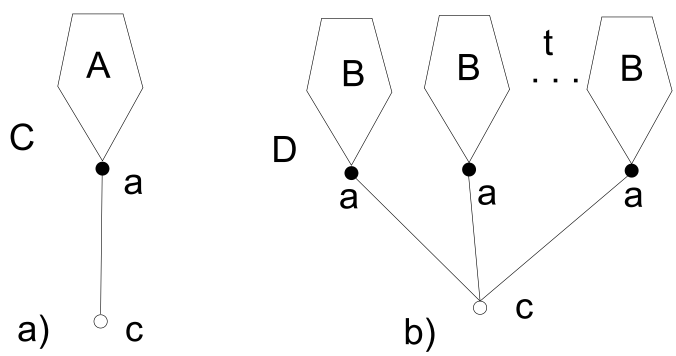

Figure 6: